Exercise Problem Sets 7

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Problem 1. Let $A \subseteq \mathbb{R}^n$. Define the sequence of sets $A^{(m)}$ as follows: $A^{(0)} = A$ and $A^{(m+1)} =$ the derived set of $A^{(m)}$ for $m \in \mathbb{N}$. Complete the following.

1. Prove that each $A^{(m)}$ for $m \in \mathbb{N}$ is a closed set; thus $A^{(1)} \supseteq A^{(2)} \supseteq \cdots$.
2. Show that if there exists some $m \in \mathbb{N}$ such that $A^{(m)}$ is a countable set, then A is countable.
3. For any given $m \in \mathbb{N}$, is there a set A such that $A^{(m)} \neq \varnothing$ but $A^{(m+1)} = \varnothing$?
4. Let A be uncountable. Then each $A^{(m)}$ is an uncountable set. Is it possible that $\bigcap\limits_{m=1}^{\infty} A^{(m)} = \varnothing$?
5. Let $A = \left\{ \dfrac{1}{m} + \dfrac{1}{k} \,\middle|\, m-1 > k(k-1), m, k \in \mathbb{N} \right\}$. Find $A^{(1)}$, $A^{(2)}$ and $A^{(3)}$.

Proof. 1. See Problem 2 for that A' is closed for all $A \subseteq M$. Moreover, $\bar{A} = A \cup A'$ so that $A \subseteq A'$ if A is closed (in fact, A is closed if and only if $A \subseteq A'$). Therefore, knowing that $A^{(m)}$ is closed for all $m \in \mathbb{N}$, we obtain that $A^{(m)} \supseteq A^{(m+1)}$ for all $m \in \mathbb{N}$.

2. Note that $A \backslash A'$ consists of all isolated points of A. For $m \in \mathbb{N}$, define $B^{(m-1)} = A^{(m-1)} \backslash A^{(m)}$. Then $B^{(m-1)}$ consists of isolated points of $A^{(m-1)}$; thus $B^{(m-1)}$ is countable for all $m \in \mathbb{N}$. Since for any subset A of M, we have

$$A \subseteq (A \backslash A') \cup A'$$

and equality holds if A is closed, 1 implies that

$$\begin{aligned} A &\subseteq (A \backslash A^{(1)}) \cup A^{(1)} = B^{(0)} \cup A^{(1)} = B^{(0)} \cup \left[\left(A^{(1)} \backslash A^{(2)} \right) \cup A^{(2)} \right] = B^{(0)} \cup B^{(1)} \cup A^{(2)} \\ &= \cdots = B^{(0)} \cup B^{(1)} \cup \cdots \cup B^{(m-1)} \cup A^{(m)} . \end{aligned}$$

If $A^{(m)}$ is countable, we find that A is a subset of a finite union of countable sets; thus A is countable.

3. For each $m \in \mathbb{N}$, define

$$A_m = \left\{ \frac{1}{i_1} + \frac{1}{i_2} + \cdots + \frac{1}{i_m} \,\middle|\, i_m \geqslant i_{m-1} \geqslant i_{m-2} \geqslant \cdots \geqslant i_1 \right\} .$$

Then $A_m' = \bigcup_{j=1}^{m-1} A_j \cup \{0\}$. To see this, let $\{x_k\}_{k=1}^{\infty}$ be a convergent sequence in A_m. W.L.O.G. we can assume that $\{x_k\}_{k=1}^{\infty}$ has distinct terms; that is, $x_k \neq x_j$ if $k \neq j$ for otherwise

(a) if finitely many terms are the same, eliminating all but one such terms from the original sequence does not change the limit of the sequence;

(b) if infinitely many terms are the same, then this term is a cluster point of the sequence; thus the sequence converges to this term which is one particular element of A_m.

If $\{x_k\}_{k=1}^{\infty}$ has distinct terms, then there exists $1 \leqslant j \leqslant m$ such that

$$\#\big\{k \in \mathbb{N} \,\big|\, i_j^{(k)}\big\} = \infty$$

that is, at least one $i_j^{(k)}$ has infinitely many

$A''_m = \bigcup_{j=1}^{m-2} A_j \cup \{0\}$, $\cdots$, $A_m^{(m-1)} = A_1 \cup \{0\}$, $A_m^{(m)} = \{0\}$, $A_m^{(m+1)} = \varnothing$.

