Exercise Problems for Advanced Calculus

MA2045, National Central University, Spring Semester 2016

Weak 1 (Feb. 12 - Feb. 18):

Problem 1. Let ℓ_2 be the collection of square sumable real sequences; that is,

$$\ell_2 \equiv \Big\{ \{x_k\}_{k=1}^\infty \subseteq \mathbb{R} \,\Big|\, \sum_{k=1}^\infty |x_k|^2 < \infty \Big\},$$

and $\|\cdot\|_2$ be defined by

$$\big\|\{x_k\}_{k=1}^\infty\big\|_2 \equiv \Big(\sum_{k=1}^\infty |x_k|^2\Big)^{\frac{1}{2}}.$$

Then $(\ell_2, \|\cdot\|_2)$ is a normed vector space.

1. Let $L : \ell_2 \to \ell_2$ be given by $L\big(\{x_k\}_{k=1}^\infty\big) = \big\{\frac{x_k}{k}\big\}_{k=1}^\infty$. Show that $L \in \mathscr{B}(\ell_2, \ell_2)$.

2. Compute $\|L\|_{\mathscr{B}(\ell_2,\ell_2)}$.

Problem 2. Let $\mathscr{P}([0,1])$ denote the collection of polynomials defined on $[0,1]$, and the norm on $\mathscr{P}([0,1])$ is given by

$$\|p\|_{\mathscr{P}([0,1])} \equiv \sup_{x\in[0,1]} \big|p(x)\big|.$$

Show that the linear map $p \mapsto p'$, where p' denotes the derivatives of p, is unbounded.

Problem 3. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces, and $L \in \mathscr{B}(X, Y)$. Show that

$$\|L\|_{\mathscr{B}(X,Y)} = \sup_{x\neq 0} \frac{\|Lx\|_Y}{\|x\|_X} = \inf\big\{M > 0 \,\big|\, \|Lx\|_Y \leqslant M\|x\|_X\big\}.$$

Problem 4. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed vector spaces. Show that $(\mathscr{B}(X, Y), \|\cdot\|_{\mathcal{B}(X,Y)})$ is also a normed vector space.

Problem 5. Let $\mathcal{M}(m, n)$ denote the collection of all $m \times n$ matrices, and $\{A_k\}_{k=1}^\infty \subseteq \mathcal{M}(m, n)$ be a sequences of $m \times n$ matrices.

1. Let $\mathbf{0}$ denote the zero matrix. Show that $A_k \to \mathbf{0}$ in $\mathscr{B}(\mathbb{R}^n, \mathbb{R}^m)$ if and only if each entry of A_k converges to 0. In other words, by writing $A_k = \big[a_{ij}^{(k)}\big]_{1\leqslant i\leqslant m, 1\leqslant j\leqslant n}$, show that $L_k \to \mathbf{0}$ in $\mathscr{B}(\mathbb{R}^n, \mathbb{R}^m)$ if and only if $\lim\limits_{k\to\infty} a_{ij}^{(k)} = 0$ for all $1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n$.

2. Show that the function $f : \mathcal{M}(n, n) \to \mathbb{R}$ given by $f(A) = \det(A)$ is continuous.

3. Show that the collection of all $n \times n$ invertible matrices (which can be viewed as $\mathrm{GL}(n)$) is open in $\mathcal{M}(n, n)$.

Weak 2 (Feb. 19 - Feb. 25):

Problem 6. Let $f : \mathcal{M}(n,n) \to \mathcal{M}(n,n)$ be defined by $f(A) = A^2$ for all $A \in \mathcal{M}(n,n)$. Show that f is differentiable at every "point" $A \in \mathcal{M}(n,n)$. Find $(Df)(A)$ (for $A \in \mathcal{M}(n,n)$).

