Exercise Problem Sets 9

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In Exercise Problem 1 through 4, we focus on another kind of connected sets, so-called path connected sets. First we introduce path connectedness in the following

Definition 0.1. Let (M, d) be a metric space. A subset $A \subseteq M$ is said to be ***path connected*** if for every $x, y \in A$, there exists a continuous map $\varphi : [0,1] \to A$ such that $\varphi(0) = x$ and $\varphi(1) = y$.

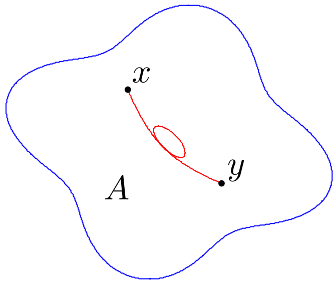

Figure 1: Path connected sets

Problem 1. Show that a convex set in a normed space is path connected.

Proof. Let C be a convex set in a normed space $(\mathcal{V}, \|\cdot\|)$, and $\boldsymbol{x}, \boldsymbol{y} \in C$. Define $\varphi : [0,1] \to \mathcal{V}$ by $\varphi(t) = (1-t)\boldsymbol{x} + t\boldsymbol{y}$. Then clearly φ is continuous on $[0,1]$ for

$$\big\|\varphi(t) - \varphi(s)\big\| = |t-s|\|\boldsymbol{x} - \boldsymbol{y}\| \qquad \forall\, t, s \in [0,1]\,.$$

Moreover, $\varphi(0) = \boldsymbol{x}$, $\varphi(1) = \boldsymbol{y}$, and the convexity (defined in Problem 4 of Exercise 7) of C implies that $\varphi([0,1]) \subseteq C$. Therefore, $\varphi : [0,1] \to C$ so that C is path connected. □

Problem 2. A set S in a vector space $\mathcal{V}$ is called ***star-shaped*** if there exists $p \in S$ such that for any $q \in S$, the line segment joining p and q lies in S. Show that a star-shaped set in a normed space is path connected.

Proof. Suppose that there exists $p \in S$ such that for any $q \in S$, the line segment joining p and q lies in S. In other words, such $p \in S$ satisfies that

$$(1-\lambda)q + \lambda p \subseteq S \qquad \forall\, \lambda \in [0,1] \text{ and } q \in S\,.$$

Let x, y in S. Define

$$\varphi(t) = \begin{cases} (1-2t)x + 2tp & \text{if } 0 \leqslant t \leqslant \dfrac{1}{2}\,, \\ (2-2t)p + (2t-1)y & \text{if } \dfrac{1}{2} < t \leqslant 1\,. \end{cases}$$

Then φ is continuous on $[0,1]$ $\big($since $\lim\limits_{t\to 0.5^+} \varphi(t) = \lim\limits_{t\to 0.5^-} \varphi(t) = p$ so that φ is continuous at $0.5\big)$. Moreover, $\varphi([0,0.5]) = \overline{xp}$ and $\varphi([0.5,1]) = \overline{py}$ so that $\varphi : [0,1] \to A$ is continuous with $\varphi(0) = x$ and $\varphi(1) = y$. Therefore, A is path connected. □

Problem 3. Let $A = \left\{\left(x, \sin\frac{1}{x}\right) \,\middle|\, x \in (0,1]\right\} \cup (\{0\} \times [-1,1])$. Show that A is connected in $(\mathbb{R}^2, \|\cdot\|_2)$, but A is not path connected.

Proof. Assume the contrary that A is path connected such that there is a continuous function $\varphi : [0,1] \to A$ such that $\varphi(0) = (x_0, y_0) \in \left\{\left(x, \sin\frac{1}{x}\right) \,\middle|\, x \in (0,1)\right\}$ and $\varphi(1) = (0,0) \in \{0\} \times [-1,1]$. Let $t_0 = \inf\left\{t \in [0,1] \,\middle|\, \varphi(t) \in \{0\} \times [-1,1]\right\}$. In other words, at $t = t_0$ the path touches $0 \times [-1,1]$ for the "first time". By the continuity of φ, $\varphi(t_0) \in \{0\} \times [-1,1]$. Since $\varphi(0) \notin \{0\} \times [-1,1]$, $\varphi([0,t_0)) \subseteq \left\{\left(x, \sin\frac{1}{x}\right) \,\middle|\, x \in (0,1)\right\}$.

