Analysis MA2050-* Final Exam

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Problem 1. (15%) For a function $g : [0, \infty) \to \mathbb{C}$ satisfying $\int_0^\infty |g(x)|\,dx < \infty$, the Fourier sine transform of g, denoted by $\mathscr{F}_{\sin}[g]$, is a function defined by

$$\mathscr{F}_{\sin}[g](\xi) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(y)\sin(y\xi)\,dy\,.$$

Show that if g is integrable on $[0,\infty)$ and $\mathscr{F}_{\sin}[g]$ is also integrable on $[0,\infty)$, then

$$\mathscr{F}_{\sin}\big[\mathscr{F}_{\sin}[g]\big](x) = g(x) \quad \text{whenever } x \in (0,\infty) \text{ and } g \text{ is continuous at } x$$

or equivalently,

$$g(x) = \frac{2}{\pi}\int_0^\infty \Big(\int_0^\infty g(y)\sin(y\xi)\,dy\Big)\sin(x\xi)\,d\xi \quad \text{whenever } x \in (0,\infty) \text{ and } g \text{ is continuous at } x.$$

Hint: Consider the odd extension of g, and make use of the Fourier inversion formula.

Problem 2. In this problem we discuss the derivative of tempered distributions. Complete the following.

1. (5%) Since $\big\langle \frac{\partial f}{\partial x_j}, g\big\rangle = -\big\langle f, \frac{\partial g}{\partial x_j}\big\rangle$ for all $f, g \in \mathscr{S}(\mathbb{R}^n)$, we define the derivatives of tempered distributions as follows: Let $T \in \mathscr{S}(\mathbb{R}^n)'$ be a tempered distribution. The partial derivative of T w.r.t. x_j, denoted by $\frac{\partial T}{\partial x_j}$, is a tempered distribution defined by

   $$\Big\langle \frac{\partial T}{\partial x_j}, \phi\Big\rangle = -\Big\langle T, \frac{\partial \phi}{\partial x_j}\Big\rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)\,.$$

   Show that $\frac{\partial T}{\partial x_j}$ is indeed a tempered distribution; that is, show that there exists a sequence $\{C_k\}_{k=1}^\infty$ such that

   $$\Big|\Big\langle \frac{\partial T}{\partial x_j}, \phi\Big\rangle\Big| \leqslant C_k p_k(\phi) \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n) \text{ and } k \gg 1\,.$$

2. (10%) Show that for $1 \leqslant j \leqslant n$,

   $$\mathscr{F}_x\Big[\frac{\partial T}{\partial x_j}\Big](\xi) = i\xi_j \widehat{T}(\xi) \qquad \text{and} \qquad \frac{\partial}{\partial x_j}\widehat{T}(\xi) = -i\mathscr{F}_x[xT(x)](\xi)$$

   or to be more precise,

   $$\Big\langle \widehat{\frac{\partial T}{\partial x_j}}, \phi\Big\rangle = \big\langle \widehat{T}(\xi), i\xi_j\phi(\xi)\big\rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)$$

   and

   $$\Big\langle \frac{\partial}{\partial \xi_j}\widehat{T}(\xi), \phi(\xi)\Big\rangle = \big\langle T(x), -ix\widehat{\phi}(x)\big\rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)\,.$$

3. (5%) Let $T \in \mathscr{S}(\mathbb{R}^n)'$ and $f \in S(\mathbb{R}^n)$. Show that the Leibniz rule holds; that is, show that

   $$\frac{\partial}{\partial x_i}(fT) = f\frac{\partial T}{\partial x_i} + \frac{\partial f}{\partial x_i}T\,.$$

Problem 3. (10%) Let sgn : $\mathbb{R} \to \mathbb{R}$ be the sign function defined by

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0\,, \\ -1 & \text{if } x < 0\,, \\ 0 & \text{if } x = 0\,. \end{cases}$$

Then clearly sgn is a tempered distribution since

$$\big|\langle \operatorname{sgn}, \phi\rangle\big| \leqslant \|\phi\|_{L^1(\mathbb{R})} \leqslant \pi p_2(\phi) \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R})\,.$$

Show that $\dfrac{d}{dx}\operatorname{sgn}(x) = 2\delta$ in $\mathscr{S}(\mathbb{R})'$, where the derivative of tempered distributions is defined in Problem 2 and δ is the Dirac delta function.

Problem 4. The Hilbert transform of a function $f : \mathbb{R} \to \mathbb{R}$, denoted by $\mathscr{H}[f]$, is a function defined (formally) by

$$\mathscr{H}[f](x) = \frac{1}{\pi} \lim_{\epsilon \to 0^+} \int_{|y-x|>\epsilon} \frac{f(y)}{x-y}\, dy\,,$$

1. (5%) Show that $\mathscr{H}[f]$ is well-defined if $f \in \mathscr{S}(\mathbb{R})$.
2. (15%) Show that $\mathscr{F}\big[\mathscr{H}[f]\big](\xi) = i\operatorname{sgn}(\xi)\widehat{f}(\xi)$ for all $f \in \mathscr{S}(\mathbb{R})$.
3. (10%) Show that $\|\mathscr{H}[f]\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$ for all $f \in \mathscr{S}(\mathbb{R})$, where $\|g\|_{L^2(\mathbb{R})} = \Big(\displaystyle\int_{\mathbb{R}} \big|g(x)\big|^2\, dx\Big)^{\frac{1}{2}}$.

