Exercise Problem Sets 8

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In the exercise section of this chapter, we first introduce the concepts of accumulation points, isolated points and derived set of a set as follows.

Definition 0.1. Let (M, d) be a normed vector space, and A be a subset of M.

1. A point $x \in M$ is called an ***accumulation point*** of A if there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in $A \backslash \{x\}$ such that $\{x_n\}_{n=1}^{\infty}$ converges to x.

2. A point $x \in A$ is called an ***isolated point*** (孤立點) (of A) if there exists no sequence in $A \backslash \{x\}$ that converges to x.

3. The ***derived set*** of A is the collection of all accumulation points of A, and is denoted by A'.

Problem 1. Let (M, d) be a metric space, and A be a subset of M.

1. Show that the collection of all isolated points of A is $A \backslash A'$.

2. Show that $A' = \bar{A} \backslash (A \backslash A')$. In other words, the derived set consists of all limit points that are not isolated points. Also show that $\bar{A} \backslash A' = A \backslash A'$.

Proof. 1. By the definition of isolated points of sets,

$$\begin{aligned} x \in A \backslash A' &\Leftrightarrow x \in A \text{ and } x \text{ is not an accumulation point of } A \\ &\Leftrightarrow x \in A \text{ and } \exists\, \varepsilon > 0 \ni B(x, \varepsilon) \cap A \backslash \{x\} = \varnothing \\ &\Leftrightarrow x \in A \text{ and } \exists\, \varepsilon > 0 \ni B(x, \varepsilon) \cap A \subseteq \{x\} \\ &\Leftrightarrow \exists\, \varepsilon > 0 \ni B(x, \varepsilon) \cap A = \{x\}\,; \end{aligned}$$

thus x is an isolated point of A if and only if $x \in A \backslash A'$.

2. First we show that $\bar{A} = A \cup A'$. To see this, let $x \in \bar{A} \backslash A$. By the fact that $A = A \backslash \{x\}$, there exists $\{x_n\}_{n=1}^{\infty} \subseteq A \backslash \{x\}$ such that $\lim_{n \to \infty} x_n = x$. Therefore, $x \in A'$ which implies that

$$\bar{A} \backslash A \subseteq A' \subseteq \bar{A}\,,$$

where we use the fact that $\bar{A} \supseteq A'$ to conclude the last inclusion. The inclusion relation above then shows that

$$\bar{A} = A \cup \bar{A} = A \cup (\bar{A} \backslash A) \subseteq A \cup A' \subseteq A \cup \bar{A} = \bar{A}\,;$$

thus we establish that $\bar{A} = A \cup A'$. This identity further shows that

$$\bar{A} \cap A^{\complement} = (A \cup A') \cap A^{\complement} = A' \cap A^{\complement} \subseteq A\,.$$

Now, using the identity $A\backslash B = A \cap B^{\complement}$ we find that

$$\begin{aligned}\bar{A}\backslash(A\backslash A') &= \bar{A} \cap \big(A \cap (A')^{\complement}\big)^{\complement} = \bar{A} \cap (A^{\complement} \cup A') = (\bar{A} \cap A^{\complement}) \cup (\bar{A} \cap A') \\ &= (\bar{A} \cap A^{\complement}) \cup A' = A' .\end{aligned}$$

Moreover, using $\bar{A} = A \cup A'$ we also have

$$\bar{A}\backslash A' = (A \cup A') \cap (A')^{\complement} = A \cap (A')^{\complement} = A\backslash A' .$$

□

Problem 2. Let A and B be subsets of a metric space (M, d). Show that

1. $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.
2. $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$.
3. $\mathrm{cl}(A \cap B) \subseteq \mathrm{cl}(A) \cap \mathrm{cl}(B)$. Find examples of that $\mathrm{cl}(A \cap B) \subsetneq \mathrm{cl}(A) \cap \mathrm{cl}(B)$.

Proof. 1. Since $\mathrm{cl}(A)$ is closed, by the definition of closed set we have $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.

2. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, we have $\mathrm{cl}(A) \subseteq \mathrm{cl}(A \cup B)$ and $\mathrm{cl}(B) \subseteq \mathrm{cl}(A \cup B)$; thus $\mathrm{cl}(A) \cup \mathrm{cl}(B) \subseteq \mathrm{cl}(A \cup B)$. On the other hand, if $x \in \mathrm{cl}(A \cup B)$, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in $A \cup B$ such that $\lim\limits_{n\to\infty} x_n = x$. Since $A \cup B$ contains infinitely many terms of $\{x_n\}_{n=1}^{\infty}$, at least one of A and B contains infinitely many terms of $\{x_n\}_{n=1}^{\infty}$. W.L.O.G., suppose that $\#\big\{n \in \mathbb{N} \,\big|\, x_n \in A\big\} = \infty$. Let

$$\big\{n \in \mathbb{N} \,\big|\, x_n \in A\big\} = \big\{n_k \in \mathbb{N} \,\big|\, n_k < n_{k+1}\big\} .$$

Then $\{x_{n_k}\}_{k=1}^{\infty} \in A$. Since $x_n \to x$ as $n \to \infty$, we must have $x_{n_k} \to x$ as $k \to \infty$; thus $x \in \mathrm{cl}(A)$. Therefore, $\mathrm{cl}(A \cup B) \subseteq \mathrm{cl}(A) \cup \mathrm{cl}(B)$.

3. Let $x \in \mathrm{cl}(A \cap B)$. Then

$$(\forall\, \varepsilon > 0)(B(x, \varepsilon) \cap (A \cap B) \neq \varnothing) .$$

Therefore, by the fact that $B(x, \varepsilon) \cap A \subseteq B(x, \varepsilon) \cap (A \cap B)$ and $B(x, \varepsilon) \cap B \subseteq B(x, \varepsilon) \cap (A \cap B)$, we have

$$(\forall\, \varepsilon > 0)(B(x, \varepsilon) \cap A \neq \varnothing) \qquad \text{and} \qquad (\forall\, \varepsilon > 0)(B(x, \varepsilon) \cap B \neq \varnothing) .$$

This implies that $x \in \bar{A} \cap \bar{B}$. Note that if $A = \mathbb{Q}$ and $B = \mathbb{Q}^{\complement}$, then $\mathrm{cl}(A \cap B) = \varnothing$, while $\bar{A} = \bar{B} = \mathbb{R}$ which provides an example of $\mathrm{cl}(A \cap B) \subsetneq \bar{A} \cap \bar{B}$.