Suppose that $\varphi(t_0) = (0, \bar{y})$ for some $\bar{y} \in [-1,1]$, and $\varphi(t) = \left(x(t), \sin\frac{1}{x(t)}\right)$ for $0 \leqslant t < t_0$. By the continuity of φ, there exists $\delta > 0$ such that if $|t - t_0| < \delta$, $|\varphi(t) - \varphi(t_0)| < 1$. In particular,

$$x(t)^2 + \left(\sin\frac{1}{x(t)} - \bar{y}\right)^2 < 1 \qquad \forall\, t \in (t_0 - \delta, t)\,.$$

On the other hand, since φ is continuous, $x(t)$ is continuous on $[0,t_0)$; thus by the fact that $[0,t_0)$ is connected, $x([0,t_0))$ is connected. Therefore, $x([0,t_0)) = (0,\bar{x}]$ for some $\bar{x} > 0$. Since $\lim\limits_{t\to t_0} x(t) = 0$, there exists $\{t_n\}_{n=1}^\infty \in [0,t_0)$ such that $t_n \to t_0$ as $n \to \infty$ and $\left|\sin\frac{1}{x(t_n)} - \bar{y}\right| \geqslant 1$. For $n \gg 1$, $t_n \in (t_0 - \delta, t_0)$ but

$$x(t_n)^2 + \left(\sin\frac{1}{x(t_n)} - \bar{y}\right)^2 \geqslant 1\,,$$

a contradiction.

On the other hand, A is the closure of the connected set $B = \left\{\left(x, \sin\frac{1}{x}\right) \,\middle|\, x \in (0,1)\right\}$ (the connectedness of B follows from the fact that the function $\psi(x) = \left(x, \sin\frac{1}{x}\right)$ is continuous on the connected set $(0,1)$). Therefore, by Problem 9 of Exercise 8, $A = \bar{B}$ is connected. □

Problem 4. Let (M,d) be a metric space, and $A \subseteq M$. Show that if A is path connected, then A is connected.

Hint: Use the fact that connected sets on $(\mathbb{R}, |\cdot|)$ are intervals and prove by contradiction.

Proof. Assume the contrary that there are non-empty sets A_1, A_2 such that $A = A_1 \cup A_2$ but $A_1 \cap \overline{A_2} = A_2 \cap \overline{A_1} = \varnothing$. Let $x \in A_1$ and $y \in A_2$. By the path connectedness of A, there exists a continuous map $\varphi : [0,1] \to A$ such that $\varphi(0) = x$ and $\varphi(1) = y$. Define $I_1 = \varphi^{-1}(A_1)$ and $I_2 = \varphi^{-1}(A_2)$. Then clearly $0 \in I_1$ and $1 \in I_2$, and $I_1 \cap I_2 = \varnothing$. Moreover,

$$[0,1] = \varphi^{-1}(A) = \varphi^{-1}(A_1 \cup A_2) = \varphi^{-1}(A_1) \cup \varphi^{-1}(A_2) = I_1 \cup I_2\,.$$

Claim: $I_1 \cap \bar{I}_2 = I_2 \cap \bar{I}_1 = \varnothing$.

Suppose the contrary that $t \in I_1 \cap \bar{I}_2$. Then $t \in \varphi(A_1)$ which shows that $\varphi(t) \in A_1$. On the other hand, $t \in \bar{I}_2$; thus there exists $\{t_n\}_{n=1}^\infty \subseteq I_2$ such that $t_n \to t$ as $n \to \infty$. By the continuity of φ,

$$\varphi(t) = \lim_{n\to\infty} \varphi(t_n) \in \overline{A_2}\,;$$

thus we find that $\varphi(t) \in A_1 \cap \overline{A_2}$, a contradiction. Therefore, $I_1 \cap \bar{I}_2 = \varnothing$. Similarly, $I_2 \cap \bar{I}_1 = \varnothing$; thus we establish the existence of non-empty sets I_1 and I_2 such that

$$[0,1] = I_1 \cup I_2\,, \quad I_1, I_2 \neq \varnothing\,, \quad I_1 \cap \bar{I}_2 = I_2 \cap \bar{I}_1 = \varnothing$$

which shows that $[0,1]$ is disconnected, a contradiction. □

Alternative proof. Assume the contrary that there are two open sets V_1 and V_2 such that

1. $A \cap V_1 \cap V_2 = \varnothing$; 2. $A \cap V_1 \neq \varnothing$; 3. $A \cap V_2 \neq \varnothing$; 4. $A \subseteq V_1 \cup V_2$.

