Exercise Problem Sets 2

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Problem 1. Complete the following.

1. Verify the Wallis's formula: if n is a non-negative integer, then

$$\int_0^{\frac{\pi}{2}} \sin^{2n+1} x\, dx = \int_0^{\frac{\pi}{2}} \cos^{2n+1} x\, dx = \frac{(2^n n!)^2}{(2n+1)!}$$

and

$$\int_0^{\frac{\pi}{2}} \sin^{2n} x\, dx = \int_0^{\frac{\pi}{2}} \cos^{2n} x\, dx = \frac{(2n)!}{(2^n n!)^2} \cdot \frac{\pi}{2}\,.$$

2. Let $I_n = \displaystyle\int_0^{\frac{\pi}{2}} \sin^n x\, dx$. Show that $\displaystyle\lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = 1$.

3. Let $s_n = \dfrac{n!}{n^{n+0.5}e^{-n}}$. Show that $\{s_n\}_{n=1}^{\infty}$ is a decreasing sequence; that is, $s_n \geqslant s_{n+1}$ for all $n \in \mathbb{N}$.

Hint:

2. Show that $I_{2n+2} \leqslant I_{2n+1} \leqslant I_{2n}$ for all $n \in \mathbb{N}$ and then apply the Sandwich lemma.

3. Consider the function $f(x) = \big(1 + \dfrac{1}{x}\big)^{x+0.5}$.

Proof. 1. Integrating by parts, we find that

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^n x\, dx &= -\sin^{n-1} x \cos x\Big|_{x=0}^{x=\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x\, dx \\ &= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x(1-\sin^2 x)\, dx \\ &= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x\, dx - (n-1)\int_0^{\frac{\pi}{2}} \sin^n x\, dx\,;\end{aligned}$$

thus

$$\int_0^{\frac{\pi}{2}} \sin^n x\, dx = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x\, dx\,.$$

Therefore,

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^{2n+1} x\, dx &= \frac{2n}{2n+1}\int_0^{\frac{\pi}{2}} \sin^{2n-1} x\, dx = \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\int_0^{\frac{\pi}{2}} \sin^{2n-3} x\, dx = \cdots \\ &= \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdot\frac{2n-4}{2n-3}\cdots\frac{2}{3}\int_0^{\frac{\pi}{2}} \sin x\, dx = \frac{2}{3}\cdot\frac{4}{5}\cdots\frac{2n}{2n+1} \\ &= \frac{2^2 4^2\cdots(2n)^2}{(2n+1)!} = \frac{(2^n n!)^2}{(2n+1)!}\end{aligned}$$

and

$$\int_0^{\frac{\pi}{2}} \sin^{2n} x\,dx = \frac{2n-1}{2n}\int_0^{\frac{\pi}{2}} \sin^{2n-2} x\,dx = \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\int_0^{\frac{\pi}{2}} \sin^{2n-4} x\,dx = \cdots$$
$$= \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdot\frac{2n-5}{2n-4}\cdots\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^0 x\,dx = \frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n}\cdot\frac{\pi}{2}$$
$$= \frac{(2n)!}{2^2 4^2\cdots(2n)^2}\cdot\frac{\pi}{2} = \frac{(2n)!}{(2^n n!)^2}\cdot\frac{\pi}{2}.$$

2. On the interval $\big[0,\frac{\pi}{2}\big]$, $0 \leqslant \sin x \leqslant 1$; thus

$$\sin^{2n+2} x \leqslant \sin^{2n+1} x \leqslant \sin^{2n} x \qquad \forall\, x \in \Big[0,\frac{\pi}{2}\Big].$$

Therefore, $I_{2n+2} \leqslant I_{2n+1} \leqslant I_{2n}$ so that

$$\frac{I_{2n+2}}{I_{2n}} \leqslant \frac{I_{2n+1}}{I_{2n}} \leqslant 1 \qquad \forall\, n \in \mathbb{N}.$$

Since $\dfrac{I_{2n+2}}{I_{2n}} = \dfrac{2n+1}{2(n+1)}$, the Sandwich Lemma implies that

$$\lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = 1.$$

3. Since $\displaystyle\lim_{n\to\infty}\Big(1+\frac{1}{n}\Big)^{n+0.5} = e$ and $\dfrac{s_n}{s_{n+1}} = \dfrac{\dfrac{n!}{n^{n+0.5}e^{-n}}}{\dfrac{(n+1)!}{(n+1)^{n+1.5}e^{-n-1}}} = \dfrac{1}{e}\Big(1+\dfrac{1}{n}\Big)^{n+0.5}$, it suffices to show that the function $f(x) \equiv \big(1+\frac{1}{x}\big)^{x+0.5}$ is increasing on $[1,\infty)$. Nevertheless, this is the same as proving that the function $g(x) \equiv (1+x)^{\frac{1}{x}+\frac{1}{2}}$ is decreasing on $(0,1]$.

