Exercise Problem Sets 11

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Problem 1. Complete the following.

1. Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) \quad \text{and} \quad \lim_{y\to 0}\lim_{x\to 0} f(x,y)$$

exist but are not equal.

2. Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that the two limits above exist and are equal but f is not continuous.

3. Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ that is continuous on every line through the origin but is not continuous.

Problem 2. Complete the following.

1. Show that the projection map $f : \begin{array}{rcl} \mathbb{R}^2 & \to & \mathbb{R} \\ (x,y) & \mapsto & x \end{array}$ is continuous.

2. Show that if $U \subseteq \mathbb{R}$ is open, then $A = \big\{(x,y) \in \mathbb{R}^2 \,\big|\, x \in U\big\}$ is open.

3. Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ and an open set $U \subseteq \mathbb{R}$ such that $f(U)$ is not open.

Problem 3. Show that $f : A \to \mathbb{R}^m$, where $A \subseteq \mathbb{R}^n$, is continuous if and only if for every $B \subseteq A$,

$$f(\mathrm{cl}(B) \cap A) \subseteq \mathrm{cl}(f(B)) .$$

Proof. "$\Rightarrow$" Let $B \subseteq A$ and $y \in f(\mathrm{cl}(B) \cap A)$. Then there exists $x \in \mathrm{cl}(B) \cap A$ such that $y = f(x)$. By the property of $\bar{B}$, there exists a sequence $\{x_n\}_{n=1}^{\infty} \subseteq B$ such that $\lim\limits_{n\to\infty} x_n = x$. Since $B \subseteq A$, $\{x_n\}_{n=1}^{\infty} \subseteq A$; thus the continuity of f (at x) implies that

$$\lim_{n\to\infty} f(x_n) = f(x) .$$

On the other hand, $\{f(x_n)\}_{n=1}^{\infty}$ is a sequence in $f(B)$, so the limit $f(x)$ must belong to $\mathrm{cl}(f(B))$. Therefore, $y = f(x) \in \mathrm{cl}(f(B))$ which shows the inclusion $f((\mathrm{cl}(B) \cap A) \subseteq \mathrm{cl}(f(B))$.

"$\Leftarrow$" Suppose the contrary that there exists a sequence $\{x_n\}_{n=1}^{\infty} \subseteq A$ with limit $x \in A \cap A'$ such that $\lim\limits_{n\to\infty} f(x_n) \neq f(x)$. Then there exists $\varepsilon > 0$ such that for all $N > 0$ there exists $n \geqslant N$ such that $\|f(x_n) - f(x)\| \geqslant \varepsilon$. Let $n_1 \in \mathbb{N}$ be such that $\big\|f(x_{n_1} - f(x)\big\| \geqslant \varepsilon$. Let $n_2 > n_1$ such that $\big\|f(x_{n_2}) - f(x)\big\| \geqslant \varepsilon$. Continuing this process, we obtain an increasing sequence $\{n_j\}_{j=1}^{\infty}$ such that

$$\big\|f(x_{n_j}) - f(x)\big\| \geqslant \varepsilon \qquad \forall\, j \in \mathbb{N} . \tag{$\star$}$$

Let $B = \{x_{n_j}\}$. Then $x \in \bar{B}$ since $\lim\limits_{n\to\infty} x_n = x$ $\big($so that $\lim\limits_{j\to\infty} x_{n_j} = x\big)$. On the other hand, $(\star)$ implies that $f(x) \notin \mathrm{cl}(f(B))$ since $B(f(x), \varepsilon) \cap f(B) = \varnothing$. Therefore,

$$f(\mathrm{cl}(B) \cap A) \nsubseteq \mathrm{cl}(f(B)),$$

a contradiction. □

Problem 4. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ satisfy $T(x + y) = T(x) + T(y)$ for all $x, y \in \mathbb{R}^n$.

1. Show that $T(rx) = rT(x)$ for all $r \in \mathbb{Q}$ and $x \in \mathbb{R}^n$.
2. Suppose that T is continuous on $\mathbb{R}^n$. Show that T is linear; that is, $T(cx + y) = cT(x) + T(y)$ for all $c \in \mathbb{R}$, $x, y \in \mathbb{R}^n$.
3. Suppose that T is continuous at some point x_0 in $\mathbb{R}^n$. Show that T is continuous on $\mathbb{R}^n$.
4. Suppose that T is bounded on some open subset of $\mathbb{R}^n$. Show that T is continuous on $\mathbb{R}^n$.
5. Suppose that T is bounded from above (or below) on some open subset of $\mathbb{R}^n$. Show that T is continuous on $\mathbb{R}^n$.
6. Construct a $T : \mathbb{R} \to \mathbb{R}$ which is discontinuous at every point of $\mathbb{R}$, but $T(x+y) = T(x)+T(y)$ for all $x, y \in \mathbb{R}$.

Proof. 1. By induction, $T(kx) = kT(x)$ for all $k \in \mathbb{N}$. Moreover, $T(0) = T(0 + 0) = T(0) + T(0)$ which implies that $T(0) = 0$; thus $T(0x) = 0T(x)$ and if $k \in \mathbb{N}$,

$$-kT(x) = -kT(x)+T(0) = -kT(x)+T(kx+(-kx)) = -kT(x)+T(kx)+T(-kx) = T(-kx).$$

Therefore, $T(kx) = kT(x)$ for all $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$. Let $r = \dfrac{q}{p}$ for some $p, q \in \mathbb{Z}$. Then for $x \in \mathbb{R}^n$,

$$pT(rx) = T(prx) = T(qx) = qT(x)$$

which implies that $T(rx) = rT(x)$ for all $r \in \mathbb{Q}$ and $x \in \mathbb{R}^n$.

