Exercise Problem Sets 4

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Problem 1. Let $a \in \mathbb{R}$. Define a_n through the iterated relation

$$a_n = a_{n-1}^2 - a_{n-1} + 1 \qquad \forall\, n > 1, a_1 = a\,.$$

For what a is the sequence $\{a_n\}_{n=1}^\infty$ (1) monotone? (2) bounded? (3) convergent? Compute the limit in the case of convergence.

Proof. For $n \in \mathbb{N}$, we have

$$a_{n+1} - a_n = a_n^2 - a_n + 1 - a_n = a_n^2 - 2a_n + 1 = (a_n - 1)^2 \geqslant 0\,;$$

thus $\{a_n\}_{n=1}^\infty$ is increasing no matter what a is. Moreover, **MSP** shows that $\{a_n\}_{n=1}^\infty$ is bounded if and only if $\{a_n\}_{n=1}^\infty$ is convergent.

Note that if $\lim\limits_{n\to\infty} a_n = x$, then

$$x = \lim_{n\to\infty} a_n = \lim_{n\to\infty} (a_{n-1}^2 - a_{n-1} + 1) = x^2 - x + 1$$

which implies that the limit of (the increasing sequence) $\{a_n\}_{n=1}^\infty$, if it converges, must be 1. On the other hand, note that the function $f(x) = x^2 - x + 1$ maps $[0,1]$ into $[0,1]$, and $f(x) > 1$ if $x \notin [0,1]$. Therefore, if $a_1 = a \in [0,1]$, then $a_n \in [0,1]$ for all $n \in \mathbb{N}$ so that $\{a_n\}_{n=1}^\infty$ converges, while if $a_1 = a \notin [0,1]$, then $a_2 = f(a_1) > 1$ which implies that $\{a_n\}_{n=1}^\infty$ diverges. □

Problem 2. Suppose that $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are two Cauchy sequences in $\mathbb{R}$. Show that the sequence $\big\{|x_n - y_n|\big\}_{n=1}^\infty$ converges.

Proof. By the completeness of $\mathbb{R}$, $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} y_n = y$ exist. Therefore,

$$\lim_{n\to\infty} |x_n - y_n| = \lim_{n\to\infty} |x - y|\,.$$

□

Problem 3. Given the following sets consisting of elements of some sequence of real numbers. Find the limsup and liminf of the sequence.

1. $\big\{\cos m \,\big|\, m = 0, 1, 2, \cdots\big\}$.
2. $\big\{\sqrt[m]{|\sin m|} \,\big|\, m = 1, 2, \cdots\big\}$.
3. $\Big\{\big(1 + \frac{1}{m}\big) \sin \frac{m\pi}{6} \,\Big|\, m = 1, 2, \cdots\Big\}$.

Hint: For 1, first show that for all irrational α, the set

$$S = \big\{x \in [0,1] \,\big|\, x = k\alpha \text{ (mod 1) for some } k \in \mathbb{N}\big\}$$

is dense in $[0,1]$; that is, for all $y \in [0,1]$ and $\varepsilon > 0$, there exists $x \in S \cap (y - \varepsilon, y + \varepsilon)$. Then choose

$\alpha = \dfrac{1}{2\pi}$ to conclude that

$$T = \big\{x \in [0, 2\pi] \,\big|\, x = k \pmod{2\pi} \text{ for some } k \in \mathbb{N}\big\}$$

is dense in $[0, 2\pi]$. To prove that S is dense in $[0, 1]$, you might want to consider the following set

$$S_k = \big\{x \in [0, 1] \,\big|\, x = \ell\alpha \pmod 1 \text{ for some } 1 \leqslant \ell \leqslant k+1\big\}$$

Note that there must be two points in S_k whose distance is less than $\dfrac{1}{k}$. What happened to (the multiples of) the difference of these two points?