4. By 2, if $A^{(m)}$ is countable for some $m \in \mathbb{N}$, then A is countable; thus if A is uncountable, $A^{(m)}$ must be uncountable for all $m \in \mathbb{N}$.
5. Similar to 3, we have $A^{(1)} = \Big\{\frac{1}{k} \,\Big|\, k \in \mathbb{N}\Big\} \cup \{0\}$, $A^{(2)} = \{0\}$ and $A^{(3)} = \varnothing$. □

Problem 2. Let (M, d) be a metric space, and A be a subset of M. Show that A', the derived set of A consisting of all accumulation points of A (defined in Exercise 6), is closed.

Proof. Let $y \notin A'$. Then there exists $\varepsilon > 0$ such that

$$B(y, \varepsilon) \cap (A \backslash \{y\}) = (B(y, \varepsilon) \backslash \{y\}) \cap A = \varnothing\,.$$

Then $A \subseteq \big(B(y, \varepsilon) \backslash \{y\}\big)^{\complement}$. Since

$$\big(B(y, \varepsilon) \backslash \{y\}\big)^{\complement} = \big(B(y, \varepsilon) \cap \{y\}^{\complement}\big)^{\complement} = B(y, \varepsilon)^{\complement} \cup \{y\}\,,$$

by the fact that $\big(B(y, \varepsilon) \backslash \{y\}\big)^{\complement}$ is closed,

$$\bar{A} \subseteq \big(B(y, \varepsilon) \backslash \{y\}\big)^{\complement} \quad \text{or equivalently,} \quad \bar{A} \cap B(y, \varepsilon) \backslash \{y\} = \varnothing\,.$$

Since $\bar{A} \subseteq A'$, the equality above implies that

$$A' \cap B(y, \varepsilon) \backslash \{y\} = \varnothing\,;$$

thus the fact that $y \notin A'$ implies that $B(y, \varepsilon) \cap A' = \varnothing$. □

Problem 3. Recall that a cluster point x of a sequence $\{x_n\}_{n=1}^{\infty}$ satisfies that

$$\forall\, \varepsilon > 0, \#\big\{n \in \mathbb{N} \,\big|\, x_n \in B(x, \varepsilon)\big\} = \infty\,.$$

Show that the collection of cluster points of a sequence (in a metric space) is closed.

Proof. Let (M, d) be a metric space, $\{x_k\}_{k=1}^{\infty}$ be a sequence in M, and A be the collection of cluster points of $\{x_k\}_{k=1}^{\infty}$. We would like to show that $A \supseteq \bar{A}$.

Let $y \in A^{\complement}$. Then y is not a cluster point of $\{x_k\}_{k=1}^{\infty}$; thus

$$\exists\, \varepsilon > 0 \ni \#\big\{n \in \mathbb{N} \,\big|\, x_n \in B(y, \varepsilon)\big\} < \infty\,.$$

For $z \in B(y, \varepsilon)$, let $r = \varepsilon - d(y, z) > 0$. Then $B(z, r) \subseteq B(y, \varepsilon)$ (see Figure 1 or check rigorously using the triangle inequality). As a consequence, $\#\big\{n \in \mathbb{N} \,\big|\, x_n \in B(z, r)\big\} < \infty$ which implies that $z \notin A$.

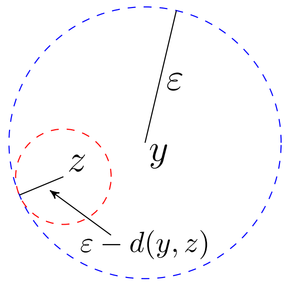

Figure 1: $B(z, \varepsilon - d(y,z)) \subseteq B(y,\varepsilon)$ if $z \in B(y,\varepsilon)$

Therefore, if $z \in B(y,\varepsilon)$ then $z \in A^{\complement}$; thus $B(y,\varepsilon) \cap A = \varnothing$. We then conclude that if $y \in A^{\complement}$ then $y \notin \bar{A}$. □

Problem 4. Let $(\mathcal{V}, \|\cdot\|)$ ba a normed vector space. A subset C of $\mathcal{V}$ is said to be convex if

$$(\forall\, \boldsymbol{x}, \boldsymbol{y} \in C \wedge \lambda \in [0,1])(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y} \in C)\,.$$

Let C be a non-empty convex set in $\mathcal{V}$.