□

Problem 3. Let A and B be subsets of a metric space (M, d). Show that

1. $\mathrm{int}(\mathrm{int}(A)) = \mathrm{int}(A)$.
2. $\mathrm{int}(A \cap B) = \mathrm{int}(A) \cap \mathrm{int}(B)$.
3. $\mathrm{int}(A \cup B) \supseteq \mathrm{int}(A) \cup \mathrm{int}(B)$. Find examples of that $\mathrm{int}(A \cup B) \supsetneq \mathrm{int}(A) \cup \mathrm{int}(B)$.

Proof. 1. Since $\text{int}(A)$ is open, by the definition of open sets we have $\text{int}(\text{int}(A)) = \text{int}(A)$.

2. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, we have $\text{int}(A \cap B) \subseteq \text{int}(A)$ and $\text{int}(A \cap B) \subseteq \text{int}(B)$; thus $\text{int}(A \cap B) \subseteq \text{int}(A) \cap \text{int}(B)$. On the other hand, let $x \in \text{int}(A) \cap \text{int}(B)$. Then $x \in \text{int}(A)$ and $x \in \text{int}(B)$; thus there exist $r_1, r_0 > 0$ such that

$$B(x, r_1) \subseteq A \qquad \text{and} \qquad B(x, r_)) \subseteq B\,.$$

Let $r = \min\{r_1, r_2\}$. Then $r > 0$, and $B(x, r) \subseteq B(x, r_1)$ and $B(x, r) \subseteq B(x, r_2)$. Therefore, $B(x, r) \subseteq A$ and $B(x, r) \subseteq B$ which further implies that $B(x, r) \subseteq A \cap B$; thus $x \in \text{int}(A \cap B)$.

3. Let $x \in \mathring{A} \cup \mathring{B}$. Then $x \in \mathring{A}$ or $x \in \mathring{B}$; thus there exists $r > 0$ such that $B(x, r) \subseteq A$ or $B(x, r) \subseteq B$. Therefore, there exists $r > 0$ such that $B(x, r) \subseteq A \cup B$ which shows that $\text{int}(A \cup B) \supseteq \text{int}(A) \cup \text{int}(B)$. Note that if $A = \mathbb{Q}$ and $B = \mathbb{Q}^\complement$, then $\text{int}(A \cup B) = \mathbb{R}$ while $\text{int}(A) = \text{int}(B) = \varnothing$; thus we obtain an example of $\text{int}(A \cup B) \supsetneq \text{int}(A) \cup \text{int}(B)$. □

Problem 4. Let (M, d) be a metric space, and A be a subset of M. Show that

$$\partial A = \big(A \cap \text{cl}(M \backslash A)\big) \cup \big(\text{cl}(A) \backslash A\big).$$

Proof. By the definition of the boundary, $\partial A = \bar{A} \cap \overline{A^\complement}$; thus

$$\begin{aligned}
\big(A \cap \text{cl}(M \backslash A)\big) \cup \big(\text{cl}(A) \backslash A\big) &= \big(A \cap \overline{A^\complement}\big) \cup \big(\bar{A} \cap A^\complement\big) \\
&= \big[A \cup \big(\bar{A} \cap A^\complement\big)\big] \cap \big[\overline{A^\complement} \cup \big(\bar{A} \cap A^\complement\big)\big] = \bar{A} \cap \big[\big(\overline{A^\complement} \cup \bar{A}\big) \cap \big(\overline{A^\complement} \cup A^\complement\big)\big] \\
&= \bar{A} \cap \big[\big(\overline{A^\complement} \cup \bar{A}\big) \cap \overline{A^\complement}\big] = \partial A \cap \big(\overline{A^\complement} \cup \bar{A}\big) = \partial A\,,
\end{aligned}$$

where the last equality follows from that $\partial A \subseteq \bar{A}$ and $\partial A \subseteq \overline{A^\complement}$. □

Problem 5. Recall that in a metric space (M, d), a subset A is said to be dense in S if subsets satisfy $A \subseteq S \subseteq \text{cl}(A)$. For example, $\mathbb{Q}$ is dense in $\mathbb{R}$.

1. Show that if A is dense in S and if S is dense in T, then A is dense in T.

2. Show that if A is dense in S and $B \subseteq S$ is open, then $B \subseteq \text{cl}(A \cap B)$.

Proof. 1. If A is dense in S and if S is dense in T, then $A \subseteq S \subseteq \bar{A}$ and $S \subseteq T \subseteq \bar{S}$. Since $S \subseteq \bar{A}$, we must have $\bar{S} \subseteq \bar{A}$; thus

$$A \subseteq S \subseteq T \subseteq \bar{S} \subseteq \bar{A}$$

which shows that A is dense in T.

2. Let $x \in B$. Since B is open, there exists $\varepsilon_0 > 0$ such that $B(x, \varepsilon_0) \subseteq B \subseteq S$. On the other hand, $x \in S$ since B is a subset of S; thus the denseness of A in S implies that

$$(\forall\, \varepsilon > 0)(B(x, \varepsilon) \cap A \neq \varnothing)\,.$$

Therefore, for a given $\varepsilon > 0$, if $\varepsilon \geqslant \varepsilon_0$, then

$$B(x,\varepsilon) \cap (A \cap B) \supseteq B(x,\varepsilon_0) \cap (A \cap B) = B(x,\varepsilon_0) \cap A \neq \varnothing)$$

while if $\varepsilon < \varepsilon_0$, then

$$B(x,\varepsilon) \cap (A \cap B) = B(x,\varepsilon) \cap A \neq \varnothing\,.$$

This implies that

$$(\forall\, \varepsilon > 0)(B(x,\varepsilon) \cap (A \cap B) \neq \varnothing)\,;$$

thus $x \in \mathrm{cl}(A \cap B)$. □

Problem 6. Let A and B be subsets of a metric space (M,d). Show that

1. $\partial(\partial A) \subseteq \partial(A)$. Find examples of that $\partial(\partial A) \subsetneq \partial A$. Also show that $\partial(\partial A) = \partial A$ if A is closed.
2. $\partial(A \cup B) \subseteq \partial A \cup \partial B \subseteq \partial(A \cup B) \cup A \cup B$. Find examples of that equalities do not hold.
3. If $\mathrm{cl}(A) \cap \mathrm{cl}(B) = \varnothing$, then $\partial(A \cup B) = \partial A \cup \partial B$.
4. $\partial(A \cap B) \subseteq \partial A \cup \partial B$. Find examples of the equalities do not hold.
5. $\partial(\partial(\partial A)) = \partial(\partial A)$.