2. Let $x, y \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then there exists $\{c_k\}_{k=1}^{\infty} \subseteq \mathbb{Q}$ such that $\lim\limits_{k\to\infty} c_k = c$. This further implies that $c_k x \to cx$ as $k \to \infty$ since

$$\lim_{k\to\infty} \|c_n x - cx\| = \lim_{k\to\infty} \|(c_k - c)x\| = \|x\| \lim_{k\to\infty} |c_k - c| = 0$$

Therefore, by the continuity of T,

$$T(cx + y) = T(cx) + T(y) = \lim_{k\to\infty} T(c_k x) + T(y) = \lim_{k\to\infty} c_k T(x) + T(y) = cT(x) + T(y).$$

3. Let $a \in \mathbb{R}^n$ and $\varepsilon > 0$ be given. By the continuity of T at x_0, there exists $\delta > 0$ such that

$$\|T(x - x_0)\| = \|T(x) - T(x_0)\| < \varepsilon \quad \text{whenever} \quad \|x - x_0\| < \delta.$$

The statement above implies that if $\|x\| < \delta$, then $\|T(x)\| < \varepsilon$. Therefore,

$$\|T(x) - T(a)\| = \|T(x - a)\| < \varepsilon \quad \text{whenever} \quad \|x - a\| < \delta$$

which shows that T is continuous at a.

4. Suppose that T is bounded on an open set U so that $T(U) \subseteq B(0, M)$. Let $x_0 \in U$. Then there exists $r > 0$ such that $B(x_0, r) \subseteq U$. Therefore, if $x \in B(0, r)$, then $x + x_0 \in B(x_0, r)$ so that

$$\|T(x)\| \leqslant \|T(x + x_0)\| + \|T(x_0)\| \leqslant M + \|T(x_0)\| \equiv R\,;$$

thus we establish that there exists r and R such that

$$\|T(x)\| \leqslant R \quad \text{whenever} \quad \|x\| < r\,.$$

Let $\varepsilon > 0$ be given. Choose $c \in \mathbb{Q}$ so that $0 < c < \dfrac{\varepsilon}{R}$. For such a fixed $c \in \mathbb{Q}$, choose $0 < \delta < cr$. If $\|x\| < \delta$, then $\left\|\dfrac{x}{c}\right\| < \dfrac{\delta}{c} < r$; thus if $\|x\| < \delta$, we have $\left\|T\big(\dfrac{x}{c}\big)\right\| \leqslant R$ so that

$$\|T(x)\| = \left\|T\Big(c\frac{x}{c}\Big)\right\| = \left\|cT\Big(\frac{x}{c}\Big)\right\| = c\left\|T\Big(\frac{x}{c}\Big)\right\| \leqslant cR < \varepsilon\,.$$

Therefore, T is continuous at 0. By 3, T is continuous on $\mathbb{R}^n$.

5. Suppose that $Tx \leqslant M$ (so that in this case $m = 1$) for all $x \in U$, where U is an open set in $\mathbb{R}^n$. Let $x_0 \in U$. Then there exists $r > 0$ such that $B(x_0, r) \subseteq U$; thus if $x \in B(0, r)$,

$$Tx = T(x + x_0) - T(x_0) \leqslant M - T(x_0) \equiv R\,.$$

Therefore, we establish that there exist r and R such that

$$T(x) \leqslant R \quad \text{whenever} \quad x \in B(0, r)\,.$$

For $x \in B(0, r)$, we must have $-x \in B(0, r)$; thus

$$-T(x) = T(-x) \leqslant R\,;$$

thus $-R \leqslant T(x)$ whenever $x \in B(0, r)$. Therefore, $|T(x)| \leqslant R$ whenever $\|x\| < r$. By 4, T is continuous on $\mathbb{R}^n$. □

Problem 5. Let (M, d) be a metric space, $A \subseteq M$, and $f : A \to \mathbb{R}$. For $a \in A'$, define

$$\liminf_{x \to a} f(x) = \lim_{r \to 0^+} \inf\big\{f(x) \,\big|\, x \in B(a, r) \cap A \backslash \{a\}\big\}\,,$$
$$\limsup_{x \to a} f(x) = \lim_{r \to 0^+} \sup\big\{f(x) \,\big|\, x \in B(a, r) \cap A \backslash \{a\}\big\}\,.$$

Complete the following.

1. Show that both $\liminf\limits_{x \to a} f(x)$ and $\limsup\limits_{x \to a} f(x)$ exist (which may be $\pm\infty$), and

$$\liminf_{x \to a} f(x) \leqslant \limsup_{x \to a} f(x)\,.$$

Furthermore, there exist sequences $\{x_n\}_{n=1}^\infty$, $\{y_n\}_{n=1}^\infty \subseteq A \backslash \{a\}$ such that $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ both converge to a, and

$$\lim_{n \to \infty} f(x_n) = \liminf_{x \to a} f(x) \qquad \text{and} \qquad \lim_{n \to \infty} f(y_n) = \limsup_{x \to a} f(x)\,.$$

2. Let $\{x_n\}_{n=1}^{\infty} \subseteq A\backslash\{a\}$ be a convergent sequence with limit a. Show that

$$\liminf_{x\to a} f(x) \leqslant \liminf_{n\to\infty} f(x_n) \leqslant \limsup_{n\to\infty} f(y_n) \leqslant \limsup_{x\to a} f(x)\,.$$

3. Show that $\lim\limits_{x\to a} f(x) = \ell$ if and only if

$$\liminf_{x\to a} f(x) = \limsup_{x\to a} f(x) = \ell\,.$$

4. Show that $\liminf\limits_{x\to a} f(x) = \ell \in \mathbb{R}$ if and only if the following two conditions hold:

   (a) for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\ell - \varepsilon < f(x)$ for all $x \in B(a,\delta) \cap A\backslash\{a\}$;

   (b) for all $\varepsilon > 0$ and $\delta > 0$, there exists $x \in B(a,\delta) \cap A\backslash\{a\}$ such that $f(x) < \ell + \varepsilon$.

   Formulate a similar criterion for limsup and for the case that $\ell = \pm\infty$.

