Exercise Problem Sets 7

Apr. 17. 2021

Problem 1. 1. Let $f : [-\pi, \pi]$ be a Riemann integrable function. Show that

$$\lim_{k\to\infty} \int_{-\pi}^{\pi} f(x) \cos kx \, dx = \lim_{k\to\infty} \int_{-\pi}^{\pi} f(x) \sin kx \, dx = 0 .$$

2. Recall that $f : [a, b] \to \mathbb{R}$ is integrable if f is Riemann measurable (that is, the collection of discontinuities of f has measure zero) and the limits $\lim\limits_{k\to\infty} \int_{-\pi}^{\pi} (f^{\pm} \wedge k)(x) \, dx$ both exist, where $f^{\pm} = \max\{\pm f, 0\}$. Show the Riemann-Lebesgue Lemma

   > If $f : [-\pi, \pi] \to \mathbb{R}$ is an integrable function, then
   > $$\lim_{k\to\infty} \int_{-\pi}^{\pi} f(x) \cos kx \, dx = \lim_{k\to\infty} \int_{-\pi}^{\pi} f(x) \sin kx \, dx = 0 .$$

   Hint: First show that for every $\varepsilon > 0$ there exists a Riemann integrable function $g : [-\pi, \pi] \to \mathbb{R}$ such that $\int_{-\pi}^{\pi} \big|f(x) - g(x)\big| \, dx < \varepsilon$, then apply the conclusion in 1.

Proof. 1. Let $\varepsilon > 0$ be given. Then by Lemma 6.63 of the lecture note, there exists $g \in \mathscr{C}([-\pi, \pi]; \mathbb{R})$ such that

$$f(x) \leqslant g(x) \leqslant \sup_{x\in[-\pi,\pi]} f(x) \quad \forall\, x \in [-\pi, \pi] \qquad \text{and} \qquad \int_{-\pi}^{\pi} f(x) \, dx > \int_{-\pi}^{\pi} g(x) \, dx - \frac{\varepsilon}{3} .$$

By the Weierstrass Theorem, there exists a polynomial p such that

$$\|g - p\|_{\infty} < \frac{\varepsilon}{6\pi} .$$

Since p is a polynomial, integrating by parts (or by Problem 3 of Exercise 6) we can show that

$$\lim_{k\to\infty} \int_{-\pi}^{\pi} p(x) \cos kx \, dx = \lim_{k\to\infty} \int_{-\pi}^{\pi} p(x) \sin kx \, dx = 0 .$$

Therefore, there exists $N > 0$ such that if $k \geqslant N$,

$$\Big|\int_{-\pi}^{\pi} p(x) \cos kx \, dx\Big| < \frac{\varepsilon}{3} \qquad \text{and} \qquad \Big|\int_{-\pi}^{\pi} p(x) \sin kx \, dx\Big| < \frac{\varepsilon}{3} .$$

Therefore, if $k \geqslant N$,

$$\begin{aligned}
&\Big|\int_{-\pi}^{\pi} f(x) \cos kx \, dx\Big| \\
&\qquad \leqslant \Big|\int_{-\pi}^{\pi} \big[f(x) - g(x)\big] \cos kx \, dx\Big| + \Big|\int_{-\pi}^{\pi} \big[g(x) - p(x)\big] \cos kx \, dx\Big| + \Big|\int_{-\pi}^{\pi} p(x) \cos kx \, dx\Big| \\
&\qquad \leqslant \int_{-\pi}^{\pi} \big|f(x) - g(x)\big| \, dx + \int_{-\pi}^{\pi} \|g - p\|_{\infty} \, dx + \frac{\varepsilon}{3} \\
&\qquad \leqslant \int_{-\pi}^{\pi} \big[g(x) - f(x)\big] \, dx + \int_{-\pi}^{\pi} \frac{\varepsilon}{6\pi} \, dx + \frac{\varepsilon}{3} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon ,
\end{aligned}$$

and similarly,

$$\Big|\int_{-\pi}^{\pi} f(x) \cos kx \, dx\Big| < \varepsilon \quad \text{whenever} \quad k \geqslant N .$$