Proof. 3. Let $x_m = \big(1 + \dfrac{1}{m}\big)\sin\dfrac{m\pi}{6}$. Since $\lim\limits_{m\to\infty}\big(1 + \dfrac{1}{m}\big) = 1 > 0$ and there are seven cluster points, $\big\{\pm 1, \pm\dfrac{\sqrt{3}}{2}, \pm\dfrac{1}{2}, 0\big\}$, of the sequence $\big\{\sin\dfrac{m\pi}{6}\big\}_{m=1}^{\infty}$, we expect that

$$\limsup_{m\to\infty}\big(1 + \frac{1}{m}\big)\sin\frac{m\pi}{6} = 1 \qquad \text{and} \qquad \liminf_{m\to\infty}\big(1 + \frac{1}{m}\big)\sin\frac{m\pi}{6} = -1\,.$$

To see that our expectation is in fact true, we let $\varepsilon > 0$ be given and observe that

$$\#\big\{m \in \mathbb{N} \,\big|\, x_m > 1 + \varepsilon\big\} \leqslant \big[\frac{1}{\varepsilon}\big] + 1 < \infty$$

while the set $\big\{m \in \mathbb{N} \,\big|\, x_m > 1 + \varepsilon\big\} \supseteq \big\{12k + 3 \,\big|\, k \in \mathbb{N}\big\}$ so that

$$\#\big\{m \in \mathbb{N} \,\big|\, x_m > 1 + \varepsilon\big\} = \infty\,.$$

Therefore, Proposition 1.98 shows that 1 is the limit superior of $\{x_m\}_{m=1}^{\infty}$. Similarly, -1 is the limit inferior of $\{x_m\}_{m=1}^{\infty}$. □

Problem 4. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences in $\mathbb{R}$. Prove the following inequalities:

$$\begin{aligned}
\liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n &\leqslant \liminf_{n\to\infty}(x_n + y_n) \leqslant \liminf_{n\to\infty} x_n + \limsup_{n\to\infty} y_n \\
&\leqslant \limsup_{n\to\infty}(x_n + y_n) \leqslant \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n\,; \\
\big(\liminf_{n\to\infty}|x_n|\big)\big(\liminf_{n\to\infty}|y_n|\big) &\leqslant \liminf_{n\to\infty}|x_n y_n| \leqslant \big(\liminf_{n\to\infty}|x_n|\big)\big(\limsup_{n\to\infty}|y_n|\big) \\
&\leqslant \limsup_{n\to\infty}|x_n y_n| \leqslant \big(\limsup_{n\to\infty}|x_n|\big)\big(\limsup_{n\to\infty}|y_n|\big)\,.
\end{aligned}$$

Give examples showing that the equalities are generally not true.

Proof. 1. Let $k \in \mathbb{N}$ be fixed. Note that for $n \geqslant k$, we have

$$\inf_{n\geqslant k}(x_n + y_n) \leqslant x_n + y_n \leqslant \sup_{n\geqslant k}(x_n + y_n)\,.$$

Note that the LHS and the RHS for functions of k and is independent of n. Therefore,

$$\inf_{n\geqslant k}\Big[\inf_{n\geqslant k}(x_n + y_n) - y_n\Big] \leqslant \inf_{n\geqslant k} x_n \leqslant \inf_{n\geqslant k}\Big[\sup_{n\geqslant k}(x_n + y_n) - y_n\Big]$$

which further shows that

$$\inf_{n\geqslant k}(x_n+y_n)-\sup_{n\geqslant k}y_n\leqslant\inf_{n\geqslant k}x_n\leqslant\sup_{n\geqslant k}(x_n+y_n)-\sup_{n\geqslant k}y_n\,.$$

Therefore,

$$\inf_{n\geqslant k}(x_n+y_n)\leqslant\inf_{n\geqslant k}x_n+\sup_{n\geqslant k}y_n\leqslant\sup_{n\geqslant k}(x_n+y_n)\qquad\forall\,k\in\mathbb{N}\,,$$

and the first inequality follows from the fact that

$$\inf_{n\geqslant k}x_n+\inf_{n\geqslant k}y_n\leqslant\inf_{n\geqslant k}(x_n+y_n)\leqslant\inf_{n\geqslant k}x_n+\sup_{n\geqslant k}y_n\leqslant\sup_{n\geqslant k}(x_n+y_n)\leqslant\sup_{n\geqslant k}x_n+\sup_{n\geqslant k}y_n$$

for each $k\in\mathbb{N}$.