Proof. 1. We note that if F is closed, then

$$\partial F = \bar{F} \cap \overline{F^{\complement}} = F \cap \overline{F^{\complement}} \subseteq F\,. \tag{$\diamond$}$$

Since ∂F is closed, we must have $\partial(\partial A) \subseteq \partial A$. Note that if $A = \mathbb{Q} \cap [0,1]$, then $\partial A = [0,1]$; thus $\partial(\partial A) = \{0,1\} \subsetneq \partial A$. Finally we show that $\partial(\partial A) = \partial A$ if A is closed. Using ($\diamond$), it suffices to show that $\partial A \subseteq \partial(\partial A)$. Using 2 of Problem 2,

$$\begin{aligned}\partial(\partial A) &= \partial A \cap \mathrm{cl}((\partial A)^{\complement}) = \partial A \cap \mathrm{cl}(A^{\complement} \cup \overline{A^{\complement}}^{\complement}) = \partial A \cap \big(\overline{A^{\complement}} \cup \mathrm{cl}(\overline{A^{\complement}}^{\complement}\big) \\ &= \big(\partial A \cap \overline{A^{\complement}}\big) \cup \big(\partial A \cap \mathrm{cl}(\overline{A^{\complement}}^{\complement})\big) \supseteq \big(\partial A \cap \overline{A^{\complement}}\big) = \partial A\,.\end{aligned}$$

2. Using 2 and 3 of Problem 2,

$$\begin{aligned}\partial(A \cup B) &= \overline{A \cup B} \cap \mathrm{cl}\big((A \cup B)^{\complement}\big) = \big(\bar{A} \cup \bar{B}\big) \cap \mathrm{cl}(A^{\complement} \cap B^{\complement}) \subseteq \big(\bar{A} \cup \bar{B}\big) \cap \big(\overline{A^{\complement}} \cap \overline{B^{\complement}}\big) \\ &= \big(\bar{A} \cap \overline{A^{\complement}} \cap \overline{B^{\complement}}\big) \cup \big(\bar{B} \cap \overline{A^{\complement}} \cap \overline{B^{\complement}}\big) \subseteq \big(\bar{A} \cap \overline{A^{\complement}}\big) \cup \big(\bar{B} \cap \overline{B^{\complement}}\big) = \partial A \cup \partial B\,.\end{aligned}$$

On the other hand, since $\partial A = \bar{A} \backslash \mathring{A}$ and $\mathring{A} \subseteq A$, we have

$$\bar{A} \subseteq A \cup \partial A \subseteq \mathring{A} \cup (\bar{A} \backslash \mathring{A}) = \bar{A}$$

which implies that $A \cup \partial A = \bar{A}$. Therefore,

$$\partial A \subseteq \bar{A} \subseteq \overline{A \cup B} = A \cup B \cup \partial(A \cup B)$$

and similarly $\partial B \subseteq A \cup B \cup \partial(A \cup B)$. Therefore,

$$\partial A \cup \partial B \subseteq \partial(A \cup B) \cup A \cup B\,.$$

Note that if $A = [-1,0] \cup \big(\mathbb{Q} \cap [0,1]\big)$ and $B = [-1,0] \cup (\mathbb{Q}^{\complement} \cap [0,1])$, then $A \cup B = [-1,1]$, $\partial A = \partial B = \{-1\} \cup [0,1]$ which implies that

$$\partial(A \cup B) = \{-1,1\} \subsetneq \partial A \cup \partial B \subsetneq A \cup B = \partial(A \cup B) \cup A \cup B\,.$$

3. By 2, it suffices to shows that $\partial A \cup \partial B \subseteq \partial(A \cup B)$ if $\bar{A} \cap \bar{B} = \varnothing$. Let $x \in \partial A \cup \partial B$. W.L.O.G., assume that $x \in \partial A$. Then $x \in \bar{A}$; thus $x \notin \bar{B}$ which further implies that there exists $\varepsilon_0 > 0$ such that $B(x,\varepsilon_0) \cap B = \varnothing$ or equivalently, $B(x,\varepsilon_0) \subseteq B^{\complement}$. Therefore, for given $r > 0$, if $r < \varepsilon_0$, then
$$B(x,r) \cap (A \cup B) \supseteq B(x,r) \cap A \neq \varnothing$$
and
$$B(x,r) \cap \big((A \cup B)^{\complement}\big) = B(x,r) \cap (A^{\complement} \cap B^{\complement}) = B(x,r) \cap A^{\complement} \neq \varnothing$$
while if $r \geqslant \varepsilon_0$, then
$$B(x,r) \cap (A \cup B) \subseteq B(x,\varepsilon_0) \cap (A \cup B) \supseteq B(x,\varepsilon_0) \cap A \neq \varnothing$$
and
$$B(x,r) \cap \big((A \cup B)^{\complement}\big) \supseteq B(x,\varepsilon_0) \cap (A^{\complement} \cap B^{\complement}) = B(x,\varepsilon_0) \cap A^{\complement} \neq \varnothing\,.$$
As a consequence, for each $r > 0$,
$$B(x,r) \cap (A \cup B) \neq \varnothing \quad \text{and} \quad B(x,r) \cap (A \cup B)^{\complement}\,;$$
thus $x \in \overline{A \cup B}$ and $x \in \mathrm{cl}\big((A \cup B)^{\complement}\big)$ which implies that $x \in \partial(A \cup B)$.

