Exercise Problem Sets 11

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Problem 1. Let $\alpha > 0$ be given. Show that the Fourier transform of the function

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t} e^{-t|x|^2} dt$$

is positive.

Proof. For $\xi \in \mathbb{R}^n$, define $g(x,t) = t^{\alpha-1} e^{-t} e^{-t|x|^2} e^{ix\cdot\xi}$. By the Tonelli Theorem,

$$\begin{aligned}\int_{\mathbb{R}^n \times (0,\infty)} \big|g(x,t)\big|\, d(x,t) &= \int_{\mathbb{R}} \int_0^\infty t^{\alpha-1} e^{-t} e^{-t|x|^2} dt dx = \int_0^\infty t^{\alpha-1} e^{-t} \Big(\int_{\mathbb{R}} e^{-t|x|^2}\, dx \Big) dt \\ &= \int_0^\infty t^{\alpha-1} e^{-t} \Big(\frac{t}{\pi}\Big)^{\frac{n}{2}} dt = \frac{1}{\sqrt{\pi}^n} \int_0^\infty t^{\frac{n}{2}+\alpha-1} e^{-t}\, dt = \frac{\Gamma(\frac{n}{2}+\alpha-1)}{\sqrt{\pi}^n} < \infty\,.\end{aligned}$$

The computation above also shows that $f \in L^1(\mathbb{R}^n)$. Therefore, the Fubini Theorem implies that

$$\begin{aligned}\Gamma(a)\widehat{f}(\xi) &= \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} \Big(\int_0^\infty t^{\alpha-1} e^{-t} e^{-t|x|^2} dt \Big) e^{-ix\cdot\xi} dx \\ &= \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} \Big(\int_0^\infty t^{\alpha-1} e^{-t} e^{-t|x|^2} e^{-ix\cdot\xi} dt \Big) dx = \int_0^\infty t^{\alpha-1} e^{-t} \Big(\frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} e^{-t|x|^2} e^{-ix\cdot\xi} dx \Big) dt \\ &= \int_0^\infty t^{\alpha-1} e^{-t} \mathscr{F}_x[e^{-t|x|^2}](\xi)\, dt = \int_0^\infty t^{\alpha-1} e^{-t} \sqrt{2t}^n e^{-\frac{|\xi|^2}{4t}}\, dt > 0\,.\end{aligned}$$

The positivity of $\widehat{f}$ then follows from the fact that $\Gamma(\alpha) > 0$. □

Problem 2. Compute the Fourier transform of the function $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(x) = |x|^\alpha$, where $-n < \alpha < 0$, by the following procedure.

1. Show that $f \notin L^1(\mathbb{R}^n)$.

2. Recall that the Gamma function $\Gamma : (0,\infty) \to \mathbb{R}$ defined by $\Gamma(x) = \displaystyle\int_0^\infty t^{x-1} e^{-t}\, dt$. Show that

$$|x|^\alpha = \frac{1}{\Gamma(-\frac{\alpha}{2})} \int_0^\infty s^{-\frac{\alpha}{2}-1} e^{-s|x|^2}\, ds \qquad \forall\, x \neq 0\,.$$

3. Assume that you can apply the Fubini Theorem to obtain that

$$\int_{\mathbb{R}^n} \Big(\int_0^\infty s^{-\frac{\alpha}{2}-1} e^{-s|x|^2} e^{-ix\cdot\xi}\, ds \Big)\, dx = \int_0^\infty \Big(\int_{\mathbb{R}^n} s^{-\frac{\alpha}{2}-1} e^{-s|x|^2} e^{-ix\cdot\xi}\, dx \Big)\, ds\,.$$

   Find that Fourier transform of f.

4. Find the Fourier transform of the function $g : \mathbb{R}^n \to \mathbb{R}$ given by $g(x) = x_1 |x|^\alpha$, where x_1 is the first component of x and $-n-2 < \alpha < -2$.

Hint: 4. For a distribution T, for each $1 \leqslant j \leqslant n$ one should treat $\dfrac{\partial T}{\partial x_j}$ as the tempered distribution defined by

$$\Big\langle \frac{\partial T}{\partial x_j}, \phi \Big\rangle = -\Big\langle T, \frac{\partial \phi}{\partial x_j} \Big\rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)\,.$$

It can be shown that the Fourier transform of the tempered distribution $\dfrac{\partial T}{\partial x_j}$ is $i\xi_j \widehat{T}(\xi)$ (you should try to prove this simple fact using of Lemma 9.11 in the lecture note). Note that $g(x) = \dfrac{1}{\alpha+2}\dfrac{\partial}{\partial x_1}|x|^{\alpha+2}$ so that you can apply the results above.