2. Let $k\in\mathbb{N}$ be fixed. Note that for $n\geqslant k$, we have

$$\inf_{n\geqslant k}\Big[|x_n|\big(|y_n|+\frac{1}{k}\big)\Big]\leqslant|x_n|\big(|y_n|+\frac{1}{k}\big)\leqslant\sup_{n\geqslant k}\Big[|x_n|\big(|y_n|+\frac{1}{k}\big)\Big]\,.$$

Note that the LHS and the RHS for functions of k and is independent of n. Therefore,

$$\inf_{n\geqslant k}\frac{\inf\limits_{n\geqslant k}\Big[|x_n|\big(|y_n|+\frac{1}{k}\big)\Big]}{|y_n|+\frac{1}{k}}\leqslant\inf_{n\geqslant k}|x_n|\leqslant\inf_{n\geqslant k}\frac{\sup\limits_{n\geqslant k}\Big[|x_n|\big(|y_n|+\frac{1}{k}\big)\Big]}{|y_n|+\frac{1}{k}}\,.$$

By the fact that

$$\inf_{n\geqslant k}\frac{1}{|y_n|+\frac{1}{k}}=\frac{1}{\sup\limits_{n\geqslant k}\big(|y_n|+\frac{1}{k}\big)}\,,$$

we find that

$$\frac{\inf\limits_{n\geqslant k}\Big[|x_n|\big(|y_n|+\frac{1}{k}\big)\Big]}{\sup\limits_{n\geqslant k}\big(|y_n|+\frac{1}{k}\big)}\leqslant\inf_{n\geqslant k}|x_n|\leqslant\inf_{n\geqslant k}\frac{\sup\limits_{n\geqslant k}\Big[|x_n|\big(|y_n|+\frac{1}{k}\big)\Big]}{\sup\limits_{n\geqslant k}\big(|y_n|+\frac{1}{k}\big)}\,;$$

thus

$$\inf_{n\geqslant k}\Big[|x_n|\big(|y_n|+\frac{1}{k}\big)\Big]\leqslant\inf_{n\geqslant k}|x_n|\sup_{n\geqslant k}\big(|y_n|+\frac{1}{k}\big)\leqslant\sup_{n\geqslant k}\Big[|x_n|\big(|y_n|+\frac{1}{k}\big)\Big]\,.$$

The second inequality follows from the fact that

$$\begin{aligned}\inf_{n\geqslant k}|x_n|\inf_{n\geqslant k}\big(|y_n|+\frac{1}{k}\big)&\leqslant\inf_{n\geqslant k}\Big[|x_n|\big(|y_n|+\frac{1}{k}\big)\Big]\leqslant\inf_{n\geqslant k}|x_n|\sup_{n\geqslant k}\big(|y_n|+\frac{1}{k}\big)\\&\leqslant\sup_{n\geqslant k}\Big[|x_n|\big(|y_n|+\frac{1}{k}\big)\Big]\leqslant\sup_{n\geqslant k}|x_n|\sup_{n\geqslant k}\big(|y_n|+\frac{1}{k}\big)\end{aligned}$$

for each $k\in\mathbb{N}$, and passing to the limit as $k\to\infty$.