4. Using 2 and 3 of Problem 2,
$$\begin{aligned}\partial(A \cap B) &= \overline{A \cap B} \cap \mathrm{cl}\big((A \cap B)^{\complement}\big) = \overline{A \cap B} \cap \mathrm{cl}(A^{\complement} \cup B^{\complement}) \subseteq \big(\bar{A} \cap \bar{B}\big) \cap \big(\overline{A^{\complement}} \cup \overline{B^{\complement}}\big)\\ &= \big[\big(\bar{A} \cap \bar{B}\big) \cap \overline{A^{\complement}}\big] \cup \big[\big(\bar{A} \cap \bar{B}\big) \cap \overline{B^{\complement}}\big] \subseteq \big(\bar{A} \cap \overline{A^{\complement}}\big) \cup \big(\bar{B} \cap \overline{B^{\complement}}\big) = \partial A \cup \partial B\,.\end{aligned}$$
Note that if $A = \mathbb{Q}$ and $B = \mathbb{Q}^{\complement}$, then $\partial A = \partial B = \mathbb{R}$ but
$$\partial(A \cap B) = \varnothing \subsetneq \mathbb{R} = \partial A \cap \partial B\,.$$

5. Since ∂A is closed, 1 implies that $\partial(\partial(\partial A)) = \partial(\partial A)$. □

Problem 7. Let (M,d) be a metric space, and A be a subset of M. Show that $A \supseteq A'$ if and only if A is closed.

Proof. "$\Leftarrow$" Note that 2 of Problem 1 implies that $\bar{A} \supseteq A'$; thus if A is closed, $A = \bar{A} \supseteq A'$.

"$\Rightarrow$" In 2 of Problem 1, we show that $\bar{A} = A \cup A'$. Therefore, if $A \supseteq A'$, we have $\bar{A} = A \cup A' = A$ which shows that A is closed. □

Problem 8. Show that the derived set of a set (in a metric space) is closed.

Proof. Let $y \notin A'$. Then there exists $\varepsilon > 0$ such that

$$B(y,\varepsilon) \cap (A\backslash\{y\}) = (B(y,\varepsilon)\backslash\{y\}) \cap A = \varnothing\,.$$

Then $A \subseteq \big(B(y,\varepsilon)\backslash\{y\}\big)^{\complement}$. Since

$$\big(B(y,\varepsilon)\backslash\{y\}\big)^{\complement} = \big(B(y,\varepsilon) \cap \{y\}^{\complement}\big)^{\complement} = B(y,\varepsilon)^{\complement} \cup \{y\}\,,$$

by the fact that $B(y,\varepsilon)^{\complement}$ is closed, $\big(B(y,\varepsilon)\backslash\{y\}\big)^{\complement}$ is closed. Therefore,

$$\bar{A} \subseteq \big(B(y,\varepsilon)\backslash\{y\}\big)^{\complement} \qquad \text{or equivalently,} \qquad \bar{A} \cap B(y,\varepsilon)\backslash\{y\} = \varnothing\,.$$

Since $\bar{A} = A \cup A'$, the equality above implies that

$$A' \cap B(y,\varepsilon)\backslash\{y\} = \varnothing\,;$$

thus the fact that $y \notin A'$ implies that $B(y,\varepsilon) \cap A' = \varnothing$. □

Problem 9. Let $A \subseteq \mathbb{R}^n$. Define the sequence of sets $A^{(m)}$ as follows: $A^{(0)} = A$ and $A^{(m+1)} =$ the derived set of $A^{(m)}$ for $m \in \mathbb{N}$. Complete the following.

1. Prove that each $A^{(m)}$ for $m \in \mathbb{N}$ is a closed set; thus $A^{(1)} \supseteq A^{(2)} \supseteq \cdots$.

2. Show that if there exists some $m \in \mathbb{N}$ such that $A^{(m)}$ is a countable set, then A is countable.

3. For any given $m \in \mathbb{N}$, is there a set A such that $A^{(m)} \neq \varnothing$ but $A^{(m+1)} = \varnothing$?

4. Let A be uncountable. Then each $A^{(m)}$ is an uncountable set. Is it possible that $\bigcap\limits_{m=1}^{\infty} A^{(m)} = \varnothing$?

5. Let $A = \Big\{\dfrac{1}{m} + \dfrac{1}{k}\,\Big|\, m-1 > k(k-1), m, k \in \mathbb{N}\Big\}$. Find $A^{(1)}$, $A^{(2)}$ and $A^{(3)}$.

Proof. 1. See Problem 8 for that A' is closed for all $A \subseteq M$. Moreover, Problem 7 shows that $A \supseteq A'$ if A is closed (in fact, A is closed if and only if $A \supseteq A'$). Therefore, knowing that $A^{(m)}$ is closed for all $m \in \mathbb{N}$, we obtain that $A^{(m)} \supseteq A^{(m+1)}$ for all $m \in \mathbb{N}$.

2. Note that $A\backslash A'$ consists of all isolated points of A. For $m \in \mathbb{N}$, define $B^{(m-1)} = A^{(m-1)}\backslash A^{(m)}$. Then $B^{(m-1)}$ consists of isolated points of $A^{(m-1)}$; thus $B^{(m-1)}$ is countable for all $m \in \mathbb{N}$ (why?). Since for any subset A of M, we have

$$A \subseteq (A\backslash A') \cup A'$$

and equality holds if A is closed, 1 implies that

$$\begin{aligned} A &\subseteq (A\backslash A^{(1)}) \cup A^{(1)} = B^{(0)} \cup A^{(1)} = B^{(0)} \cup \big[\big(A^{(1)}\backslash A^{(2)}\big) \cup A^{(2)}\big] = B^{(0)} \cup B^{(1)} \cup A^{(2)} \\ &= \cdots = B^{(0)} \cup B^{(1)} \cup \cdots \cup B^{(m-1)} \cup A^{(m)}\,. \end{aligned}$$

If $A^{(m)}$ is countable, we find that A is a subset of a finite union of countable sets; thus A is countable.

4. By 2, if $A^{(m)}$ is countable for some $m \in \mathbb{N}$, then A is countable; thus if A is uncountable, $A^{(m)}$ must be uncountable for all $m \in \mathbb{N}$.