2. Let $g_k(x) = (f^+ \wedge k)(x) - (f^- \wedge k)(x)$. Then

$$\begin{aligned}\int_{-\pi}^{\pi} \big|f(x) - g_k(x)\big|\, dx &= \int_{-\pi}^{\pi} \big|f^+(x) - f^-(x) - g_k(x)\big|\, dx \\ &\leqslant \int_{-\pi}^{\pi} \big|f^+(x) - (f^+ \wedge k)(x)\big|\, dx + \int_{-\pi}^{\pi} \big|f^-(x) - (f^- \wedge k)(x)\big|\, dx\,;\end{aligned}$$

thus by the fact that

$$\lim_{k\to\infty} \int_{-\pi}^{\pi} (f^+ \wedge k)(x)\, dx = \int_{-\pi}^{\pi} f^+(x)\, dx \quad \text{and} \quad \lim_{k\to\infty} \int_{-\pi}^{\pi} (f^- \wedge k)(x)\, dx = \int_{-\pi}^{\pi} f^-(x)\, dx\,,$$

we find that there exists $K > 0$ such that

$$\int_{-\pi}^{\pi} \big|f(x) - g_k(x)\big|\, dx < \frac{\varepsilon}{2} \qquad \text{whenever} \quad k \geqslant K\,.$$

Let $h = g_K$. Note that h is Riemann integrable on $[-\pi, \pi]$; thus part 1 implies that there exists $N > 0$ such that if $k \geqslant N$,

$$\Big|\int_{-\pi}^{\pi} h(x) \cos kx\, dx\Big| < \frac{\varepsilon}{2} \qquad \text{and} \qquad \Big|\int_{-\pi}^{\pi} h(x) \sin kx\, dx\Big| < \frac{\varepsilon}{2}\,.$$

Therefore, if $k \geqslant N$,

$$\begin{aligned}\Big|\int_{-\pi}^{\pi} f(x) \cos kx\, dx\Big| &= \Big|\int_{-\pi}^{\pi} \big[f(x) - h(x)\big] \cos kx\, dx\Big| + \Big|\int_{-\pi}^{\pi} h(x) \cos kx\, dx\Big| \\ &\leqslant \int_{-\pi}^{\pi} \big|f(x) - h(x)\big|\, dx + \Big|\int_{-\pi}^{\pi} h(x) \cos kx\, dx\Big| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

and similarly,

$$\Big|\int_{-\pi}^{\pi} f(x) \sin kx\, dx\Big| < \varepsilon \qquad \text{whenever} \quad k \geqslant N\,.$$

□

Problem 2. Let $\alpha \in (0, 1]$ and I be an interval. A function $f : I \to \mathbb{R}$ is said to be Hölder continuous with exponent α if

$$\sup_{x,y\in I, x\neq y} \frac{\big|f(x) - f(y)\big|}{|x - y|^\alpha} < \infty\,.$$

1. Show that f is uniformly continuous.

2. Show that the function $f(x) = |x|^\alpha$ is Hölder continuous with exponent α.

Proof. 1. Let $M = \sup\limits_{\substack{x,y\in I\\ x\neq y}} \dfrac{\big|f(x) - f(y)\big|}{|x - y|^\alpha}$. Then $M < \infty$, and

$$\big|f(x) - f(y)\big| \leqslant M|x - y|^\alpha \qquad \forall\, x, y \in I\,.$$

Therefore, f is uniformly continuous since for a given $\varepsilon > 0$ we can choose $\delta = (M^{-1}\varepsilon)^{\frac{1}{\alpha}}$ so that $\delta > 0$ and

$$\big|f(x) - f(y)\big| < \varepsilon \qquad \text{whenever} \quad |x - y| < \delta \text{ and } x, y \in I\,.$$

2. It suffices to show that $\sup\limits_{x>y>0} \dfrac{\big||x|^\alpha - |y|^\alpha\big|}{|x-y|^\alpha} < \infty$ since $\sup\limits_{x>y=0} \dfrac{\big||x|^\alpha - |y|^\alpha\big|}{|x-y|^\alpha} = 1$,

$$\sup_{x>0>y} \frac{\big||x|^\alpha - |y|^\alpha\big|}{|x-y|^\alpha} \leqslant \sup_{x>y>0} \frac{\big||x|^\alpha - |y|^\alpha\big|}{|x-y|^\alpha} \quad \text{and} \quad \sup_{x>y>0} \frac{\big||x|^\alpha - |y|^\alpha\big|}{|x-y|^\alpha} = \sup_{x<y<0} \frac{\big||x|^\alpha - |y|^\alpha\big|}{|x-y|^\alpha}.$$