5. Compute the liminf and limsup of the following functions at any point of $\mathbb{R}$.

   (a) $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}^{\complement}\,, \\ \dfrac{1}{p} & \text{if } x = \dfrac{q}{p} \text{ with } (p,q) = 1,\ q > 0,\ p \neq 0\,. \end{cases}$

   (b) $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}\,, \\ -x & \text{if } x \in \mathbb{Q}^{\complement}\,. \end{cases}$

Proof. For $r > 0$, define $m, M : A' \to R^*$ by

$$m(r) = \inf\big\{f(x)\,\big|\,x \in B(a,r) \cap A\backslash\{a\}\big\} \quad \text{and} \quad M(r) = \sup\big\{f(x)\,\big|\,x \in B(a,r) \cap A\backslash\{a\}\big\}\,.$$

We remark that it is possible that $m(r) = -\infty$ or $M(r) = \infty$. Note that m is decreasing and M is increasing in $(0,\infty)$.

1. By the monotonicity of m and M, $\lim\limits_{r\to 0^+} m(r)$ and $\lim\limits_{r\to 0^+} M(r)$ "exist" (which may be $\pm\infty$). Moreover, $m(r) \leqslant M(r)$ for all $r > 0$; thus $\lim\limits_{r\to 0^+} m(r) \leqslant \lim\limits_{r\to 0^+} M(r)$ so we conclude that

$$\liminf_{x\to a} f(x) = \lim_{r\to 0^+} m(r) \leqslant \lim_{r\to 0^+} M(r) = \limsup_{x\to a} f(x)\,.$$

   Since $\liminf\limits_{x\to a} f(x) = -\limsup\limits_{x\to a}(-f)(x)$, it suffices to consider the case of the limit superior.

   (a) If $\limsup\limits_{x\to a} f(x) = \infty$, then for each $n \in \mathbb{N}$ there exists $0 < \delta_n < \dfrac{1}{n}$ such that

$$M(r) \geqslant n \quad \textit{whenever} \quad 0 < r < \delta_n\,.$$

   By the definition of the supremum, for each $n \in \mathbb{N}$ there exists $x_n \in B\big(a, \frac{\delta_n}{2}\big) \cap A\backslash\{a\}$ such that $f(x_n) \geqslant n - 1$.

(b) If $\limsup\limits_{x\to a} f(x) = L$, then for each $n \in \mathbb{N}$ there exists $0 < \delta_n < \dfrac{1}{n}$ such that

$$\big|M(r) - L\big| < \frac{1}{n} \quad \textit{whenever} \quad 0 < r < \delta_n\,.$$

By the definition of the supremum, for each $n \in \mathbb{N}$ there exists $x_n \in B\big(a, \dfrac{\delta_n}{2}\big) \cap A\backslash\{a\}$ such that

$$L - \frac{1}{n} < f(x_n) < L + \frac{1}{n}\,.$$

Since $\delta_n \to 0$ as $n \to \infty$, we find that $\{x_n\}_{n=1}^{\infty} \subseteq A\backslash\{a\}$ converges to a and $\lim\limits_{n\to\infty} f(x_n) = \limsup\limits_{x\to a} f(x)$.

2. It suffices to show the case of the limit inferior. Let $\{x_n\}_{n=1}^{\infty} \subseteq A\backslash\{a\}$ and $x_n \to a$ as $n \to \infty$. For every $k \in \mathbb{N}$, there exists $N_k > 0$ such that $0 < d(x_n, a) < \dfrac{1}{k}$ whenever $n \geqslant N_k$. W.L.O.G., we can assume that $N_k \geqslant k$ and $N_{k+1} > N_k$ for all $k \in \mathbb{N}$. By the definition of infimum,

$$m\Big(\frac{1}{k}\Big) \leqslant f(x_n) \quad \textit{whenever} \quad n \geqslant N_k$$

which further implies that

$$m\Big(\frac{1}{k}\Big) \leqslant \inf_{n\geqslant N_k} f(x_n)\,.$$

Note that $\lim\limits_{r\to 0^+} m(r) = \lim\limits_{k\to\infty} m\Big(\dfrac{1}{k}\Big)$ and $\lim\limits_{k\to\infty} \inf\limits_{n\geqslant N_k} f(x_k) = \lim\limits_{k\to\infty} \inf\limits_{n\geqslant k} f(x_k)$, we conclude that

$$\liminf_{x\to a} f(x) = \lim_{r\to 0^+} m(r) = \lim_{k\to\infty} m\Big(\frac{1}{k}\Big) \leqslant \lim_{k\to\infty} \inf_{n\geqslant N_k} f(x_n) = \lim_{k\to\infty} \inf_{n\geqslant k} f(x_n) = \liminf_{n\to\infty} f(x_n)\,.$$

3. $(\Rightarrow)$ Let $\varepsilon > 0$ be given. There exists $\delta > 0$ such that

$$|f(x) - \ell| < \varepsilon \qquad \text{whenever} \quad x \in B(a, \delta) \cap A\backslash\{a\}\,.$$

Therefore,

$$\ell - \varepsilon < f(x) < \ell + \varepsilon \qquad \text{whenever} \quad x \in B(a, \delta) \cap A\backslash\{a\}$$

which implies that

$$\ell - \varepsilon \leqslant m(\delta) \leqslant M(\delta) \leqslant \ell + \varepsilon\,.$$

By the monotonicity of m and M, the inequality above implies that

$$\ell - \varepsilon \leqslant m(\delta) \leqslant m(r) \leqslant M(r) \leqslant M(\delta) \leqslant \ell + \varepsilon \quad \forall\, 0 < r < \delta\,.$$

Passing to the limit as $r \to 0^+$, we find that

$$\ell - \varepsilon \leqslant \liminf_{x\to a} f(x) \leqslant \limsup_{x\to a} f(x) \leqslant \ell + \varepsilon\,.$$

Since $\varepsilon > 0$ is chosen arbitrary, we conclude that $\liminf\limits_{x\to a} f(x) = \limsup\limits_{x\to a} f(x) = \ell$.