Hint: In this problem you can use the conclusion (without proving again) in Problem 5 of Exercise 11. Consider the tempered distribution T defined in Problem 5(2) of Exercise 11 by

$$\langle T, \varphi\rangle = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}\setminus[-\epsilon,\epsilon]} \frac{\varphi(x)}{x}\, dx = \lim_{\epsilon \to 0^+} \Big(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\Big) \frac{\varphi(x)}{x}\, dx \qquad \forall\, \varphi \in \mathscr{S}(\mathbb{R})\,.$$

1. Show that $\mathscr{H}[f] = \langle T, \tau_x f\rangle$ for all $f \in \mathscr{S}(\mathbb{R})$, where τ_x is a translation operator.
2. Show that the tempered distribution S defined by $\langle S, \phi\rangle = \langle T(x), x\phi(x)\rangle$ is indeed the same as the tempered distribution

   $$\phi \mapsto \int_{\mathbb{R}} \phi(x)\, dx = \langle 1, \phi\rangle\,.$$

   Use Problem 2 to show that $\dfrac{d}{d\xi}\widehat{T}(\xi) = -\sqrt{\dfrac{\pi}{2}}\, i \dfrac{d}{d\xi}\operatorname{sgn}(\xi)$, where sgn is given in Problem 3. Use the fact that $\dfrac{dT}{dx} = 0$ if and only if there exists C such that $\langle T, \phi\rangle = \langle C, \phi\rangle$ for all $\phi \in \mathscr{S}(\mathbb{R})$ to conclude that

   $$\widehat{T}(\xi) = -\sqrt{\frac{\pi}{2}}\, i\operatorname{sgn}(\xi) + C$$

   for some constant C. Find the constant C and also show that $\mathscr{H}[f] = \dfrac{1}{\pi} T * f = \sqrt{\dfrac{2}{\pi}}\, T \star f$.
3. Use the Plancherel formula.

Problem 5. (25%) Let ω be a positive real number, and $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \dfrac{\sin(\omega|x|)}{|x|} & \text{if } x \neq 0\,, \\ \omega & \text{if } x = 0\,, \end{cases}$$

where $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ if $x = (x_1, x_2, x_3)$. Then $f \in \mathscr{S}(\mathbb{R}^3)'$ since f is bounded. Show that the Fourier transform of f is given by

$$\langle \widehat{f}, \varphi \rangle = \sqrt{\frac{\pi}{2}} \frac{1}{\omega} \int_{\partial B(0,\omega)} \varphi \, dS \equiv \sqrt{\frac{\pi}{2}} \frac{1}{\omega} \int_0^{\pi} \int_0^{2\pi} \varphi(\omega \cos\theta \sin\phi, \omega \sin\theta \sin\phi, \omega \cos\phi) \omega^2 \, sin\phi \, d\theta d\phi$$

for all $\varphi \in \mathscr{S}(\mathbb{R}^3)$, where $\displaystyle\int_{\partial B(0,\omega)} \varphi \, dS$ is the surface integral of φ on the sphere $\partial B(0,\omega)$.

Hint: You can show part 2 through the following procedures:

Step 1: By the definition of the Fourier transform of the tempered distributions,

$$\langle \widehat{f}, \varphi \rangle = \langle f, \widehat{\varphi} \rangle = \lim_{m \to \infty} \int_{B(0,m)} f(x) \Big(\frac{1}{\sqrt{2\pi}^3} \int_{\mathbb{R}^3} \varphi(\xi) e^{-ix\cdot\xi} \, d\xi \Big) \, dx$$

and the Fubini Theorem implies that

$$\langle \widehat{f}, \varphi \rangle = \frac{1}{\sqrt{2\pi}^3} \lim_{m \to \infty} \int_{\mathbb{R}^3} \Big(\int_{B(0,m)} f(x) e^{-ix\cdot\xi} \, dx \Big) \varphi(\xi) d\xi \, .$$

We focus on the inner integral first. Show that for each 3×3 orthonormal matrix O,

$$\int_{B(0,m)} f(x) e^{-ix\cdot\xi} \, dx = \int_{B(0,m)} \frac{\sin(\omega |y|)}{|y|} \, e^{-i(\mathrm{O}^{\mathrm{T}}\xi)\cdot y} \, dy \, .$$

Step 2: For each $\xi \in \mathbb{R}^3$, choose a 3×3 orthonormal matrix O such that $\mathrm{O}^{\mathrm{T}}\xi = (0, 0, |\xi|)$. Using the spherical coordinate $y = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$ to show that

$$\int_{B(0,m)} f(x) e^{-ix\cdot\xi} \, dx = \int_0^m \frac{2\sin(\omega\rho)\sin(|\xi|\rho)}{|\xi|} \, d\rho$$

so that we conclude that

$$\langle \widehat{f}, \varphi \rangle = \frac{1}{\sqrt{2\pi}^3} \lim_{m \to \infty} \int_{\mathbb{R}^3} \Big(\int_0^m \frac{2\sin(\omega\rho)\sin(|\xi|\rho)}{|\xi|} \varphi(\xi) \, d\rho \Big) d\xi \, .$$

Step 3: For each $r > 0$, define

$$\psi(r) = \int_{\partial B(0,r)} \varphi \, dS \equiv \int_0^{\pi} \int_0^{2\pi} \varphi(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi) r^2 \sin\phi \, d\theta d\phi \, .$$

Using the spherical coordinate $\xi = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)$ to show that

$$\langle \widehat{f}, \varphi \rangle = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \Big(\int_0^{\infty} \sin(\omega\rho)\sin(r\rho) \frac{2\psi(r)}{r} \, dr \Big) d\rho \, .$$

Step 4: Apply the conclusion in Problem 1.