Since A is path connected, for $x \in A \cap V_1$ and $y \in A \cap V_2$, there exists a continuous map $\varphi : [0,1] \to A$ such that $\varphi(0) = x$ and $\varphi(1) = y$. Since V_1 and V_2 are open, there exist U_1 and U_2 open in $(\mathbb{R}, |\cdot|)$ such that $\varphi^{-1}(V_1) = U_1 \cap [0,1]$ and $\varphi^{-1}(V_2) = U_2 \cap [0,1]$. Therefore,

$$[0,1] = \varphi^{-1}(A) \subseteq \varphi^{-1}(V_1) \cup \varphi^{-1}(V_2) \subseteq U_1 \cup U_2\,.$$

Since $0 \in U_1$, $1 \in U_2$, and $[0,1] \cap U_1 \cap U_2 = \varphi^{-1}(A \cap V_1 \cap V_2) = \varnothing$, we conclude that $[0,1]$ is disconnected, a contradiction. □

Problem 5. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ satisfy $T(x+y) = T(x) + T(y)$ for all $x, y \in \mathbb{R}^n$.

1. Show that $T(rx) = rT(x)$ for all $r \in \mathbb{Q}$ and $x \in \mathbb{R}^n$.
2. Suppose that T is continuous on $\mathbb{R}^n$. Show that T is linear; that is, $T(cx+y) = cT(x) + T(y)$ for all $c \in \mathbb{R}$, $x, y \in \mathbb{R}^n$.
3. Suppose that T is continuous at some point x_0 in $\mathbb{R}^n$. Show that T is continuous on $\mathbb{R}^n$.
4. Suppose that T is bounded on some open subset of $\mathbb{R}^n$. Show that T is continuous on $\mathbb{R}^n$.
5. Suppose that T is bounded from above (or below) on some open subset of $\mathbb{R}^n$. Show that T is continuous on $\mathbb{R}^n$.
6. Construct a $T : \mathbb{R} \to \mathbb{R}$ which is discontinuous at every point of $\mathbb{R}$, but $T(x+y) = T(x) + T(y)$ for all $x, y \in \mathbb{R}$.

Proof. 1. By induction, $T(kx) = kT(x)$ for all $k \in \mathbb{N}$. Moreover, $T(0) = T(0+0) = T(0) + T(0)$ which implies that $T(0) = 0$; thus $T(0x) = 0T(x)$ and if $k \in \mathbb{N}$,

$$-kT(x) = -kT(x) + T(0) = -kT(x) + T(kx + (-kx)) = -kT(x) + T(kx) + T(-kx) = T(-kx)\,.$$

Therefore, $T(kx) = kT(x)$ for all $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$. Let $r = \dfrac{q}{p}$ for some $p, q \in \mathbb{Z}$. Then for $x \in \mathbb{R}^n$,

$$pT(rx) = T(prx) = T(qx) = qT(x)$$

which implies that $T(rx) = rT(x)$ for all $r \in \mathbb{Q}$ and $x \in \mathbb{R}^n$.

2. Let $x, y \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then there exists $\{c_k\}_{k=1}^\infty \subseteq \mathbb{Q}$ such that $\lim\limits_{k\to\infty} c_k = c$. This further implies that $c_k x \to cx$ as $k \to \infty$ since

$$\lim_{k\to\infty} \|c_n x - cx\| = \lim_{k\to\infty} \|(c_k - c)x\| = \|x\| \lim_{k\to\infty} |c_k - c| = 0$$

Therefore, by the continuity of T,

$$T(cx + y) = T(cx) + T(y) = \lim_{k\to\infty} T(c_k x) + T(y) = \lim_{k\to\infty} c_k T(x) + T(y) = cT(x) + T(y)\,.$$

3. Let $a \in \mathbb{R}^n$ and $\varepsilon > 0$ be given. By the continuity of T at x_0, there exists $\delta > 0$ such that

$$\|T(x - x_0)\| = \|T(x) - T(x_0)\| < \varepsilon \quad \text{whenever} \quad \|x - x_0\| < \delta\,.$$

The statement above implies that if $\|x\| < \delta$, then $\|T(x)\| < \varepsilon$. Therefore,

$$\|T(x) - T(a)\| = \|T(x - a)\| < \varepsilon \quad \text{whenever} \quad \|x - a\| < \delta$$

which shows that T is continuous at a.

