Exercise Problem Sets 3

Oct. 9. 2020

Problem 1. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an Archimedean ordered field, and $f : \mathbb{F} \to \mathbb{F}$ be a function so that

$$|f(x) - f(y)| \leqslant \frac{|x-y|}{2} \qquad \forall\, x, y \in \mathbb{F}\,.$$

Pick an arbitrary $x_1 \in \mathbb{F}$, and define $x_{k+1} = f(x_k)$ for all $k \in \mathbb{N}$. Show that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{F}$.

Proof. Let $x_1 \in \mathbb{F}$ be given, and $x_{k+1} = f(x_k)$ for all $k \in \mathbb{N}$. Then

$$\begin{aligned} |x_{k+1} - x_k| &= \big|f(x_k) - f(x_{k-1})\big| \leqslant \frac{|x_k - x_{k-1}|}{2} = \frac{1}{2}\big|f(x_{k-1}) - f(x_{k-2})\big| \\ &\leqslant \frac{1}{2^2}\big|x_{k-1} - x_{k-2}\big| \leqslant \cdots \leqslant \frac{1}{2^{k-1}}|x_2 - x_1| = \frac{1}{2^{k-1}}\big|f(x_1) - x_1\big|\,. \end{aligned}$$

Let $\varepsilon > 0$ be given. By the Archimedean Property, there exists $N > 0$ such that

$$\frac{1}{2^{N-2}}\big|f(x_1) - x_1\big| < \varepsilon\,.$$

Therefore, if $n > m \geqslant N$,

$$\begin{aligned} |x_n - x_m| &\leqslant |x_m - x_{m+1}| + |x_{m+1} - x_n| \leqslant |x_m - x_{m+1}| + |x_{m+1} - x_{m+2}| + |x_{m+2} - x_n| \leqslant \cdots \\ &\leqslant |x_m - x_{m+1}| + |x_{m+1} - x_{m+2}| + \cdots + |x_{n-1} - x_n| \\ &\leqslant \frac{1}{2^{m-1}}\big|f(x_1) - x_1\big| + \frac{1}{2^m}\big|f(x_1) - x_1\big| + \cdots + \frac{1}{2^{n-2}}\big|f(x_1) - x_1\big| \\ &\leqslant \big|f(x_1) - x_1\big|\Big(\frac{1}{2^{m-1}} + \frac{1}{2^m} + \cdots + \frac{1}{2^{n-2}}\Big) \leqslant \frac{1}{2^{m-2}}\big|f(x_1) - x_1\big| \\ &\leqslant \frac{1}{2^{N-2}}\big|f(x_1) - x_1\big| < \varepsilon\,; \end{aligned}$$

thus $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. □

Problem 2. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field satisfying the monotone sequence property, $b \in \mathbb{F}$ and $b > 1$.

1. Show the law of exponents holds (for rational exponents); that is, show that

   (a) if r, s in $\mathbb{Q}$, then $b^{r+s} = b^r \cdot b^s$.

   (b) if r, s in $\mathbb{Q}$, then $b^{r\cdot s} = (b^r)^s$.

2. For $x \in \mathbb{F}$, let $B(x) = \big\{b^t \in \mathbb{F} \,\big|\, t \in \mathbb{Q}, t \leqslant x\big\}$. Show that $\sup B(x)$ exists for all $x \in \mathbb{F}$, and $b^r = \sup B(r)$ if $r \in \mathbb{Q}$.

3. Define $b^x = \sup B(x)$ for $x \in \mathbb{F}$. Show that $B(x) > 0$ for all $x \in \mathbb{F}$ and the law of exponents (for exponents in $\mathbb{F}$)

(a) if x, y in $\mathbb{F}$, then $b^{x+y} = b^x \cdot b^y$, (b) if $x, y > 0$, then $b^{x \cdot y} = (b^x)^y$,

are also valid.

4. Show that if $x_1, x_2 \in \mathbb{F}$ and $x_1 < x_2$, then $b^{x_1} < b^{x_2}$. This implies that if x_1, x_2 are two numbers in $\mathbb{F}$ satisfying $b^{x_1} = b^{x_2}$, then $x_1 = x_2$.

5. Let $y > 0$ be given. Show that if $u, v \in \mathbb{F}$ such that $b^u < y$ and $b^v > y$, then $b^{u+1/n} < y$ and $b^{v-1/n} > y$ for sufficiently large n.

6. Let $y > 0$ be given, and $A \subseteq \mathbb{F}$ be the set of all w such that $b^w < y$. Show that $\sup A$ exists and $x = \sup A$ satisfies $b^x = y$. The number x (the uniqueness is guaranteed by 4) satisfying $b^x = y$ is called the logarithm of y to the base b, and is denoted by $\log_b y$.