Proof. 1. By the change of variables formula,

$$\int_{\mathbb{R}^n} |x|^\alpha\, dx = \int_{\mathbb{S}^{n-1}} \int_0^\infty r^\alpha r^{n-1} dr dS = \omega_{n-1} \int_0^\infty r^{\alpha+n-1}\, dr = \infty\,.$$

Therefore, $f \notin L^1(\mathbb{R}^n)$.

2. By the substitution of variable $s|x|^2 = t$ (for $x \neq 0$),

$$\int_0^\infty s^{-\frac{\alpha}{2}-1} e^{-s|x|^2}\, ds = \int_0^\infty |x|^{\alpha+2} t^{-\frac{\alpha}{2}-1} e^{-t} |x|^{-2}\, dt = |x|^\alpha \int_0^\infty t^{-\frac{\alpha}{2}-1} e^{-t}\, dt = |x|^\alpha \Gamma\big(-\frac{\alpha}{2}\big)\,.$$

Therefore, $|x|^\alpha = \dfrac{1}{\Gamma\big(-\frac{\alpha}{2}\big)} \displaystyle\int_0^\infty s^{-\frac{\alpha}{2}-1} e^{-s|x|^2}\, ds.$

3. Assume that

$$\int_{\mathbb{R}^n} \Big(\int_0^\infty s^{-\frac{\alpha}{2}-1} e^{-s|x|^2} e^{-ix\cdot\xi}\, ds \Big)\, dx = \int_0^\infty \Big(\int_{\mathbb{R}^n} s^{-\frac{\alpha}{2}-1} e^{-s|x|^2} e^{-ix\cdot\xi}\, dx \Big)\, ds\,.$$

Using the expression of $|x|^\alpha$ in 2, we find that

$$\begin{aligned}
\Gamma\big(-\frac{\alpha}{2}\big)\mathscr{F}_x[|x|^\alpha](\xi) &= \frac{1}{\sqrt{2\pi}^{\,n}} \int_{\mathbb{R}^n} \Big(\int_0^\infty s^{-\frac{\alpha}{2}-1} e^{-s|x|^2} e^{-ix\cdot\xi}\, ds \Big)\, dx \\
&= \frac{1}{\sqrt{2\pi}^{\,n}} \int_0^\infty \Big(\int_{\mathbb{R}^n} s^{-\frac{\alpha}{2}-1} e^{-s|x|^2} e^{-ix\cdot\xi}\, dx \Big)\, ds = \int_0^\infty s^{-\frac{\alpha}{2}-1} \Big(\frac{1}{\sqrt{2\pi}^{\,n}} \int_{\mathbb{R}^n} e^{-s|x|^2} e^{-ix\cdot\xi}\, dx \Big)\, ds \\
&= 2^{-\frac{n}{2}} \int_0^\infty s^{-\frac{n+\alpha}{2}-1} e^{-\frac{|\xi|^2}{4s}}\, ds
\end{aligned}$$

and the substitution of variable $t = \dfrac{|\xi|^2}{4s}$ implies that

$$\begin{aligned}
\Gamma\big(-\frac{\alpha}{2}\big)\mathscr{F}_x[|x|^\alpha](\xi) &= 2^{-\frac{n}{2}} \int_0^\infty s^{-\frac{n+\alpha}{2}-1} e^{-\frac{|\xi|^2}{4s}}\, ds = 2^{-\frac{n}{2}} \int_0^\infty (4t)^{\frac{n+\alpha}{2}+1} |\xi|^{-n-\alpha-2} e^{-t} \frac{|\xi|^2}{4t^2}\, dt \\
&= 2^{\frac{n}{2}+\alpha} |\xi|^{-\alpha-n} \int_0^\infty t^{\frac{n+\alpha}{2}-1} e^{-t}\, dt = 2^{\frac{n}{2}+\alpha} |\xi|^{-\alpha-n} \Gamma\big(\frac{n+\alpha}{2}\big)\,.
\end{aligned}$$

