Exercise Problem Sets 2

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Problem 1. For a function $f : [a,b] \to \mathbb{R}$, define the ***total variation*** of f on $[a,b]$ by

$$V_a^b(f) = \sup\Big\{ \sum_{k=1}^n \big|f(x_k) - f(x_{k-1})\big| \,\Big|\, \{a = x_0 < \cdots < x_n = b\} \text{ is a partition of } [a,b] \Big\}.$$

Sometimes $V_a^b(f)$ is written as $\|f\|_{\mathrm{TV}([a,b])}$.

A function $f : [a,b]$ is said to have ***bounded variation*** on $[c,d]$ or be ***of bounded variation*** on $[c,d]$, where $[c,d] \subseteq [a,b]$, if $V_c^d(f) < \infty$. Complete the following.

1. Let $\mathrm{BV}([a,b]) = \big\{ f : [a,b] \to \mathbb{R} \,\big|\, V_a^b(f) < \infty \big\}$, called the space of functions of bounded variation (on $[a,b]$). Show that $\mathrm{BV}([a,b])$ is a vector space.

2. Is V_a^b a norm on $\mathrm{BV}([a,b])$; that is, does $\|\cdot\| : \mathrm{BV}([a,b]) \to \mathbb{R}$ defined by $\|f\| \equiv V_a^b(f)$ satisfy Definition 2.15 in the lecture note?

3. Recall that $\mathscr{C}^1([a,b];\mathbb{R}) \equiv \big\{ f : [a,b] \to \mathbb{R} \,\big|\, f' \text{ is continuous on } [a,b] \big\}$. Show that if $f \in \mathscr{C}^1([a,b];\mathbb{R})$, then f is of bounded variation.

4. Show that if $f \in \mathscr{C}^1([a,b];\mathbb{R})$, then $V_a^b(f) = \displaystyle\int_a^b \big|f'(x)\big|\,dx$.

5. Show that if $V_a^b(f) < \infty$ (f is not necessarily differentiable everywhere), then

$$V_a^b(f) = \sup\Big\{ \int_a^b f(x)\phi'(x)\,dx \,\Big|\, \phi \in \mathscr{C}^1([a,b];\mathbb{R}), |\phi(x)| \leqslant 1 \text{ for all } x \in [a,b], \phi(a) = \phi(b) = 0 \Big\}.$$

Proof. For a partition $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$, define

$$V(f,\mathcal{P}) = \sum_{k=1}^n \big|f(x_k) - f(x_{k-1})\big|.$$

We note that the triangle inequality implies that

$$V(f,\mathcal{P}) \leqslant V(f,\mathcal{P}') \quad \text{whenever} \quad \mathcal{P}' \text{ is a refinement of } \mathcal{P}. \tag{0.1}$$

2. V_a^b is not a norm since any constant function has zero variation. This violates property (b) in Definition 2.15 in the lecture note.

3. Suppose that f is continuously differentiable on $[a,b]$. By the Extreme Value Theorem, $\sup\limits_{x\in[a,b]} \big|f'(x)\big| < \infty$. Therefore, for each partition $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a,b]$, the Mean Value Theorem implies that

$$\begin{aligned} V(f,\mathcal{P}) &= \sum_{k=1}^n \big|f(x_k) - f(x_{k-1})\big| \leqslant \sum_{k=1}^n \sup_{x\in[x_{k-1},x_k]} \big|f'(x)\big|(x_k - x_{k-1}) \\ &\leqslant \sup_{x\in[a,b]} \big|f'(x)\big| \sum_{k=1}^n (x_k - x_{k-1}) = (b-a) \sup_{x\in[a,b]} \big|f'(x)\big| < \infty; \end{aligned}$$

thus $f \in \mathrm{BV}([a,b])$.

4. Suppose that f is continuously differentiable on $[a,b]$. Then f' is continuous on $[a,b]$; thus $|f'|$ is also continuous on $[a,b]$. Therefore, $I = \int_a^b \big|f'(x)\big|\,dx$ exists. Next we show that $V_a^b(f) = I$. Let $\varepsilon > 0$. By the definition of total variation, there exists a partition $\mathcal{P}_1$ of $[a,b]$ such that

$$V_a^b(f) - \frac{\varepsilon}{2} < V(f, \mathcal{P}_1)\,.$$

By the definition of integrals, there exists a partition $\mathcal{P}_2$ of $[a,b]$ such that