Hint: Make use of Problem 4 in Exercise 1.

Proof. We note that $\mathbb{F}$ also satisfies the Archimedean Property and the least upper bound property because of a Proposition and a Theorem that we talked about in class.

2. First we show that $x \in \mathbb{F}$, $B(x)$ is non-empty and bounded from above. By the Archimedean Property, there exists $n \in \mathbb{N}$ such that $-x < n$. Therefore, there exists a rational number $-n$ such that $-n < x$; thus $b^{-n} \in B(x)$ which implies that $B(x)$ is non-empty.

   On the other hand, the Archimedean Property implies that there exists $m \in \mathbb{N}$ such that $x < m$. By the fact that
   $$b^t \leqslant b^s \quad \text{whenever} \quad t \leqslant s \text{ and } t, s \in \mathbb{Q}, \tag{$*$}$$
   we conclude that b^m is an upper bound for $B(x)$. Therefore, $B(x)$ is bounded from above. By the least upper bound property, we conclude that $\sup B(x)$ exists for all $x \in \mathbb{F}$.

   Next we show that $b^r = \sup B(r)$ if $r \in \mathbb{Q}$. To see this, we note that $b^r \in B(r)$ if $r \in \mathbb{Q}$. On theother hand, $(*)$ implies that b^r is an upper bound for $B(r)$; thus $\sup B(r) = b^r$.

3. We first show that
   $$\sup(cA) = c \cdot \sup A \qquad \forall\, c > 0, \tag{$\star$}$$
   where $cA = \{c \cdot x \,|\, x \in A\}$. To see $(\star)$, we observe that
   $$x \in A \Rightarrow x \leqslant \sup A \Rightarrow c \cdot x \leqslant c \cdot \sup A \quad \text{(by the compatibility of $\cdot$ and $\leqslant$)};$$
   thus every element in cA is bounded from above by $c \cdot \sup A$. Therefore,
   $$\sup(cA) \leqslant c \cdot \sup A.$$
   On the other hand, let $\varepsilon > 0$ be given. Then there exists $x \in A$ and $x > \sup A - \frac{\varepsilon}{c}$. Therefore, $c \cdot x > c \cdot \sup A - \varepsilon$; thus
   $$\sup(cA) \geqslant c \cdot x > c \cdot \sup A - \varepsilon.$$

Since $\varepsilon > 0$ is given arbitrarily, we find that $\sup(cA) \geqslant c \cdot \sup A$; thus $(\star)$ is concluded.

Next we show that

$$\sup\left\{b^t \,\middle|\, t \in \mathbb{Q}, t \leqslant x\right\} = \inf\left\{b^s \,\middle|\, s \in \mathbb{Q}, s \geqslant x\right\}. \qquad (\diamond)$$

Let $S(x) = \left\{b^s \,\middle|\, s \in \mathbb{Q}, s \geqslant x\right\}$. If $b^t \in B(x)$, then b^t is a lower bound for $S(x)$. Therefore, $B(x)$ is a subset of the collection of all lower bounds for $S(x)$. By Problem 3 of Exercise 2,

$$\sup B(x) \leqslant \sup\left\{y \,\middle|\, y \text{ is a lower bound for } S(x)\right\} = \inf S(x).$$

Suppose that $\sup B(x) < \inf S(x)$. Since $b^{\frac{1}{n}} \searrow 1$ as $n \to \infty$ (Problem 4 of Exercise 1), there exists $n \in \mathbb{N}$ such that $\inf S(x) > b^{\frac{1}{n}} \sup B(x)$. By the fact that there exists $r \in \mathbb{Q}$ and $x \leqslant r \leqslant x + \dfrac{1}{n}$, we find that

$$\begin{aligned}\inf S(x) > b^{\frac{1}{n}} \sup B(x) &= \sup\left\{b^{r+\frac{1}{n}} \,\middle|\, r \in \mathbb{Q}, r \leqslant x\right\} = \sup\left\{b^s \,\middle|\, s \in \mathbb{Q}, s \leqslant x + \frac{1}{n}\right\} \\ &\geqslant b^r \geqslant \inf\left\{b^s \,\middle|\, s \in \mathbb{Q}, s \geqslant x\right\} = \inf S(x),\end{aligned}$$

a contradiction. Observe that

$$\sup A^{-1} = \big(\inf A\big)^{-1} \qquad \text{for every subset } A \text{ of } (0, \infty),$$

where $A^{-1} = \left\{t^{-1} \,\middle|\, t \in A\right\}$ and $(0, \infty)$ is the collection consisting of positive elements in $\mathbb{F}$. Therefore, $(\diamond)$ implies that for $x \in \mathbb{F}$,

$$\begin{aligned}b^{-x} &= \sup\left\{b^t \,\middle|\, t \in \mathbb{Q}, t \leqslant -x\right\} = \sup\left\{b^{-t} \,\middle|\, t \in \mathbb{Q}, t \geqslant x\right\} = \Big[\inf\left\{b^t \,\middle|\, t \in \mathbb{Q}, t \geqslant x\right\}\Big]^{-1} \\ &= (b^x)^{-1}.\end{aligned}$$

