Exercise Problem Sets 1

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Problem 1. Let $A \subseteq \mathbb{R}^n$ be a bounded set, and $f, g : A \to \mathbb{R}$ be functions. Show that

$$\underline{\int_A} f(x)\,dx \leqslant \underline{\int_A} g(x)\,dx \qquad \text{and} \qquad \overline{\int_A} f(x)\,dx \leqslant \overline{\int_A} g(x)\,dx\,.$$

Proof. By the fact that $\overline{f}^A \leqslant \overline{g}^A$ on $\mathbb{R}^n$, we find that

$$U(f,\mathcal{P}) \leqslant U(g,\mathcal{P}) \quad \text{and} \quad L(f,\mathcal{P}) \leqslant L(g,\mathcal{P}) \qquad \forall \text{ partitions } \mathcal{P} \text{ of } A.$$

Since $\overline{\int_A} f(x)\,dx$ is a lower bound for $\big\{U(f,\mathcal{P})\,\big|\,\mathcal{P} \text{ is a partition of } A\big\}$ and $\underline{\int_A} g(x)\,dx$ is an upper bound for $\big\{L(g,\mathcal{P})\,\big|\,\mathcal{P} \text{ is a partition of } A\big\}$, we find that

$$\overline{\int_A} f(x)\,dx \leqslant U(f,\mathcal{P}) \leqslant U(g,\mathcal{P}) \;\text{ and }\; L(f,\mathcal{P}) \leqslant L(g,\mathcal{P}) \leqslant \underline{\int_A} g(x)\,dx \quad \forall \text{ partitions } \mathcal{P} \text{ of } A.$$

The inequalities above shows that $\overline{\int_A} f(x)\,dx$ is a lower bound for $\big\{U(g,\mathcal{P})\,\big|\,\mathcal{P} \text{ is a partition of } A\big\}$ and $\underline{\int_A} g(x)\,dx$ is an upper bound for $\big\{L(f,\mathcal{P})\,\big|\,\mathcal{P} \text{ is a partition of } A\big\}$; thus we conclude that

$$\underline{\int_A} f(x)\,dx \leqslant \underline{\int_A} g(x)\,dx \qquad \text{and} \qquad \overline{\int_A} f(x)\,dx \leqslant \overline{\int_A} g(x)\,dx\,. \qquad \square$$

Problem 2. Let $A \subseteq \mathbb{R}^n$ be a bounded set, and $f : [a,b] \to \mathbb{R}$ be a function.

1. Show that if f is Riemann integrable on A, then f is bounded.
2. Show that if f is Darboux integrable on A, then f is bounded.

Note that we in some sense use these properties in the proof of the equivalence between the Riemann integrability and the Darboux integrability, so you'd better no use this equivalence in the proof.

Hint:

1. Let $\mathrm{I} = (R)\int_A f(x)\,dx$. There exists $\delta > 0$ such that if $\mathcal{P}$ is a partition of A, then any Riemann sum of f for $\mathcal{P}$ locates in $(\mathrm{I}-1, \mathrm{I}+1)$. Let $\mathcal{P} = \{\Delta_1, \Delta_2, \cdots, \Delta_N\}$ be a partition of A satisfying $\|\mathcal{P}\| < \delta$, and for each $1 \leqslant k \leqslant N$, let c_k be the center of Δ_k. Show that for each $1 \leqslant \ell \leqslant N$,
$$\frac{1}{\nu(\Delta_\ell)}\Big[\mathrm{I}-1-\sum_{1\leqslant \ell\leqslant N, k\neq \ell} \overline{f}^A(c_k)\nu(\Delta_k)\Big] < \overline{f}^A(x) < \frac{1}{\nu(\Delta_k)}\Big[\mathrm{I}+1-\sum_{1\leqslant \ell\leqslant N, k\neq \ell} \overline{f}^A(c_k)\nu(\Delta_k)\Big] \quad \forall\, x \in \Delta_\ell\,.$$

2. Note that if $\mathcal{P}$ be a partition of A and $\Delta \in \mathcal{P}$, then
$$-\infty < \sup_{x\in\Delta} \overline{f}^A(x) \leqslant \infty \qquad \text{and} \qquad -\infty \leqslant \inf_{x\in\Delta} \overline{f}^A(x) < \infty\,.$$
What happened to $U(f,\mathcal{P})$ and $L(f,\mathcal{P})$ if f is not bounded?

Proof. Since f is Riemann integrable on A, there exists $I \in \mathbb{R}$ and $\delta > 0$ such that if $\mathcal{P}$ is a partition of A satisfying $\|\mathcal{P}\| < \delta$, then any Riemann sum of f for $\mathcal{P}$ locates in $(\mathrm{I}-1, \mathrm{I}+1)$. Let $\mathcal{P} = \{\Delta_1, \Delta_2, \cdots, \Delta_N\}$ be a partition of A satisfying $\|\mathcal{P}\| < \delta$. For each $1 \leqslant k \leqslant N$, let c_k be the center of Δ_k. Then for each $1 \leqslant \ell \leqslant N$,

$$\mathrm{I}-1 < \overline{f}^A(x)\nu(\Delta_\ell) + \sum_{1\leqslant k\leqslant N, k\neq \ell} \overline{f}^A(c_k)\nu(\Delta_k) < \mathrm{I}+1 \qquad \forall\, x \in \Delta_\ell$$

