Exercise Problem Sets 4

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Problem 1. Let (M,d) and (N,ρ) be metric spaces, $A \subseteq M$, and $f_k : A \to N$ be a sequence of functions such that for some function $f : A \to N$, we have that for all $x \in A$, if $\{x_k\}_{k=1}^{\infty} \subseteq A$ and $x_k \to x$ as $k \to \infty$, then

$$\lim_{k\to\infty} f_k(x_k) = f(x)\,.$$

Show that

1. $\{f_k\}_{k=1}^{\infty}$ converges pointwise to f.
2. If $\{f_{k_j}\}_{j=1}^{\infty}$ is a subsequence of $\{f_k\}_{k=1}^{\infty}$, and $\{x_j\}_{j=1}^{\infty} \subseteq A$ is a convergent sequence satisfying that $\lim\limits_{j\to\infty} x_j = x$, then
$$\lim_{j\to\infty} f_{k_j}(x_j) = f(x)\,.$$
3. Show that if in addition A is compact and f is continuous on A, then $\{f_k\}_{k=1}^{\infty}$ converges uniformly f on A.

Proof. 1. Let $x \in A$ be given. Define $\{x_k\}_{k=1}^{\infty}$ by $x_k = x$ for all $k \in \mathbb{N}$. Then $\lim\limits_{k\to\infty} x_k = x$; thus

$$\lim_{k\to\infty} f_k(x) = \lim_{k\to\infty} f_k(x_k) = f(x)$$

which shows that $\{f_k\}_{k=1}^{\infty}$ converges pointwise to f.

2. Let $\{f_{k_j}\}_{j=1}^{\infty}$ be a subsequence of $\{f_k\}_{k=1}^{\infty}$, and $\{x_j\}_{j=1}^{\infty}$ be a convergent sequence with limits x. Define a new sequence $\{y_\ell\}_{\ell=1}^{\infty}$ by

$$y_1, \cdots, y_{k_1} = x_1\,,\ y_{k_1+1}, \cdots, y_{k_2} = x_2\,,\ \cdots\,,\ y_{k_\ell+1}, \cdots, y_{k_{\ell+1}} = x_{\ell+1}\,, \cdots\,;$$

that is, the first k_1 terms of $\{y_\ell\}_{\ell=1}^{\infty}$ is x_1, the next $(k_2 - k_1)$ terms of $\{y_\ell\}_{\ell=1}^{\infty}$ is x_2, and so on. Then $\{y_\ell\}_{\ell=1}^{\infty}$ converges to x;

$$\lim_{\ell\to\infty} f_\ell(y_\ell) = f(x)\,.$$

Since $\{f_{k_j}(x_j)\}_{j=1}^{\infty}$ is a subsequence of $\{f_\ell(y_\ell)\}_{\ell=1}^{\infty}$, $\lim\limits_{j\to\infty} f_{k_j}(x_j) = f(x)$.

3. Suppose the contrary that $\{f_k\}_{k=1}^{\infty}$ does not converge uniformly to f on A. Then there exists $\varepsilon > 0$ such that for each $k > 0$ there exist $n_k \geqslant k$ (W.L.O.G. we can assume that $n_{k+1} > n_k$ for all $k \in \mathbb{N}$) and $x_k \in A$ such that

$$\rho\big(f_{n_k}(x_k), f(x_k)\big) \geqslant \varepsilon\,.$$

By the compactness of A, there exists a convergent subsequence $\{x_{k_j}\}_{j=1}^{\infty}$ of $\{x_k\}_{k=1}^{\infty}$. Suppose that $\lim\limits_{j\to\infty} x_{k_j} = x$. Since

$$\rho\big(f_{n_{k_j}}(x_{k_j}), f(x_{k_j})\big) \geqslant \varepsilon \qquad \forall\, j \in \mathbb{N}\,,$$

by the fact that $\lim_{j\to\infty} f_{n_{k_j}}(x_{k_j}) = f(x)$ and that f is continuous at x, we obtain that

$$\begin{aligned}\rho\big(f(x), f(x)\big) &= \lim_{j\to\infty} \rho\big(f(x_{k_j}), f(x)\big) \geqslant \lim_{j\to\infty} \Big[\rho\big(f_{n_{k_j}}(x_{k_j}), f(x_{k_j})\big) - \rho\big(f_{n_{k_j}}(x_{k_j}), f(x)\big)\Big] \\ &= \lim_{j\to\infty} \rho\big(f_{n_{k_j}}(x_{k_j}), f(x_{k_j})\big) \geqslant \frac{\varepsilon}{2},\end{aligned}$$

a contradiction. □

Remark. Using the inequality

$$\rho\big(f_k(x_k), f(x)\big) \leqslant \rho\big(f(x_k), f(x)\big) + \sup_{x\in A} \rho\big(f_k(x), f(x)\big),$$

we find that if $\{f_k\}_{k=1}^{\infty}$ converges uniformly to a continuous function f, then $\lim_{k\to\infty} f_k(x_k) = f(x)$ as long as $\lim_{k\to\infty} x_k = x$. Together with the conclusion in 3, we conclude that