Now we show the law of exponential

$$b^x \cdot b^y = b^{x+y} \qquad \forall\, x, y \in \mathbb{F}. \qquad (\star\star)$$

Let $x, y \in \mathbb{F}$ be given. If $t, s \in \mathbb{Q}$ and $t \leqslant x$, $s \leqslant y$, then $t + s \in \mathbb{Q}$ and $t + s \leqslant x + y$; thus

$$b^t \cdot b^s = b^{t+s} \leqslant \sup B(x+y) = b^{x+y}.$$

For any given rational $t \leqslant x$, taking the supremum of the left-hand side over all rational $s \leqslant y$ and using $(\star)$ we find that

$$b^t \cdot b^y = b^t \cdot \sup\left\{b^s \,\middle|\, s \in \mathbb{Q}, s \leqslant y\right\} \leqslant b^{x+y}.$$

Taking the supremum of the left-hand side over all rational $t \leqslant x$, using $(\star)$ again we find that

$$b^y \cdot b^x = b^y \cdot \sup\left\{b^t \,\middle|\, t \in \mathbb{Q}, t \leqslant x\right\} \leqslant b^{x+y};$$

thus we establish that

$$b^x \cdot b^y \leqslant b^{x+y} \qquad \forall\, x, y \in \mathbb{F} \qquad (\diamond\diamond)$$

Now, note that ($\diamond\diamond$) implies that for all $x, y \in \mathbb{F}$,

$$b^y = b^{-x+x+y} \geqslant b^{-x} \cdot b^{x+y} = (b^x)^{-1} \cdot b^{x+y} \geqslant (b^x)^{-1} \cdot b^x \cdot b^y = b^y\,.$$

The inequality above is indeed an equality and we obtain that

$$b^y = b^{-x} b^{x+y} \qquad \forall\, x, y \in \mathbb{F}\,.$$

This is indeed ($\star\star$) because of that $b^{-x} = (b^x)^{-1}$.

Next we show that $(b^x)^y = \sup B(x \cdot y)$ for all $x > 0$ and $y \in \mathbb{F}$. For $z > 0$, define $A(z) = \big\{s \in \mathbb{F} \,\big|\, s \in \mathbb{Q}, 0 < s \leqslant z\big\}$. Note that if $z > 0$, then $b^z = \sup A(z)$. Since for $x > 0$, we have $b^x > 1$; thus for $x, y > 0$,

$$(b^x)^y = \sup\big\{(b^x)^t \,\big|\, t \in \mathbb{Q}, 0 < t \leqslant y\big\} = \sup_{t \in A(y)} (b^x)^t = \sup_{t \in A(y)} \Big(\sup_{s \in A(x)} b^s\Big)^t\,.$$

By Problem 4 of Exercise 2,

$$\sup_{t \in A(y)} \Big(\sup_{s \in A(x)} b^s\Big)^t = \sup_{(t,s) \in A(y) \times A(x)} (b^s)^t = \sup_{(t,s) \in A(y) \times A(x)} b^{st} = b^{\sup_{(t,s) \in A(y) \times A(x)} ts} = b^{xy}\,.$$

4. Let $x_1 < x_2$ be given. Then **AP** implies that there exists $r, s \in \mathbb{Q}$ such that $x_1 < r < s < x_2$. Therefore, $B(x_1) \subseteq B(r) \subseteq B(s) \subseteq B(x_2)$; thus

$$b^{x_1} = \sup B(x_1) \leqslant \sup B(r) \leqslant \sup B(s) \leqslant \sup B(x_2) = b^{x_2}\,.$$

Since $B(r) = b^r$ and $B(s) = b^s$, we must have $B(r) < B(s)$; thus 4 is concluded.