Problem 7. Let $\mathscr{C}([0,1];\mathbb{R})$ denote the collection of real-valued continuous functions defined on $[0,1]$, and $\|\cdot\| : \mathscr{C}([0,1];\mathbb{R}) \to \mathbb{R}$ be defined by

$$\|f\| = \sup_{x\in[0,1]} \big|f(x)\big| .$$

Then $\big(\mathscr{C}([0,1];\mathbb{R}), \|\cdot\|\big)$ is a normed vector space. Consider the map $\delta : \mathscr{C}([0,1];\mathbb{R}) \to \mathbb{R}$ defined by $\delta(f) = f(0)$. Show that δ is differentiable at every "point" $f \in \mathscr{C}([0,1];\mathbb{R})$. Find $(D\delta)(f)$ (for $f \in \mathscr{C}([0,1];\mathbb{R})$).

Problem 8. Let $I : \mathscr{C}([0,1];\mathbb{R}) \to \mathbb{R}$ be defined by

$$I(f) = \int_0^1 f(x)^2 dx .$$

Show that I is differentiable at every "point" $f \in \mathscr{C}([0,1];\mathbb{R})$.
Hint: Figure out what $(DI)(f)$ is by computing $I(f+h) - I(f)$, where $h \in \mathscr{C}([0,1];\mathbb{R})$ is a "small" continuous function.

Problem 9. Investigate the differentiability of function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) , \\ 0 & \text{if } (x,y) = (0,0) , \end{cases}$$

at point $(0,0)$.

Problem 10. Investigate the differentiability of function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{x+y^2} & \text{if } x+y^2 \neq 0 , \\ 0 & \text{if } x+y^2 = 0 , \end{cases}$$

at point $(0,0)$.

Weak 3 (Feb. 26 - Mar. 4):

Problem 11. Let $f : D(a,r) \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function such that $\dfrac{\partial f}{\partial x_j}(x) = 0$ for all $x \in D(a,r)$. Show that $f(x) = f(a)$ for all $x \in D(a,r)$.

Problem 12. Let $r > 0$ and $\alpha > 1$. Suppose that $f : D(0,r) \to \mathbb{R}$ satisfies $|f(x)| \leqslant \|x\|^\alpha$ for all $x \in D(0,r)$. Show that f is differentiable at 0. What happens if $\alpha = 1$?

Problem 13. Suppose that $f, g : \mathbb{R} \to \mathbb{R}^m$ are differentiable at a and there is a $\delta > 0$ such that $g(x) \neq 0$ for all $0 < |x - a| < \delta$. If $f(a) = g(a) = 0$ and $(Dg)(a) \neq 0$, show that

$$\lim_{x \to a} \frac{\|f(x)\|}{\|g(x)\|} = \frac{\|(Df)(a)\|}{\|(Dg)(a)\|} .$$

Problem 14. Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, and $f : \mathcal{U} \to \mathbb{R}$. Suppose that the partial derivatives $\dfrac{\partial f}{\partial x_1}, \cdots, \dfrac{\partial f}{\partial x_n}$ are bounded on $\mathcal{U}$; that is, there exists a real number $M > 0$ such that

$$\left|\frac{\partial f}{\partial x_j}(x)\right| \leqslant M \qquad \forall\, x \in \mathcal{U} \text{ and } j = 1, \cdots, n .$$

Show that f is continuous on $\mathcal{U}$.

Hint: Mimic the proof of Theorem 4.32 in the lecture note.

Problem 15. Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, and $f : \mathcal{U} \to \mathbb{R}$. Show that f is differentiable at $a \in \mathcal{U}$ if and only if there exists a vector-valued function $\varepsilon : \mathcal{U} \to \mathbb{R}^n$ such that

$$f(x) - f(a) - \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(a)(x_j - a_j) = \varepsilon(x) \cdot (x - a)$$

and $\varepsilon(x) \to 0$ as $x \to a$.

Weak 4 (Mar. 5 - Mar. 11):

Problem 16. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{x^3 y}{x^4 + y^2} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

1. Show that the directional derivative of f at the origin exists in all direction, and

$$(D_u f)(0,0) = \Big(\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0)\Big) \cdot u \qquad \forall\, u \in \mathbb{R}^n\,, \|u\| = 1\,.$$

2. Is f differentiable at $(0,0)$?

Problem 17. Show (4.5.2) in the lecture note.