1. Show that $\bar{C}$ is convex.
2. Show that if $\boldsymbol{x} \in \mathring{C}$ and $\boldsymbol{y} \in \bar{C}$, then $(1-\lambda)\boldsymbol{x} + \lambda \boldsymbol{y} \in \mathring{C}$ for all $\lambda \in (0,1)$. This result is sometimes called the ***line segment principle***.
3. Show that $\mathring{C}$ is convex (you may need the conclusion in 2 to prove this).
4. Show that $\text{cl}(\mathring{C}) = \text{cl}(C)$.
5. Show that $\text{int}(\bar{C}) = \text{int}(C)$.

Hint: 2. Prove by contradiction.
3 and 4. Use the line segment principle.
5. Show that $\boldsymbol{x} \in \text{int}(\bar{C})$ can be written as $(1-\lambda)\boldsymbol{y} + \lambda \boldsymbol{z}$ for some $\boldsymbol{y} \in \mathring{C}$ and $\boldsymbol{z} \in B(\boldsymbol{x},\varepsilon) \subseteq \bar{C}$.

Proof. 1. Let $\boldsymbol{x}, \boldsymbol{y} \in \bar{C}$ and $0 \leqslant \lambda \leqslant 1$. Then there exist sequences $\{\boldsymbol{x}_k\}_{k=1}^{\infty}$ and $\{\boldsymbol{y}_k\}_{k=1}^{\infty}$ in C such that $\boldsymbol{x}_k \to \boldsymbol{x}$ and $\boldsymbol{y}_k \to \boldsymbol{y}$ as $k \to \infty$. Since C is convex, $(1-\lambda)\boldsymbol{x}_k + \lambda \boldsymbol{y}_k \in C$ for each $k \in \mathbb{N}$; thus by the fact that $C \subseteq \bar{C}$, $(1-\lambda)\boldsymbol{x}_k + \lambda \boldsymbol{y}_k \in \bar{C}$ for each $k \in \mathbb{N}$. Since $(1-\lambda)\boldsymbol{x}_k + \lambda \boldsymbol{y}_k \to (1-\lambda)\boldsymbol{x} + \lambda \boldsymbol{y}$ as $k \to \infty$ and $\bar{C}$ is closed, we must have $(1-\lambda)\boldsymbol{x} + \lambda \boldsymbol{y} \in \bar{C}$; thus $\bar{C}$ is convex if C is convex.

2. Suppose the contrary that there exists $\lambda \in (0,1)$ such that $(1-\lambda)\boldsymbol{x} + \lambda \boldsymbol{y} \notin \mathring{C}$. Then for each $k \in \mathbb{N}$, there exists $\boldsymbol{z}_k \notin C$ such that

$$\big\|(1-\lambda)\boldsymbol{x} + \lambda \boldsymbol{y} - \boldsymbol{z}_k\big\| < \frac{1}{k} \qquad \forall\, k \in \mathbb{N}\,.$$

Since $\boldsymbol{y} \in \bar{C}$, there exists a sequence $\{\boldsymbol{y}_k\}_{k=1}^{\infty} \in C$ satisfying

$$\|\boldsymbol{y}_k - \boldsymbol{y}\| < \frac{1}{\lambda k} \qquad \forall\, k \in N\,.$$

Therefore, if $k \in N$,

$$\big\|(1-\lambda)\boldsymbol{x}+\lambda\boldsymbol{y}_k-\boldsymbol{z}_k\big\| \leqslant \big\|(1-\lambda)\boldsymbol{x}+\lambda\boldsymbol{y}-\boldsymbol{z}_k\big\| + \|\lambda(\boldsymbol{y}-\boldsymbol{y}_k)\| < \frac{2}{k}\,;$$

thus

$$\Big\|\boldsymbol{x}-\frac{\boldsymbol{z}_k-\lambda\boldsymbol{y}_k}{1-\lambda}\Big\| < \frac{2}{k(1-\lambda)} \qquad \forall\, k\in\mathbb{N}\,.$$