> Let (M, d), (N, ρ) be metric spaces, $K \subseteq M$ be a compact set, $f_k : K \to N$ be a function for each $k \in \mathbb{N}$, and $f : K \to N$ be continuous. The sequence $\{f_k\}_{k=1}$ converges uniformly to f if and only if $\lim_{k\to\infty} f_k(x_k) = f(x)$ whenever sequence $\{x_k\}_{k=1}^{\infty} \subseteq K$ converges to x.

Problem 2. Let (M, d) be a metric space, $A \subseteq M$, (N, ρ) be a complete metric space, and $f_k : A \to N$ be a sequence of functions (not necessary continuous) which converges uniformly on A. Suppose that $a \in \mathrm{cl}(A)$ and

$$\lim_{x\to a} f_k(x) = L_k$$

exists for all $k \in \mathbb{N}$. Show that $\{L_k\}_{k=1}^{\infty}$ converges, and

$$\lim_{x\to a} \lim_{k\to\infty} f_k(x) = \lim_{k\to\infty} \lim_{x\to a} f_k(x).$$

Proof. Let $\varepsilon > 0$ be given. Since $\{f_k\}_{k=1}^{\infty}$ converges uniformly, there exists $N_1 > 0$ such that

$$\rho\big(f_k(x), f_\ell(x)\big) < \frac{\varepsilon}{3} \quad \textit{whenever} \quad k, \ell \geqslant N_1 \text{ and } x \in A. \tag{$\star$}$$

If $a \in \mathrm{cl}(A)$, then the inequality above implies that

$$\rho(L_k, L_\ell) = \lim_{x\to a} \rho\big(f_k(x), f_\ell(x)\big) \leqslant \frac{\varepsilon}{3} < \varepsilon \quad \textit{whenever} \quad k, \ell \geqslant N_1;$$

thus $\{L_k\}_{k=1}^{\infty}$ is a Cauchy sequence in (N, ρ). Therefore, $\{L_k\}_{k=1}^{\infty}$ converges. Suppose that $\lim_{k\to\infty} L_k = L$ and $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f. There exists $N_2 > 0$ such that $\rho(L_k, L) < \frac{\varepsilon}{3}$ whenever $k \geqslant N_2$. Moreover, passing to the limit as $\ell \to \infty$ in $(\star)$, we obtain that

$$\rho\big(f_k(x), f(x)\big) \leqslant \frac{\varepsilon}{3} \quad \textit{whenever} \quad k \geqslant N_1 \text{ and } x \in A.$$

Let $n = \max\{N_1, N_2\}$. Since $\lim_{x\to a} f_n(x) = L_n$, there exists $\delta > 0$ such that

$$\rho\big(f_n(x), L_n\big) < \frac{\varepsilon}{3} \quad \textit{whenever} \quad x \in B(a, \delta) \cap A\backslash\{a\}.$$

Then if $x \in B(a,\delta) \cap A \backslash \{a\}$,

$$\rho\big(f(x), L\big) \leqslant \rho\big(f(x), f_n(x)\big) + \rho\big(f_n(x), L_n\big) + \rho(L_n, L) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\,.$$

Therefore, $\lim\limits_{x\to a} f(x) = L$ which shows that $\lim\limits_{x\to a}\lim\limits_{k\to\infty} f_k(x) = \lim\limits_{k\to\infty}\lim\limits_{x\to a} f_k(x)$. ▫

Problem 3. Prove the Dini theorem:

> Let K be a compact set, and $f_k : K \to \mathbb{R}$ be continuous for all $k \in \mathbb{N}$ such that $\{f_k\}_{k=1}$ converges pointwise to a continuous function $f : K \to \mathbb{R}$. Suppose that $f_k \leqslant f_{k+1}$ for all $k \in \mathbb{N}$. Then $\{f_k\}_{k=1}^\infty$ converges uniformly to f on K.

Hint: Mimic the proof of showing that $\{c_k\}_{k=1}^\infty$ converges to 0 in Lemma 6.64 in the lecture note.