Differentiate g, we find that

$$g'(x) = g(x)\frac{\big[\ln(1+x)+\frac{2+x}{1+x}\big]2x - 2(2+x)\ln(1+x)}{4x^2}$$
$$= \frac{2x+x^2-2(1+x)\ln(1+x)}{2x^2(1+x)}.$$

To see the sign of the denominator $h(x) = 2x+x^2-2(1+x)\ln(1+x)$ on $(0,1]$, we differentiate h and find that

$$h'(x) = 2+2x-2\ln(1+x)-2 = 2\big[x-\ln(1+x)\big]$$

and one more differentiation shows that

$$h''(x) = 1-\frac{1}{1+x} = \frac{x}{1+x} > 0 \qquad \forall\, x \in (0,1].$$

Therefore, h' in increasing on $(0,1]$ which implies that $h'(x) \geqslant h'(0) = 0$ for all $x \in (0,1]$. This further implies that $h(x) \geqslant h(0) = 0$ for all $x \in (0,1]$; thus $g'(x) \geqslant 0$ for all $x \in \big(0,1\big]$. □

Problem 2. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field with Archimedean property, $I \subseteq \mathbb{F}$ be an interval, and $f : I \to \mathbb{F}$ be a function.

1. f is said to have a limit at c or we say that the limit of f at c exists if

   (a) there exists a sequence $\{x_n\}_{n=1}^{\infty} \subseteq I \backslash \{c\}$ with limit c, and

   (b) $\lim\limits_{n \to \infty} f(x_n)$ exists for all convergent sequences $\{x_n\}_{n=1}^{\infty} \subseteq I \backslash \{c\}$ with limit c.

   Show that the limit of f at c exists if and only if there exists $L \in \mathbb{F}$ satisfying that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\big|f(x) - L\big| < \varepsilon \qquad \text{whenever} \quad 0 < |x - c| < \delta \text{ and } x \in I\,.$$

2. f is said to be continuous at a point $c \in I$ if

$$\lim_{n \to \infty} f(x_n) = f(c) \quad \text{for all convergent sequences } \{x_n\}_{n=1}^{\infty} \subseteq I \text{ with limit } c\,.$$

   Show that f is continuous at c if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\big|f(x) - f(c)\big| < \varepsilon \qquad \text{whenever} \quad |x - c| < \delta \text{ and } x \in I\,.$$

Proof. 1. ("$\Rightarrow$") Suppose that the limit of f at c exists.

Claim: If $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \subseteq I \backslash \{c\}$ and $\lim\limits_{n \to \infty} x_n = \lim\limits_{n \to \infty} y_n = c$, then $\lim\limits_{n \to \infty} f(x_n) = \lim\limits_{n \to \infty} f(y_n)$.

Proof of claim: Define z_n by

$$z_n = \begin{cases} x_{\frac{n+1}{2}} & \text{if } n \text{ is odd}\,, \\ y_{\frac{n}{2}} & \text{if } n \text{ is even}\,. \end{cases}$$

Then $\lim\limits_{n \to \infty} z_n = c$; thus by the assumption that the limit of f at c exists, we find that $\lim\limits_{n \to \infty} f(z_n)$ exists. On the other hand, since $\lim\limits_{n \to \infty} f(x_n)$ and $\lim\limits_{n \to \infty} f(y_n)$ both exist, we must have

$$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} f(z_n) = \lim_{n \to \infty} f(y_n)\,. \qquad \square$$

Having established the claim, we find that there exists $L \in \mathbb{F}$ such that $\lim\limits_{n \to \infty} f(x_n) = L$ whenever $\{x_n\}_{n=1}^{\infty} \subseteq I \backslash \{c\}$ is a convergent sequence with limit c.

Suppose the contrary that there exists $\varepsilon > 0$ such that for each $\delta > 0$ there exists $x \in I$ satisfying $0 < |x - c| < \delta$ and $\big|f(x) - L\big| \geqslant \varepsilon$. In particular, for each $n \in \mathbb{N}$, there exists $x_n \in I$ satisfying

$$0 < |x_n - c| < \frac{1}{n} \qquad \text{and} \qquad \big|f(x_n) - L\big| \geqslant \varepsilon\,.$$

Then $\{x_n\}_{n=1}^{\infty} \subseteq I \backslash \{c\}$ and Archimedean Property implies that $\lim\limits_{n \to \infty} x_n = c$. Therefore, the claim shows that $\lim\limits_{n \to \infty} f(x_n) = L$ which contradicts to the inequality $\big|f(x_n) - L\big| \geqslant \varepsilon$.