Since $\boldsymbol{x}\in\mathring{C}$, there exists $N>0$ such that $B\big(\boldsymbol{x},\dfrac{2}{(1-\lambda)N}\big)\subseteq C$; thus $\dfrac{\boldsymbol{z}_k-\lambda\boldsymbol{y}_k}{1-\lambda}\in C$ whenever $k\geqslant N$. By the convexity of C,

$$\boldsymbol{z}_k=(1-\lambda)\frac{\boldsymbol{z}_k-\lambda\boldsymbol{y}_k}{1-\lambda}+\lambda\boldsymbol{y}_k\in C\,,$$

a contradiction.

3. Let $\boldsymbol{x},\boldsymbol{y}\in\mathring{C}$. By the line segment principle, $(1-\lambda)\boldsymbol{x}+\lambda\boldsymbol{y}\in\mathring{C}$ for all $\lambda\in(0,1)$ (since $\mathring{C}\subseteq\bar{C}$). This further implies that $(1-\lambda)\boldsymbol{x}+\lambda\boldsymbol{y}\in\mathring{C}$ for all $\lambda\in[0,1]$ since $\boldsymbol{x},\boldsymbol{y}\in\mathring{C}$; thus $\mathring{C}$ is convex.

4. It suffices to show that $\mathrm{cl}(\mathring{C})\supseteq\mathrm{cl}(C)$. Let $\boldsymbol{x}\in\mathrm{cl}(C)$. Pick any $\boldsymbol{y}\in\mathring{C}$. By the line segment principle,

$$\boldsymbol{x}_k\equiv\big(1-\frac{1}{k}\big)\boldsymbol{x}+\frac{1}{k}\boldsymbol{y}\in\mathring{C} \qquad \forall\, k\geqslant 2\,.$$

Since $\boldsymbol{x}_k\to\boldsymbol{x}$ as $k\to\infty$, we find that $\boldsymbol{x}\in\mathrm{cl}(\mathring{C})$.

5. It suffices to show that $\mathrm{int}(\bar{C})\subseteq\mathrm{int}(C)$. Let $\boldsymbol{x}\in\mathrm{int}(\bar{C})$. Then there exists $\varepsilon>0$ such that $B(\boldsymbol{x},\varepsilon)\subseteq\bar{C}$. Let $\boldsymbol{y}\in\mathrm{int}(C)$. If $\boldsymbol{y}=\boldsymbol{x}$, then $\boldsymbol{x}\in\mathrm{int}(C)$. If $\boldsymbol{y}\neq\boldsymbol{x}$, define $\boldsymbol{z}=\boldsymbol{x}+\alpha(\boldsymbol{x}-\boldsymbol{y})$, where

$$\alpha=\frac{\varepsilon}{2\|\boldsymbol{x}-\boldsymbol{y}\|}\,.$$

Then $\|\boldsymbol{x}-\boldsymbol{z}\|=\dfrac{\varepsilon}{2}$; thus $\boldsymbol{z}\in B(\boldsymbol{x},\varepsilon)$ which further implies that $\boldsymbol{z}\in\bar{C}$. By the line segment principle implies that $(1-\lambda)\boldsymbol{y}+\lambda\boldsymbol{z}\in\mathring{C}$ for all $\lambda\in(0,1)$. Taking $\lambda=\dfrac{1}{1+\alpha}$, we find that

$$(1-\lambda)\boldsymbol{y}+\lambda\boldsymbol{z}=\frac{\alpha}{1+\alpha}\boldsymbol{y}+\frac{1}{1+\alpha}\big(\boldsymbol{x}+\alpha(\boldsymbol{x}-\boldsymbol{y})\big)=\boldsymbol{x}$$

which shows that $\boldsymbol{x}\in\mathrm{int}(C)$. □

Problem 5. Let $(\mathcal{V},\|\cdot\|)$ be a normed vector space. Show that for all $\boldsymbol{x}\in\mathcal{V}$ and $r>0$,