3. Let $x_n=2+\sin n$ and $y_n=2+\cos n$. Then $x_n,y_n>0$, and

$$\liminf_{n\to\infty}x_n=\liminf_{n\to\infty}y_n=1\,,\quad\limsup_{n\to\infty}x_n=\limsup_{n\to\infty}y_n=3\,.$$

By Problem 3, the set $\big\{x \in [0, 2\pi] \,\big|\, x = k \pmod{2\pi} \text{ for some } k \in \mathbb{N}\big\}$ is dense in $[0, 2\pi]$; thus for each $\theta \in [0, 2\pi]$ there exists an increasing sequence $\{k_j\}_{j=1}^\infty \subseteq \mathbb{N}$ such that $x_{k_j} = k_j \pmod{2\pi}$ and $\big\{x_{k_j}\big\}_{j=1}^\infty$ converges to θ. This implies that for each $\theta \in [-1, 1]$, there exists a subsequence $\{\cos k_j\}_{j=1}^\infty$ such that

$$\lim_{j\to\infty} \cos n_j = \cos\theta \quad \text{and} \quad \lim_{j\to\infty} \sin n_j = \sin\theta\,.$$

Therefore, we have

$$\liminf_{n\to\infty}(x_n + y_n) = 4 - \sqrt{2}\,, \quad \limsup_{n\to\infty}(x_n + y_n) = 4 + \sqrt{2}\,,$$

and

$$\liminf_{n\to\infty} x_n y_n = \frac{9}{2} - 2\sqrt{2}\,, \quad \limsup_{n\to\infty} x_n y_n = \frac{9}{2} + 2\sqrt{2}\,.$$

Therefore,

$$\begin{aligned}\liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n < \liminf_{n\to\infty}(x_n + y_n) &< \liminf_{n\to\infty} x_n + \limsup_{n\to\infty} y_n \\ &< \limsup_{n\to\infty}(x_n + y_n) < \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n\end{aligned}$$

and

$$\begin{aligned}\liminf_{n\to\infty} x_n \cdot \liminf_{n\to\infty} y_n < \liminf_{n\to\infty}(x_n y_n) &< \liminf_{n\to\infty} x_n \cdot \limsup_{n\to\infty} y_n \\ &< \limsup_{n\to\infty}(x_n y_n) < \limsup_{n\to\infty} x_n \cdot \limsup_{n\to\infty} y_n\,.\end{aligned}$$

Therefore, the equalities are generally not true. □

Problem 5. Prove that

$$\liminf_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} \leqslant \liminf_{n\to\infty} \sqrt[n]{|x_n|} \leqslant \limsup_{n\to\infty} \sqrt[n]{|x_n|} \leqslant \limsup_{n\to\infty} \frac{|x_{n+1}|}{|x_n|}\,.$$

Give examples to show that the equalities are not true in general. Is it true that $\lim\limits_{n\to\infty} \sqrt[n]{|x_n|}$ exists implies that $\lim\limits_{n\to\infty} \dfrac{|x_{n+1}|}{|x_n|}$ also exists?

Proof. Let $a = \liminf_{n\to\infty} \dfrac{|x_{n+1}|}{|x_n|}$ and $b = \limsup\limits_{n\to\infty} \dfrac{|x_{n+1}|}{|x_n|}$, and $\varepsilon > 0$ be given. W.L.O.G. we can assume that $a \neq -\infty$ and $b \neq \infty$. Then there exists $N > 0$ such that

$$a - \varepsilon < \frac{|x_{n+1}|}{|x_n|} < b + \varepsilon \qquad \forall\, n \geqslant N\,.$$

Therefore,

$$(a - \varepsilon)|x_n| < |x_{n+1}| < (b + \varepsilon)|x_n| \qquad \forall\, n \geqslant N$$

which implies that if $n > N$,

$$|x_n| > (a - \varepsilon)|x_{n-1}| > (a - \varepsilon)^2|x_{n-2}| > \cdots > (a - \varepsilon)^{n-N}|x_N|$$

and

$$|x_n| < (b+\varepsilon)|x_{n-1}| < (b+\varepsilon)^2|x_{n-2}| < \cdots < (b+\varepsilon)^{n-N}|x_N|\,.$$