Proof. Suppose the contrary that there exist $\varepsilon > 0$ such that

$$\limsup_{k\to\infty} \sup_{x\in K} \big|f_k(x) - f(x)\big| \geqslant 2\varepsilon\,.$$

Then there exists $1 \leqslant k_1 < k_2 < \cdots$ such that

$$\max_{x\in K} \big|f_{k_j}(x) - f(x)\big| = \sup_{x\in K} \big|f_{k_j}(x) - f(x)\big| > \varepsilon\,.$$

In other words, for some $\varepsilon > 0$ and strictly increasing sequence $\{k_j\}_{j=1}^\infty \subseteq \mathbb{N}$,

$$F_j \equiv \big\{x \in K \,\big|\, f(x) - f_{k_j}(x) \geqslant \varepsilon\big\} \neq \varnothing \qquad \forall\, j \in \mathbb{N}\,.$$

Note that since $f_k \leqslant f_{k+1}$ for all $k \in \mathbb{N}$, $F_j \supseteq F_{j+1}$ for all $j \in \mathbb{N}$. Moreover, by the continuity of f_k and f, F_j is a closed subset of K; thus F_j is compact. Therefore, the nested set property for compact sets implies that $\bigcap_{j=1}^\infty F_j$ is non-empty. In other words, there exists $x \in K$ such that $f(x) - f_{k_j}(x) \geqslant \varepsilon$ for all $j \in \mathbb{N}$ which contradicts to the fact that $f_k \to f$ p.w. on K. ▫

Problem 4. Let (M,d) and (N,ρ) be metric spaces, $A \subseteq M$, and $f_k : A \to N$ be uniformly continuous functions, and $\{f_k\}_{k=1}^\infty$ converges uniformly to $f : A \to N$ on A. Show that f is uniformly continuous on A.

Proof. Let $\varepsilon > 0$ be given. Since $\{f_k\}_{k=1}^\infty$ converges uniformly to f, there exists $N > 0$ such that

$$\rho\big(f_k(x), f(x)\big) < \frac{\varepsilon}{3} \quad \text{whenever} \quad k \geqslant N \text{ and } x \in A\,.$$

Since f_N is uniformly continuous, there exists $\delta > 0$ such that

$$\rho\big(f_N(x_1), f_N(x_2)\big) < \frac{\varepsilon}{3} \quad \text{whenever} \quad x_1, x_2 \in A \text{ and } d(x_1,x_2) < \delta\,.$$

Therefore, if $x_1, x_2 \in A$ satisfying $d(x_1,x_2) < \delta$, we have

$$\begin{aligned}\rho\big(f(x_1), f(x_2)\big) &\leqslant \rho\big(f(x_1), f_N(x_1)\big) + \rho\big(f_N(x_1), f_N(x_2)\big) + \rho\big(f_N(x_2), f(x_2)\big)\\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\,;\end{aligned}$$

thus f is uniformly continuous on A. ▫

Problem 5. Complete the following.

1. Suppose that $f_k, f, g : [0,\infty) \to \mathbb{R}$ are functions such that

   (a) $\forall\, R > 0$, f_k and g are Riemann integrable on $[0, R]$;

   (b) $|f_k(x)| \leqslant g(x)$ for all $k \in \mathbb{N}$ and $x \in [0,\infty)$;

   (c) $\forall\, R > 0$, $\{f_k\}_{k=1}^{\infty}$ converges to f uniformly on $[0, R]$;

   (d) $\displaystyle\int_0^\infty g(x)dx \equiv \lim_{R\to\infty}\int_0^R g(x)dx < \infty$.

   Show that $\displaystyle\lim_{k\to\infty}\int_0^\infty f_k(x)dx = \int_0^\infty f(x)dx$; that is,

$$\lim_{k\to\infty}\lim_{R\to\infty}\int_0^R f_k(x)dx = \lim_{R\to\infty}\lim_{k\to\infty}\int_0^R f_k(x)dx\,.$$

2. Let $f_k(x)$ be given by $f_k(x) = \begin{cases} 1 & \text{if } k-1 \leqslant x < k\,, \\ 0 & \text{otherwise.}\end{cases}$ Find the (pointwise) limit f of the sequence $\{f_k\}_{k=1}^{\infty}$, and check whether $\displaystyle\lim_{k\to\infty}\int_0^\infty f_k(x)dx = \int_0^\infty f(x)dx$ or not. Briefly explain why one can or cannot apply 1.

3. Let $f_k : [0,\infty) \to \mathbb{R}$ be given by $f_k(x) = \dfrac{x}{1+kx^4}$. Find $\displaystyle\lim_{k\to\infty}\int_0^\infty f_k(x)dx$.

Proof. 1. First we note that since $|f_k(x)| \leqslant g(x)$ for all $x \in \mathbb{R}$, passing to the limit as $k \to \infty$ shows that $|f(x)| \leqslant g(x)$ for all $x \in \mathbb{R}$.