Problem 18. Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, and for each $1 \leqslant i, j \leqslant n$, $a_{ij} : \mathcal{U} \to \mathbb{R}$ be differentiable functions. Define $A = [a_{ij}]$ and $J = \det(A)$. Show that

$$\frac{\partial J}{\partial x_k} = \operatorname{tr}\big(\operatorname{Adj}(A)\frac{\partial A}{\partial x_k}\big) \qquad \forall\, 1 \leqslant k \leqslant n\,,$$

where for a square matrix $M = [m_{ij}]$, $\operatorname{tr}(M)$ denotes the trace of M, $\operatorname{Adj}(M)$ denotes the adjoint matrix of M, and $\dfrac{\partial M}{\partial x_k}$ denotes the matrix whose (i,j)-th entry is given by $\dfrac{\partial m_{ij}}{\partial x_k}$.

Hint: Show that

$$\frac{\partial J}{\partial x_k} = \begin{vmatrix} \frac{\partial a_{11}}{\partial x_k} & a_{12} & \cdots & a_{1n} \\ \frac{\partial a_{21}}{\partial x_k} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ \frac{\partial a_{n1}}{\partial x_k} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & \frac{\partial a_{12}}{\partial x_k} & a_{13} & \cdots & a_{1n} \\ a_{21} & \frac{\partial a_{22}}{\partial x_k} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \vdots \\ a_{n1} & \frac{\partial a_{n2}}{\partial x_k} & a_{n3} & \cdots & a_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} a_{11} & \cdots & a_{(n-1)1} & \frac{\partial a_{1n}}{\partial x_k} \\ a_{21} & \cdots & a_{(n-1)2} & \frac{\partial a_{2n}}{\partial x_k} \\ \vdots & & & \vdots \\ a_{n1} & \cdots & a_{(n-1)n} & \frac{\partial a_{n1}}{\partial x_k} \end{vmatrix}$$

and rewrite this identity in the form which is asked to prove.

Problem 19. Let $\psi : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable function such that $\dfrac{\partial^2 \psi_k}{\partial x_i \partial x_j} = \dfrac{\partial}{\partial x_i}\big(\dfrac{\partial \psi_k}{\partial x_j}\big)$ exists and is continuous in $\mathbb{R}^n$ for each $1 \leqslant i, j, k \leqslant n$. Suppose that $(D\psi)(x) \in \mathrm{GL}(n)$ for all $x \in \mathbb{R}^n$, and define $A = (D\psi)^{-1}$ (or in terms of their matrix representation, $[A] = [D\psi]^{-1}$). Let $\psi = (\psi_1, \cdots, \psi_n)$ and $[A] = [a_{ij}]$.

1. Show that $\sum\limits_{k=1}^{n} a_{ik} \dfrac{\partial \psi_k}{\partial x_j} = \sum\limits_{k=1}^{n} \dfrac{\partial \psi_i}{\partial x_k} a_{kj} = \delta_{ij}$, where δ_{ij} is the Kronecker delta; that is, $\delta_{ij} = 1$ if $i = j$ or $\delta_{ij} = 0$ if $i \neq j$.

2. Show that for each $1 \leqslant i, j, k \leqslant n$, $a_{ij} : \mathbb{R}^n \to \mathbb{R}$ is differentiable, and

$$\frac{\partial a_{ij}}{\partial x_k} = -\sum_{r,s=1}^{n} a_{ir} \frac{\partial^2 \psi_r}{\partial x_k \partial x_s} a_{sj}\,.$$

Problem 20. Let (r, θ, φ) be the spherical coordinate of $\mathbb{R}^3$ so that

$$x = r\cos\theta\sin\varphi, y = r\sin\theta\sin\varphi, z = r\cos\varphi\,.$$

1. Find the Jacobian matrices of the map $(x,y,z) \mapsto (r,\theta,\varphi)$ and the map $(r,\theta,\varphi) \mapsto (x,y,z)$.