$$\mathrm{int}\big(B[\boldsymbol{x},r]\big)=B(\boldsymbol{x},r)\,.$$

Proof. Let $\boldsymbol{y}\in\mathcal{V}$ such that $\|\boldsymbol{x}-\boldsymbol{y}\|=r$. Then $\boldsymbol{x}+\lambda(\boldsymbol{y}-\boldsymbol{x})\in B[\boldsymbol{x},r]^{\complement}$ for all $|\lambda|>1$. In particular, $\boldsymbol{y}_n\equiv\boldsymbol{x}+\big(1+\frac{1}{n}\big)(\boldsymbol{y}-\boldsymbol{x})\in B[\boldsymbol{x},r]^{\complement}$ for all $n\in\mathbb{N}$. Moreover,

$$\|\boldsymbol{y}_n-\boldsymbol{y}\|=\frac{1}{n}\|\boldsymbol{x}-\boldsymbol{y}\|=\frac{r}{n}\to 0 \quad \text{as} \quad n\to\infty\,.$$

Therefore, $\lim_{n\to\infty} \boldsymbol{y}_n = \boldsymbol{y}$ which implies that $\boldsymbol{y} \in \partial B[\boldsymbol{x}, r]$ (since $\boldsymbol{y} \in B[\boldsymbol{x}, r]$ and $\boldsymbol{y}$ is the limit of a sequence from $B[\boldsymbol{x}, r]^\complement$); thus

$$\big\{\boldsymbol{y} \in \mathcal{V} \,\big|\, \|\boldsymbol{x} - \boldsymbol{y}\| = r\big\} \subseteq \partial B[\boldsymbol{x}, r] .$$

On the other hand, $B(\boldsymbol{x}, r)$ is open and

$$B[\boldsymbol{x}, r] = B(\boldsymbol{x}, r) \cup \big\{\boldsymbol{y} \in \mathcal{V} \,\big|\, \|\boldsymbol{x} - \boldsymbol{y}\| = r\big\} .$$

Therefore, $B(x, r)$ is the largest open set contained inside $B[\boldsymbol{x}, r]$; thus $B(\boldsymbol{x}, r) = \text{int}(B[\boldsymbol{x}, r])$. ▫

Problem 6. Let $\mathcal{M}_{n\times n}$ denote the collection of all $n \times n$ square real matrices, and $(\mathcal{M}_{n\times n}, \|\cdot\|)$ be a normed space with norm $\|\cdot\|$ given in Problem 3 of Exercise 5 (with $p = q = 2$). Show that the set

$$\text{GL}(n) \equiv \big\{A \in \mathcal{M}_{n\times n} \,\big|\, \det(A) \neq 0\big\}$$

is an open set in $\mathcal{M}_{n\times n}$. The set $\text{GL}(n)$ is called the general linear group.

Proof. Let $A \in \text{GL}(n)$ be given. Then $A^{-1} \in \mathcal{M}_{n\times n}$ exists. We show that

$$\forall\, B \in B\big(A, \frac{1}{\|A^{-1}\|_{2,2}}\big) , \det(B) \neq 0 .$$

By the definition of the norm, for all $\boldsymbol{x} \in \mathbb{R}^n$ we have

$$\|\boldsymbol{x}\|_2 \leqslant \|A^{-1}A\boldsymbol{x}\|_2 \leqslant \|A^{-1}\|_{2,2}\|A\boldsymbol{x}\|_2 ;$$

thus for all $\boldsymbol{x} \in \mathbb{R}^n$,

$$\frac{1}{\|A^{-1}\|_{2,2}}\|\boldsymbol{x}\|_2 \leqslant \|A\boldsymbol{x}\|_2 \leqslant \|(A - B)\boldsymbol{x}\|_2 + \|B\boldsymbol{x}\|_2 \leqslant \|A - B\|_{2,2}\|\boldsymbol{x}\|_2 + \|B\boldsymbol{x}\|_2$$

which implies that

$$\|B\boldsymbol{x}\|_2 \geqslant \Big(\frac{1}{\|A^{-1}\|_{2,2}} - \|A - B\|_{2,2}\Big)\|\boldsymbol{x}\|_2 \qquad \forall\, \boldsymbol{x} \in \mathbb{R}^n .$$