The inequality above implies that

$$(a-\varepsilon)^{1-\frac{N}{n}}\sqrt[n]{|x_N|} < \sqrt[n]{|x_n|} < (b+\varepsilon)^{1-\frac{N}{n}}\sqrt[n]{|x_N|}\,;$$

thus

$$\liminf_{n\to\infty}\Big[(a-\varepsilon)^{1-\frac{N}{n}}\sqrt[n]{|x_N|}\Big] \leqslant \liminf_{n\to\infty}\sqrt[n]{|x_n|} \leqslant \limsup_{n\to\infty}\sqrt[n]{|x_n|} \leqslant \limsup_{n\to\infty}\Big[(b+\varepsilon)^{1-\frac{N}{n}}\sqrt[n]{|x_N|}\Big]\,.$$

By Problem 4 of Exercise 1, $\lim\limits_{n\to\infty} b^{\frac{1}{n}} = 1$ for all $b>0$. Therefore,

$$\liminf_{n\to\infty}\Big[(a-\varepsilon)^{1-\frac{N}{n}}\sqrt[n]{|x_N|}\Big] = \lim_{n\to\infty}(a-\varepsilon)^{1-\frac{N}{n}}\sqrt[n]{|x_N|} = a-\varepsilon = \liminf_{n\to\infty}\frac{|x_{n+1}|}{|x_n|} - \varepsilon$$

and

$$\limsup_{n\to\infty}\Big[(b+\varepsilon)^{1-\frac{N}{n}}\sqrt[n]{|x_N|}\Big] = \lim_{n\to\infty}(b+\varepsilon)^{1-\frac{N}{n}}\sqrt[n]{|x_N|} = b+\varepsilon = \limsup_{n\to\infty}\frac{|x_{n+1}|}{|x_n|} + \varepsilon\,.$$

Since the inequality above holds for all $\varepsilon>0$, we conclude that

$$\liminf_{n\to\infty}\frac{|x_{n+1}|}{|x_n|} \leqslant \liminf_{n\to\infty}\sqrt[n]{|x_n|} \leqslant \limsup_{n\to\infty}\sqrt[n]{|x_n|} \leqslant \limsup_{n\to\infty}\frac{|x_{n+1}|}{|x_n|}\,.$$

Let $\{x_n\}_{n=1}^{\infty}$ be a real sequence defined by

$$x_n = \begin{cases} 2^{-n} & \text{if } n \text{ is odd}\,,\\ 4^{-n} & \text{if } n \text{ is even}\,,\end{cases}$$

or $x_n = (3+(-1)^n)^{-n}$. Then $\sqrt[n]{|x_n|} = 3+(-1)^n$ which shows that

$$\liminf_{n\to\infty}\sqrt[n]{|x_n|} = \frac{1}{4} \qquad\text{and}\qquad \limsup_{n\to\infty}\sqrt[n]{|x_n|} = \frac{1}{2}\,.$$

To compute the limit superior and limit inferior of $\dfrac{|x_{n+1}|}{|x_n|}$, we define

$$y_n = \frac{|x_{n+1}|}{|x_n|} = \frac{(3+(-1)^{n+1})^{-n-1}}{(3+(-1)^n)^{-n}} = \frac{1}{3-(-1)^n}\Big(\frac{3-(-1)^n}{3+(-1)^n}\Big)^{-n}$$

and observe that $\lim\limits_{n\to\infty} y_{2n} = 0$ and $\lim\limits_{n\to\infty} y_{2n+1} = \infty$. Since $y_n \in [0,\infty)$, we conclude that 0 is the smallest cluster point of $\{y_n\}_{n=1}^{\infty}$ and ∞ is the largest "cluster point" of $\{y_n\}_{n=1}^{\infty}$. This shows that

$$\liminf_{n\to\infty}\frac{|x_{n+1}|}{|x_n|} = 0 \qquad\text{and}\qquad \limsup_{n\to\infty}\frac{|x_{n+1}|}{|x_n|} = \infty\,.$$

□