2. Suppose that $f(x,y,z)$ is a differential function in $\mathbb{R}^3$. Express $|\nabla f|^2$ in terms of the spherical coordinate.

Weak 5 (Mar. 12 - Mar. 18):

Problem 21. Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable functions. Suppose that $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} g(x) = 0$, $g'(x) \neq 0$ for all $x \in \mathbb{R}$, and the limit $\lim\limits_{x\to\infty} \dfrac{f'(x)}{g'(x)}$ exists. Show that the limit $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)}$ also exists, and

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}\,.$$

Problem 22. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \leqslant 0\,, \\ e^{-\frac{1}{x}} & \text{if } x > 0\,, \end{cases}$$

is k-times differentiable (at 0) for all $k \in \mathbb{N}$.

Problem 23. Let $f, g : (a, b) \to \mathbb{R}$ be differentiable functions. Show that if $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^+} g(x) = \infty$, $g'(x) \neq 0$ for all $x \in (a, b)$, and the limit $\lim\limits_{x\to a^+} \dfrac{f'(x)}{g'(x)}$ exists, then $\lim\limits_{x\to a^+} \dfrac{f(x)}{g(x)}$ exists and

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}\,.$$

Hint: Let $L = \lim\limits_{x\to a^+} \dfrac{f'(x)}{g'(x)}$ and $\epsilon > 0$ be given. Choose $c \in (a, b)$ such that

$$\Big|\frac{f'(x)}{g'(x)} - L\Big| < \frac{\epsilon}{2} \qquad \forall\, a < x < c\,.$$

Then for $a < x < c$, the Cauchy mean value theorem implies that for some $\xi \in (x, c)$ such that

$$\frac{f(x) - f(c)}{g(x) - g(c)} = \frac{f'(\xi)}{g'(\xi)}\,.$$

Show that there exists $\delta > 0$ such that $a + \delta < c$ and

$$\Big|\frac{f(x) - f(c)}{g(x) - g(c)} - \frac{f(x)}{g(x)}\Big| < \frac{\epsilon}{2} \qquad \forall\, x \in (a, a + \delta)$$

and then conclude $(\star)$.

Problem 24. Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open and convex, and $f : \mathcal{U} \to \mathbb{R}^m$ be differentiable on $\mathcal{U}$. Show that for each $a, b \in \mathcal{U}$ and vector $v \in \mathbb{R}^m$, there exists c on the line segment joining a and b such that

$$v \cdot \big[f(b) - f(a)\big] = v \cdot D(f)(c)(b - a)\,.$$

Problem 25. Let $f : \mathbb{R} \to \mathbb{R}$ be a polynomial, and f has a double root at a and b. Show that $f'(x)$ has at least three roots in $[a, b]$.

Weak 6 (Mar. 19 - Mar. 25):

Problem 26. Let $f(x, y, z) = (x^2 + 1)\cos(yz)$, and $a = (0, \frac{\pi}{2}, 1)$, $u = (1, 0, 0)$, $v = (0, 1, 0)$ and $w = (2, 0, 1)$.

1. Compute $(Df)(a)(u)$.
2. Compute $(D^2 f)(a)(v)(u)$.
3. Compute $(D^3 f)(a)(w)(v)(u)$.

Problem 27. 1. If $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $g : B \subseteq \mathbb{R}^m \to \mathbb{R}^\ell$ are twice differentiable and $f(A) \subseteq B$, then for $x_0 \in A$, $u, v \in \mathbb{R}^n$, show that

$$\begin{aligned} &D^2(g \circ f)(x_0)(u, v) \\ &\quad = (D^2 g)(f(x_0))\big((Df)(x_0)(u), Df(x_0)(v)\big) + (Dg)(f(x_0))\big((D^2 f)(x_0)(u, v)\big) . \end{aligned}$$