($\Leftarrow$) Let $\{x_n\}_{n=1}^{\infty} \subseteq A\backslash\{a\}$ be a sequence with limit a. Then 2 and the assumption that $\liminf_{x \to a} f(x) = \limsup_{x \to a} f(x) = \ell$ imply that $\liminf_{n \to \infty} f(x_n) = \limsup_{n \to \infty} f(x_n) = \ell$. Therefore, $\lim_{n \to \infty} f(x_n) = \ell$.

4. ($\Rightarrow$) This direction is proved by contradiction.

   (a) Suppose the contrary that there exists $\varepsilon > 0$ such that for each $n \in \mathbb{N}$, there exists $x_n \in B\big(a, \frac{1}{n}\big) \cap A\backslash\{a\}$ such that $f(x_n) \leqslant \ell - \varepsilon$. Then $\{x_n\}_{n=1}^{\infty} A\backslash\{a\}$ and $\lim_{n \to \infty} x_n = a$; however,

   $$\liminf_{n \to \infty} f(x_n) \leqslant \ell - \varepsilon < \ell = \liminf_{x \to a} f(x)\,,$$

   a contradiction to 2.

   (b) Suppose the contrary that there exist $\varepsilon > 0$ and $\delta > 0$ such that

   $$f(x) \geqslant \ell + \varepsilon \qquad \forall\, x \in B(a, \delta) \cap A\backslash\{a\}\,.$$

   Then $m(\delta) \geqslant \ell + \varepsilon$; thus the monotonicity of m implies that

   $$\ell + \varepsilon \leqslant m(\delta) \leqslant m(r) \quad \text{whenever} \quad 0 < r < \delta\,.$$

   Passing to the limit as $r \to 0^+$, we conclude that

   $$\ell + \varepsilon \leqslant \lim_{r \to 0^+} m(r) = \liminf_{x \to a} f(x)\,,$$

   a contradiction.

   ($\Leftarrow$) Let $\{x_n\}_{n=1}^{\infty} \subseteq A\backslash\{a\}$ be a sequence with limit a, and $\varepsilon > 0$ be given. Then (a) provides $\delta > 0$ such that $f(x) > \ell - \varepsilon$ whenever $x \in B(a, \delta) \cap A\backslash\{a\}$. For such $\delta > 0$, there exists $N > 0$ such that $0 < d(x_n, a) < \delta$ for all $n \geqslant N$. Therefore, if $n \geqslant N$, $f(x_n) > \ell - \varepsilon$ which implies that $\liminf_{n \to \infty} f(x_n) \geqslant \ell - \varepsilon$. Since $\varepsilon > 0$ is chosen arbitrary, we conclude that

   $$\liminf_{n \to \infty} f(x_n) \geqslant \ell \text{ for every convergent sequence } \{x_n\}_{n=1}^{\infty} \subseteq A\backslash\{a\} \text{ with limit } a.$$

   On the other hand, using (b) we find that for each $n \in \mathbb{N}$, there exists $x_n \in B\big(a, \frac{1}{n}\big) \cap A\backslash\{a\}$ such that $f(x_n) < \ell + \frac{1}{n}$. Then $\liminf_{n \to \infty} f(x_n) \leqslant \ell$, and (i) further implies that $\liminf_{n \to \infty} f(x_n) = \ell$; thus we establish that there exists a convergent sequence $\{x_n\}_{n=1}^{\infty} \subseteq A\backslash\{a\}$ with limit a such that $\liminf_{n \to \infty} f(x_n) = \ell$.
   By 1 and 2, we conclude that $\ell = \liminf_{x \to a} f(x)$.

5. (a) $\liminf_{x \to a} f(x) = \limsup_{x \to a} f(x) = 0$ for all $a \in \mathbb{R}$.

   (b) $\liminf_{x \to a} f(x) = -|a|$, $\limsup_{x \to a} f(x) = |a|$. In particular, $\lim_{x \to 0} f(x) = 0$. □

Problem 6. Let (M, d) be a metric space, and $A \subseteq M$. A function $f : A \to \mathbb{R}$ is called ***lower semi-continuous*** / ***upper semi-continuous*** at $a \in A$ if either $a \in A\backslash A'$ or $\liminf_{x \to a} f(x) \geqslant f(a)$, / $\limsup_{x \to a} f(x) \leqslant f(a)$, and is called lower/upper semi-continuous on A if f is lower/uppser semi-continuous at a for all $a \in A$.

1. Show that $f: A \to \mathbb{R}$ is lower semi-continuous on A if and only if $f^{-1}((-\infty, r])$ is closed relative to A. Also show that $f: A \to \mathbb{R}$ is upper semi-continuous on A if and only if $f^{-1}([r, \infty))$ is closed relative to A.

2. Show that f is lower semi-continuous on A if and only if for all convergent sequences $\{x_n\}_{n=1}^{\infty} \subseteq A$ and $\{s_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ satisfying $f(x_n) \leqslant s_n$ for all $n \in \mathbb{N}$, we have
$$f\big(\lim_{n\to\infty} x_n\big) \leqslant \lim_{n\to\infty} s_n \,.$$

3. Let $\{f_\alpha\}_{\alpha\in I}$ be a family of lower semi-continuous functions on A. Prove that $f(x) = \sup\limits_{\alpha\in I} f_\alpha(x)$ is lower semi-continuous on A.

4. Let A be a perfect set (that is, A contains no isolated points) and $f: A \to \mathbb{R}$ be given. Define
$$f^*(x) = \limsup_{y\to x} f(y) \qquad \text{and} \qquad f_*(x) = \liminf_{y\to x} f(y)\,.$$
Show that f^* is upper semi-continuous and f_* is lower semi-continuous, and $f_*(x) \leqslant f(x) \leqslant f^*(x)$ for all $x \in A$. Moreover, if g is a lower semi-continuous function on A such that $g(x) \leqslant f(x)$ for all $x \in A$, then $g \leqslant f_*$.