Therefore, $\mathscr{F}_x[|x|^\alpha](\xi) = \dfrac{\Gamma\big(\frac{n+\alpha}{2}\big)}{\Gamma\big(-\frac{\alpha}{2}\big)} 2^{\frac{n}{2}+\alpha} |\xi|^{-\alpha-n}.$

A rigorous approach is given as follows. For a given Schwartz function $\phi \in \mathscr{S}(\mathbb{R}^n)$, define $g(x,s) = s^{-\frac{\alpha}{2}-1} e^{-s|x|^2} \widehat{\phi}(x)$ and $h(\xi,s) = s^{-\frac{n}{2}-\frac{\alpha}{2}-1} e^{-\frac{|\xi|^2}{4s}} \phi(\xi)$. Then

$$\begin{aligned}\int_{\mathbb{R}^n\times(0,\infty)} \big|g(x,s)\big|\,d(x,s) &= \int_{\mathbb{R}^n} \Big(\int_0^\infty s^{-\frac{\alpha}{2}-1} e^{-s|x|^2} \big|\widehat{\phi}(x)\big|\,ds\Big)\,dx \\ &= \int_{\mathbb{R}^n} |x|^\alpha \big|\widehat{\phi}(x)\big|\,dx = \int_{\mathbb{S}^{n-1}} \Big(\int_0^\infty r^{n+\alpha-1}\big|\widehat{\phi}(r\omega)\big|\,dr\Big) dS\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^n\times(0,\infty)} \big|h(\xi,s)\big|\,d(\xi,s) \\ &\quad= \int_{\mathbb{R}^n} \Big(\int_0^\infty s^{-\frac{n}{2}-\frac{\alpha}{2}-1} e^{-\frac{|\xi|^2}{4s}} \big|\phi(\xi)\big|\,ds\Big)\,d\xi = \int_{\mathbb{R}^n} \Big(\int_0^\infty s^{-\frac{n}{2}-\frac{\alpha}{2}-1} e^{-\frac{|\xi|^2}{4s}}\,ds\Big) \big|\phi(\xi)\big|\,d\xi \\ &\quad= \int_{\mathbb{R}^n} \Big(\int_0^\infty (4t)^{\frac{n}{2}+\frac{\alpha}{2}+1} |\xi|^{-n-\alpha-2} e^{-t} \frac{|\xi|^2}{4t^2}\,dt\Big) \big|\phi(\xi)\big|\,d\xi \\ &\quad= 2^{n+\alpha} \int_{\mathbb{R}^n} \Big(\int_0^\infty t^{\frac{n+\alpha}{2}-1} e^{-t}\,dt\Big) |\xi|^{-n-\alpha} \big|\phi(\xi)\big|\,d\xi \\ &\quad= 2^{n+\alpha}\Gamma\big(\frac{n+\alpha}{2}\big) \int_{\mathbb{R}^n} |\xi|^{-n-\alpha} \big|\phi(\xi)\big|\,d\xi\,.\end{aligned}$$

Since

$$\begin{aligned}\int_0^\infty r^{n+\alpha-1}\big|\widehat{\phi}(r\omega)\big|\,dr &\leqslant \|\widehat{\phi}\|_\infty \int_0^1 r^{n+\alpha-1} dr + \sup_{x\in\mathbb{R}^n} \big(|x|^n \big|\widehat{\phi}(x)\big|\big) \int_1^\infty r^{\alpha-1} dr \\ &\leqslant \frac{\|\phi\|_{L^1(\Omega)}}{n+\alpha} + \frac{1}{-\alpha} \sup_{x\in\mathbb{R}^n} \big(|x|^n\big|\widehat{\phi}(x)\big|\big) < \infty\end{aligned}$$

and

$$\int_{\mathbb{R}^n} |\xi|^{-n-\alpha}\big|\phi(\xi)\big|\,d\xi \leqslant \int_{\mathbb{R}^n} \langle\xi\rangle^{-n-1}\langle\xi\rangle^{1-\alpha}\big|\phi(\xi)\big|\,d\xi \leqslant \|\langle\xi\rangle\|_{L^1(\mathbb{R}^n)} \sup_{\xi\in\mathbb{R}^n} \langle\xi\rangle^{1-\alpha}\big|\phi(\xi)\big| < \infty\,,$$

we find that g and h are integrable on $\mathbb{R}^n \times (0,\infty)$. By the definition of the Fourier transform of tempered distributions,