2. If $p : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map plus some constant; that is, $p(x) = Lx + c$ for some $L \in \mathscr{B}(\mathbb{R}^n, \mathbb{R}^m)$, and $f : A \subseteq \mathbb{R}^m \to \mathbb{R}^s$ is k-times differentiable, prove that

$$D^k(f \circ p)(x_0)(u^{(1)}, \cdots, u^{(k)}) = (D^k f)\big(p(x_0)\big)\big((Dp)(x_0)(u^{(1)}), \cdots, (Dp)(x_0)(u^{(k)}\big) .$$

Problem 28. Let $f(x, y)$ be a real-valued function on $\mathbb{R}^2$. Suppose that f is of class $\mathscr{C}^1$ (that is, all first partial derivatives are continuous on $\mathbb{R}^2$) and $\dfrac{\partial^2 f}{\partial x \partial y}$ exists and is continuous. Show that $\dfrac{\partial^2 f}{\partial y \partial x}$ exists and $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$.

Hint: Mimic the proof of Theorem 4.84 in the Lecture note.

Problem 29. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable, and Df is a constant map in $\mathscr{B}(\mathbb{R}^n, \mathbb{R}^m)$; that is, $(Df)(x_1)(u) = (Df)(x_2)(u)$ for all $x_1, x_2 \in \mathbb{R}^n$ and $u \in \mathbb{R}^n$. Show that f is a linear term plus a constant and that the linear part of f is the constant value of Df.

Problem 30. Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, and $\psi : \mathcal{U} \to \mathbb{R}^n$ be of class $\mathscr{C}^2$ (which is the same as saying that the second partial derivatives are continuous) such that $D\psi : \mathcal{U} \to \mathscr{B}(\mathbb{R}^n, \mathbb{R}^n)$ satisfies $(D\psi)(x) \in \mathrm{GL}(n)$ for all $x \in \mathcal{U}$. Define $J = \det([D\psi])$ and $A = [D\psi]^{-1}$. With a_{ij} denoting the (i, j)-th entry of A, show the Piola identity

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(J a_{ij})(x) = 0 \qquad \forall\, 1 \leqslant j \leqslant n \text{ and } x \in \mathcal{U} .$$

Hint: Check Problem 18 and 19 for the derivatives of J and a_{ij}.

Weak 7 (Mar. 26 - Apr. 1):

Problem 31. Let $f : (a, b) \to \mathbb{R}$ be k-times differentiable, and $c \in (a, b)$. Let $h_k : (a, b) \to \mathbb{R}$ be given by

$$h_k(x) = f(x) - \sum_{j=0}^{k} \frac{f^{(j)}(c)}{j!}(x - c)^j .$$

1. Show that $\lim_{x \to c} \frac{h_k(x)}{(x - c)^k} = 0$.
2. Suppose that in addition $f : (a, b) \to \mathbb{R}$ is $(k + 1)$-times continuously differentiable. Show that for each $x \in (a, b)$,

$$h_k(x) = \int_c^x \frac{f^{(k+1)}(t)}{k!}(x - t)^k \, dt .$$

Problem 32. Let $f(x, y) = x^3 + x - 4xy + 2y^2$,

1. Find all critical points of f.
2. Find the corresponding quadratic from $(D^2 f)(x, y)\big((h, k), (h, k)\big)$ at these critical points, and determine which of them is positive definite.
3. Find all relative extrema and saddle points.
4. Find the maximal value of f on the set

$$A = \big\{(x, y) \,\big|\, 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1, x + y \leqslant 1\big\}.$$

Problem 33. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = \begin{cases} x^2 + y^2 - 2x^2 y - \dfrac{4x^6 y^2}{(x^4 + y^2)^2} & \text{if } (x, y) \neq (0, 0) , \\ 0 & \text{if } (x, y) = (0, 0) . \end{cases}$$

1. Show that f is continuous (at $(0, 0)$) by showing that for all $(x, y) \in \mathbb{R}^2$,

$$4x^4 y^2 \leqslant (x^4 + y^2)^2 .$$

2. For $0 \leqslant \theta \leqslant 2\pi$, $-\infty < t < \infty$, define

$$g_\theta(t) = f(t \cos\theta, t \sin\theta) .$$

   Show that each g_θ has a strict local minimum at $t = 0$. In other words, the restriction of f to each straight line through $(0, 0)$ has a strict local minimum at $(0, 0)$.

