# Exercise Problem Sets 12

Dec. 25. 2020

**Problem 1.** Investigate the differentiability of

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

*Solution.* First we note that

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0 \qquad \text{and} \qquad f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = 0\,.$$

For $(x,y) \neq (0,0)$,

$$\frac{\big|f(x,y)-f(0,0)-f_x(0,0)x-f_y(0,0)y\big|}{\sqrt{x^2+y^2}} = \frac{xy}{x^2+y^2}$$

whose limit, as $(x,y) \to (0,0)$, does not exist. Therefore, $f$ is not differentiable at $(0,0)$.

On the other hand, for $(x,y) \neq (0,0)$,

$$f_x(x,y) = \frac{y\sqrt{x^2+y^2} - \dfrac{x^2 y}{\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{y^3}{(x^2+y^2)^{\frac{3}{2}}}$$

and similarly, $f_y(x,y) = \dfrac{x^3}{(x^2+y^2)^{\frac{3}{2}}}$. Clearly $f_x$ and $f_y$ are continuous on $\mathbb{R}^2\backslash\{(0,0)\}$; thus $f$ is differentiable on $\mathbb{R}^2\backslash\{(0,0)\}$. □

**Problem 2.** Investigate the differentiability of

$$f(x,y) = \begin{cases} \dfrac{xy}{x+y^2} & \text{if } x+y^2 \neq 0\,, \\ 0 & \text{if } x+y^2 = 0\,. \end{cases}$$

*Solution.* For $x+y^2 \neq 0$,

$$f_x(x,y) = \frac{y(x+y^2)-xy}{(x+y^2)^2} = \frac{y^3}{(x+y^2)^2} \quad \text{and} \quad f_y(x,y) = \frac{x(x+y^2)-2xy^2}{(x+y^2)^2} = \frac{x^2-xy^2}{(x+y^2)^2}\,.$$

Clearly $f_x$ and $f_y$ are continuous on $\mathbb{R}^2\backslash\big\{(x,y)|x+y^2=0\big\}$; thus $f$ is differentiable at point $(x,y)$ satisfying $x+y^2 \neq 0$ (by Theorem 5.40 of the lecture note).

Now we consider the differentiability of $f$ at $(a,b)$ when $a+b^2=0$. First we note that

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h} = \lim_{h\to 0} \frac{(a+h)b}{h(a+h+b^2)} = \begin{cases} 0 & (a,b) = (0,0)\,, \\ \text{D.N.E.} & (a,b) \neq (0,0)\,; \end{cases}$$

thus $f$ is not differentiable at $(a,b)$ if $a+b^2=0$ and $(a,b) \neq (0,0)$ (because of Theorem 5.27).

Finally we justify the differentiability of $f$ at $(0,0)$. Note that

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = 0\,.$$

For $x = y^2$ with $y \neq 0$, we have

$$\frac{\big|f(x,y) - f(0,0) - f_x(0,0)x - f_y(0,0)y\big|}{\sqrt{x^2+y^2}} = \frac{|y^3|}{2y^2\sqrt{y^4+y^2}} = \frac{1}{2\sqrt{y^2+1}}$$

whose limit, as $y \to 0$, cannot be zero; thus

$$\lim_{(x,y)\to(0,0)} \frac{\big|f(x,y) - f(0,0) - f_x(0,0)x - f_y(0,0)y\big|}{\sqrt{x^2+y^2}} \neq 0\,.$$

Therefore, $f$ is not differentiable at $(0,0)$. □

**Problem 3.** Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\dfrac{1}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

Discuss the differentiability of $f$.

*Solution.* If $(x,y) \neq (0,0)$, then

$$\begin{aligned} f_x(x,y) &= 2x\sin\frac{1}{\sqrt{x^2+y^2}} + (x^2+y^2)\cos\frac{1}{\sqrt{x^2+y^2}} \cdot \frac{-x}{(x^2+y^2)^{\frac{3}{2}}} \\ &= 2x\sin\frac{1}{\sqrt{x^2+y^2}} - \frac{1}{\sqrt{x^2+y^2}}\cos\frac{1}{\sqrt{x^2+y^2}} \end{aligned}$$

and similarly,

$$f_y(x,y) = 2y\sin\frac{1}{\sqrt{x^2+y^2}} - \frac{1}{\sqrt{x^2+y^2}}\cos\frac{1}{\sqrt{x^2+y^2}}\,.$$

Clearly $f_x$ and $f_y$ are continuous on $\mathbb{R}^2 \backslash \{(0,0)\}$; thus $f$ is differentiable at point $(x,y) \neq (0,0)$ (by Theorem 5.40 of the lecture note).