$$\langle |x|^\alpha, \widehat{\phi}\rangle = \int_{\mathbb{R}^n} |x|^\alpha \widehat{\phi}(x)\,dx = \frac{1}{\Gamma(-\frac{\alpha}{2})} \int_{\mathbb{R}^n} \Big(\int_0^\infty s^{-\frac{\alpha}{2}-1} e^{-s|x|^2}\,ds\Big) \widehat{\phi}(x)\,dx$$

and the Fubini Theorem (which can be applied since g is integrable on $\mathbb{R}^n \times (0,\infty)$) implies that

$$\begin{aligned}\Gamma\big(-\frac{\alpha}{2}\big)\langle |x|^\alpha, \widehat{\phi}\rangle &= \int_{\mathbb{R}^n} \Big(\int_0^\infty s^{-\frac{\alpha}{2}-1} e^{-s|x|^2}\,ds\Big)\widehat{\phi}(x)\,dx = \int_0^\infty \Big(\int_{\mathbb{R}^n} s^{-\frac{\alpha}{2}-1} e^{-s|x|^2} \widehat{\phi}(x)\,dx\Big)\,ds \\ &= \int_0^\infty s^{-\frac{\alpha}{2}-1} \langle e^{-s|x|^2}, \widehat{\phi}(x)\rangle\,ds = \int_0^\infty s^{-\frac{\alpha}{2}-1} \langle \mathscr{F}_x[e^{-s|x|^2}](\xi), \phi(\xi)\rangle\,ds \\ &= \int_0^\infty s^{-\frac{\alpha}{2}-1} \Big(\int_{\mathbb{R}^n} (2s)^{-\frac{n}{2}} e^{-\frac{|\xi|^2}{4s}} \phi(\xi)\,d\xi\Big)\,ds \\ &= 2^{-\frac{n}{2}} \int_0^\infty s^{-\frac{n}{2}-\frac{\alpha}{2}-1} \Big(\int_{\mathbb{R}^n} e^{-\frac{|\xi|^2}{4s}} \phi(\xi)\,d\xi\Big)\,ds\,.\end{aligned}$$

By the integrability of h on $\mathbb{R}^n \times (0,\infty)$, we can apply the Fubini Theorem to obtain that

$$\begin{aligned}\Gamma\big(-\frac{\alpha}{2}\big)\langle |x|^\alpha, \widehat{\phi}\rangle &= 2^{-\frac{n}{2}} \int_0^\infty s^{-\frac{n}{2}-\frac{\alpha}{2}-1}\Big(\int_{\mathbb{R}^n} e^{-\frac{|\xi|^2}{4s}}\phi(\xi)\,d\xi\Big)\,ds \\ &= 2^{-\frac{n}{2}} \int_{\mathbb{R}^n}\Big(\int_0^\infty s^{-\frac{n}{2}-\frac{\alpha}{2}-1} e^{-\frac{|\xi|^2}{4s}}\,ds\Big)\phi(\xi)\,d\xi \\ &= 2^{\frac{n}{2}+\alpha}\Gamma\big(\frac{n+\alpha}{2}\big)\int_{\mathbb{R}^n}|\xi|^{-n-\alpha}\phi(\xi)\,d\xi \\ &= 2^{\frac{n}{2}+\alpha}\Gamma\big(\frac{n+\alpha}{2}\big)\langle |\xi|^{-n-\alpha}, \phi(\xi)\rangle .\end{aligned}$$

Therefore, $\mathscr{F}_x[|x|^\alpha](\xi) = \dfrac{\Gamma\big(\frac{n+\alpha}{2}\big)}{\Gamma\big(-\frac{\alpha}{2}\big)} 2^{\frac{n}{2}+\alpha}|\xi|^{-\alpha-n}$.