4. Suppose that T is bounded on an open set U so that $T(U) \subseteq B(0, M)$. Let $x_0 \in U$. Then there exists $r > 0$ such that $B(x_0, r) \subseteq U$. Therefore, if $x \in B(0, r)$, then $x + x_0 \in B(x_0, r)$ so that

$$\|T(x)\| \leqslant \|T(x + x_0)\| + \|T(x_0)\| \leqslant M + \|T(x_0)\| \equiv R\,;$$

thus we establish that there exists r and R such that

$$\|T(x)\| \leqslant R \quad \text{whenever} \quad \|x\| < r\,.$$

Let $\varepsilon > 0$ be given. Choose $c \in \mathbb{Q}$ so that $0 < c < \dfrac{\varepsilon}{R}$. For such a fixed $c \in \mathbb{Q}$, choose $0 < \delta < cr$. If $\|x\| < \delta$, then $\|\frac{x}{c}\| < \dfrac{\delta}{c} < r$; thus if $\|x\| < \delta$, we have $\|T(\frac{x}{c})\| \leqslant R$ so that

$$\|T(x)\| = \Big\|T\Big(c\frac{x}{c}\Big)\Big\| = \Big\|cT\Big(\frac{x}{c}\Big)\Big\| = c\Big\|T\Big(\frac{x}{c}\Big)\Big\| \leqslant cR < \varepsilon\,.$$

Therefore, T is continuous at 0. By 3, T is continuous on $\mathbb{R}^n$.

5. Suppose that $Tx \leqslant M$ (so that in this case $m = 1$) for all $x \in U$, where U is an open set in $\mathbb{R}^n$. Let $x_0 \in U$. Then there exists $r > 0$ such that $B(x_0, r) \subseteq U$; thus if $x \in B(0, r)$,

$$Tx = T(x + x_0) - T(x_0) \leqslant M - T(x_0) \equiv R\,.$$

Therefore, we establish that there exist r and R such that

$$T(x) \leqslant R \quad \text{whenever} \quad x \in B(0, r)\,.$$

For $x \in B(0, r)$, we must have $-x \in B(0, r)$; thus

$$-T(x) = T(-x) \leqslant R\,;$$

thus $-R \leqslant T(x)$ whenever $x \in B(0, r)$. Therefore, $|T(x)| \leqslant R$ whenever $\|x\| < r$. By 4, T is continuous on $\mathbb{R}^n$. $\square$

Problem 6. Let (M, d) be a metric space, $A \subseteq M$, and $f : A \to \mathbb{R}$. For $a \in A'$, define

$$\liminf_{x \to a} f(x) = \lim_{r \to 0^+} \inf \left\{ f(x) \,\middle|\, x \in B(a, r) \cap A \backslash \{a\} \right\},$$
$$\limsup_{x \to a} f(x) = \lim_{r \to 0^+} \sup \left\{ f(x) \,\middle|\, x \in B(a, r) \cap A \backslash \{a\} \right\}.$$

Complete the following.

1. Show that both $\liminf\limits_{x \to a} f(x)$ and $\limsup\limits_{x \to a} f(x)$ exist (which may be $\pm\infty$), and

$$\liminf_{x \to a} f(x) \leqslant \limsup_{x \to a} f(x).$$

Furthermore, there exist sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty} \subseteq A \backslash \{a\}$ such that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ both converge to a, and

$$\lim_{n \to \infty} f(x_n) = \liminf_{x \to a} f(x) \qquad \text{and} \qquad \lim_{n \to \infty} f(y_n) = \limsup_{x \to a} f(x).$$

2. Let $\{x_n\}_{n=1}^{\infty} \subseteq A \backslash \{a\}$ be a convergent sequence with limit a. Show that

$$\liminf_{x \to a} f(x) \leqslant \liminf_{n \to \infty} f(x_n) \leqslant \limsup_{n \to \infty} f(y_n) \leqslant \limsup_{x \to a} f(x).$$

3. Show that $\lim\limits_{x \to a} f(x) = \ell$ if and only if

$$\liminf_{x \to a} f(x) = \limsup_{x \to a} f(x) = \ell.$$

4. Show that $\liminf\limits_{x \to a} f(x) = \ell \in \mathbb{R}$ if and only if the following two conditions hold:

   (a) for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\ell - \varepsilon < f(x)$ for all $x \in B(a, \delta) \cap A \backslash \{a\}$;

   (b) for all $\varepsilon > 0$ and $\delta > 0$, there exists $x \in B(a, \delta) \cap A \backslash \{a\}$ such that $f(x) < \ell + \varepsilon$.