5. Since $\dfrac{y}{b^u} > 1$ and $\dfrac{b^v}{y} > 1$, by the fact that $b^{\frac{1}{n}} \to 1$ as $n \to \infty$, there exist $N_1, N_2 > 0$ such that

$$\big|b^{\frac{1}{n}} - 1\big| < \frac{y}{b^u} - 1 \;\text{ whenever }\; n \geqslant N_1 \quad\text{and}\quad \big|b^{\frac{1}{n}} - 1\big| < \frac{b^v}{y} - 1 \;\text{ whenever }\; n \geqslant N_2\,.$$

Let $N = \max\{N_1, N_2\}$. For $n \geqslant N$, we have $b^{\frac{1}{n}} < \dfrac{y}{b^u}$ and $b^{\frac{1}{n}} < \dfrac{b^v}{y}$ or equivalently,

$$b^{u+\frac{1}{n}} < y \quad\text{and}\quad b^{v-\frac{1}{n}} > y \qquad \forall\, n \geqslant N\,.$$

6. Let $A = \big\{w \in \mathbb{F} \,\big|\, b^w < y\big\}$. Since $b > 1$, 2 of Problem 4 in Exercise 1 implies that

$$b^n > 1 + n(b-1) \quad\text{ whenever }\quad n \geqslant 2\,. \tag{$\star\star\star$}$$

By **AP**, there exists $N \geqslant 2$ such that $1 + N(b-1) > y$; thus A is bounded from above by N. Moreover, there exists $M \geqslant 2$ such that

$$1 + M(b-1) > \frac{1}{y}\,;$$

thus ($\star\star\star$) implies that $b^{-M} < y$ or $-N \in A$. Therefore, A is non-empty. By **LUBP**, we conclude that $\sup A$ exists.

Let $x = \sup A$. Then $x + \dfrac{1}{n} \notin A$; thus $b^{x+\frac{1}{n}} \geqslant y$ for all $n \in \mathbb{N}$. Since $b^{\frac{1}{n}} \to 1$ sa $n \to \infty$, we find that

$$b^x = b^x \lim_{n\to\infty} b^{\frac{1}{n}} = \lim_{n\to\infty} b^{x+\frac{1}{n}} \geqslant y\,.$$

On the other hand, 4 implies that $x - \dfrac{1}{n} \in A$; thus $b^{x-\frac{1}{n}} > y$ for all $n \in \infty$ and we have

$$b^x = b^x \lim_{n\to\infty} b^{-\frac{1}{n}} = \lim_{n\to\infty} b^{x-\frac{1}{n}} \leqslant y\,.$$

Therefore, $b^x = y$. □

Problem 3. Let $(\mathbb{F}, +\cdot, \leqslant)$ be an ordered field satisfying the monotone sequence property. In this problem we prove the Intermediate Value Theorem:

> Let $a, b \in \mathbb{F}$, $a < b$ and $f : [a, b] \to \mathbb{F}$ be continuous (at every point of $[a, b]$); that is,
>
> $$\lim_{n\to\infty} f(x_n) = f\big(\lim_{n\to\infty} x_n\big) \qquad \text{for all convergent sequence } \{x_n\}_{n=1}^{\infty} \subseteq [a, b].$$
>
> If $f(a)f(b) < 0$, then there exists $c \in [a, b]$ such that $f(c) = 0$.

Complete the following.

1. W.L.O.G, we can assume that $f(a) < 0$. Define the set $S = \{x \in [a, b] \,|\, f(x) > 0\}$. Show that $\inf S$ exists.

2. Let $c = \inf S$. Show that $f(c) \geqslant 0$.

3. Conclude that $f(c) \leqslant 0$ as well.

Hint:

1. Show that S is non-empty and bounded from below and note that **MSP** $\Leftrightarrow$ **LUBP**.

2. Show that there exists a sequence $\{c_n\}_{n=1}^{\infty}$ in S such that $c_n \to c$ as $n \to \infty$.

3. Show that there exists a sequence $\{c_n\}_{n=1}^{\infty}$ in $[a, c)$ such that $c_n \to c$ as $n \to \infty$.

Proof. 1. Since $f(b) > 0$, $b \in S$. Moreover, a is a lower bound for S; thus S is non-empty and bounded from below. Since **MSP** $\Leftrightarrow$ **LUBP**, $\inf S \in \mathbb{F}$ exists.