Proof. We first note that by 1, 2 and 4 of Problem 5,

$$\begin{aligned} &f: A \to \mathbb{R} \text{ is lower semi-continuous at } a \\ &\quad\Leftrightarrow \text{for all } \varepsilon > 0, \text{ there exists } \delta > 0 \text{ such that } f(a) - \varepsilon < f(x) \text{ for all } x \in B(a,\delta) \cap A \qquad (\diamond)\\ &\quad\Leftrightarrow \text{for all convergent sequence } \{x_n\}_{n=1}^{\infty} \subseteq A \text{ with limit } a,\ f(a) \leqslant \liminf_{n\to\infty} f(x_n)\,. \end{aligned}$$

We note that the first statement implies the second one because of 4(a) in Problem 5, the second statement implies the third one because of $x_n \in B(a,\delta) \cap A$ when $n \gg 1$, and the third statement implies the first one because of 1 in Problem 5.

1. ($\Rightarrow$) It suffices to prove the case for limit inferior since $\limsup\limits_{x\to a} f(x) = -\liminf\limits_{x\to a}(-f)(x)$. We note that E is closed relative to A if and only if $E \cap A$ is a closed set in the metric space (A, d). Therefore, a subset of E of A is closed relative to A if and only if

   every sequence $\{x_n\}_{n=1}^{\infty} \subseteq E$ that converges to a point in A must also has limit in E.

   Let $r \in \mathbb{R}$ and $\{x_n\}_{n=1}^{\infty}$ be a sequence in $E \equiv f^{-1}((-\infty, r])$ such that $\{x_n\}_{n=1}^{\infty}$ converges to a point $a \in A$. Then $f(a) \leqslant \liminf\limits_{n\to\infty} f(x_n) \leqslant r$ which implies that $a \in f^{-1}((-\infty, r])$.

   ($\Leftarrow$) Let $a \in A$ and $\varepsilon > 0$ be given. Define $r = f(a) - \varepsilon$. Then $V = f^{-1}((r, \infty))$ is open relative to A (since $f^{-1}((-\infty, r])$ is closed relative to A). Since $a \in V$, there exists $\delta > 0$ such that $B(a,\delta) \cap A \subseteq V$. This implies that
$$f(a) - \varepsilon < f(x) \qquad \forall\, x \in B(a,\delta) \cap A\,.$$
   Therefore, the equivalence $(\diamond)$ shows that f is lower semi-continuous at a.

2. ($\Rightarrow$) Let $\{x_n\}_{n=1}^{\infty}$ be a convergent sequence in A with limit a, $\{s_n\}_{n=1}^{\infty}$ be a real sequence with limit s, and $f(x_n) \leqslant s_n$ for all $n \in \mathbb{N}$. Suppose that $f(a) > s$. Let $\varepsilon = \dfrac{f(a)-s}{2}$. Since f is lower semi-continuous at a, $\liminf\limits_{x\to a} f(x) \geqslant f(a)$; thus there exists $\delta > 0$ such that

$$f(a) - \varepsilon < f(x) \qquad \forall\, x \in B(a,\delta) \cap A\,.$$

On the other hand, there exists $N > 0$ such that $x_n \in B(a,\delta) \cap A$ and $s_n < s+\varepsilon$ whenever $n \geqslant N$. Therefore, if $n \geqslant N$,

$$s_n < s + \varepsilon = f(a) - \varepsilon < f(x_n)\,,$$

a contradiction.

($\Leftarrow$) Let $a \in A$, and $\{x_n\}_{n=1}^{\infty} \subseteq A$ be a sequence with limit a. Let $\{x_{n_j}\}_{j=1}^{\infty}$ be a subsequence of $\{x_n\}_{n=1}^{\infty}$ such that $\lim\limits_{j\to\infty} f(x_{n_j}) = \liminf\limits_{n\to\infty} f(x_n)$. Define $s_j = f\big(x_{n_j}\big)$. Then clearly $f\big(x_{n_j}\big) \leqslant s_j$ for all $j \in \mathbb{N}$; thus by assumption

$$f(a) \leqslant \lim_{j\to\infty} s_j = \liminf_{n\to\infty} f(x_n)\,.$$

3. Let $a \in A \cap A'$ and $\{x_n\}_{n=1}^{\infty} \subseteq A\backslash\{a\}$ be a sequence with limit a. Then $f_\alpha(x_n) \leqslant f(x_n)$ for all $n \in \mathbb{N}$ and $\alpha \in I$. Since f_α is lower semi-continuous for each $\alpha \in I$, for $\alpha \in I$ we have

$$f_\alpha(a) \leqslant \liminf_{x\to a} f_\alpha(x) \leqslant \liminf_{x\to a} f(x)\,.$$

The inequality above implies that

$$f(a) = \sup_{\alpha\in I} f_\alpha(a) \leqslant \liminf_{x\to a} f(x)\,;$$

thus f is lower semi-continuous at a. □

Problem 7. Complete the following.

1. Show that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous, and $B \subseteq \mathbb{R}^n$ is bounded, then $f(B)$ is bounded.

2. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $K \subseteq \mathbb{R}$ is compact, is $f^{-1}(K)$ necessarily compact?

3. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $C \subseteq \mathbb{R}$ is connected, is $f^{-1}(C)$ necessarily connected?

Solution. 1. Since B is bounded, $\bar{B}$ is closed and bounded; thus the Heine-Borel Theorem implies that $\bar{B}$ is compact. Since $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous, $f(\bar{B})$ is also compact; thus bounded. The boundedness of $f(B)$ then follows from the fact that $f(B) \subseteq f(\bar{B})$.

2. No. For example, consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sin x$ and $K = [-1,1]$. Then K is compact but $f^{-1}(K)$ is the whole real line so that $f^{-1}(K)$ is not compact.

3. No. For example, consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ and $C = [1,4]$. Then C is connected since it is an interval but $f^{-1}(C) = [-2,-1] \cup [1,2]$ which is clearly disconnected. □

Problem 8. Consider a compact set $K \subseteq \mathbb{R}^n$ and let $f : K \to \mathbb{R}^m$ be continuous and one-to-one. Show that the inverse function $f^{-1} : f(K) \to K$ is continuous. How about if K is not compact but connected?