   Formulate a similar criterion for limsup and for the case that $\ell = \pm\infty$.

5. Compute the liminf and limsup of the following functions at any point of $\mathbb{R}$.

   (a) $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}^{\complement}, \\ \dfrac{1}{p} & \text{if } x = \dfrac{q}{p} \text{ with } (p, q) = 1,\ q > 0,\ p \neq 0. \end{cases}$

   (b) $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ -x & \text{if } x \in \mathbb{Q}^{\complement}. \end{cases}$

Proof. For $r > 0$, define $m, M : A' \to R^*$ by

$$m(r) = \inf \left\{ f(x) \,\middle|\, x \in B(a, r) \cap A \backslash \{a\} \right\} \quad \text{and} \quad M(r) = \sup \left\{ f(x) \,\middle|\, x \in B(a, r) \cap A \backslash \{a\} \right\}.$$

We remark that it is possible that $m(r) = -\infty$ or $M(r) = \infty$. Note that m is decreasing and M is increasing in $(0, \infty)$.

1. By the monotonicity of m and M, $\lim_{r\to 0^+} m(r)$ and $\lim_{r\to 0^+} M(r)$ exist (which may be $\pm\infty$). Moreover, $m(r) \leqslant M(r)$ for all $r > 0$; thus $\lim_{r\to 0^+} m(r) \leqslant \lim_{r\to 0^+} M(r)$ so we conclude that

$$\liminf_{x\to a} f(x) = \lim_{r\to 0^+} m(r) \leqslant \lim_{r\to 0^+} M(r) = \limsup_{x\to a} f(x)\,.$$

Since $\liminf_{x\to a} f(x) = -\limsup_{x\to a}(-f)(x)$, it suffices to consider the case of the limit superior.

   (a) If $\limsup_{x\to a} f(x) = \infty$, then for each $n \in \mathbb{N}$ there exists $0 < \delta_n < \frac{1}{n}$ such that

   $$M(r) \geqslant n \quad \textit{whenever} \quad 0 < r < \delta_n\,.$$

   By the definition of the supremum, for each $n \in \mathbb{N}$ there exists $x_n \in B\big(a, \frac{\delta_n}{2}\big) \cap A\backslash\{a\}$ such that $f(x_n) \geqslant n - 1$.

   (b) If $\limsup_{x\to a} f(x) = L$, then for each $n \in \mathbb{N}$ there exists $0 < \delta_n < \frac{1}{n}$ such that

   $$\big|M(r) - L\big| < \frac{1}{n} \quad \textit{whenever} \quad 0 < r < \delta_n\,.$$

   By the definition of the supremum, for each $n \in \mathbb{N}$ there exists $x_n \in B\big(a, \frac{\delta_n}{2}\big) \cap A\backslash\{a\}$ such that

   $$L - \frac{1}{n} < f(x_n) < L + \frac{1}{n}\,.$$

Since $\delta_n \to 0$ as $n \to \infty$, we find that $\{x_n\}_{n=1}^{\infty} \subseteq A\backslash\{a\}$ converges to a and $\lim_{n\to\infty} f(x_n) = \limsup_{x\to a} f(x)$.