$$U\big(|f'|, \mathcal{P}_2\big) < I + \frac{\varepsilon}{2}\,.$$

Let $\mathcal{P}_3 = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be the common refinement of $\mathcal{P}_1$ and $\mathcal{P}_2$. By the Mean Value Theorem, for each $1 \leqslant k \leqslant n$ there exists $\xi_k \in (x_{k-1}, x_k)$ such that

$$f(x_k) - f(x_{k-1)} = f'(\xi_k)(x_k - x_{k-1})\,;$$

thus (0.1) implies that

$$\begin{aligned} V_a^b(f) - \frac{\varepsilon}{2} &< V(f, \mathcal{P}) = \sum_{k=1}^{n} \big|f(x_k) - f(x_{k-1})\big| = \sum_{k=1}^{n} \big|f'(\xi_k)\big|(x_k - x_{k-1}) \\ &\leqslant \sum_{k=1}^{n} \sup_{x \in [x_{k-1}, x_k]} \big|f'(x)\big|(x_k - x_{k-1}) = U\big(|f'|, \mathcal{P}\big) \leqslant U\big(|f'|, \mathcal{P}_2\big) < I + \frac{\varepsilon}{2}\,. \end{aligned}$$

Therefore,

$$V_a^b(f) < I + \varepsilon\,. \tag{0.2}$$

On the other hand, by the uniform continuity, there exists $\delta > 0$ such that

$$\Big|\big|f'(x)\big| - \big|f'(y)\big|\Big| < \frac{\varepsilon}{2(b-a)} \qquad \text{whenever} \qquad |x-y| < \delta \text{ and } x, y \in [a,b]\,.$$

Let $\mathcal{P}_4 = \{a = y_0 < y_1 < \cdots < y_m = b\}$ be a refinement of $\mathcal{P}_2$ such that $\|\mathcal{P}_2\| < \delta$. The Mean Value Theorem implies that for each $1 \leqslant k \leqslant m$, there exists $\eta_k \in (y_{k-1}, y_k)$ such that

$$f(y_k) - f(y_{k-1}) = f'(\eta_k)(y_k - y_{k-1})\,.$$

Then for each $1 \leqslant k \leqslant m$,

$$\sup_{y \in [y_{k-1}, y_k]} \big|f'(y)\big| \leqslant \big|f'(\eta_k)\big| + \frac{\varepsilon}{2(b-a)}\,.$$

The inequality above further implies that

$$\begin{aligned} I \leqslant U\big(|f'|, \mathcal{P}_4\big) &= \sum_{k=1}^{m} \sup_{y \in [y_{k-1}, y_k]} \big|f'(y)\big|(y_k - y_{k-1}) \\ &\leqslant \sum_{k=1}^{m} \Big(\big|f'(\eta_k)\big| + \frac{\varepsilon}{2(b-a)}\Big)(y_k - y_{k-1}) \leqslant \sum_{k=1}^{n} \big|f(y_k) - f(y_{k-1})\big| + \frac{\varepsilon}{2} \\ &< V_a^b(f) + \varepsilon\,. \end{aligned}$$

Therefore, together with (0.2), we conclude that

$$\big|V_a^b(f) - I\big| < \varepsilon\,.$$

Since $\varepsilon > 0$ is given arbitrary, we find that $V_a^b(f) = I$. □

Problem 2. Complete the following.

1. Show that if A is a set of volume zero, then A has measure zero. Is it true that if A has measure zero, then A also has volume zero?

2. Let $a, b \in \mathbb{R}$ and $a < b$. Show that the interval $[a, b]$ does not have measure zero (in $\mathbb{R}$).

3. Let $A \subseteq [a, b]$ be a set of measure zero (in $\mathbb{R}$). Show that $[a, b]\backslash A$ does not have measure zero (in $\mathbb{R}$).

4. Show that the Cantor set (defined in Problem 9 of Exercise 7 in the fall semester) has volume zero.

Proof. 1. No. The set $\mathbb{Q} \cap [0, 1]$ has measure zero; however, it does not have volume since Dirichlet function is not Riemann integrable on $[0, 1]$.

2. This is a direct consequence of Corollary 6.25 in the lecture note.

3. Suppose the contrary that $[a, b]\backslash A$ has measure zero. By the fact that countable union of measure zero sets has measure zero (Theorem 6.26 in the lecture note), we conclude that

$$[a, b] = A \cup ([a, b]\backslash A)$$

has measure zero, a contradiction to Corollary 6.25 in the lecture note.