Proof. Let F be a closed subset of K. Then 1 of Problem 1 in Exercise 9 implies that F is compact. Therefore, $f(F)$ is compact since f is continuous. Since $f(F) = (f^{-1})^{-1}(F)$, we conclude that the pre-image of F under f^{-1} is compact; hence $(f^{-1})^{-1}(F)$ is closed in $f(K)$ for all closed sets $F \subseteq K$. Therefore, Theorem 4.14 in the lecture note shows that $f^{-1} : f(K) \to K$ is continuous.

However, $f^{-1} : f(K) \to K$ is not necessarily continuous if K is connected. For example, consider $f : [0, 2\pi) \to \mathbb{R}^2$ given by $f(t) = (\cos t, \sin t)$. Then f is one-to-one but $f^{-1} : f([0, 2\pi)) \to [0, 2\pi)$ is not continuous at $f(0) = (1, 0)$ since the sequences $\{\boldsymbol{x}_n\}_{n=1}^{\infty}, \{\boldsymbol{y}_n\}_{n=1}^{\infty}$ given by

$$\boldsymbol{x}_n = \big(\cos\frac{1}{n}, \sin\frac{1}{n}\big) \qquad \text{and} \qquad \boldsymbol{y}_n = \big(\cos\big(2\pi - \frac{1}{n}\big), \sin\big(2\pi - \frac{1}{n}\big)\big)$$

both converges to $(1, 0)$ but $f^{-1}(\boldsymbol{x}_n) = \dfrac{1}{n}$ and $f^{-1}(\boldsymbol{y}_n) = 2\pi - \dfrac{1}{n}$ so that

$$\lim_{n\to\infty} f^{-1}(\boldsymbol{x}_n) = 0 \neq 2\pi = \lim_{n\to\infty} f^{-1}(\boldsymbol{y}_n). \qquad \square$$

Problem 9. Let (M, d) be a metric space, $K \subseteq M$ be compact, and $f : K \to \mathbb{R}$ be lower semi-continuous (see Problem 6 for the definition). Show that f attains its minimum on K.

Proof. **Claim**: there exists a sequence $\{x_n\}_{n=1}^{\infty}$ such that $\lim\limits_{n\to\infty} f(x_n) = \inf\limits_{x\in K} f(x)$.

Proof of claim: If $\inf\limits_{x\in K} f(x) \in \mathbb{R}$, for each $n \in \mathbb{N}$ there exists $x_n \in K$ such that

$$\inf_{x\in K} f(x) \leqslant f(x_n) \leqslant \inf_{x\in K} f(x) + \frac{1}{n}.$$

If $\inf\limits_{x\in K} f(x) = -\infty$, for each $n \in \mathbb{N}$ there exists $x_n \in K$ such that $f(x_n) < -n$. In either case, $\lim\limits_{n\to\infty} f(x_n) = \inf\limits_{x\in K} f(x)$. $\square$

W.L.O.G. we can assume that $f(x_n) > \inf\limits_{x\in K} f(x)$ for all $n \in \mathbb{N}$ (for otherwise we find that f attains its minimum at some x_n). Let $n_1 = 1$, and for given n_k choose $n_{k+1} > n_k$ such that $f(x_{n_k}) > f(x_{n_{k+1}})$. In this way we obtain a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ satisfying that

$$\lim_{k\to\infty} f(x_{n_k}) = \inf_{x\in K} f(x) \qquad \text{and} \qquad f(x_{n_k}) \geqslant f(x_{n_{k+1}}) \quad \forall\, k \in \mathbb{N}.$$

Since $\{x_{n_k}\}_{k=1}^{\infty} \subseteq K$, by the compactness of K there exists a convergent subsequence $\{x_{n_{k_\ell}}\}_{\ell=1}^{\infty}$ of $\{x_{n_k}\}_{k=1}^{\infty}$. Suppose that $\lim\limits_{\ell\to\infty} x_{n_{k_\ell}} = a$. Then by the fact that $x_{n_k} \neq x_{n_\ell}$ for all $k \neq \ell$, we have

$$\#\big\{\ell \in \mathbb{N} \,\big|\, x_{n_{k_\ell}} = a\big\} \leqslant 1.$$

Therefore, up to deleting one term in the sequence we can assume that $\big\{x_{n_{k_\ell}}\big\}_{\ell=1}^{\infty} \subseteq K \backslash \{a\}$. In such a case the lower semi-continuity of f implies that

$$\liminf_{\ell\to\infty} f\big(x_{n_{k_\ell}}\big) \geqslant \liminf_{x\to a} f(x) \geqslant f(a).$$

Since $\lim\limits_{n\to\infty} f(x_n) = \inf\limits_{x\in K} f(x)$, the inequality above implies that

$$\inf_{x\in K} f(x) = \liminf_{\ell\to\infty} f\big(x_{n_{k_\ell}}\big) \geqslant \liminf_{x\to a} f(x) \geqslant f(a) \geqslant \inf_{x\in K} f(x)\,;$$

thus $f(a) = \inf\limits_{x\in K} f(x)$. ▫

Problem 10. Let (M, d) be a metric space. Show that a subset $A \subseteq M$ is connected if and only if every continuous function defined on A whose range is a subset of $\{0, 1\}$ is constant.

Proof. "$\Rightarrow$" Assume that A is connected and $f : A \to \{0, 1\}$ is a continuous function, and $\delta = 1/2$. Suppose the contrary that $f^{-1}(\{0\}) \neq \varnothing$ and $f^{-1}(\{1\}) \neq \varnothing$. Then $A = f^{-1}((-\delta, \delta))$ and $B = f^{-1}((1 - \delta, 1 + \delta))$ are non-empty set. Moreover, the continuity of f implies that A and B are open relative to A; thus there exist open sets U and V such that

$$f^{-1}((-\delta, \delta)) = U \cap A \qquad \text{and} \qquad f^{-1}((1 - \delta, 1 + \delta)) = V \cap A\,.$$

Then

(1) $A \cap U \cap V = f^{-1}((-\delta, \delta)) \cap f^{-1}((1 - \delta, 1 + \delta)) = \varnothing\,,$

(2) $A \cap U \neq \varnothing$ and $A \cap V \neq \varnothing\,,$

(3) $A \subseteq U \cup V$ since the range of f is a subset of $\{0, 1\}\,;$

thus A is disconnect, a contradiction.