On the other hand, if $x > y > 0$, letting $\theta = \dfrac{y}{x}$ we find that

$$\sup_{x>y>0} \frac{\big||x|^\alpha - |y|^\alpha\big|}{|x-y|^\alpha} = \sup_{0<\theta<1} \frac{1-\theta^\alpha}{(1-\theta)^\alpha}.$$

Let $f(\theta) = \dfrac{1-\theta^\alpha}{(1-\theta)^\alpha}$. Then

$$f'(\theta) = \frac{-\alpha\theta^{\alpha-1}\cdot(1-\theta)^\alpha + (1-\theta^\alpha)\cdot\alpha(1-\theta)^{\alpha-1}}{(1-\theta)^{2\alpha}} = \frac{\alpha(1-\theta^{\alpha-1})}{(1-\theta)^{\alpha+1}}$$

so that $f'(\theta) < 0$ for all $0 < \theta < 1$. Therefore,

$$\sup_{x>y>0} \frac{\big||x|^\alpha - |y|^\alpha\big|}{|x-y|^\alpha} = \sup_{x>y>0} \frac{\big||x|^\alpha - |y|^\alpha\big|}{|x-y|^\alpha} = \lim_{\theta\to 0^+} \frac{1-\theta^\alpha}{(1-\theta)^\alpha} = 1.$$ □

Problem 3. Suppose that $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$; that is, f is 2π-periodic Hölder continuous function with exponent α for some $\alpha \in (0,1]$. Show that (without using the Berstein Theorem) the Fourier series of f converges pointwise to f, by completing the following.

1. Explain why it is enough to show that $s_n(f,0) \to f(0)$ as $n \to \infty$. Also explain why we can assume that $f(0) = 0$.

2. Show that
$$\lim_{n\to\infty} \Big(s_n(f,0) - \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\frac{\sin nx}{x}\,dx\Big) = 0.$$
Therefore, it suffices to show that $\lim\limits_{n\to\infty} \displaystyle\int_{-\pi}^{\pi} f(x)\frac{\sin nx}{x}\,dx = 0$ if $f(0) = 0$.

3. Show that if $f \in \mathscr{C}^{0,\alpha}(\mathbb{R})$ and $f(0) = 0$, then the function $y = \dfrac{f(x)}{x}$ is integrable. Apply the Riemann-Lebesgue Lemma to conclude that $s_n(f,0) \to 0$ as $n \to \infty$.

Proof. 1. Suppose that one can show that if g is a 2π-periodic Hölder continuous function with exponent $\alpha \in (0,1]$, then $s_n(g,0) \to g(0)$ as $n \to \infty$. If f is 2π-periodic Hölder continuous function with exponent $\alpha \in (0,1]$ and $a \in \mathbb{R}$, let $g(x) = f(x+a)$. Then g is a 2π-periodic Hölder continuous function with exponent α; thus $s_n(g,0) \to g(0)$ as $n \to \infty$.

On the other hand, let $\{c_k\}_{k=0}^\infty$ and $\{s_k\}_{k=1}^\infty$ be the Fourier coefficients of f and $\{\bar{c}_k\}_{k=0}^\infty$ and $\{\bar{s}_k\}_{k=1}^\infty$ be the Fourier coefficients of g. Then

$$\begin{aligned}
\bar{c}_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+a)\cos kx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos k(x-a)\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\big(\cos kx\cos ka + \sin kx\sin ka\big)\,dx \\
&= c_k\cos ka + s_k\sin ka.
\end{aligned}$$

Note that

$$s_n(g,0) = \frac{\bar{c}_0}{2} + \sum_{k=1}^{n} \left[\bar{c}_k \cos(k\cdot 0) + \bar{s}_k \sin(k\cdot 0)\right] = \sum_{k=1}^{n} \big(c_k \cos ka + s_k \sin ka\big) = s_n(f,a)\,;$$

thus the fact that $g(0) = f(a)$ implies that $s_n(f,a) \to f(a)$ as $n \to \infty$. Moreover, if $f(0) \neq 0$, we consider the function $h(x) = f(x) - f(0)$. Then $h(0) = 0$ and $s_n(f,x) = s_n(h,x) + f(0)$ so that if the $s_n(h,0)$ converges to 0, then $s_n(f,0)$ converges to $f(0)$. In other words, we can further assume that $f(0) = 0$.