2. It suffices to show the case of the limit inferior. Let $\{x_n\}_{n=1}^{\infty} \subseteq A\backslash\{a\}$ and $x_n \to a$ as $n \to \infty$. For every $k \in \mathbb{N}$, there exists $N_k > 0$ such that $0 < d(x_n, a) < \frac{1}{k}$ whenever $n \geqslant N_k$. W.L.O.G., we can assume that $N_k \geqslant k$ and $N_{k+1} > N_k$ for all $k \in \mathbb{N}$. By the definition of infimum,

$$m\big(\frac{1}{k}\big) \leqslant f(x_n) \quad \textit{whenever} \quad n \geqslant N_k$$

which further implies that

$$m\big(\frac{1}{k}\big) \leqslant \inf_{n\geqslant N_k} f(x_n)\,.$$

Note that $\lim_{r\to 0^+} m(r) = \lim_{k\to\infty} m\big(\frac{1}{k}\big)$ and $\lim_{k\to\infty} \inf_{n\geqslant N_k} f(x_k) = \lim_{k\to\infty} \inf_{n\geqslant k} f(x_k)$, we conclude that

$$\liminf_{x\to a} f(x) = \lim_{r\to 0^+} m(r) = \lim_{k\to\infty} m\big(\frac{1}{k}\big) \leqslant \lim_{k\to\infty} \inf_{n\geqslant N_k} f(x_n) = \lim_{k\to\infty} \inf_{n\geqslant k} f(x_n) = \liminf_{n\to\infty} f(x_n)\,.$$

3. ($\Rightarrow$) Let $\varepsilon > 0$ be given. There exists $\delta > 0$ such that

$$|f(x) - \ell| < \varepsilon \quad \text{whenever} \quad x \in B(a,\delta) \cap A\backslash\{a\}\,.$$

   Therefore,

$$\ell - \varepsilon < f(x) < \ell + \varepsilon \quad \text{whenever} \quad x \in B(a,\delta) \cap A\backslash\{a\}$$

which implies that

$$\ell - \varepsilon \leqslant m(\delta) \leqslant M(\delta) \leqslant \ell + \varepsilon\,.$$

By the monotonicity of m and M, the inequality above implies that

$$\ell - \varepsilon \leqslant m(\delta) \leqslant m(r) \leqslant M(r) \leqslant M(\delta) \leqslant \ell + \varepsilon \quad \forall\, 0 < r < \delta\,.$$

Passing to the limit as $r \to 0^+$, we find that

$$\ell - \varepsilon \leqslant \liminf_{x\to a} f(x) \leqslant \limsup_{x\to a} f(x) \leqslant \ell + \varepsilon\,.$$

Since $\varepsilon > 0$ is chosen arbitrary, we conclude that $\liminf\limits_{x\to a} f(x) = \limsup\limits_{x\to a} f(x) = \ell$.

($\Leftarrow$) Let $\{x_n\}_{n=1}^\infty \subseteq A\backslash\{a\}$ be a sequence with limit a. Then 2 and the assumption that $\liminf\limits_{x\to a} f(x) = \limsup\limits_{x\to a} f(x) = \ell$ imply that $\liminf\limits_{n\to\infty} f(x_n) = \limsup\limits_{n\to\infty} f(x_n) = \ell$. Therefore, $\lim\limits_{n\to\infty} f(x_n) = \ell$.

4. ($\Rightarrow$) This direction is proved by contradiction.

   (a) Suppose the contrary that there exists $\varepsilon > 0$ such that for each $n \in \mathbb{N}$, there exists $x_n \in B\big(a, \frac{1}{n}\big) \cap A\backslash\{a\}$ such that $f(x_n) \leqslant \ell - \varepsilon$. Then $\{x_n\}_{n=1}^\infty A\backslash\{a\}$ and $\lim\limits_{n\to\infty} x_n = a$; however,

   $$\liminf_{n\to\infty} f(x_n) \leqslant \ell - \varepsilon < \ell = \liminf_{x\to a} f(x)\,,$$

   a contradiction to 2.

   (b) Suppose the contrary that there exist $\varepsilon > 0$ and $\delta > 0$ such that

   $$f(x) \geqslant \ell + \varepsilon \qquad \forall\, x \in B(a,\delta) \cap A\backslash\{a\}\,.$$

   Then $m(\delta) \geqslant \ell + \varepsilon$; thus the monotonicity of m implies that

   $$\ell + \varepsilon \leqslant m(\delta) \leqslant m(r) \quad \text{whenever} \quad 0 < r < \delta\,.$$

   Passing to the limit as $r \to 0^+$, we conclude that

   $$\ell + \varepsilon \leqslant \lim_{r\to 0^+} m(r) = \liminf_{x\to a} f(x)\,,$$

   a contradiction.