Let $\varepsilon > 0$ be given. Since $\displaystyle\lim_{R\to\infty}\int_0^R g(x)\,dx = \int_0^\infty g(x)\,dx$ exists, there exists $M > 0$ such that

$$\int_R^\infty g(x)\,dx = \Big|\int_0^R g(x)\,dx - \int_0^\infty g(x)\,dx\Big| < \frac{\varepsilon}{3} \qquad \forall\, R \geqslant M\,.$$

Since $\{f_k\}_{k=1}^{\infty}$ converges uniformly on $[0, M]$, $\displaystyle\lim_{k\to\infty}\int_0^M f_k(x)\,dx = \int_0^M f(x)\,dx$; thus there exists $N \geqslant 0$ such that

$$\Big|\int_0^M f_k(x)\,dx - \int_0^M f(x)\,dx\Big| < \frac{\varepsilon}{3} \quad \text{whenever} \quad k \geqslant N\,.$$

Therefore, if $k \geqslant N$, we have

$$\begin{aligned}
&\Big|\int_0^\infty f_k(x)\,dx - \int_0^\infty f(x)\,dx\Big| \\
&\leqslant \Big|\int_0^M f_k(x)\,dx - \int_0^M f(x)\,dx\Big| + \int_M^\infty |f(x)|\,dx + \int_M^\infty |f_k(x)|\,dx \\
&< \frac{\varepsilon}{3} + 2\int_M^\infty g(x)\,dx < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon\,.
\end{aligned}$$

thus $\lim_{k\to\infty}\int_0^\infty f_k(x)\,dx = \int_0^\infty f(x)\,dx$. This implies that

$$\lim_{k\to\infty}\lim_{R\to\infty}\int_0^R f_k(x)dx = \lim_{k\to\infty}\int_0^\infty f_k(x)\,dx = \int_0^\infty f(x)\,dx = \lim_{R\to\infty}\int_0^R f(x)\,dx$$
$$= \lim_{R\to\infty}\lim_{k\to\infty}\int_0^R f_k(x)dx\,.$$

2. If $x \in [0,\infty)$, we have $x \leqslant N$ for some $N \in \mathbb{N}$ (by the Archimedean property); thus for $k \geqslant N$ we have $f_k(x) = 0$. In other words, $\{f_k\}_{k=1}^\infty$ converges pointwise to the zero function. Let f be the zero function. Then
$$\int_0^\infty f_k(x)\,dx = \int_{k-1}^k 1\,dx = 1$$
so that $\lim_{k\to\infty}\int_0^\infty f_k(x)\,dx = 1 \neq 0 = \int_0^\infty f(x)\,dx$. This is because we cannot find an integrable g satisfying that $|f_k(x)| \leqslant g(x)$ for all $x \in [0,\infty)$. In fact, if $|f_k(x)| \leqslant g(x)$ for all $x \in [0,\infty)$, then $g(x) \geqslant 1$ for all $x \in [0,\infty)$.

3. Let $g(x) = \dfrac{x}{1+x^4}$. Then $|f_k(x)| \leqslant g(x)$ for all $x \in [0,\infty)$ and $k \in \mathbb{N}$. Since $g(x) \leqslant x$ for $x \in [0,1]$ and $g(x) \leqslant \frac{1}{x^3}$ for $x \geqslant 1$, we find that
$$\int_0^\infty g(x)\,dx \leqslant \int_0^1 x\,dx + \int_1^\infty \frac{1}{x^3}\,dx = \frac{1}{2} + \frac{1}{2} = 1 < \infty\,.$$
Moreover,
$$f_k'(x) = \frac{1+kx^4-4kx^4}{(1+kx^4)^2} = \frac{1-3kx^4}{(1+kx^4)^2}$$
which implies that for each $R > 0$,
$$\sup_{x\in[0,R]}|f_k(x)| \leqslant |f_k(0)| + |f_k(R)| + \left|\frac{(3k)^{-\frac{1}{4}}}{1+k\cdot\frac{1}{3k}}\right| = \frac{R}{1+kR^4} + \frac{3}{4}\Big(\frac{1}{3k}\Big)^{\frac{1}{4}}\,.$$
Therefore, the Sandwich Lemma implies that $\lim_{k\to\infty}\sup_{x\in[0,R]}|f_k(x)| = 0$ which shows that $\{f_k\}_{k=1}^\infty$ converges uniformly to the zero function on $[0,R]$ for every $R > 0$. By 1,
$$\lim_{k\to\infty}\int_0^\infty f_k(x)\,dx = 0\,. \qquad \square$$