Therefore, if $B \in B\Big(A, \frac{1}{\|A^{-1}\|_{2,2}}\Big)$, then $B\boldsymbol{x} = \mathbf{0}$, then $\boldsymbol{x} = \mathbf{0}$. This shows that B is invertible if $B \in B\Big(A, \frac{1}{\|A^{-1}\|_{2,2}}\Big)$; thus $B\Big(A, \frac{1}{\|A^{-1}\|_{2,2}}\Big) \subseteq \text{GL}(n)$. ▫

Problem 7. Show that every open set in $\mathbb{R}$ is the union of at most countable collection of disjoint open intervals; that is, if $U \subseteq \mathbb{R}$ is open, then

$$U = \bigcup_{k\in\mathcal{I}} (a_k, b_k) ,$$

where $\mathcal{I}$ is countable, and $(a_k, b_k) \cap (a_\ell, b_\ell) = \varnothing$ if $k \neq \ell$.
Hint: For each point $x \in U$, define $L_x = \big\{y \in \mathbb{R} \,\big|\, (y, x) \subseteq U\big\}$ and $R_x = \big\{y \in \mathbb{R} \,\big|\, (x, y) \subseteq U\big\}$. Define $I_x = (\inf L_x, \sup R_x)$. Show that $I_x = I_y$ if $(x, y) \in U$.

Proof. As suggested in the hint, for each point $x \in U$ we define $L_x = \{y \in \mathbb{R} \,|\, (y, x) \subseteq U\}$ and $R_x = \{y \in \mathbb{R} \,|\, (x, y) \subseteq U\}$. We note that $a \equiv \inf L_x \notin U$ since if $a \in U$, by the openness of U there exists $r > 0$ such that $(a - r, a + r) \subseteq U$ which implies that $(a - r, x) \subseteq U$ so that $a - r \in L_x$, a contradiction to the fact that $a = \inf L_x$. Similarly, $\sup R_x \notin U$. Therefore, $I_x = (\inf L_x, \sup L_x)$ is the maximal connected subset of U containing x.

If $x, y \in U$ and $(x, y) \subseteq U$, then $(L_x, y) = (L_x, x) \cup \{x\} \cup x, y) \subseteq U$ which implies that $L_x \subseteq L_y$. On the other hand, if $z \in L_y$, then $z \leqslant x$ and $(z, x) \subseteq U$; thus $L_y \subseteq L_x$ which implies that $L_x = L_y$ if $x, y \in U$ and $(x, y) \subseteq U$. This shows that $I_x = I_y$ if $x, y \in U$ and $(x, y) \subseteq U$. Moreover, if $x, y \in U$ but $(x, y) \nsubseteq U$, then there exists $x < z < y$ such that $z \notin U$; thus $\sup R_x \leqslant z \leqslant \inf L_y$ which implies that $I_x \cap I_y = \varnothing$. Therefore, we establish that

1. if $x, y \in U$ and $(x, y) \subseteq U$, then $I_x = I_y$.
2. if $x, y \in U$ and $(x, y) \nsubseteq U$, then $I_x \cap I_y = \varnothing$.

This implies that U is the union of disjoint open intervals. Since every such open interval contains a rational number, we can denote each such open interval as I_k, where k belongs to a countable index set $\mathcal{I}$. Write $I_k = (a_k, b_k)$, then $U = \bigcup_{k \in \mathcal{I}} (a_k, b_k)$. □

Problem 8. In class we introduce the normed vector space $(\ell^\infty, \|\cdot\|_\infty)$:

$$\ell^\infty = \Big\{\{x_n\}_{n=1}^\infty \subseteq \mathbb{R} \,\Big|\, \exists\, M > 0 \ni |x_n| \leqslant M \text{ for all } n \in \mathbb{N}\Big\}$$

equipped with

$$\big\|\{x_n\}_{n=1}^\infty\big\|_\infty = \sup_{n \in \mathbb{N}} |x_n| .$$

Complete the following.