4. Let E_n be the set defined in Problem 9 of Exercise 7 in the fall semester. Then E_n is the union of finite intervals whose volumes sum to $\frac{2^n}{3^n}$. Therefore, for each $\varepsilon > 0$ there exist finite rectangles $S_1, S_2, \cdots, S_N$ whose disjoint union is E_N and $\sum_{k=1}^{N} \nu(S_k) = \frac{2^N}{3^N} < \varepsilon$. This shows that the Cantor set has volume zero.

□

Problem 3. Let $A = \bigcup_{k=1}^{\infty} B\big(\frac{1}{k}, \frac{1}{2^k}\big) = \bigcup_{k=1}^{\infty} \big(\frac{1}{k} - \frac{1}{2^k}, \frac{1}{k} + \frac{1}{2^k}\big)$ be a subset of $\mathbb{R}$. Does A have volume?

Proof. We first show that $\bar{A} = \{0\} \cup \bigcup_{k=N+1}^{\infty} \big[\frac{1}{k} - \frac{1}{2^k}, \frac{1}{k} + \frac{1}{2^k}\big]$.

1. Clearly $\bigcup_{k=N+1}^{\infty} \big[\frac{1}{k} - \frac{1}{2^k}, \frac{1}{k} + \frac{1}{2^k}\big] \subseteq \bar{A}$. In fact, we have $\bigcup_{\alpha \in I} \overline{B_\alpha} \subseteq \mathrm{cl}\Big(\bigcup_{\alpha \in I} B_\alpha\Big)$: if $x \in \bigcup_{\alpha \in I} \overline{B_\alpha}$, then $x \in \overline{B_\alpha}$ for some $\alpha \in I$ which implies that there exists $\alpha \in I$ and $\{x_\ell\}_{\ell=1}^{\infty} \in B_\alpha \subseteq \bigcup_{\alpha \in I} B_\alpha$ such that $x_\ell \to x$ as $\ell \to \infty$. Therefore, $x \in \mathrm{cl}\Big(\bigcup_{\alpha \in I} B_\alpha\Big)$.

2. Suppose that $x \in \bar{A}$. Then there exists $\{x_\ell\}_{\ell=1}^{\infty} \subseteq A$ such that $x_\ell \to x$ as $\ell \to \infty$. Since every element in A is positive, we conclude that $x \geqslant 0$.

(a) **the case** $x = 0$: Since $\{x_\ell\}_{\ell=1}^\infty$ defined by $x_\ell = \dfrac{1}{\ell}$ is a sequence in A, we conclude that $0 \in \bar{A}$ since $\lim\limits_{\ell\to\infty} x_\ell = 0$.

(b) **the case** $x > 0$: By the definition of the limit of sequences, there exists $N > 0$ such that $x_\ell \in \big(\frac{x}{2}, \frac{3x}{2}\big)$ for all $\ell \geqslant N$. Since $\lim\limits_{k\to\infty} \frac{1}{k} + \frac{1}{2^k} = 0$, there exists $M > 0$ such that $\frac{1}{k} + \frac{1}{2^k} < \frac{x}{2}$ for all $k \geqslant M$. Therefore,

$$A \cap \big(\frac{x}{2}, \frac{3x}{2}\big) = \bigcup_{k=1}^{M-1} \big(\frac{1}{k} - \frac{1}{2^k}, \frac{1}{k} + \frac{1}{2^k}\big)\,;$$

thus there exists $1 \leqslant j \leqslant M-1$ such that

$$\#\big\{\ell \in \mathbb{N} \,\big|\, x_\ell \in \big(\frac{1}{j} - \frac{1}{2^j}, \frac{1}{j} + \frac{1}{2^j}\big) = \infty\,.$$

Let $\big\{x_{\ell_k}\big\}_{k=1}^\infty$ be a subsequence of $\{x_\ell\}_{\ell=1}^\infty$ satisfying that $\big\{x_{\ell_k}\big\}_{k=1}^\infty \subseteq \big(\frac{1}{j} - \frac{1}{2^j}, \frac{1}{j} + \frac{1}{2^j}\big)$, we conclude that $x \in \big[\frac{1}{j} - \frac{1}{2^j}, \frac{1}{j} + \frac{1}{2^j}\big]$ since $\lim\limits_{k\to\infty} x_{\ell_k} = x$.