5. For each $k \in \mathbb{N}$, let $B_k = \left\{\frac{1}{m} + \frac{1}{k} \,\middle|\, m - 1 > k(k-1), m, k \in \mathbb{N}\right\}$. Then $A = \bigcup_{k=1}^{\infty} B_k$. Moreover, for each $k \in \mathbb{N}$,

$$\sup B_k = \frac{1}{k(k-1)+2} + \frac{1}{k} \qquad \text{and} \qquad \inf B_k = \frac{1}{k}\,;$$

thus $\sup B_{k+1} < \inf B_k$ for each $k \in \mathbb{N}$. Therefore, B_{k+1} is on the left of B_k for each $k \in \mathbb{N}$. We also note that every element in A is an isolated point of A.

Suppose that $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence in A.

(a) Suppose that there exists $k \in \mathbb{N}$ such that $\{n \in \mathbb{N} \,|\, x_n \in B_k\} = \infty$. Then $\lim_{n\to\infty} x_n \in \overline{B_k}$.

(b) Suppose that for all $k \in \mathbb{N}$ we have $\{n \in \mathbb{N} \,|\, x_n \in B_k\} < \infty$. Then there exists a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ satisfying that $x_{n_{j+1}} < x_{n_j}$ for all $j \in \mathbb{N}$. Such a subsequence must converge to 0 since for each $k \in \mathbb{N}$ only finitely many terms of x_{n_j} belongs to the set $B_1 \cup B_2 \cup \cdots \cup B_k$ while the supremum of the rest of the subsequence is not greater than $\inf B_k$.

Therefore, by the fact that $\overline{B_k} = B_k \cup \{\frac{1}{k}\}$, we find that

$$\bar{A} = A \cup \left\{\frac{1}{k} \,\middle|\, k \in \mathbb{N}\right\} \cup \{0\}\,.$$

Then the fact that every point in A is an isolated point of A implies that

$$A' = \bar{A} \backslash \text{ collection of isolated point of } A = \left\{\frac{1}{k} \,\middle|\, k \in \mathbb{N}\right\} \cup \{0\}\,.$$

Noting that every point of A' except $\{0\}$ is an isolated point of A', we have $A^{(2)} = \{0\}$ so that $A^{(3)} = \varnothing$.

3. Following 5, we have a clear picture how to construct such a set. Let

$$A_m = \left\{\frac{1}{i_1} + \frac{1}{i_2} + \cdots + \frac{1}{i_m} \,\middle|\, i_j \in \mathbb{N} \text{ and } i_{j+1} - 1 > i_j(i_j - 1) \text{ for all } 1 \leqslant j \leqslant m\right\}.$$

Then $A_m' = A_{m-1} \cup \{0\}$, $A_m^{(2)} = A_{m-2} \cup \{0\}$, $\cdots$, $A_m^{(k)} = A_{m-k} \cup \{0\}$ if $m > k$, $A_m^{(m)} = \{0\}$ and $A_m^{(m+1)} = \varnothing$. □

Problem 10. Recall that a cluster point x of a sequence $\{x_n\}_{n=1}^{\infty}$ satisfies that

$$\forall\, \varepsilon > 0, \#\{n \in \mathbb{N} \,|\, x_n \in B(x, \varepsilon)\} = \infty\,.$$

Show that the collection of cluster points of a sequence (in a metric space) is closed.

Proof. Let (M,d) be a metric space, $\{x_k\}_{k=1}^\infty$ be a sequence in M, and A be the collection of cluster points of $\{x_k\}_{k=1}^\infty$. We would like to show that $A \supseteq \bar{A}$.

Let $y \in A^\complement$. Then y is not a cluster point of $\{x_k\}_{k=1}^\infty$; thus

$$\exists\, \varepsilon > 0 \ni \#\big\{n \in \mathbb{N} \,\big|\, x_n \in B(y,\varepsilon)\big\} < \infty\,.$$

For $z \in B(y,\varepsilon)$, let $r = \varepsilon - d(y,z) > 0$. Then $B(z,r) \subseteq B(y,\varepsilon)$ (see Figure 1 or check rigorously using the triangle inequality). As a consequence, $\#\big\{n \in \mathbb{N} \,\big|\, x_n \in B(z,r)\big\} < \infty$ which implies that $z \notin A$.

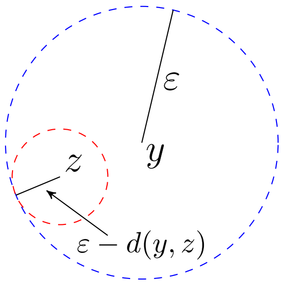

Figure 1: $B(z, \varepsilon - d(y,z)) \subseteq B(y,\varepsilon)$ if $z \in B(y,\varepsilon)$

Therefore, if $z \in B(y,\varepsilon)$ then $z \in A^\complement$; thus $B(y,\varepsilon) \cap A = \varnothing$. We then conclude that if $y \in A^\complement$ then $y \notin \bar{A}$. □

Problem 11. Let $(\mathcal{V}, \|\cdot\|)$ ba a normed vector space. A set C in $\mathcal{V}$ is called ***convex*** if for all $x, y \in C$, the line segment joining x and y, denoted by $\overline{xy}$, lies in C. Let C be a non-empty convex set in $\mathcal{V}$.

1. Show that $\bar{C}$ is convex.

2. Show that if $\boldsymbol{x} \in \mathring{C}$ and $\boldsymbol{y} \in \bar{C}$, then $(1-\lambda)\boldsymbol{x} + \lambda\boldsymbol{y} \in \mathring{C}$ for all $\lambda \in (0,1)$. This result is sometimes called the ***line segment principle***.

3. Show that $\mathring{C}$ is convex (you may need the conclusion in 2 to prove this).

4. Show that $\text{cl}(\mathring{C}) = \text{cl}(C)$.

5. Show that $\text{int}(\bar{C}) = \text{int}(C)$.

Hint: 2. Prove by contradiction.
3 and 4. Use the line segment principle.
5. Show that $\boldsymbol{x} \in \text{int}(\bar{C})$ can be written as $(1-\lambda)\boldsymbol{y} + \lambda\boldsymbol{z}$ for some $\boldsymbol{y} \in \mathring{C}$ and $\boldsymbol{z} \in B(\boldsymbol{x},\varepsilon) \subseteq \bar{C}$.