2. Note that $s_n(f,x) = (D_n \star f)(x)$; thus

$$s_n(f,0) = \int_{-\pi}^{\pi} f(x) \frac{\sin(n+\frac{1}{2})x}{2\pi \sin\frac{x}{2}}\, dx\,.$$

Therefore,

$$\begin{aligned} s_n(f,0) - \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\frac{\sin nx}{x}\,dx &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\Big[\frac{\sin(n+\frac{1}{2})x}{2\sin\frac{x}{2}} - \frac{\sin nx}{x}\Big]\,dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\Big(\frac{\sin nx \cos\frac{x}{2} + \sin\frac{x}{2}\cos nx}{2\sin\frac{x}{2}} - \frac{\sin nx}{x}\Big)\,dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\Big(\frac{\cos\frac{x}{2}}{2\sin\frac{x}{2}} - \frac{1}{x}\Big)\sin nx\,dx\,. \end{aligned}$$

Note that

$$\lim_{x\to 0}\Big(\frac{\cos\frac{x}{2}}{2\sin\frac{x}{2}} - \frac{1}{x}\Big) = \lim_{x\to 0}\frac{x\cos\frac{x}{2} - 2\sin\frac{x}{2}}{2x\sin\frac{x}{2}} = \lim_{x\to 0}\frac{x(1-\frac{x^2}{8}) - 2(\frac{x}{2} - \frac{x^3}{48})}{2x\cdot\frac{x}{2}} = 0\,;$$

thus the function $y = f(x)\Big(\frac{\cos\frac{x}{2}}{2\sin\frac{x}{2}} - \frac{1}{x}\Big)$ is continuous on $[-\pi,\pi]$. By the Riemann-Lebesgue Lemma,

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\Big(\frac{\cos\frac{x}{2}}{2\sin\frac{x}{2}} - \frac{1}{x}\Big)\sin nx\,dx = 0\,.$$

Therefore,

$$\lim_{n\to\infty}\Big(s_n(f,0) - \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\frac{\sin nx}{x}\,dx\Big) = 0\,.$$

3. Since $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$ for some $\alpha \in (0,1]$,

$$M \equiv \sup_{x\neq y}\frac{\big|f(x) - f(y)\big|}{|x-y|^\alpha} < \infty\,.$$

In particular, if $x \neq 0$,

$$\frac{\big|f(x)\big|}{|x|^\alpha} = \frac{\big|f(x) - f(0)\big|}{|x-0|^\alpha} \leqslant \sup_{x\neq y}\frac{\big|f(x) - f(y)\big|}{|x-y|^\alpha} = M < \infty\,;$$

thus

$$\Big|\frac{f(x)}{x}\Big| \leqslant M|x|^{\alpha-1} \qquad \forall\, x \neq 0\,.$$

Therefore, the comparison test implies that the function $y = \dfrac{f(x)}{x}$ is integrable on $[-\pi,\pi]$ since

$$\int_0^\pi x^{\alpha-1}\,dx = \lim_{\varepsilon\to 0^+} \frac{1}{\alpha}x^\alpha\Big|_{x=\varepsilon}^{\pi} = \frac{\pi^\alpha}{\alpha}$$

and the change of varialbe $x \mapsto -x$ shows that

$$\int_{-\pi}^0 |x|^{\alpha-1}\,dx = \int_0^\pi x^{\alpha-1}\,dx = \frac{\pi^\alpha}{\alpha}\,.$$

The Riemann-Lebesgue Lemma then implies that $\displaystyle\lim_{n\to\infty}\int_{-\pi}^{\pi}\frac{f(x)}{x}\sin nx\,dx = 0.$ □

Problem 4. 1. Let $f:\mathbb{R}\to\mathbb{R}$ be 2π-periodic such that f is Riemann integrable on $[-\pi,\pi]$. Show that

$$\widehat{f}_k = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f\big(x+\frac{\pi}{k}\big)e^{-ikx}\,dx$$

and hence

$$\widehat{f}_k = \frac{1}{4\pi}\int_{-\pi}^{\pi}\big[f(x)-f\big(x+\frac{\pi}{k}\big)\big]e^{-ikx}\,dx\,.$$

2. Show that if $f\in\mathscr{C}^{0,\alpha}(\mathbb{T})$; that is, f is 2π-periodic Hölder continuous function with exponent α for some $\alpha\in(0,1]$, then the Fourier coefficients $\widehat{f}_k$ satisfies $|\widehat{f}_k| \leqslant \dfrac{\pi^\alpha\|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}}{2k^\alpha}$.