Now we justify the differentiability of $f$ at $(0,0)$. First we compute $f_x(0,0)$ and $f_y(0,0)$ and find that

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} h\sin\frac{1}{|h|} = 0$$

and

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} k\sin\frac{1}{|k|} = 0\,,$$

where the limits above are obtained by the Sandwich Lemma. For $(x,y) \neq (0,0)$, we have

$$\frac{\big|f(x,y) - f(0,0) - 0\cdot(x-0) - 0\cdot(y-0)\big|}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2}\sin\frac{1}{\sqrt{x^2+y^2}} \leqslant \sqrt{x^2+y^2}\,;$$

thus the Sandwich Lemma implies that

$$\lim_{(x,y)\to(0,0)} \frac{\big|f(x,y) - f(0,0) - 0\cdot(x-0) - 0\cdot(y-0)\big|}{\sqrt{x^2+y^2}} = 0\,.$$

Therefore, $f$ is also differentiable at $(0,0)$; thus $f$ is differentiable on $\mathbb{R}^2$. □

**Problem 4.** Let
$$f(x,y) = \begin{cases} \dfrac{x^3y}{x^4+y^2} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$
and $u \in \mathbb{R}^2$ be a unit vector. Show that the directional derivative of $f$ at the origin exists in all direction, and
$$(D_u f)(0,0) = \Big(\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0)\Big) \cdot u\,.$$
Is $f$ differentiable at $(0,0)$?

*Solution.* Let $u = (\cos\theta, \sin\theta)$ be a unit vector. Then the directional derivative of $f$ at $(0,0)$ in direction $u$ is
$$\begin{aligned}(D_u f)(0,0) &= \lim_{t\to 0^+} \frac{f(t\cos\theta, t\sin\theta) - f(0,0)}{t} = \lim_{t\to 0^+} \frac{t^4\cos^3\theta\sin\theta}{t(t^4\cos^4\theta + t^2\sin^2\theta)} \\ &= \lim_{t\to 0^+} \frac{t\cos^3\theta\sin\theta}{t^2\cos^4\theta + \sin^2\theta} = 0\,.\end{aligned}$$
On the other hand,
$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0 \quad \text{and} \quad f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = 0\,;$$
thus we conclude that $(D_u f)(0,0) = \big(f_x(0,0), f_y(0,0)\big) \cdot u$.

Since $f_x(0,0) = f_y(0,0) = 0$, if $f$ is differentiable at $(0,0)$, we must have
$$\lim_{(x,y)\to(0,0)} \frac{\big|f(x,y) - f(0,0) - 0\cdot(x-0) - 0\cdot(y-0)\big|}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{|x^3y|}{\sqrt{x^2+y^2}(x^4+y^2)} = 0\,;$$
however, by passing to the limit as $(x,y) \to (0,0)$ along the curve $y = x^2$, we find that
$$0 = \lim_{x\to 0} \frac{|x^3\cdot x^2|}{\sqrt{x^2+x^4}(x^4+x^4)} = \lim_{x\to 0} \frac{1}{2\sqrt{1+x^2}} = \frac{1}{2}\,,$$
a contradiction. Therefore, $f$ is not differentiable at $(0,0)$. □

**Problem 5.** Let $U \subseteq \mathbb{R}^n$ be open, and $f : U \to \mathbb{R}$. Suppose that the partial derivatives $\dfrac{\partial f}{\partial x_1}, \cdots, \dfrac{\partial f}{\partial x_n}$ are bounded on $U$; that is, there exists a real number $M > 0$ such that
$$\Big|\frac{\partial f}{\partial x_j}(x)\Big| \leqslant M \qquad \forall\, x \in U \text{ and } j = 1, \cdots, n\,.$$
Show that $f$ is continuous on $U$.

**Hint:** Mimic the proof of the sufficient condition for differentiability.

*Proof.* Assume that $\Big|\dfrac{\partial f}{\partial x_i}(x)\Big| \leqslant M$ for all $x \in U$ and $1 \leqslant i \leqslant n$. Let $a \in U$ be given. Then there exists $r > 0$ such that $B(a,r) \subseteq U$. For $x \in B(a,r)$, let $k = x - a$. Then
$$\begin{aligned}\big|f(x) - f(a)\big| &= \big|f(a_1+k_1, a_2+k_2, \cdots, a_n+k_n) - f(a_1, a_2, \cdots, a_n)\big| \\ &= \Big|\sum_{j=1}^n \big[f(a_1, \cdots, a_{j-1}, a_j+k_j, \cdots, a_n+k_n) - f(a_1, \cdots, a_j, a_{j+1}+k_{j+1}, \cdots, a_n+k_n)\big]\Big| \\ &\leqslant \sum_{j=1}^n \Big|f(a_1, \cdots, a_{j-1}, a_j+k_j, \cdots, a_n+k_n) - f(a_1, \cdots, a_j, a_{j+1}+k_{j+1}, \cdots, a_n+k_n)\Big|\,.\end{aligned}$$