"$\Leftarrow$" Suppose the contrary that A is disconnected so that there exist open sets U and V such that

(1) $A \cap U \cap V = \varnothing\,,$ (2) $A \cap U \neq \varnothing\,,$ (3) $A \cap V \neq \varnothing\,,$ (4) $A \subseteq U \cup V\,.$

Let $f : A \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in A \cap U\,, \\ 1 & \text{if } x \in A \cap V\,. \end{cases}$$

We first prove that f is continuous on A. Let $a \in A$. Then $a \in A \cap U$ or $a \in A \cap V$. Suppose that $a \in A \cap U$. In particular $a \in U$; thus the openness of U provides $r > 0$ such that $B(a, r) \subseteq U$. Note that if $x \in B(a, r) \cap A$, then $x \in A \subseteq U$; thus

$$\big|f(x) - f(a)\big| = 0 \qquad \forall\, x \in B(a, r) \cap A$$

which shows the continuity of f at a. Similar argument can be applied to show that f is continuous at $a \in A \cap V$. ▫

Problem 11. Let $\mathcal{D} \subseteq \mathbb{R}^n$ be an open connected set, where $n > 1$. If a, b and c are any three points in $\mathcal{D}$, show that there is a path in G which connects a and b without passing through c. In particular, this shows that $\mathcal{D}$ is path connected and $\mathcal{D}$ is not homeomorphic to any subset of $\mathbb{R}$.

In Exercise Problem 12 through 15, we focus on another kind of connected sets, so-called path connected sets. First we introduce path connectedness in the following

Definition 0.1. Let (M, d) be a metric space. A subset $A \subseteq M$ is said to be ***path connected*** if for every $x, y \in A$, there exists a continuous map $\varphi : [0,1] \to A$ such that $\varphi(0) = x$ and $\varphi(1) = y$.

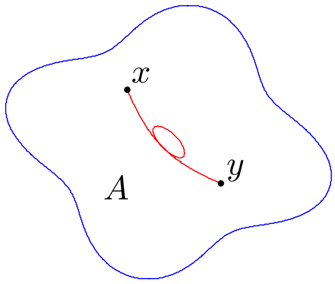

Figure 1: Path connected sets

Problem 12. Recall that a set A in a vector space $\mathcal{V}$ is called ***convex*** if for all $x, y \in A$, the line segment joining x and y, denoted by $\overline{xy}$, lies in A. Show that a convex set in a normed space is path connected.

Proof. Let C be a convex set in a normed space $(\mathcal{V}, \|\cdot\|)$, and $\boldsymbol{x}, \boldsymbol{y} \in C$. Define $\varphi : [0,1] \to \mathcal{V}$ by $\varphi(t) = (1-t)\boldsymbol{x} + t\boldsymbol{y}$. Then $\varphi([0,1]) = \overline{xy}$; thus the convexity of C implies that $\varphi : [0,1] \to C$. $\square$

Problem 13. A set S in a vector space $\mathcal{V}$ is called ***star-shaped*** if there exists $p \in S$ such that for any $q \in S$, the line segment joining p and q lies in S. Show that a star-shaped set in a normed space is path connected.

Proof. Suppose that there exists $p \in S$ such that for any $q \in S$, the line segment joining p and q lies in S. In other words, such $p \in S$ satisfies that

$$(1-\lambda)q + \lambda p \subseteq S \qquad \forall\, \lambda \in [0,1] \text{ and } q \in S\,.$$

Let x, y in S. Define

$$\varphi(t) = \begin{cases} (1-2t)x + 2tp & \text{if } 0 \leqslant t \leqslant \dfrac{1}{2}\,, \\ (2-2t)p + (2t-1)y & \text{if } \dfrac{1}{2} < t \leqslant 1\,. \end{cases}$$

Then φ is continuous on $[0,1]$ $\big($since $\lim\limits_{t\to 0.5^+} \varphi(t) = \lim\limits_{t\to 0.5^-} \varphi(t) = p$ so that φ is continuous at 0.5$\big)$. Moreover, $\varphi([0,0.5]) = \overline{xp}$ and $\varphi([0.5,1]) = \overline{py}$ so that $\varphi : [0,1] \to A$ is continuous with $\varphi(0) = x$ and $\varphi(1) = y$. Therefore, A is path connected. $\square$

Problem 14. Let $A = \Big\{\Big(x, \sin\dfrac{1}{x}\Big) \,\Big|\, x \in (0,1]\Big\} \cup (\{0\} \times [-1,1])$. Show that A is connected in $(\mathbb{R}^2, \|\cdot\|_2)$, but A is not path connected.

Proof. Assume the contrary that A is path connected such that there is a continuous function $\varphi : [0,1] \to A$ such that $\varphi(0) = (x_0, y_0) \in \Big\{\Big(x, \sin\dfrac{1}{x}\Big) \,\Big|\, x \in (0,1)\Big\}$ and $\varphi(1) = (0,0) \in \{0\} \times [-1,1]$. Let $t_0 = \inf\big\{t \in [0,1] \,\big|\, \varphi(t) \in \{0\} \times [-1,1]\big\}$. In other words, at $t = t_0$ the path touches $0 \times [-1,1]$ for the "first time". By the continuity of φ, $\varphi(t_0) \in \{0\} \times [-1,1]$. Since $\varphi(0) \notin \{0\} \times [-1,1]$, $\varphi([0,t_0)) \subseteq \Big\{\Big(x, \sin\dfrac{1}{x}\Big) \,\Big|\, x \in (0,1)\Big\}$.