3. Show that $(0, 0)$ is not a local minimum for f.

Weak 9 (Apr. 9 - Apr. 15):

Problem 34. Let $f, g : [a, b] \to \mathbb{R}$ be functions, where g is continuous, and f is non-negative, bounded, Riemann integrable over $[a, b]$. Show that fg is Riemann integrable over $[a, b]$.

Problem 35. Let $f : [a, b] \to \mathbb{R}$ be differentiable and assume that f' is Riemann integrable. Prove that $\int_a^b f'(x)\,dx = f(b) - f(a)$.

Hint: Use the Mean Value Theorem.

Problem 36. Suppose that $f : [a, b] \to \mathbb{R}$ is Riemann integrable, $m \leqslant f(x) \leqslant M$ for all $x \in [a, b]$, and $\varphi : [m, M] \to \mathbb{R}$ is continuous. Show that $\varphi \circ f$ is Riemann integrable on $[a, b]$.

Problem 37. Complete the following.

1. Let $f : [a, b] \to \mathbb{R}$ be bounded and monotone. Show that f is Riemann integrable over $[a, b]$.
2. Let $f : [0, 1] \times [0, 1] \to \mathbb{R}$ be a bounded function such that $f(x, y) \leqslant f(x, z)$ if $y < z$ and $f(x, y) \leqslant f(t, z)$ if $x < t$. In other words, $f(x, \cdot)$ and $f(\cdot, y)$ are both non-decreasing functions for fixed $x, y \in [0, 1]$. Show that f is Riemann integrable over $[0, 1] \times [0, 1]$.

Problem 38. Let $f : [a, b] \to \mathbb{R}$ be a bounded function, and $\mathcal{P}_n$ denote the division of $[a, b]$ into 2^n equal sub-intervals. Show that f is Riemann integrable over $[a, b]$ if and only if

$$\lim_{n\to\infty} U(f, \mathcal{P}_n) = \lim_{n\to\infty} L(f, \mathcal{P}_n)\,.$$

Problem 39. Let $f : [a, b] \to \mathbb{R}$ be bounded and Riemann integrable. Show that the graph of f has volume zero by considering the difference of the upper and lower sums of f.

Weak 10 (Apr. 16 - Apr. 22):

Problem 40. Let $A \subseteq \mathbb{R}^n$ be bounded, and $f : A \to \mathbb{R}$ be bounded. Show that if f is Riemann integrable over A, then $|f|$ is also Riemann integrable over A. Is it true that if $|f|$ is Riemann integrable over A, then f is also Riemann integrable?

Problem 41. Let $\varphi : [a, b] \to \mathbb{R}$ be a continuous function such that it is differentiable on (a, b). Suppose that f is continuous on the range of φ. Show that

$$\frac{d}{dx} \int_{\varphi(a)}^{\varphi(x)} f(t)\, dt = f(\varphi(x))\varphi'(x) \, .$$

Problem 42. Let $S = \Big\{ \big(\frac{p}{m}, \frac{k}{m}\big) \in [0,1] \times [0,1] \,\Big|\, m, p, k \in \mathbb{N}\,, \gcd(m, p) = 1 \text{ and } 1 \leqslant k \leqslant m-1 \Big\}$. Show that

$$\int_0^1 \Big(\int_0^1 \mathbf{1}_S(x,y)\, dy \Big) dx = \int_0^1 \Big(\int_0^1 \mathbf{1}_S(x,y)\, dx \Big) dy = 0$$

but $\mathbf{1}_S$ is not Riemann integrable over $[0,1] \times [0,1]$.