By the Mean Value Theorem, for each $1 \leqslant j \leqslant n$ there exists $\theta_j \in (0,1)$ such that

$$\begin{aligned}&\big|f(a_1,\cdots,a_{j-1},a_j+k_j,\cdots,a_n+k_n)-f(a_1,\cdots,a_j,a_{j+1}+k_{j+1},\cdots,a_n+k_n)\\&\quad=\frac{\partial f}{\partial x_j}(a_1,\cdots,a_{j-1},a_j+\theta_j k_j,a_{j+1}+k_{j+1},\cdots,a_n+k_n)k_j\,;\end{aligned}$$

thus

$$\big|f(a_1,\cdots,a_{j-1},a_j+k_j,\cdots,a_n+k_n)-f(a_1,\cdots,a_j,a_{j+1}+k_{j+1},\cdots,a_n+k_n)\big|\leqslant M|k_j|\,.$$

Therefore, if $x \in B(a,r)$,

$$\big|f(x)-f(a)\big|=\sum_{j=1}^{n}M|k_j|\leqslant M\sqrt{n}\Big(\sum_{j=1}^{n}|k_j|^2\Big)^{\frac{1}{2}}=\sqrt{n}M\|x-a\|_{\mathbb{R}^n}\,.$$

This shows that $f$ is continuous at $a$. □

**Problem 6.** Let $U \subseteq \mathbb{R}^n$ be open, and $f : U \to \mathbb{R}$. Show that $f$ is differentiable at $a \in U$ if and only if there exists a vector-valued function $\varepsilon : U \to \mathbb{R}^n$ such that

$$f(x)-f(a)-\sum_{j=1}^{n}\frac{\partial f}{\partial x_j}(a)(x_j-a_j)=\varepsilon(x)\cdot(x-a)$$

and $\varepsilon(x) \to 0$ as $x \to a$.

*Proof.* "⇒" Suppose that $f$ is differentiable at $a$. Define $\varepsilon : U \to \mathbb{R}^n$ by

$$\varepsilon(x)=\begin{cases}\Big[f(x)-f(a)-\sum\limits_{j=1}^{n}\dfrac{\partial f}{\partial x_j}(a)(x_j-a_j)\Big]\dfrac{x-a}{\|x-a\|^2} & \text{if } x\neq a\,,\\ 0 & \text{if } x=a\,.\end{cases}$$

Then for $x \neq a$,

$$\big|\varepsilon(x)\big|\leqslant\frac{\Big|f(x)-f(a)-\sum\limits_{j=1}^{n}\dfrac{\partial f}{\partial x_j}(a)(x_j-a_j)\Big|}{\|x-a\|}$$

which, by the differentiability of $f$ at $a$, implies that

$$\lim_{x\to a}\big|\varepsilon(x)\big|=0\,.$$

Moreover,

$$\varepsilon(x)\cdot(x-a)=f(x)-f(a)-\sum_{j=1}^{n}\frac{\partial f}{\partial x_j}(a)(x_j-a_j)\,.$$

"⇐" Suppose that there exists a vector-valued function $\varepsilon : U \to \mathbb{R}^n$ such that

$$f(x)-f(a)-\sum_{j=1}^{n}\frac{\partial f}{\partial x_j}(a)(x_j-a_j)=\varepsilon(x)\cdot(x-a)$$

and $\varepsilon(x) \to 0$ as $x \to a$. Then for $x \neq a$, the Cauchy-Schwarz inequality implies that

$$\frac{\Big|f(x) - f(a) - \sum\limits_{j=1}^{n} \frac{\partial f}{\partial x_j}(a)(x_j - a_j)\Big|}{\|x - a\|} = \frac{|\varepsilon(x) \cdot (x - a)|}{\|x - a\|} \leqslant \|\varepsilon(x)\|\,;$$

thus

$$\lim_{x \to a} \frac{\Big|f(x) - f(a) - \sum\limits_{j=1}^{n} \frac{\partial f}{\partial x_j}(a)(x_j - a_j)\Big|}{\|x - a\|} = 0\,.$$

Therefore, $f$ is differentiable at $a$ with $[(Df)(a)] = \Big[\frac{\partial f}{\partial x_1}(a), \cdots, \frac{\partial f}{\partial x_n}(a)\Big]$. □