("$\Leftarrow$") Let $\{x_n\}_{n=1}^{\infty} \subseteq I \backslash \{c\}$ be a convergent sequence with limit c, and $\varepsilon > 0$ be given. By assumption, there exists $\delta > 0$ such that

$$\big|f(x) - L\big| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta \text{ and } x \in I\,.$$

By the fact that $\lim_{n\to\infty} x_n = c$, there exists $N > 0$ such that

$$|x_n - c| < \delta \quad \text{whenever} \quad n \geqslant N\,.$$

Therefore, if $n \geqslant N$, then $0 < |x_n - c| < \delta$ and $x_n \in I$ so that $\big|f(x_n) - L\big| < \varepsilon$. This implies that $\lim_{n\to\infty} f(x_n) = L$; thus

$$\lim_{n\to\infty} f(x_n) \text{ exists for all convergent sequences } \{x_n\}_{n=1}^{\infty} \subseteq I\backslash\{c\} \text{ with limit } c.$$

2. ("$\Rightarrow$") Suppose that f is continuous at a point $c \in I$; that is,

$$\lim_{n\to\infty} f(x_n) = f(c) \quad \text{for all convergent sequences } \{x_n\}_{n=1}^{\infty} \subseteq I \text{ with limit } c\,.$$

In particular, for all convergent sequences $\{x_n\}_{n=1}^{\infty} \subseteq I\backslash\{c\}$ with limit c we have $\lim_{n\to\infty} f(x_n) = f(c)$. Therefore, 1 implies that

$$(\forall\, \varepsilon > 0)(\exists\, \delta > 0)\Big(\big|f(x) - f(c)\big| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta \text{ and } x \in I\Big)\,.$$

We note that we must have $\big|f(c) - f(c)\big| < \varepsilon$; thus the statement above implies that

$$(\forall\, \varepsilon > 0)(\exists\, \delta > 0)\Big(\big|f(x) - f(c)\big| < \varepsilon \quad \text{whenever} \quad |x - c| < \delta \text{ and } x \in I\Big)\,.$$

("$\Leftarrow$") We note that the assumption in particular implies that

$$(\forall\, \varepsilon > 0)(\exists\, \delta > 0)\Big(\big|f(x) - f(c)\big| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta \text{ and } x \in I\Big)\,;$$

thus 1 implies that

$$\lim_{n\to\infty} f(x_n) = f(c) \text{ for all convergent sequences } \{x_n\}_{n=1}^{\infty} \subseteq I\backslash\{c\} \text{ with limit } c. \tag{0.1}$$

Now suppose the contrary that there exists a convergent sequence $\{x_n\}_{n=1}^{\infty} \subseteq I$ with limit c but $\lim_{n\to\infty} f(x_n) \neq f(c)$. Then (0.1) implies that

$$\#\{n \in \mathbb{N} \,|\, x_n = c\} = \infty\,.$$

(a) If $\#\{n \in \mathbb{N} \,|\, x_n \neq c\} < \infty$, then there exists $N > 0$ such that $x_n = c$ for all $n \geqslant N$. This implies that $\big|f(x_n) - f(c)\big| = 0 < \varepsilon$ whenever $n \geqslant$N, a contradiction to that $\lim_{n\to\infty} f(x_n) \neq f(c)$.

(b) If $\#\{n \in \mathbb{N} \,|\, x_n \neq c\} = \infty$, then $\{n \in \mathbb{N} \,|\, x_n \neq c\} = \{n_j \in \mathbb{N} \,|\, j \in \mathbb{N}, n_j < n_{j+1}\}$ and $\{x_{n_j}\}_{j=1}^{\infty} \subseteq I\backslash\{c\}$ is a convergent sequence with limit c. Therefore, (0.1) implies that

$$\lim_{j\to\infty} f\big(x_{n_j}\big) = f(c)\,.$$

Let $\varepsilon > 0$ be given. The limit above shows that there exists $J > 0$ such that $\big|f(x_{n_j}) - f(c)\big| < \varepsilon$ whenever $j \geqslant J$. Let $N = n_J$. Then for all $n \geqslant N$, we have either $x_n = c$ or $x_n = x_{n_j}$ for some $j \geqslant J$; thus

$$\big|f(x_n) - f(c)\big| < \varepsilon \quad \text{whenever} \quad n \geqslant N\,,$$

a contradiction to that $\lim_{n\to\infty} f(x_n) \neq f(c)$. □