4. Since $\mathscr{F}\big[\dfrac{\partial T}{\partial x_j}\big](\xi) = i\xi_j \widehat{T}(\xi)$, and $g(x) = \dfrac{1}{\alpha+2}\dfrac{\partial}{\partial x_1}|x|^{\alpha+2}$, by the fact that $|x|^{\alpha+2}$ is a tempered distribution for $-n < \alpha+2 < 0$, we conclude that if $-n-2 < \alpha < -2$, we have

$$\widehat{g}(\xi) = \frac{1}{\alpha+2} i\xi_1 \mathscr{F}_x\big[|x|^{\alpha+2}\big](\xi) = i\frac{\Gamma\big(\frac{n+\alpha+2}{2}\big)}{\Gamma\big(-\frac{\alpha+2}{2}\big)}\frac{2^{\frac{n}{2}+\alpha+2}}{\alpha+2}\xi_1|\xi|^{-\alpha-n-2}\,.$$ ◻

Problem 3. Let $f \in L^1(\mathbb{R})$. Show that the function $y = \int_{-\infty}^x f(t)\,dt$ can be written as the convolution of f and a function $\varphi \in L^1_{\rm loc}(\mathbb{R})$.

Proof. Let φ be the characteristic function of the set $(0,\infty)$, or

$$\varphi(x) = \begin{cases} 1 & \text{if } x > 0\,, \\ 0 & \text{if } x \leqslant 0\,. \end{cases}$$

Then $\varphi \in L^1_{\rm loc}(\mathbb{R})$, and

$$(\varphi * f)(x) = \int_{\mathbb{R}} \varphi(x-y)f(y)\,dy = \int_{-\infty}^x f(y)\,dy$$

which is the anti-derivative of f. ◻

Problem 4. In this problem we use symbolic computation to find the Fourier transform of the function

$$f(x) = \begin{cases} \dfrac{\sin(\omega x)}{x} & \text{if } x \neq 0\,, \\ \omega & \text{if } x = 0\,, \end{cases}$$

without knowing that it is the Fourier transform of the function $y = \sqrt{\frac{\pi}{2}}\chi_{(-\omega,\omega)}(x)$ (where $\chi_{(-\omega,\omega)}$ is the characteristic/indicator function of the set $(-\omega,\omega)$). Complete the following.

1. In class we have shown that $f \notin L^1(\mathbb{R})$ but $f \in \mathscr{S}(\mathbb{R})'$. Let $\widehat{f}$ be the Fourier transform of f (in the sense of the Fourier transform of tempered distributions). Formally we can write $\widehat{f}(\xi) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^\infty \frac{\sin(\omega x)}{x}e^{-ix\xi}\,dx$ and assume that we can differentiate $\widehat{f}$ using

$$\widehat{f}\,'(\xi) = \frac{d}{d\xi}\Big(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\sin(\omega x)}{x} e^{-ix\xi}\,dx\Big) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\partial}{\partial \xi}\Big(\frac{\sin(\omega x)}{x} e^{-ix\xi}\Big)\,dx\,.$$

Find the "derivative" of $\widehat{f}$.

2. Suppose that you can use the Fundamental Theorem of Calculus so that

$$\widehat{f}(\xi) - \widehat{f}(0) = \int_0^{\xi} \widehat{f}\,'(t)\,dt\,.$$

Use the fact that $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi$ (and treating $\delta_{\pm\omega}$ as the evaluation operation at $\pm\omega$) to find $\widehat{f}(\xi)$ (for $\xi \neq \pm\omega$).

Hint: 1. Recall that we have shown in class that $\mathscr{F}_x[\sin(\omega x)](\xi) = \dfrac{\sqrt{2\pi}}{2i}(\delta_\omega - \delta_{-\omega})$.

Proof. 1. Using

$$\widehat{f}\,'(\xi) = \frac{d}{d\xi}\Big(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\sin(\omega x)}{x} e^{-ix\xi}\,dx\Big) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\partial}{\partial \xi}\Big(\frac{\sin(\omega x)}{x} e^{-ix\xi}\Big)\,dx\,,$$

we find that

$$\widehat{f}\,'(\xi) = \frac{-i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \sin(\omega x) e^{-ix\xi}\,dx = -i\mathscr{F}_x\big[\sin(\omega x)\big](\xi) = -\sqrt{\frac{\pi}{2}}(\delta_\omega - \delta_{-\omega})\,.$$

A rigorous approach is given as follows. Let $\phi \in \mathscr{S}(\mathbb{R})$. Then by the "definition" of the derivative of tempered distributions,

$$\begin{aligned}\langle \widehat{f}\,', \phi\rangle = -\langle \widehat{f}, \phi'\rangle = -\langle f, \widehat{\phi'}\rangle = -\langle f(x), ix\widehat{\phi}(x)\rangle = -i\langle \sin(\omega x), \widehat{\phi}(x)\rangle \\ = -i\langle \mathscr{F}_x[\sin(\omega x)](\xi), \phi(\xi)\rangle\end{aligned}$$

which shows that

$$\widehat{f}\,'(\xi) = -i\mathscr{F}_x[\sin(\omega x)](\xi) = -\sqrt{\frac{\pi}{2}}(\delta_\omega - \delta_{-\omega})\,.$$