Proof. 1. Let $\boldsymbol{x}, \boldsymbol{y} \in \bar{C}$ and $0 \leqslant \lambda \leqslant 1$ be given. Then there exist sequences $\{\boldsymbol{x}_k\}_{k=1}^\infty$ and $\{\boldsymbol{y}_k\}_{k=1}^\infty$ in C such that $\boldsymbol{x}_k \to \boldsymbol{x}$ and $\boldsymbol{y}_k \to \boldsymbol{y}$ as $k \to \infty$. Since C is convex, $(1-\lambda)\boldsymbol{x}_k + \lambda\boldsymbol{y}_k \in C$ for each $k \in \mathbb{N}$; thus by the fact that $C \subseteq \bar{C}$, $(1-\lambda)\boldsymbol{x}_k + \lambda\boldsymbol{y}_k \in \bar{C}$ for each $k \in \mathbb{N}$. Since $(1-\lambda)\boldsymbol{x}_k + \lambda\boldsymbol{y}_k \to (1-\lambda)\boldsymbol{x} + \lambda\boldsymbol{y}$ as $k \to \infty$ and $\bar{C}$ is closed, we must have $(1-\lambda)\boldsymbol{x} + \lambda\boldsymbol{y} \in \bar{C}$; thus $\bar{C}$ is convex if C is convex.

2. Suppose the contrary that there exists $\lambda \in (0,1)$ such that $(1-\lambda)\boldsymbol{x} + \lambda \boldsymbol{y} \notin \mathring{C}$. Then for each $k \in \mathbb{N}$, there exists $\boldsymbol{z}_k \notin C$ such that

$$\big\|(1-\lambda)\boldsymbol{x} + \lambda \boldsymbol{y} - \boldsymbol{z}_k\big\| < \frac{1}{k} \qquad \forall\, k \in \mathbb{N}\,.$$

Since $\boldsymbol{y} \in \bar{C}$, there exists a sequence $\{\boldsymbol{y}_k\}_{k=1}^{\infty} \in C$ satisfying

$$\|\boldsymbol{y}_k - \boldsymbol{y}\| < \frac{1}{\lambda k} \qquad \forall\, k \in N\,.$$

Therefore, if $k \in N$,

$$\big\|(1-\lambda)\boldsymbol{x} + \lambda \boldsymbol{y}_k - \boldsymbol{z}_k\big\| \leqslant \big\|(1-\lambda)\boldsymbol{x} + \lambda \boldsymbol{y} - \boldsymbol{z}_k\big\| + \|\lambda(\boldsymbol{y} - \boldsymbol{y}_k)\| < \frac{2}{k}\,;$$

thus

$$\Big\|\boldsymbol{x} - \frac{\boldsymbol{z}_k - \lambda \boldsymbol{y}_k}{1-\lambda}\Big\| < \frac{2}{k(1-\lambda)} \qquad \forall\, k \in \mathbb{N}\,.$$

Since $\boldsymbol{x} \in \mathring{C}$, there exists $N > 0$ such that $B\big(\boldsymbol{x}, \dfrac{2}{(1-\lambda)N}\big) \subseteq C$; thus $\dfrac{\boldsymbol{z}_k - \lambda \boldsymbol{y}_k}{1-\lambda} \in C$ whenever $k \geqslant N$. By the convexity of C,

$$\boldsymbol{z}_k = (1-\lambda)\frac{\boldsymbol{z}_k - \lambda \boldsymbol{y}_k}{1-\lambda} + \lambda \boldsymbol{y}_k \in C\,,$$

a contradiction.

3. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathring{C}$. By the line segment principle, $(1-\lambda)\boldsymbol{x} + \lambda \boldsymbol{y} \in \mathring{C}$ for all $\lambda \in (0,1)$ (since $\mathring{C} \subseteq \bar{C}$ so that $y \in \bar{C}$). This further implies that $(1-\lambda)\boldsymbol{x} + \lambda \boldsymbol{y} \in \mathring{C}$ for all $\lambda \in [0,1]$ since $\boldsymbol{x}, \boldsymbol{y} \in \mathring{C}$; thus $\mathring{C}$ is convex.

4. It suffices to show that $\text{cl}(\mathring{C}) \supseteq \text{cl}(C)$. Let $\boldsymbol{x} \in \text{cl}(C)$. Pick any $\boldsymbol{y} \in \mathring{C}$. By the line segment principle,

$$\boldsymbol{x}_k \equiv \big(1 - \frac{1}{k}\big)\boldsymbol{x} + \frac{1}{k}\boldsymbol{y} \in \mathring{C} \qquad \forall\, k \geqslant 2\,.$$

Since $\boldsymbol{x}_k \to \boldsymbol{x}$ as $k \to \infty$, we find that $\boldsymbol{x} \in \text{cl}(\mathring{C})$.

5. It suffices to show that $\text{int}(\bar{C}) \subseteq \text{int}(C)$. Let $\boldsymbol{x} \in \text{int}(\bar{C})$. Then there exists $\varepsilon > 0$ such that $B(\boldsymbol{x}, \varepsilon) \subseteq \bar{C}$. Let $\boldsymbol{y} \in \text{int}(C)$. If $\boldsymbol{y} = \boldsymbol{x}$, then $\boldsymbol{x} \in \text{int}(C)$. If $\boldsymbol{y} \neq \boldsymbol{x}$, define $\boldsymbol{z} = \boldsymbol{x} + \alpha(\boldsymbol{x} - \boldsymbol{y})$, where

$$\alpha = \frac{\varepsilon}{2\|\boldsymbol{x} - \boldsymbol{y}\|}\,.$$

Then $\|\boldsymbol{x} - \boldsymbol{z}\| = \dfrac{\varepsilon}{2}$; thus $\boldsymbol{z} \in B(\boldsymbol{x}, \varepsilon)$ which further implies that $\boldsymbol{z} \in \bar{C}$. The line segment principle implies that $(1-\lambda)\boldsymbol{y} + \lambda \boldsymbol{z} \in \mathring{C}$ for all $\lambda \in (0,1)$. Taking $\lambda = \dfrac{1}{1+\alpha}$, we find that

$$(1-\lambda)\boldsymbol{y} + \lambda \boldsymbol{z} = \frac{\alpha}{1+\alpha}\boldsymbol{y} + \frac{1}{1+\alpha}\big(\boldsymbol{x} + \alpha(\boldsymbol{x} - \boldsymbol{y})\big) = \boldsymbol{x}$$