Problem 43. Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 1 & \text{if } y \in \mathbb{Q}\,, \\ x & \text{if } y \notin \mathbb{Q}\,. \end{cases}$$

Prove that f is not Riemann integrable over $[0,1] \times [0,1]$ using the Fubini theorem.

Problem 44. Let $A \subseteq \mathbb{R}^n$ be bounded such that ∂A has volume zero, and $f, g : A \to \mathbb{R}$ be bounded, Riemann integrable functions. Show that the functions $f \wedge g$ and $f \vee g$ defined by

$$(f \wedge g)(x) = \min\big\{f(x), g(x)\big\} \quad \text{and} \quad (f \vee g)(x) = \max\big\{f(x), g(x)\big\}$$

are also Riemann integrable over A.

Problem 45. Suppose that $\mathcal{U} \subseteq \mathbb{R}^n$ is open and $f : \mathcal{U} \to \mathbb{R}$ is continuous. Prove that if $\int_A f(x)\, dx = 0$ for all $A \subseteq \mathcal{U}$ satisfying ∂A has volume zero, then $f = 0$ on $\mathcal{U}$.

Weak 11 (Apr. 23 - Apr. 29):

Problem 46.

1. Draw the region corresponding to the integral $\int_0^1 \Big(\int_1^{e^x} (x+y)dy \Big) dx$ and evaluate.
2. Change the order of integration of the integral in 1 and check if the answer is unaltered.

Problem 47. Evaluate the integral $\int_0^1 \Big[\int_0^{1-x} \Big(\int_y^1 \frac{\sin(\pi z)}{z(2-z)}\, dz \Big) dy \Big] dx$. (Hint: Change of order of integration to $dxdydz$)

Problem 48. Find the volume of the region under the surface $z = x^2 \sin(y^4)$ and above the triangle in the xy-plane with vertices $(0,0)$, $(0, \pi^{1/4})$ and $(\pi^{1/4}, \pi^{1/4})$.

Problem 49. Evaluate the integral $\int_R e^{x+y}\, d(x,y)$, where R is the region $\big\{(x,y) \,\big|\, |x| + |y| \leqslant a\big\}$.

Problem 50. Find the volume of the region lying inside all three of the circular cylinders $x^2+y^2 = a^2$, $x^2+z^2 = a^2$ and $y^2+z^2 = a^2$.

Problem 51. Let T be the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$. Evaluate the integral $\int_T e^{(y-x)/(y+x)} d(x,y)$

1. by transforming to polar coordinates, and
2. by using the transformation $u = y - x$ and $v = y + x$.

Problem 52. Find $\int_R x\, d(x,y,z)$ and $\int_R z\, d(x,y,z)$, where R is the region given by

$$R = \left\{(x,y,z) \in \mathbb{R}^3 \,\middle|\, x \geqslant 0, y \geqslant 0, 0 \leqslant z \leqslant h \times \left(1 - \frac{\sqrt{x^2+y^2}}{a}\right)\right\}.$$

Weak 13 (Apr. 30 - May 6):

Problem 53. Let (M, d) be a metric space, $A \subseteq M$, and $f_k : A \to \mathbb{R}$ be a sequence of functions (not necessary continuous) which converges uniformly on A. Suppose that $a \in \text{cl}(A)$ and

$$\lim_{x \to a} f_k(x) = A_k$$

exists for all $k \in \mathbb{N}$. Show that $\{A_k\}_{k=1}^{\infty}$ converges, and

$$\lim_{x \to a} \lim_{k \to \infty} f_k(x) = \lim_{k \to \infty} \lim_{x \to a} f_k(x).$$

Problem 54. Complete the following.