Suppose that $\varphi(t_0) = (0, \bar{y})$ for some $\bar{y} \in [-1, 1]$, and $\varphi(t) = \left(x(t), \sin \frac{1}{x(t)}\right)$ for $0 \leqslant t < t_0$. By the continuity of φ, there exists $\delta > 0$ such that if $|t - t_0| < \delta$, $|\varphi(t) - \varphi(t_0)| < 1$. In particular,

$$x(t)^2 + \left(\sin \frac{1}{x(t)} - \bar{y}\right)^2 < 1 \qquad \forall\, t \in (t_0 - \delta, t)\,.$$

On the other hand, since φ is continuous, $x(t)$ is continuous on $[0, t_0)$; thus by the fact that $[0, t_0)$ is connected, $x([0, t_0))$ is connected. Therefore, $x([0, t_0)) = (0, \bar{x}]$ for some $\bar{x} > 0$. Since $\lim\limits_{t \to t_0} x(t) = 0$, there exists $\{t_n\}_{n=1}^{\infty} \in [0, t_0)$ such that $t_n \to t_0$ as $n \to \infty$ and $\left|\sin \frac{1}{x(t_n)} - \bar{y}\right| \geqslant 1$. For $n \gg 1$, $t_n \in (t_0 - \delta, t_0)$ but

$$x(t_n)^2 + \left(\sin \frac{1}{x(t_n)} - \bar{y}\right)^2 \geqslant 1\,,$$

a contradiction.

On the other hand, A is the closure of the connected set $B = \left\{\left(x, \sin \frac{1}{x}\right) \,\Big|\, x \in (0, 1)\right\}$ (the connectedness of B follows from the fact that the function $\psi(x) = \left(x, \sin \frac{1}{x}\right)$ is continuous on the connected set $(0, 1)$). Therefore, by Problem 12 in Exercise 9, $A = \bar{B}$ is connected. □

Problem 15. Let (M, d) be a metric space, and $A \subseteq M$. Show that if A is path connected, then A is connected.

Hint: Apply Theorem 3.68 in the lecture note and prove by contradiction.

Proof. Assume the contrary that there are non-empty sets A_1, A_2 such that $A = A_1 \cup A_2$ but $A_1 \cap \overline{A_2} = A_2 \cap \overline{A_1} = \varnothing$. Let $x \in A_1$ and $y \in A_2$. By the path connectedness of A, there exists a continuous map $\varphi : [0, 1] \to A$ such that $\varphi(0) = x$ and $\varphi(1) = y$. Define $I_1 = \varphi^{-1}(A_1)$ and $I_2 = \varphi^{-1}(A_2)$. Then clearly $0 \in I_1$ and $1 \in I_2$, and $I_1 \cap I_2 = \varnothing$. Moreover,

$$[0, 1] = \varphi^{-1}(A) = \varphi^{-1}(A_1 \cup A_2) = \varphi^{-1}(A_1) \cup \varphi^{-1}(A_2) = I_1 \cup I_2\,.$$

Claim: $I_1 \cap \bar{I}_2 = I_2 \cap \bar{I}_1 = \varnothing$.

Suppose the contrary that $t \in I_1 \cap \bar{I}_2$. Then $t \in \varphi(A_1)$ which shows that $\varphi(t) \in A_1$. On the other hand, $t \in \bar{I}_2$; thus there exists $\{t_n\}_{n=1}^{\infty} \subseteq I_2$ such that $t_n \to t$ as $n \to \infty$. By the continuity of φ,

$$\varphi(t) = \lim_{n \to \infty} \varphi(t_n) \in \overline{A_2}\,;$$

thus we find that $\varphi(t) \in A_1 \cap \overline{A_2}$, a contradiction. Therefore, $I_1 \cap \bar{I}_2 = \varnothing$. Similarly, $I_2 \cap \bar{I}_1 = \varnothing$; thus we establish the existence of non-empty sets I_1 and I_2 such that

$$[0, 1] = I_1 \cup I_2\,, \quad I_1, I_2 \neq \varnothing\,, \quad I_1 \cap \bar{I}_2 = I_2 \cap \bar{I}_1 = \varnothing$$

which shows that $[0, 1]$ is disconnected, a contradiction. □

Alternative proof. Assume the contrary that there are two open sets V_1 and V_2 such that

1. $A \cap V_1 \cap V_2 = \varnothing$; 2. $A \cap V_1 \neq \varnothing$; 3. $A \cap V_2 \neq \varnothing$; 4. $A \subseteq V_1 \cup V_2\,.$

Since A is path connected, for $x \in A \cap V_1$ and $y \in A \cap V_2$, there exists a continuous map $\varphi : [0,1] \to A$ such that $\varphi(0) = x$ and $\varphi(1) = y$. By Theorem 4.14 in the lecture note, there exist U_1 and U_2 open in $(\mathbb{R}, |\cdot|)$ such that $\varphi^{-1}(V_1) = U_1 \cap [0,1]$ and $\varphi^{-1}(V_2) = U_2 \cap [0,1]$. Therefore,

$$[0,1] = \varphi^{-1}(A) \subseteq \varphi^{-1}(V_1) \cup \varphi^{-1}(V_2) \subseteq U_1 \cup U_2 .$$

Since $0 \in U_1$, $1 \in U_2$, and $[0,1] \cap U_1 \cap U_2 = \varphi^{-1}(A \cap V_1 \cap V_2) = \varnothing$, we conclude that $[0,1]$ is disconnected, a contradiction to Theorem 3.68 in the lecture note. □

Problem 16. Let (M,d), (N,ρ) be metric spaces, A be a subset of M, and $f : A \to N$ be a continuous map. Show that if $C \subseteq A$ is path connected, so is $f(C)$.

Proof. Let $y_1, y_2 \in f(C)$. Then $\exists\, x_1, x_2 \in C$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since C is path connected, $\exists\, r : [0,1] \to C$ such that r is continuous on $[0,1]$ and $r(0) = x_1$ and $r(1) = x_2$. Let $\varphi : [0,1] \to f(C)$ be defined by $\varphi = f \circ r$. By Corollary 4.24 in the lecture note φ is continuous on $[0,1]$, and $\varphi(0) = y_1$ and $\varphi(1) = y_2$. □