Advanced Calculus MA-2047 Take Home Midterm

National Central University, Nov. 7 2017 (Due: Nov. 14. 2017)

Problem 1. Let A be a rectangle in $\mathbb{R}^n$, and $f_k : A \to \mathbb{R}$ be a decreasing sequence of bounded functions. Show that if $\lim_{k\to\infty} f_k(x) = 0$ for all $x \in A$, then

$$\lim_{k\to\infty} \int_A f_k(x)\,dx = 0\,.$$

Problem 2. Let $f : [2,\infty) \to \mathbb{R}$ be given by $f(x) = x^{-1}(\log x)^{-p}$. Show that f is integrable over $[2,\infty)$ if and only if $p > 1$.

Problem 3. Define the Beta function $B(x,y) = \displaystyle\int_0^1 t^{x-1}(1-t)^{y-1}dt$ whenever the integral makes sense.

1. Show that B is well-defined for $x, y > 0$.
2. Show that $B(x,y) = B(y,x)$.
3. Find $B(2,2)$ and $B(4,3)$.

Problem 4. Suppose that $f : (0,b] \to \mathbb{R}$ is continuous, positive, integrable over $(0,b]$, and that $f(x)$ increases monotonically to ∞ as x approaches 0 from the right. Show that $\lim_{x\to 0^+} xf(x) = 0$.

Problem 5. Prove that $\displaystyle\lim_{n\to\infty} \frac{(n!)^{\frac{1}{n}}}{n} = e^{-1}$ by considering Riemann sums for $\displaystyle\int_0^1 \log x\,dx$ based on the partition $\big\{\frac{1}{n}, \frac{2}{n}, \cdots, 1\big\}$.

Problem 6. Let $A \subseteq \mathbb{R}^n$ be a Riemann measurable set, and $f : A \to \mathbb{R}$ be a Riemann measurable function. Show that if $f : A \to \mathbb{R}$ is integrable over A, so is αf for all $\alpha \in \mathbb{R}$.

Problem 7. Let $f : A \times B \to \mathbb{R}$ be non-negative, uniformly continuous and integrable over $A \times B$. Define $F(x) = \displaystyle\int_B f(x,y)\,dy$.

1. Show that if B is bounded, then $F : A \to \mathbb{R}$ is continuous. How about if B is not bounded?
2. Let f have the additional property that for each $\varepsilon > 0$, there exists $K > 0$ such that

   $$\Big|\int_{B\cap D(0,k)} (f \wedge k)(x,y)\,dy - \int_B f(x,y)\,dy\Big| < \varepsilon \qquad \forall\, k \geqslant K \text{ and } x \in A\,.$$

   Show that F is continuous on A. In particular, show that if $f(x,y) \leqslant g(y)$ for all $(x,y) \in A \times B$, and g is integrable over B, then F is continuous.