Having shown that $\bar{A} = \{0\} \cup \bigcup\limits_{k=N+1}^{\infty} \big[\frac{1}{k} - \frac{1}{2^k}, \frac{1}{k} + \frac{1}{2^k}\big]$, we conclude that

$$\partial A = \bar{A} \backslash \mathring{A} = \bar{A} \backslash A \subseteq \{0\} \cup \Big\{\frac{1}{k} - \frac{1}{2^k} \,\Big|\, k \in \mathbb{N}\Big\} \cup \Big\{\frac{1}{k} + \frac{1}{2^k} \,\Big|\, k \in \mathbb{N}\Big\}\,;$$

thus ∂A has measure zero. This implies that A has volume. □

Problem 4. Prove the following statements.

1. The function $f(x) = \sin\dfrac{1}{x}$ is Riemann integrable on $(0,1)$.
2. Let $f : (0,1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \dfrac{1}{p} & \text{if } x = \dfrac{q}{p} \in \mathbb{Q},\ (p,q) = 1\,, \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

   Then f is Riemann integrable on $(0,1]$. Find $\displaystyle\int_{(0,1]} f(x)dx$ as well.
3. Let $A \subseteq \mathbb{R}^n$ be a bounded set, and $f : A \to \mathbb{R}$ is Riemann integrable. Then f^k (f 的 k 次方) is integrable for all $k \in \mathbb{N}$.

Proof. 1. Note that $(0,1)$ has volume, f is bounded on $(0,1)$ and f is continuous on $(0,1)$. Therefore, the Lebesgue Theorem (or its corollary) implies that f is Riemann integrable on $(0,1)$.

2. In Calculus we have shown that f is continuous on $\mathbb{Q}^\complement \cap (0,1]$ so that the collection of discontinuities of $\overline{f}^{(0,1]}$ is $\mathbb{Q} \cap (0,1]$. Since $\mathbb{Q} \cap (0,1]$ is countable, we find that the collection of discontinuities of $\overline{f}^{(0,1]}$ has measure zero. Therefore, f is Riemann integrable on $(0,1]$.

Let $\mathcal{P}$ be a partition of $(0,1]$. Then $L(f,\mathcal{P})=0$ since

$$\inf_{x\in\Delta} \overline{f}^{(0,1]}(x)=0 \qquad \forall\,\Delta\in\mathcal{P}\,.$$

Therefore, $\underline{\int_A} f(x)\,dx = 0$; thus the fact that f is Riemann integrable on $(0,1]$ shows that $\int_{(0,1]} f(x)\,dx = 0$.

3. First we note that the fact that f is Riemann integrable on A implies that f is bounded on A. Therefore, f^k is bounded on A. Moreover, the Lebesgue Theorem implies that the collection D of discontinuities of $\overline{f}^A$ has measure zero. Since $\overline{f^k}^A = \big(\overline{f}^A\big)^k$, we find that the collection of discontinuities of $\overline{f^k}^A$ is exactly D; thus has measure zero. The Lebesgue Theorem then implies that f^k is Riemann integrable on A. □

Problem 5. Suppose that $f:[a,b]\to\mathbb{R}$ is Riemann integrable on $[a,b]$, and the set $\big\{x\in[a,b]\,\big|\,f(x)\neq 0\big\}$ has measure zero. Show that $\int_a^b f(x)\,dx = 0$.

Proof. First we note that for each $[c,d]\subseteq[a,b]$, then there exists $x\in[c,d]$ such that $f(x)=0$ for otherwise $f(x)\neq 0$ for all $x\in[c,d]$ so that

$$[c,d]\subseteq\big\{x\in[a,b]\,\big|\,f(x)\neq 0\big\}$$

and this implies that $[c,d]$ is a set of measure zero, a contradiction to Corollary 6.25 in the lecture note. Therefore, $L(|f|,\mathcal{P})\leqslant 0$ and $U(f,\mathcal{P})\geqslant 0$ for all partitions $\mathcal{P}$ of $[a,b]$ which shows that $\underline{\int_a^b} f(x)\,dx \leqslant 0$ and $\overline{\int_a^b} f(x)\,dx \geqslant 0$. Since f is Riemann integrable on $[a,b]$, we conclude that $\int_a^b f(x)\,dx = 0$. □

Problem 6. Find an example of the inequality

$$\underline{\int_A} f(x)\,dx + \underline{\int_A} g(x)\,dx < \underline{\int_A} (f+g)(x)\,dx < \overline{\int_A} (f+g)(x)\,dx < \overline{\int_A} f(x)\,dx + \overline{\int_A} g(x)\,dx\,.$$