2. Note that

$$\widehat{f}(0) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \frac{\sin(\omega x)}{x} e^{ix\cdot 0}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\sin y}{y}\,dy = \sqrt{\frac{\pi}{2}}\,;$$

thus the Fundamental Theorem of Calculus implies that

$$\widehat{f}(\xi) = \widehat{f}(0) + \int_0^{\xi} \widehat{f}'(t)\,dt = \sqrt{\frac{\pi}{2}}\Big[1 - \int_0^{\xi} \big[\delta_\omega(t) - \delta_{-\omega}(t)\big]\,dt\Big]\,.$$

(a) If $\xi < 0$, then

$$\int_0^{\xi} \big[\delta_\omega(t) - \delta_{-\omega}(t)\big]\,dt = -\int_{\mathbb{R}} \big[\delta_\omega(t) - \delta_{-\omega}(t)\big]\mathbf{1}_{[\xi,0]}(t)\,dt = \mathbf{1}_{[\xi,0]}(-\omega)\,;$$

thus

$$\int_0^{\xi} \big[\delta_\omega(t) - \delta_{-\omega}(t)\big]\,dt = \begin{cases} 0 & \text{if } -\omega < \xi < 0\,, \\ 1 & \text{if } \xi < -\omega\,. \end{cases}$$

(b) If $\xi > 0$, then

$$\int_0^\xi \big[\delta_\omega(t) - \delta_{-\omega}(t)\big]\,dt = \int_{\mathbb{R}} \big[\delta_\omega(t) - \delta_{-\omega}(t)\big]\mathbf{1}_{[0,\xi]}(t)\,dt = \mathbf{1}_{[0,\xi]}(\omega)\,;$$

thus

$$\int_0^\xi \big[\delta_\omega(t) - \delta_{-\omega}(t)\big]\,dt = \begin{cases} 0 & \text{if } 0 < \xi < \omega\,, \\ 1 & \text{if } \xi > \omega\,. \end{cases}$$

Therefore,

$$\int_0^\xi \big[\delta_\omega(t) - \delta_{-\omega}(t)\big]\,dt = \begin{cases} 0 & \text{if } -\omega < \xi < \omega\,, \\ 1 & \text{if } |\xi| > \omega\,, \end{cases}$$

which shows that $\widehat{f}(\xi) = \sqrt{\frac{\pi}{2}}\mathbf{1}_{(-\omega,\omega)}(\xi)$. □

Problem 5. 1. Show that the function $R : \mathbb{R} \to \mathbb{R}$ given by

$$\mathrm{R}(x) = \begin{cases} x & \text{if } x \geqslant 0\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

is a tempered distribution.

2. Let T be a generalized function defined by

$$\langle T, \varphi\rangle = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}\setminus[-\epsilon,\epsilon]} \frac{\varphi(x)}{x}\,dx = \lim_{\epsilon\to 0^+}\Big(\int_{-\infty}^{-\epsilon} + \int_\epsilon^\infty\Big)\frac{\varphi(x)}{x}\,dx \qquad \forall\,\varphi \in \mathscr{C}_c^\infty(\mathbb{R})\,.$$

Show that $T \in \mathscr{S}(\mathbb{R})'$.

3. Let H be the Heaviside function given by

$$H(x) = \begin{cases} 0 & \text{if } x \leqslant 0\,, \\ 1 & \text{if } x > 0\,. \end{cases}$$

Show that $\widehat{H} = \frac{-i}{\sqrt{2\pi}}T + \sqrt{\frac{\pi}{2}}\,\delta$, here δ is the Dirac delta function.

Hint: 3. Let $G(x) = \exp\big(-\frac{x^2}{2}\big)$. For each $\phi \in \mathscr{S}(\mathbb{R})$, define $\psi = \phi - \phi(0)G$ (which belongs to $\mathscr{S}(\mathbb{R})$). Use the identity

$$\langle \widehat{H}, \phi\rangle = \langle H, \widehat{\psi}\rangle - \phi(0)\langle H, \widehat{G}\rangle$$

to make the conclusion.