which shows that $\boldsymbol{x} \in \text{int}(C)$. □

Problem 12. Let $(\mathcal{V}, \|\cdot\|)$ be a normed vector space. Show that for all $\boldsymbol{x} \in \mathcal{V}$ and $r > 0$,

$$\operatorname{int}\big(B[\boldsymbol{x}, r]\big) = B(\boldsymbol{x}, r)\,.$$

Proof. Let $\boldsymbol{y} \in \mathcal{V}$ such that $\|\boldsymbol{x}-\boldsymbol{y}\| = r$. Then $\boldsymbol{x} + \lambda(\boldsymbol{y}-\boldsymbol{x}) \in B[\boldsymbol{x}, r]^{\complement}$ for all $|\lambda| > 1$. In particular, $\boldsymbol{y}_n \equiv \boldsymbol{x} + \big(1 + \dfrac{1}{n}\big)(\boldsymbol{y}-\boldsymbol{x}) \in B[\boldsymbol{x}, r]^{\complement}$ for all $n \in \mathbb{N}$. Moreover,

$$\|\boldsymbol{y}_n - \boldsymbol{y}\| = \frac{1}{n}\|\boldsymbol{x}-\boldsymbol{y}\| = \frac{r}{n} \to 0 \quad \text{as} \quad n \to \infty\,.$$

Therefore, $\lim\limits_{n\to\infty} \boldsymbol{y}_n = \boldsymbol{y}$ which implies that $\boldsymbol{y} \in \partial B[\boldsymbol{x}, r]$ $\big($since $\boldsymbol{y} \in B[\boldsymbol{x}, r]$ and $\boldsymbol{y}$ is the limit of a sequence from $B[\boldsymbol{x}, r]^{\complement}\big)$; thus

$$\big\{\boldsymbol{y} \in \mathcal{V} \,\big|\, \|\boldsymbol{x}-\boldsymbol{y}\| = r\big\} \subseteq \partial B[\boldsymbol{x}, r]\,.$$

On the other hand, $B(\boldsymbol{x}, r)$ is open and $B[\boldsymbol{x}, r] = B(\boldsymbol{x}, r) \cup \big\{\boldsymbol{y} \in \mathcal{V} \,\big|\, \|\boldsymbol{x}-\boldsymbol{y}\| = r\big\}$. Therefore, $B(x, r)$ is the largest open set contained inside $B[\boldsymbol{x}, r]$; thus $B(\boldsymbol{x}, r) = \operatorname{int}(B[\boldsymbol{x}, r])$. □

Problem 13. Let $\mathcal{M}_{n\times n}$ denote the collection of all $n \times n$ square real matrices, and $(\mathcal{M}_{n\times n}, \|\cdot\|_{p,q})$ be a normed space with norm $\|\cdot\|_{p,q}$ given in Problem 4 of Exercise 6. Show that the set

$$\mathrm{GL}(n) \equiv \big\{A \in \mathcal{M}_{n\times n} \,\big|\, \det(A) \neq 0\big\}$$

is an open set in $\mathcal{M}_{n\times n}$. The set $\mathrm{GL}(n)$ is called the general linear group.

Proof. Let $A \in \mathrm{GL}(n)$ be given. Then $A^{-1} \in \mathcal{M}_{n\times n}$ exists; thus

$$\|A^{-1}\boldsymbol{x}\|_2 \leqslant \|A^{-1}\|_{2,2}\|\boldsymbol{x}\|_2 \qquad \forall\, \boldsymbol{x} \in \mathbb{R}^n$$

which, using the fact that $A : \mathbb{R}^n \xrightarrow[\text{onto}]{1-1} \mathbb{R}^n$, implies that

$$\frac{1}{\|A^{-1}\|_{2,2}}\|\boldsymbol{x}\|_2 \leqslant \|A\boldsymbol{x}\|_2 \qquad \forall\, \boldsymbol{x} \in \mathbb{R}^n\,.$$

Let $r = \dfrac{1}{\|A^{-1}\|_{2,2}}$. For $B \in B(A, r)$, we have $\|A - B\|_{2,2} < r$; thus for each $\boldsymbol{x} \in \mathbb{R}^n$,

$$r\|\boldsymbol{x}\|_2 = \frac{1}{\|A^{-1}\|_{2,2}}\|\boldsymbol{x}\|_2 \leqslant \|A\boldsymbol{x}\|_{\mathbb{R}^n} \leqslant \|(A-B)\boldsymbol{x}\|_2 + \|B\boldsymbol{x}\|_2 \leqslant \|A-B\|_{2,2}\|\boldsymbol{x}\|_{\mathbb{R}^n} + \|B\boldsymbol{x}\|_2$$

which further implies that

$$\|B\boldsymbol{x}\|_2 \geqslant \big(r - \|A-B\|_{2,2}\big)\|\boldsymbol{x}\|_2 \qquad \forall\, \boldsymbol{x} \in \mathbb{R}^n\,.$$

Therefore, $B\boldsymbol{x} = \mathbf{0}$ if and only if $\boldsymbol{x} = \mathbf{0}$ which shows that B is invertible; thus we established that

for each $A \in \mathrm{GL}(n)$, there exists $r = \dfrac{1}{\|A^{-1}\|_{2,2}} > 0$ such that $B(A, r) \subseteq \mathrm{GL}(n)$.

This shows that $\mathrm{GL}(n)$ is open. □

Problem 14. Show that every open set in $\mathbb{R}$ is the union of at most countable collection of disjoint open intervals; that is, if $U \subseteq \mathbb{R}$ is open, then

$$U = \bigcup_{k\in\mathcal{I}} (a_k, b_k)\,,$$

where $\mathcal{I}$ is countable, and $(a_k, b_k) \cap (a_\ell, b_\ell) = \varnothing$ if $k \neq \ell$.