($\Leftarrow$) Let $\{x_n\}_{n=1}^\infty \subseteq A\backslash\{a\}$ be a sequence with limit a, and $\varepsilon > 0$ be given. Then (a) provides $\delta > 0$ such that $f(x) > \ell - \varepsilon$ whenever $x \in B(a,\delta) \cap A\backslash\{a\}$. For such $\delta > 0$, there exists $N > 0$ such that $0 < d(x_n, a) < \delta$ for all $n \geqslant N$. Therefore, if $n \geqslant N$, $f(x_n) > \ell - \varepsilon$ which implies that $\liminf\limits_{n\to\infty} f(x_n) \geqslant \ell - \varepsilon$. Since $\varepsilon > 0$ is chosen arbitrary, we conclude that

$$\liminf_{n\to\infty} f(x_n) \geqslant \ell \text{ for every convergent sequence } \{x_n\}_{n=1}^\infty \subseteq A\backslash\{a\} \text{ with limit } a.$$

On the other hand, using (b) we find that for each $n \in \mathbb{N}$, there exists $x_n \in B\big(a, \frac{1}{n}\big) \cap A\backslash\{a\}$ such that $f(x_n) < \ell + \frac{1}{n}$. Then $\liminf\limits_{n\to\infty} f(x_n) \leqslant \ell$, and (i) further implies that $\liminf\limits_{n\to\infty} f(x_n) =$

ℓ; thus we establish that there exists a convergent sequence $\{x_n\}_{n=1}^\infty \subseteq A\backslash\{a\}$ with limit a such that $\liminf\limits_{n\to\infty} f(x_n) = \ell$.

By 1 and 2, we conclude that $\ell = \liminf\limits_{x\to a} f(x)$.

5. (a) $\liminf\limits_{x\to a} f(x) = \limsup\limits_{x\to a} f(x) = 0$ for all $a \in \mathbb{R}$.

   (b) $\liminf\limits_{x\to a} f(x) = -|a|$, $\limsup\limits_{x\to a} f(x) = |a|$. In particular, $\lim\limits_{x\to 0} f(x) = 0$. □

Problem 7. Let (M,d) be a metric space, and $A \subseteq M$. A function $f : A \to \mathbb{R}$ is called ***lower semi-continuous*** / ***upper semi-continuous*** at $a \in A$ if either $a \in A\backslash A'$ or $\liminf\limits_{x\to a} f(x) \geqslant f(a)$, / $\limsup\limits_{x\to a} f(x) \leqslant f(a)$, and is called lower/upper semi-continuous on A if f is lower/uppser semi-continuous at a for all $a \in A$.

1. Show that $f : A \to \mathbb{R}$ is lower semi-continuous on A if and only if $f^{-1}((-\infty, r])$ is closed relative to A. Also show that $f : A \to \mathbb{R}$ is upper semi-continuous on A if and only if $f^{-1}([r,\infty))$ is closed relative to A.

2. Show that f is lower semi-continuous on A if and only if for all convergent sequences $\{x_n\}_{n=1}^\infty \subseteq A$ and $\{s_n\}_{n=1}^\infty \subseteq \mathbb{R}$ satisfying $f(x_n) \leqslant s_n$ for all $n \in \mathbb{N}$, we have
$$f\big(\lim_{n\to\infty} x_n\big) \leqslant \lim_{n\to\infty} s_n\,.$$

3. Let $\{f_\alpha\}_{\alpha\in I}$ be a family of lower semi-continuous functions on A. Prove that $f(x) = \sup\limits_{\alpha\in I} f_\alpha(x)$ is lower semi-continuous on A.

4. Let A be a perfect set (that is, A contains no isolated points) and $f : A \to \mathbb{R}$ be given. Define
$$f^*(x) = \limsup_{y\to x} f(y) \qquad \text{and} \qquad f_*(x) = \liminf_{y\to x} f(y)\,.$$
Show that f^* is upper semi-continuous and f_* is lower semi-continuous, and $f_*(x) \leqslant f(x) \leqslant f^*(x)$ for all $x \in A$. Moreover, if g is a lower semi-continuous function on A such that $g(x) \leqslant f(x)$ for all $x \in A$, then $g \leqslant f_*$.

Proof. We first note that by 1, 2 and 4 of Problem 6,

$f : A \to \mathbb{R}$ is lower semi-continuous at a

$\Leftrightarrow$ for all $\varepsilon > 0$, there exists $\delta > 0$ such that $f(a) - \varepsilon < f(x)$ for all $x \in B(a,\delta) \cap A$

$\Leftrightarrow$ for all convergent sequence $\{x_n\}_{n=1}^\infty \subseteq A$ with limit a, $f(a) \leqslant \liminf\limits_{n\to\infty} f(x_n)$.