Proof. 1. By substitution of variables,

$$\widehat{f}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iky}\,dy \overset{\text{“}y=x+\frac{\pi}{k}\text{”}}{=} \frac{1}{2\pi}\int_{-\pi-\frac{\pi}{k}}^{\pi-\frac{\pi}{k}} f\big(x+\frac{\pi}{k}\big)e^{-ikx}e^{-i\pi}\,dx$$

so that the periodicity of f and the function $y = e^{-ikx}$ implies that

$$\widehat{f}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\big(x+\frac{\pi}{k}\big)e^{-ikx}e^{-i\pi}\,dx = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f\big(x+\frac{\pi}{k}\big)e^{-ikx}\,dx\,.$$

2. Suppose that $f\in\mathscr{C}^{0,\alpha}(\mathbb{T})$ for some $\alpha\in(0,1]$. Then

$$\big|f(x)-f(y)\big| \leqslant \|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}|x-y|^\alpha \qquad \forall\, x,y\in\mathbb{R}\,.$$

Therefore,

$$\big|f(x+\frac{\pi}{k}-f(x)\big| \leqslant \|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}\frac{\pi^\alpha}{k^\alpha}$$

so that

$$\big|\widehat{f}_k\big| \leqslant \frac{1}{4\pi}\int_{-\pi}^{\pi}\big|f(x)-f\big(x+\frac{\pi}{k}\big)\big|\,dx \leqslant \frac{\|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}\pi^\alpha}{4\pi k^\alpha}\int_{-\pi}^{\pi}dx = \frac{\pi^\alpha\|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}}{2k^\alpha}\,.$$ □

Problem 5. 1. Let $\{a_k\}_{k=1}^\infty$ be a sequence, and $\{b_n\}_{n=1}^\infty$ be the Cesàro mean of $\{a_k\}_{k=1}^\infty$; that is, $b_n = \dfrac{1}{n}\sum\limits_{k=1}^n a_k$. Show that if $\{a_k\}_{k=1}^\infty$ converges to a, then $\{b_n\}_{n=1}^\infty$ converges to a.

2. Let $\{f_k\}_{k=1}^\infty$ be a sequence of bounded real-valued functions defined on A, and $\{g_n\}_{n=1}^\infty$ be the Cesàro mean of $\{f_k\}_{k=1}^\infty$; that is, $g_n = \dfrac{1}{n}\sum\limits_{k=1}^n f_k$. Show that if $\{f_k\}_{k=1}^\infty$ converges uniformly to f on $B\subseteq A$ and f is bounded on B, then $\{g_n\}_{n=1}^\infty$ converges uniformly to f on B.

Proof. 1. Let $\varepsilon > 0$ be given. Since $\lim\limits_{k\to\infty} a_k = a$, there exists $N_1 > 0$ such that

$$|a_k - a| < \frac{\varepsilon}{2} \quad \text{whenever} \quad k \geqslant N_1 .$$

Since $\lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{k=1}^{N_1} |a_k - a| = 0$, there exists $N_2 > 0$ such that

$$\frac{1}{n}\sum_{k=1}^{N_1} |a_k - a| < \frac{\varepsilon}{2} \quad \text{whenever} \quad n \geqslant N_2 .$$

Let $N = \max\{N_1, N_2\}$. Then if $n \geqslant N$,

$$\begin{aligned}|b_n - a| &= \Big|\frac{1}{n}\sum_{k=1}^{n} a_k - a\Big| \leqslant \frac{1}{n}\sum_{k=1}^{n} |a_k - a| \leqslant \frac{1}{n}\sum_{k=1}^{N_1} |a_k - a| + \frac{1}{n}\sum_{k=N_1}^{n} |a_k - a| \\ &< \frac{\varepsilon}{2} + \frac{1}{n}\sum_{k=N_1}^{n} \frac{\varepsilon}{2} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\frac{n - N_1 + 1}{n} < \varepsilon .\end{aligned}$$