Proof. 1. Let $\phi \in \mathscr{S}(\mathbb{R}^n)$. Then

$$\begin{aligned}\big|\langle R, \phi\rangle\big| &= \Big|\int_0^\infty x\phi(x)\,dx\Big| \leqslant \Big(\int_0^\infty |x|\langle x\rangle^{-3}\,dx\Big)\sup_{x\in\mathbb{R}}\langle x\rangle^3\big|\phi(x)\big| \\ &\leqslant \Big(\int_0^\infty \frac{1}{1+x^2}\,dx\Big)p_3(\phi) = \frac{\pi}{2}\,p_3(\phi)\,;\end{aligned}$$

thus

$$\big|\langle R, \phi\rangle\big| \leqslant \frac{\pi}{2}\,p_k(\phi) \qquad \forall\,k \geqslant 3\,.$$

Therefore, R is a tempered distribution.

2. For $\varphi \in \mathscr{S}(\mathbb{R})$, define $\psi : \mathbb{R} \to \mathbb{R}$ by

$$\psi(x) = \begin{cases} \dfrac{\phi(x) - \phi(0)}{x} & \text{if } x \neq 0\,, \\ \phi'(0) & \text{if } x = 0\,. \end{cases}$$

Then clearly ψ is continuous on $\mathbb{R}$, and

$$\sup_{x\in[-1,1]} \big|\psi(x)\big| \leqslant \sup_{x\in[-1,1]} \big|\phi'(x)\big| \leqslant p_1(\phi)\,.$$

By the fact that

$$\int_{-1}^{1} \psi(x)\,dx = \lim_{\epsilon\to 0^+} \int_{[-1,1]\backslash(-\epsilon,\epsilon)} \psi(x)\,dx\,,$$

we find that

$$\begin{aligned}\langle T, \phi\rangle &= \lim_{\epsilon\to 0^+} \int_{\mathbb{R}\backslash(-\epsilon,\epsilon)} \frac{\phi(x)}{x}\,dx = \int_{\mathbb{R}\backslash[-1,1]} \frac{\phi(x)}{x}\,dx + \lim_{\epsilon\to 0^+} \int_{[-1,1]\backslash(-\epsilon,\epsilon)} \frac{\phi(x)}{x}\,dx \\ &= \int_{\mathbb{R}\backslash[-1,1]} \frac{\phi(x)}{x}\,dx + \lim_{\epsilon\to 0^+} \int_{[-1,1]\backslash(-\epsilon,\epsilon)} \frac{\phi(x)-\phi(0)}{x}\,dx \\ &= \int_{\mathbb{R}\backslash[-1,1]} \frac{\phi(x)}{x}\,dx + \int_{-1}^{1} \psi(x)\,dx\,.\end{aligned}$$

Therefore, $\langle T, \phi\rangle \in \mathbb{C}$ for all $\phi \in \mathscr{S}(\mathbb{R})$. Moreover,

$$\begin{aligned}\big|\langle T, \phi\rangle\big| &\leqslant \int_{\mathbb{R}\backslash[-1,1]} \Big|\frac{\phi(x)}{x}\Big|\,dx + \int_{-1}^{1} \big|\psi(x)\big|\,dx \leqslant \int_{\mathbb{R}\backslash[-1,1]} |x|^{-2}|x|\big|\phi(x)\big|\,dx + 2p_1(\phi) \\ &\leqslant \Big(2 + \int_{\mathbb{R}\backslash[-1,1]} |x|^{-2}\,dx\Big)p_1(\phi) = 4p_1(\phi)\,;\end{aligned}$$

thus $\big|\langle T, \phi\rangle\big| \leqslant 4p_k(\phi)$ for all $k \geqslant 1$. This implies that T is a tempered distribution.

3. Define $H_n(x) = \chi_{(0,n)}(x)$. For a Schwartz function $\phi \in \mathscr{S}(\mathbb{R})$, define $\psi = \phi - \phi(0)G$. Then $\psi \in \mathscr{S}(\mathbb{R})$, and

$$\begin{aligned}\langle \widehat{H}, \varphi\rangle &= \langle \widehat{H}, \psi\rangle + \varphi(0)\langle \widehat{H}, G\rangle = \langle H, \widehat{\psi}\rangle + \varphi(0)\langle H, \widehat{G}\rangle \\ &= \lim_{n\to\infty} \langle H_n, \widehat{\psi}\rangle + \varphi(0)\langle H, G\rangle \\ &= \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \int_0^n \Big(\int_{-\infty}^{\infty} \psi(x)e^{-ix\xi}dx\Big)d\xi + \sqrt{\frac{\pi}{2}}\varphi(0)\end{aligned}$$

where we have used the fact that $\langle H, G\rangle = \sqrt{\dfrac{\pi}{2}}$ to conclude the last equality.