We note that the first statement implies the second one because of 4(a) in Problem 6, the second statement implies the third one because of $x_n \in B(a,\delta) \cap A$ when $n \ll 1$, and the third statement implies the first one because of 1 in Problem 6.

1. ($\Rightarrow$) It suffices to prove the case for limit inferior since $\limsup_{x\to a} f(x) = -\liminf_{x\to a}(-f)(x)$. We note that E is closed relative to A if and only if $E \cap A$ is a closed set in the metric space (A, d). Therefore, a subset of E of A is closed relative to A if and only if

   every sequence $\{x_n\}_{n=1}^{\infty} \subseteq E$ that converges to a point in A must also has limit in E.

   Let $r \in \mathbb{R}$ and $\{x_n\}_{n=1}^{\infty}$ be a sequence in $E \equiv f^{-1}((-\infty, r])$ such that $\{x_n\}_{n=1}^{\infty}$ converges to a point $a \in A$. Then $f(a) \leqslant \liminf_{n\to\infty} f(x_n) \leqslant r$ which implies that $a \in f^{-1}((-\infty, r])$.

   ($\Leftarrow$) Let $a \in A$ and $\varepsilon > 0$ be given. Define $r = f(a) - \varepsilon$. Then $V = f^{-1}((r, \infty))$ is open relative to A (since $f^{-1}((-\infty, r])$ is closed relative to A). Since $a \in U$, there exists $\delta > 0$ such that $B(a, \delta) \cap A \subseteq V$. This implies that

$$f(a) - \varepsilon < f(x) \qquad \forall\, x \in B(a, \delta) \cap A\,.$$

2. ($\Rightarrow$) Let $\{x_n\}_{n=1}^{\infty}$ be a convergent sequence in A with limit a, $\{s_n\}_{n=1}^{\infty}$ be a real sequence with limit s, and $f(x_n) \leqslant s_n$ for all $n \in \mathbb{N}$. Suppose that $f(a) > s$. Let $\varepsilon = \dfrac{f(a) - s}{2}$. Since f is lower semi-continuous at a, $\liminf_{x\to a} f(x) \geqslant f(a)$; thus there exists $\delta > 0$ such that

$$f(a) - \varepsilon < f(x) \qquad \forall\, x \in B(a, \delta) \cap A\,.$$

   On the other hand, there exists $N > 0$ such that $x_n \in B(a, \delta) \cap A$ and $s_n < s + \varepsilon$ whenever $n \geqslant N$. Therefore, if $n \geqslant N$,

$$s_n < s + \varepsilon = f(a) - \varepsilon < f(x_n)\,,$$

   a contradiction.

   ($\Leftarrow$) Let $a \in A$, and $\{x_n\}_{n=1}^{\infty} \subseteq A$ be a sequence with limit a. Let $\{x_{n_j}\}_{j=1}^{\infty}$ be a subsequence of $\{x_n\}_{n=1}^{\infty}$ such that $\lim_{j\to\infty} f(x_{n_j}) = \liminf_{n\to\infty} f(x_n)$. Define $s_j = f(x_{n_j})$. Then clearly $f(x_{n_j}) \leqslant s_j$ for all $j \in \mathbb{N}$; thus by assumption

$$f(a) \leqslant \lim_{j\to\infty} s_j = \liminf_{n\to\infty} f(x_n)\,.$$

3. Let $a \in A \cap A'$ and $\{x_n\}_{n=1}^{\infty} \subseteq A\backslash\{a\}$ be a sequence with limit a. Then $f_\alpha(x_n) \leqslant f(x_n)$ for all $n \in \mathbb{N}$ and $\alpha \in I$. Since f_α is lower semi-continuous for each $\alpha \in I$, for $\alpha \in I$ we have

$$f_\alpha(a) \leqslant \liminf_{x\to a} f_\alpha(x) \leqslant \liminf_{x\to a} f(x)\,.$$

   The inequality above implies that

$$f(a) = \sup_{\alpha\in I} f_\alpha(a) \leqslant \liminf_{x\to a} f(x)\,;$$

   thus f is lower semi-continuous at a. □