Hint: For each point $x \in U$, define $L_x = \{y \in \mathbb{R} \,|\, (y, x) \subseteq U\}$ and $R_x = \{y \in \mathbb{R} \,|\, (x, y) \subseteq U\}$. Define $I_x = (\inf L_x, \sup R_x)$. Show that $I_x = I_y$ if $(x, y) \in U$ and if $(x, y) \nsubseteq U$ then $I_x \cap I_y = \varnothing$

Proof. As suggested in the hint, for each point $x \in U$ we define $L_x = \{y \in \mathbb{R} \,|\, (y, x) \subseteq U\}$ and $R_x = \{y \in \mathbb{R} \,|\, (x, y) \subseteq U\}$. We note that $a \equiv \inf L_x \notin U$ since if $a \in U$, by the openness of U there exists $r > 0$ such that $(a - r, a + r) \subseteq U$ which implies that $(a - r, x) \subseteq U$ so that $a - r \in L_x$, a contradiction to the fact that $a = \inf L_x$. Similarly, $\sup R_x \notin U$. Therefore, $I_x = (\inf L_x, \sup L_x)$ is the maximal connected subset of U containing x.

Suppose that $x, y \in U$ and $(x, y) \subseteq U$. If $z \in L_x$ (so $(z, x) \subseteq U$), by the fact that $(z, y) = (z, x) \cup \{x\} \cup (x, y)$, we find that $z \in L_y$. Therefore, $L_x \subseteq L_y$ which implies that $\inf L_y \leqslant \inf L_x$. Moreover, if $\inf L_y < \inf L_x$, then there exists $z \in L_y$ such that $\inf L_y \leqslant z < \inf L_x$. Since $z \in L_y$, $(z, y) \subseteq U$; thus $(z, x) \subseteq U$ which shows that $z \in L_x$, a contradiction to that $z < \inf L_x$. Therefore, $\inf L_y = \inf L_x$. Similarly, $\sup R_y = \sup R_x$ so we conclude that $I_x = I_y$.

On the other hand, if that $x, y \in U$ but $(x, y) \nsubseteq U$, then there exists $x < z < y$ with $z \notin U$ which results in that $\sup R_x \leqslant z \leqslant \inf L_y$ so that $I_x \cap I_y = \varnothing$. Therefore, we establish that

1. if $x, y \in U$ and $(x, y) \subseteq U$, then $I_x = I_y$.

2. if $x, y \in U$ and $(x, y) \nsubseteq U$, then $I_x \cap I_y = \varnothing$.

This implies that U is the union of disjoint open intervals. Since every such open interval contains a rational number, we can denote each such open interval as I_k, where k belongs to a countable index set $\mathcal{I}$. Write $I_k = (a_k, b_k)$, then $U = \bigcup\limits_{k\in\mathcal{I}} (a_k, b_k)$. □

Problem 15. Let (M, d) be a metric space. A set $A \subseteq M$ is said to be ***perfect*** if $A = A'$ (so that there is no isolated points). The Cantor set is constructed by the following procedure: let $E_0 = [0, 1]$. Remove the segment $\left(\frac{1}{3}, \frac{2}{3}\right)$, and let E_1 be the union of the intervals

$$\left[0, \frac{1}{3}\right], \left[\frac{2}{3}, 1\right].$$

Remove the middle thirds of these intervals, and let E_2 be the union of the intervals

$$\left[0, \frac{1}{9}\right], \left[\frac{2}{9}, \frac{3}{9}\right], \left[\frac{6}{9}, \frac{7}{9}\right], \left[\frac{8}{9}, 1\right].$$

Continuing in this way, we obtain a sequence of closed set E_k such that

(a) $E_1 \supseteq E_2 \supseteq E_2 \supseteq \cdots$;

(b) E_n is the union of 2^n intervals, each of length 3^{-n}.

The set $C = \bigcap_{n=1}^{\infty} E_n$ is called the ***Cantor set***.

1. Show that C is a perfect set.

2. Show that C is uncountable.

3. Find $\text{int}(C)$.

Proof. 1. Let $x \in C$. Then $x \in E_N$ for some $N \in \mathbb{N}$. For each $n \in \mathbb{N}$, E_n is the union of disjoint closed intervals with length $\frac{1}{3^n}$, and ∂E_n consists of the end-points of these disjoint closed intervals whose union is E_n. Therefore, there exists $x_n \in \partial E_{N+n-1} \backslash \{x\}$ such that $|x_n - x| < \frac{1}{3^{N-1+n}}$. Since $\partial E_n \subseteq C$ for each $n \in \mathbb{N}$, we find that $\{x_n\}_{n=1}^{\infty} \in C \backslash \{x\}$. Moreover, $\lim_{n \to \infty} x_n = x$; thus $x \in C'$ which shows $C \subseteq C'$. Since C is the intersection of closed sets, C is closed; thus

$$C \subseteq C' \subseteq \bar{C} = C$$

so we establish that $C' = C$.

2. For $x \in [0, 1]$, write x in ternary expansion (三進位展開); that is,

$$x = 0.d_1 d_2 d_3 \cdots \cdots .$$

Here we note that repeated 2's are chosen by preference over terminating decimals. For example, we write $\frac{1}{3}$ as $0.02222 \cdots$ instead of 0.1. Define

$$A = \left\{ x = 0.d_1 d_2 d_3 \cdots \,\middle|\, d_j \in \{0, 2\} \text{ for all } j \in \mathbb{N} \right\}.$$

Note each point in ∂E_n belongs to A; thus $A \subseteq C$. On the other hand, A has a one-to-one correspondence with $[0, 1]$ $\big(x = 0.d_1 d_2 \cdots \in A \Leftrightarrow y = 0.\frac{d_1}{2}\frac{d_2}{2} \cdots \in [0, 1]$, where y is expressed in binary expansion (二進位展開) with repeated 1's instead of terminating decimals$\big)$. Since $[0, 1]$ is uncountable, A is uncountable; thus C is uncountable.

3. If $\text{int}(C)$ is non-empty, then by the fact that $\text{int}(C)$ is open in $(R, |\cdot|)$, by Problem 7 the Cantor set C contains at least one interval (x, y). Note that there exists $N > 0$ such that $|x - y| < \frac{1}{3^n}$ for all $n \geqslant N$. Since the length of each interval in E_n has length $\frac{1}{3^n}$, we find that if $n \geqslant N$, the interval (x, y) is not contained in any interval of E_n. In other words, there must be $z \in (x, y)$ such that $z \in E_n^{\complement}$ which shows that $(x, y) \nsubseteq \bigcap_{n=1}^{\infty} E_n$. Therefore, $\text{int}(C) = \varnothing$. □