1. Show that $\|\cdot\|_\infty$ is indeed a norm.
2. Show that $(\ell^\infty, \|\cdot\|_\infty)$ is a Banach space; that is, show that $(\ell^\infty, \|\cdot\|_\infty)$ is complete.
3. Show that the set $A = \left\{\{x_n\}_{n=1}^\infty \in \ell^\infty \,\middle|\, |x_n| \leqslant \frac{1}{n} \text{ for all } n \in \mathbb{N}\right\}$ is closed.

Problem 9. Let (M, d) be a metric space. A set $A \subseteq M$ is said to be ***perfect*** if $A = A'$ (so that there is no isolated points). The Cantor set is constructed by the following procedure: let $E_0 = [0, 1]$. Remove the segment $\big(\frac{1}{3}, \frac{2}{3}\big)$, and let E_1 be the union of the intervals

$$\big[0, \frac{1}{3}\big], \big[\frac{2}{3}, 1\big].$$

Remove the middle thirds of these intervals, and let E_2 be the union of the intervals

$$\big[0, \frac{1}{9}\big], \big[\frac{2}{9}, \frac{3}{9}\big], \big[\frac{6}{9}, \frac{7}{9}\big], \big[\frac{8}{9}, 1\big].$$

Continuing in this way, we obtain a sequence of closed set E_k such that

(a) $E_1 \supseteq E_2 \supseteq E_2 \supseteq \cdots$;

(b) E_n is the union of 2^n intervals, each of length 3^{-n}.

The set $C = \bigcap_{n=1}^{\infty} E_n$ is called the ***Cantor set***.

1. Show that C is a perfect set.
2. Show that C is uncountable.
3. Find $\mathrm{int}(C)$.

Proof. 1. Let $x \in C$. Then $x \in E_N$ for some $N \in \mathbb{N}$. For each $n \in \mathbb{N}$, E_n is the union of disjoint closed intervals with length $\frac{1}{3^n}$, and ∂E_n consists of the end-points of these disjoint closed intervals whose union is E_n. Therefore, there exists $x_n \in \partial E_{N+n-1} \backslash \{x\}$ such that $|x_n - x| < \frac{1}{3^{N-1+n}}$. Since $\partial E_n \subseteq C$ for each $n \in \mathbb{N}$, we find that $\{x_n\}_{n=1}^{\infty} \in C \backslash \{x\}$. Moreover, $\lim\limits_{n\to\infty} x_n = x$; thus $x \in C'$ which shows $C \subseteq C'$. Since C is the intersection of closed sets, C is closed; thus

$$C \subseteq C' \subseteq \bar{C} = C$$

so we establish that $C' = C$.

2. For $x \in [0,1]$, write x in ternary expansion (三進位展開); that is,

$$x = 0.d_1 d_2 d_3 \cdots\cdots .$$

Here we note that repeated 2's are chosen by preference over terminating decimals. For example, we write $\frac{1}{3}$ as $0.02222\cdots$ instead of 0.1. Define

$$A = \left\{ x = 0.d_1 d_2 d_3 \cdots \,\middle|\, d_j \in \{0, 2\} \text{ for all } j \in \mathbb{N} \right\}.$$

Note each point in ∂E_n belongs to A; thus $A \subseteq C$. On the other hand, A has a one-to-one correspondence with $[0,1]$ $\big(x = 0.d_1 d_2 \cdots \in A \Leftrightarrow y = 0.\frac{d_1}{2}\frac{d_2}{2}\cdots \in [0,1]$, where y is expressed in binary expansion (二進位展開) with repeated 1's instead of terminating decimals$\big)$. Since $[0,1]$ is uncountable, A is uncountable; thus C is uncountable.

3. If $\mathrm{int}(C)$ is non-empty, then by the fact that $\mathrm{int}(C)$ is open in $(R, |\cdot|)$, by Problem 7 the Cantor set C contains at least one interval (x, y). Note that there exists $N > 0$ such that $|x - y| < \frac{1}{3^n}$ for all $n \geqslant N$. Since the length of each interval in E_n has length $\frac{1}{3^n}$, we find that if $n \geqslant N$, the interval (x, y) is not contained in any interval of E_n. In other words, there must be $z \in (x, y)$ such that $z \in E_n^{\complement}$ which shows that $(x, y) \nsubseteq \bigcap_{n=1}^{\infty} E_n$. Therefore, $\mathrm{int}(C) = \varnothing$. □