since $\overline{f}^A(x)\nu(\Delta_\ell) + \sum_{1\leqslant k\leqslant N, k\neq \ell} \overline{f}^A(c_k)\nu(\Delta_k)$ is a Riemann sum of f for $\mathcal{P}$. In particular,

$$\mathrm{I}-1 < f(x)\nu(\Delta_\ell) + \sum_{1\leqslant k\leqslant N, k\neq \ell} \overline{f}^A(c_k)\nu(\Delta_k) < \mathrm{I}+1 \qquad \forall\, x \in \Delta_\ell \cap A\,.$$

which further implies that

$$\frac{1}{\nu(\Delta_\ell)}\Big[\mathrm{I}-1-\sum_{1\leqslant k\leqslant N, k\neq \ell} \overline{f}^A(c_k)\nu(\Delta_k)\Big] < f(x) < \frac{1}{\nu(\Delta_\ell)}\Big[\mathrm{I}+1-\sum_{1\leqslant k\leqslant N, k\neq \ell} \overline{f}^A(c_k)\nu(\Delta_k)\Big]$$

Since f is real-valued, $\overline{f}^A(c_k)$ is a real number. The numbers M and m defined by

$$\begin{aligned} M &\equiv \max\Big\{\frac{1}{\nu(\Delta_\ell)}\Big[\mathrm{I}+1-\sum_{1\leqslant k\leqslant N, k\neq \ell} \overline{f}^A(c_k)\nu(\Delta_k)\Big] \,\Big|\, 1 \leqslant \ell \leqslant N\Big\}\,, \\ m &\equiv \min\Big\{\frac{1}{\nu(\Delta_\ell)}\Big[\mathrm{I}-1-\sum_{1\leqslant k\leqslant N, k\neq \ell} \overline{f}^A(c_k)\nu(\Delta_k)\Big] \,\Big|\, 1 \leqslant \ell \leqslant N\Big\}\,, \end{aligned}$$

are both real numbers. Moreover, $m \leqslant f(x) \leqslant M$ for all $x \in A$; thus f is bounded.

2. Let $\mathcal{P}$ be a partition of A, and $\Delta \in \mathcal{P}$. Since f is real-valued, we must have

$$-\infty < \sup_{x\in\Delta} \overline{f}^A(x) \leqslant \infty \qquad \text{and} \qquad -\infty \leqslant \inf_{x\in\Delta} \overline{f}^A(x) < \infty\,.$$

The fact above implies that

(a) if f is not bounded from above, then $U(f, \mathcal{P}) = \infty$ for all partitions $\mathcal{P}$ of A;

(b) if f is not bounded from below, then $L(f, \mathcal{P}) = -\infty$ for all partitions $\mathcal{P}$ of A.

Therefore, if f is not bounded, either $\overline{\int_A} f(x)\,dx = \infty$ or $\underline{\int_A} f(x)\,dx = -\infty$; thus if f is Darboux integrable on A, then f must be bounded. □

Problem 3. 1. Let $f : [0,1] \to \mathbb{R}$ be a bounded monotone function. Show that f is Riemann integrable on $[0,1]$.

2. Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be a bounded function such that $f(x, y) \leqslant f(x, z)$ if $y < z$ and $f(x, y) \leqslant f(t, z)$ if $x < t$. In other words, $f(x, \cdot)$ and $f(\cdot, y)$ are both non-decreasing functions for fixed $x, y \in [0,1]$. Show that f is Riemann integrable on $[0,1] \times [0,1]$.

Problem 4. Let $f : [a, b] \to \mathbb{R}$ be differentiable. Show that if f' is Riemann integrable on $[a, b]$, then $\int_a^b f'(x)\,dx = f(b) - f(a)$.

Hint: Use the Mean Value Theorem.

Proof. Let $\mathrm{I} = \int_a^b f'(x)\,dx$, and $\varepsilon > 0$ be given. Since f' is Riemann integrable on $[a, b]$, there exists $\delta > 0$ such that if $\mathcal{P}$ is a partition of $[a, b]$ satisfying $\|\mathcal{P}\| < \delta$, then any Riemann sum of f for $\mathcal{P}$ locates in $(\mathrm{I} - \varepsilon, \mathrm{I} + \varepsilon)$. Let $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be such a partition. Then the mean value theorem implies that for each $1 \leqslant k \leqslant n$ there exists $c_k \in (x_{k-1}, x_k)$ such that $f(x_k) - f(x_{k-1}) = f'(c_k)(x_k - x_{k-1})$; thus

$$f(b) - f(a) = \sum_{k=1}^{n} \big[f(x_k) - f(x_{k-1})\big] = \sum_{k=1}^{n} f'(c_k)(x_k - x_{k-1})\,.$$

Note that the right-hand side is a Riemann sum of f for $\mathcal{P}$; thus $f(b) - f(a) \in (\mathrm{I} - \varepsilon, \mathrm{I} + \varepsilon)$ or

$$\mathrm{I} - \varepsilon < f(b) - f(a) < \mathrm{I} + \varepsilon\,.$$

Since $\varepsilon > 0$ is given arbitrarily, we conclude that $\mathrm{I} = f(b) - f(a)$. □