2. Suppose that $\big|f_k(x)\big| \leqslant M_k$ and $\big|f(x)\big| \leqslant M$ for all $x \in B$. Since $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f on B, there exists $N_1 > 0$ such that

$$\big|f_k(x) - f(x)\big| < \frac{\varepsilon}{2} \qquad \forall\, k \geqslant N_1 \text{ and } x \in B .$$

If $x \in B$, by the fact that

$$\sum_{k=1}^{N_1} \big|f_k(x) - f(x)\big| \leqslant \sum_{k=1}^{N_1} (M_k + M) < \infty ,$$

we find that $\lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{k=1}^{N_1} \|f_k - f\|_\infty = 0$; thus there exists $N_2 > 0$ such that

$$\frac{1}{n}\sum_{k=1}^{N_1} \big|f_k(x) - f(x)\big| < \frac{\varepsilon}{2} \quad \text{whenever} \quad n \geqslant N_2 \text{ and } x \in B .$$

Let $N = \max\{N_1, N_2\}$. Then if $n \geqslant N$ and $x \in B$,

$$\begin{aligned}\big|g_n(x) - f(x)\big| &= \Big|\frac{1}{n}\sum_{k=1}^{n} f_k(x) - f(x)\Big| \leqslant \frac{1}{n}\sum_{k=1}^{N_1} \big|f_k(x) - f(x)\big| + \frac{1}{n}\sum_{k=N_1}^{n} \big|f_k(x) - f(x)\big| \\ &< \frac{\varepsilon}{2} + \frac{1}{n}\sum_{k=N_1}^{n} \frac{\varepsilon}{2} < \varepsilon ;\end{aligned}$$

thus $\{g_n\}_{n=1}^{\infty}$ converges uniformly to f on B. □

Problem 6. Let $f \in \mathscr{C}(\mathbb{T})$, and $\{c_k\}_{k=0}^{\infty}$, $\{s_k\}_{k=1}^{\infty}$ be the Fourier coefficients. Show that if

$$\sum_{k=0}^{\infty} |c_k| < \infty \qquad \text{and} \qquad \sum_{k=1}^{\infty} |s_k| < \infty ,$$

then the Fourier series of f converges uniformly to f on $\mathbb{R}$.

Proof. Let $M_k = |c_k| + |s_k|$ and $|c_0| + \sum\limits_{k=1}^{\infty} \big(|c_k| + |s_k|\big) = M$. Then $\big|s_n(f,x)\big| \leqslant M$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Moreover,

$$\big|c_k \cos kx + s_k \sin kx\big| \leqslant M_k \quad \forall\, x \in \mathbb{R} \qquad \text{and} \qquad \sum_{k=1}^{\infty} M_k = \sum_{k=1}^{\infty} \big(|c_k| + |s_k|\big) \leqslant M < \infty\,.$$

Therefore, the Weierstrass M-test implies that the Fourier series converges uniformly on $\mathbb{R}$. Suppose that the Fourier series converges uniformly to g. Then $\big|g(x)\big| \leqslant M$ for all $x \in \mathbb{R}$; thus Problem 5 implies that the Cesàro mean of $\{s_k(f,\cdot)\}_{k=1}^{\infty}$ converges uniformly to g on $\mathbb{R}$. Since $f \in \mathscr{C}(\mathbb{T})$, the Cesàro mean of the Fourier series of f converges uniformly to f on $\mathbb{R}$; thus $f = g$. □

Problem 7. Let f be a 2π-periodic Lipchitz function. Show that for $n \geqslant 2$,

$$\|f - F_{n-1} \star f\|_\infty \leqslant \frac{1 + 2\log n}{2n} \pi \|f\|_{\mathscr{C}^{0,1}(\mathbb{T})}\,. \tag{0.1}$$

Hint: For (0.1), apply the estimate

$$F_n(x) \leqslant \min\Big\{\frac{n+1}{2\pi}, \frac{\pi}{2(n+1)x^2}\Big\}$$

in the following inequality:

$$\big|f(x) - F_{n+1} \star f(x)\big| \leqslant \Big[\int_{-\delta}^{\delta} + \int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\Big] \big|f(x+y) - f(x)\big| F_{n+1}(y)\, dy$$

with $\delta = \dfrac{\pi}{n+1}$.