Solution. Let $f,g:[0,2]\to\mathbb{R}$ be defined by

$$f(x)=\begin{cases} 1 & \text{if } x\in\mathbb{Q}\cap[0,2]\,,\\ 0 & \text{otherwise}\,,\end{cases} \qquad\text{and}\qquad g(x)=\begin{cases} 1 & \text{if } x\in\mathbb{Q}^\complement\cap[0,1]\,,\\ 0 & \text{otherwise}\,.\end{cases}\,.$$

Then for $A=[0,2]$,

$$\underline{\int_A} f(x)\,dx = \underline{\int_A} g(x)\,dx = 0\,,\qquad \overline{\int_A} f(x)\,dx = 2 \quad\text{and}\quad \overline{\int_A} g(x)\,dx = 1\,.$$

Moreover,

$$(f+g)(x)=\begin{cases} 1 & \text{if } x\in[0,1]\cup(\mathbb{Q}\cap[1,2])\,,\\ 0 & \text{otherwise.}\end{cases}$$

so that

$$\underline{\int_A} (f+g)(x)\,dx = 1 \qquad\text{and}\qquad \overline{\int_A} (f+g)(x)\,dx = 2\,.$$

Therefore, f and g satisfy the desired inequality.

Another example is given as follows: let $f, g : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1]\,, \\ 0 & \text{if } x \in \mathbb{Q}^{\complement} \cap [0,1]\,, \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \cap [0,1]\,, \\ 2 & \text{if } x \in \mathbb{Q}^{\complement} \cap [0,1]\,, \end{cases}$$

Then

$$(f+g)(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1]\,, \\ 2 & \text{if } x \in \mathbb{Q}^{\complement} \cap [0,1]\,, \end{cases}$$

so that we have $\underline{\int}_{[0,1]} f(x)\,dx = \underline{\int}_{[0,1]} g(x)\,dx = 0$, $\overline{\int}_{[0,1]} f(x)\,dx = \underline{\int}_{[0,1]} (f+g)(x)\,dx = 1$, and $\overline{\int}_{[0,1]} g(x)\,dx = \overline{\int}_{[0,1]} (f+g)(x)\,dx = 2$. □

Problem 7. Let $A \subseteq \mathbb{R}^n$ be a bounded set, and $f : A \to \mathbb{R}$ be a bounded function. Show that if f is Riemann integrable on A, then $|f|$ is also Riemann integrable on A.

Proof. **Method 1**: Since f is Riemnn integrable on A, the Lebesgue Theorem implies that the collection of discontinuities of $\overline{f}^A$ has measure zero. Note that if $\overline{f}^A$ is continuous at $a \in A$, then $\overline{|f|}^A$ is also continuous at a since $\overline{|f|}^A = \big|\overline{f}^A\big|$. Therefore, the collection of discontinuities of $\overline{|f|}^A$ is a subset of a measure zero set, the collection of discontinuities of $\overline{f}^A$; thus the collection of discontinuities of $\overline{|f|}^A$ has measure zero. The Lebesgue Theorem then shows that $|f|$ is Riemann integrable on A.

Method 2: Let $\varepsilon > 0$ be given. Since f is Riemann integrable on A, by Riemann's condition there exists a partition $\mathcal{P}$ of A such that

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon\,.$$

Note that for each $\Delta \in P$,

$$\sup_{x \in \Delta} \big|\overline{f}^A(x)\big| - \inf_{x \in \Delta} \big|\overline{f}^A(x)\big| \leqslant \sup_{x \in \Delta} \overline{f}^A(x) - \inf_{x \in \Delta} \overline{f}^A(x)\,;$$

thus

$$\begin{aligned} U(|f|, \mathcal{P}) - L(|f|, \mathcal{P}) &= \sum_{\Delta \in \mathcal{P}} \Big(\sup_{x \in \Delta} \big|\overline{f}^A(x)\big| - \inf_{x \in \Delta} \big|\overline{f}^A(x)\big| \Big) \nu(\Delta) \\ &\leqslant \sum_{\Delta \in \mathcal{P}} \Big(\sup_{x \in \Delta} \overline{f}^A(x) - \inf_{x \in \Delta} \overline{f}^A(x) \Big) \nu(\Delta) = U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon\,. \end{aligned}$$

By Riemann's condition, we conclude that $|f|$ is Riemann integrable on A. □