Define f by $f(x) = \dfrac{\psi(x)}{x}$ or to be more precise, $f(x) = \begin{cases} \dfrac{\psi(x)}{x} & \text{if } x \neq 0\,, \\ \psi'(0) & \text{if } x = 0\,, \end{cases}$. Then f is a Schwartz function. In fact, we have $\psi(x) = xf(x)$ for all $x \in \mathbb{R}$ and the Lebnitz rule implies that for $j \geqslant 0$,

$$xf^{(j)}(x) = \psi^{(j)}(x) - jf^{(j-1)}(x)$$

which implies that

$$|x|^k \big|f^{(j)}(x)\big| \leqslant |x|^k \big|\psi^{(j)}(x)\big| + k|x|^{k-1}\big|f^{(j-1)}(x)\big|$$

so that the boundedness of $|x|^k \big|f^{(j)}(x)\big|$ can be proved by induction.

By Fubini's Theorem,

$$\int_0^n \Big(\int_{-\infty}^{\infty} \psi(x) e^{-ix\xi} dx\Big) d\xi = \int_{-\infty}^{\infty} \Big(\int_0^n \psi(x) e^{-ix\xi} d\xi\Big) dx\,;$$

thus

$$\begin{aligned}
\langle \widehat{H}, \varphi\rangle &= \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x) \Big(\int_0^n e^{-ix\xi} d\xi\Big) dx + \sqrt{\frac{\pi}{2}}\,\langle \delta, \varphi\rangle \\
&= \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x) \frac{1-e^{-inx}}{ix} dx + \sqrt{\frac{\pi}{2}}\,\langle \delta, \varphi\rangle \\
&= \frac{1}{\sqrt{2\pi} i} \int_{-\infty}^{\infty} \frac{\psi(x)}{x} \psi(x)\, dx + \sqrt{\frac{\pi}{2}}\,\langle \delta, \varphi\rangle + i \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-inx} f(x) dx \\
&= \frac{-i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\psi(x)}{x}\, dx + \sqrt{\frac{\pi}{2}}\,\langle \delta, \varphi\rangle + i \lim_{n\to\infty} \widehat{f}(n)\,.
\end{aligned}$$

Since $f \in \mathscr{S}(\mathbb{R})$, $\widehat{f} \in \mathscr{S}(\mathbb{R})$; thus $\lim\limits_{n\to\infty} \widehat{f}(n) = 0$. Therefore, by the fact G is an even function, we conclude that

$$\begin{aligned}
\langle \widehat{H}, \varphi\rangle &= \lim_{\epsilon\to 0^+} \lim_{R\to\infty} \frac{-i}{\sqrt{2\pi}} \int_{[-R,R]\backslash(-\epsilon,\epsilon)} \frac{\psi(x)}{x}\, dx + \sqrt{\frac{\pi}{2}}\,\langle \delta, \varphi\rangle \\
&= \lim_{\epsilon\to 0^+} \lim_{R\to\infty} \frac{-i}{\sqrt{2\pi}} \int_{[-R,R]\backslash(-\epsilon,\epsilon)} \frac{\phi(x)}{x}\, dx + \sqrt{\frac{\pi}{2}}\,\langle \delta, \varphi\rangle \\
&= \lim_{\epsilon\to 0} \frac{-i}{\sqrt{2\pi}} \int_{\mathbb{R}\backslash(-\epsilon,\epsilon)} \frac{\phi(x)}{x}\, dx + \sqrt{\frac{\pi}{2}}\,\langle \delta, \varphi\rangle = \langle T, \varphi\rangle + \sqrt{\frac{\pi}{2}}\,\langle \delta, \varphi\rangle\,,
\end{aligned}$$

which shows that $\widehat{H} = \dfrac{-i}{\sqrt{2\pi}} T + \sqrt{\dfrac{\pi}{2}}\,\delta$. □