Proof. Recall that the Fejér kernel F_n is given by

$$F_n(x) = \begin{cases} \dfrac{1}{2\pi(n+1)} \dfrac{\sin^2 \frac{(n+1)x}{2}}{\sin^2 \frac{x}{2}} & \text{if } x \notin \big\{2k\pi \,\big|\, k \in \mathbb{Z}\big\}\,, \\ \dfrac{n+1}{2\pi} & \text{if } x \in \big\{2k\pi \,\big|\, k \in \mathbb{Z}\big\}\,. \end{cases}$$

Therefore, by the fact that $\sin|x| \geqslant \dfrac{2}{\pi}|x|$ for $|x| < \dfrac{\pi}{2}$, we find that

$$F_n(x) \leqslant \min\Big\{\frac{n+1}{2\pi}, \frac{\pi}{2(n+1)x^2}\Big\}\,.$$

By the fact that $\displaystyle\int_{-\pi}^{\pi} F_{n-1}(x)\, dx = 0$ for all $n \geqslant 2$, we find that if $n \geqslant 2$ and $0 < \delta < \pi$,

$$\begin{aligned}
\big|f(x) - F_{n-1} \star f(x)\big| &= \Big|\int_{-\pi}^{\pi} f(x) F_{n-1}(x-y)\, dy - \int_{-\pi}^{\pi} f(y) F_{n-1}(x-y)\, dy\Big| \\
&= \Big|\int_{-\pi}^{\pi} \big[f(x) - f(y)\big] F_{n-1}(x-y)\, dy\Big| \\
&= \Big|\int_{-\pi}^{\pi} \big[f(x) - f(x-y)\big] F_{n-1}(y)\, dy\Big| \\
&= \Big|\Big(\int_{-\delta}^{\delta} + \int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\Big) \big[f(x) - f(x-y)\big] F_{n-1}(y)\, dy\Big|\,.
\end{aligned}$$

Let $\delta = \dfrac{\pi}{n}$. Then

$$\left|\int_{-\delta}^{\delta} \big[f(x) - f(x-y)\big] F_{n-1}(y)\,dy\right| \leqslant \int_{-\delta}^{\delta} \|f\|_{\mathscr{C}^{0,1}(\mathbb{T})} |y| \cdot \frac{n}{2\pi}\,dy = \frac{n\|f\|_{\mathscr{C}^{0,1}(\mathbb{T})}}{\pi} \int_0^{\delta} y\,dy$$
$$= \frac{n\|f\|_{\mathscr{C}^{0,1}(\mathbb{T})}}{2\pi} \frac{\pi^2}{n^2} = \frac{\pi\|f\|_{\mathscr{C}^{0,1}(\mathbb{T})}}{2n}\,.$$

Moreover,

$$\left|\int_{\delta\leqslant|y|\leqslant\pi} \big[f(x) - f(x-y)\big] F_{n-1}(y)\,dy\right| \leqslant \int_{\delta\leqslant|y|\leqslant\pi} \|f\|_{\mathscr{C}^{0,1}(\mathbb{T})} |y| \cdot \frac{\pi}{2ny^2}\,dy = \frac{\pi\|f\|_{\mathscr{C}^{0,1}(\mathbb{T})}}{n} \int_{\delta}^{\pi} \frac{1}{y}\,dy$$
$$= \frac{\pi\|f\|_{\mathscr{C}^{0,1}(\mathbb{T})}}{n} \log\frac{\pi}{\delta} = \frac{\pi\|f\|_{\mathscr{C}^{0,1}(\mathbb{T})} \log n}{n}\,.$$

Therefore,

$$\big|f(x) - F_{n-1} \star f(x)\big| \leqslant \frac{\pi\|f\|_{\mathscr{C}^{0,1}(\mathbb{T})}}{2n} + \frac{\pi\|f\|_{\mathscr{C}^{0,1}(\mathbb{T})} \log n}{n} = \frac{1+\log n}{2n} \pi \|f\|_{\mathscr{C}^{0,1}(\mathbb{T})}\,.$$

□