2. Let $c = \inf S$. For each $n \in \mathbb{N}$, there exists $c_n < c + \dfrac{1}{n}$ and $c_n \in S$. Then $f(c_n) > 0$ for all $n \in \mathbb{N}$ and

$$c \leqslant c_n < c + \frac{1}{n} \qquad \forall\, n \in \mathbb{N}\,.$$

Then the Sandwich Lemma implies that $c_n \to c$ as $n \to \infty$. By the continuity of f,

$$f(c) = f\big(\lim_{n\to\infty} c_n\big) = \lim_{n\to\infty} f(c_n) \geqslant 0\,.$$

3. Consider the sequence $\{c_n\}_{n=N}^{\infty}$ defined by $c_n = c - \dfrac{1}{n}$, where N is chosen large enough so that $c_N \geqslant a$. Since $c = \inf S$ and $c_n < c$, $c_n \notin S$ for all $n \geqslant N$. Therefore, $f(c_n) < 0$ for all $n \in \mathbb{N}$. Since $c_n \to c$ as $n \to \infty$, by the continuity of f we find that

$$f(c) = f\big(\lim_{n\to\infty} c_n\big) = \lim_{n\to\infty} f(c_n) \leqslant 0\,. \qquad \square$$

Problem 4. Let $(\mathbb{F}, +\cdot, \leqslant)$ be an ordered field satisfying the monotone sequence property. In this problem we prove the Extreme Value Theorem:

> Let $a, b \in \mathbb{F}$, $a < b$ and $f : [a, b] \to \mathbb{F}$ be continuous (at every point of $[a, b]$); that is,
>
> $$\lim_{n\to\infty} f(x_n) = f\big(\lim_{n\to\infty} x_n\big) \qquad \text{for all convergent sequence } \{x_n\}_{n=1}^{\infty} \subseteq [a, b].$$
>
> Then there exist $c, d \in [a, b]$ such that $f(c) = \sup\limits_{x\in[a,b]} f(x)$ and $f(d) = \inf\limits_{x\in[a,b]} f(x)$.

Complete the following.

1. Show that there exist sequences $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ in $[a, b]$ such that

$$\lim_{n\to\infty} f(c_n) = \sup_{x\in[a,b]} f(x) \qquad \text{and} \qquad \lim_{n\to\infty} f(d_n) = \inf_{x\in[a,b]} f(x)\,.$$

2. Extract convergent subsequences $\{x_{n_k}\}_{k=1}^{\infty}$ and $\{y_{n_k}\}_{k=1}^{\infty}$ with limit c and d, respectively. Show that $c, d \in [a, b]$.

3. Show that $f(c) = \sup\limits_{x\in[a,b]} f(x)$ and $f(d) = \inf\limits_{x\in[a,b]} f(x)$.

Hint: For 2, note that **MSP** $\Rightarrow$ **BWP**.

Proof. It suffices to show the case of $\sup\limits_{x\in[a,b]} f(x)$ since $\inf\limits_{x\in[a,b]} f(x) = -\sup\limits_{x\in[a,b]} (-f)(x)$ by Problem 1 of Exercise 2.

1. Suppose that $f([a, b])$ is bounded from above. Then $M = \sup f([a, b]) = \sup\limits_{x\in[a,b]} f(x)$ exists. For each $n \in \mathbb{F}$, there exists $c_n \in [a, b]$ such that

$$M - \frac{1}{n} < f(c_n) \leqslant M\,.$$

By the Sandwich Lemma, $\lim\limits_{n\to\infty} f(c_n) = M = \sup\limits_{x\in[a,b]} f(x)$.

On the other hand, if $f([a, b])$ is not bounded from above, then $\sup f([a, b]) = \sup\limits_{x\in[a,b]} f(x) = \infty$. Moreover, for each $n \in \mathbb{F}$ there exists $c_n \in [a, b]$ such that

$$f(c_n) > n\,.$$

Then $\lim\limits_{n\to\infty} f(c_n) = \infty = \sup\limits_{x\in[a,b]} f(x)$. In either case, there exists $\{c_n\}_{n=1}^{\infty} \subseteq [a, b]$ such that $\lim\limits_{n\to\infty} f(c_n) = \sup\limits_{x\in[a,b]} f(x)$.

2. Since $\{c_n\}_{n=1}^{\infty} \subseteq [a, b]$, $\{c_n\}_{n=1}^{\infty}$ is bounded. By the fact that **MSP** $\Rightarrow$ **BWP**, there exists a convergent subsequence $\{c_{n_k}\}_{k=1}^{\infty}$ of $\{c_n\}_{n=1}^{\infty}$ with limit c. Since $a \leqslant c_{n_k} \leqslant b$ for all $k \in \mathbb{N}$, by a Proposition that we talked about in class we conclude that $a \leqslant c \leqslant b$.

3. Since $c_{n_k} \to c$ as $k \to \infty$, the continuity of f implies that

$$f(c) = f(\lim_{n\to\infty} c_n) = \lim_{n\to\infty} f(c_n) = \sup_{x\in[a,b]} f(x)\,. \qquad \square$$