(a) Suppose that $f_k, f, g : [0, \infty) \to \mathbb{R}$ are functions such that

1. $\forall R > 0$, f_k and g are Riemann integrable on $[0, R]$;
2. $|f_k(x)| \leqslant g(x)$ for all $k \in \mathbb{N}$ and $x \in [0, \infty)$;
3. $\forall R > 0$, $\{f_k\}_{k=1}^{\infty}$ converges to f uniformly on $[0, R]$;
4. $\displaystyle\int_0^{\infty} g(x)dx \equiv \lim_{R \to \infty} \int_0^R g(x)dx < \infty$.

Show that $\displaystyle\lim_{k \to \infty} \int_0^{\infty} f_k(x)dx = \int_0^{\infty} f(x)dx$; that is,

$$\lim_{k \to \infty} \lim_{R \to \infty} \int_0^R f_k(x)dx = \lim_{R \to \infty} \lim_{k \to \infty} \int_0^R f_k(x)dx.$$

(b) Let $f_k(x)$ be given by $f_k(x) = \begin{cases} 1 & \text{if } k-1 \leqslant x < k, \\ 0 & \text{otherwise.} \end{cases}$ Find the (pointwise) limit f of the sequence $\{f_k\}_{k=1}^{\infty}$, and check whether $\displaystyle\lim_{k \to \infty} \int_0^{\infty} f_k(x)dx = \int_0^{\infty} f(x)dx$ or not. Briefly explain why one can or cannot apply (a).

(c) Let $f_k : [0, \infty) \to \mathbb{R}$ be given by $f_k(x) = \dfrac{x}{1 + kx^4}$. Find $\displaystyle\lim_{k \to \infty} \int_0^{\infty} f_k(x)dx$.

Problem 55. Let $A = [a, b]$ be a closed interval in $\mathbb{R}$, and $f_k : A \to \mathbb{R}$ be a sequence of non-decreasing (not necessarily continuous) functions such that $\{f_k\}_{k=1}^{\infty}$ converges pointwise to f on A. If f is continuous, show that $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f on A.

Problem 56. Show that the series

$$\sum_{k=1}^{\infty} (-1)^k \frac{x^2 + k}{k^2}$$

converges uniformly on every bounded interval.

Problem 57. Consider the function

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{1 + k^2 x}.$$

On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges? If f bounded?

Problem 58. Determine which of the following real series $\sum_{k=1}^{\infty} g_k$ converge (pointwise or uniformly). Check the continuity of the limit in each case.

1. $g_k(x) = \begin{cases} 0 & \text{if } x \leqslant k\,, \\ (-1)^k & \text{if } x > k\,. \end{cases}$

2. $g_k(x) = \begin{cases} \dfrac{1}{k^2} & \text{if } |x| \leqslant k\,, \\ \dfrac{1}{x^2} & \text{if } |x| > k\,. \end{cases}$

3. $g_k(x) = \left(\frac{(-1)^k}{\sqrt{k}}\right)\cos(kx)$ on $\mathbb{R}$.

4. $g_k(x) = x^k$ on $(0, 1)$.

Weak 14 (May 14 - May 20):

Problem 59. In the following series of functions defined on $\mathbb{R}$, find its domain of convergence (classify it into domain of absolute and conditional convergence). If the series is a power series, find its radius of convergence. Also discuss whether the series is uniformly convergent in every compact subsets of its domain of convergence. Determine which series can be differentiated or integrated term by term in its domain of convergence.

(1) $\sum\limits_{k=1}^{\infty} \dfrac{x}{k^\alpha + k^\beta x^2}$, $\alpha \geqslant 0$, $\beta > 0$;

(2) $\sum\limits_{k=1}^{\infty} \dfrac{1}{2^k}\sqrt{1-x^{2k}}$;

(3) $\sum\limits_{k=1}^{\infty} \dfrac{1\cdot 3\cdots(2k-1)}{2\cdot 4\cdots(2k)}\big(1+\dfrac{1}{2}+\cdots+\dfrac{1}{k}\big)x^{2k}$;

(4) $\sum\limits_{k=1}^{\infty} \dfrac{(-1)^{k-1}}{k\log(k+1)}x^{k!}$;

(5) $\sum\limits_{k=1}^{\infty} a_k x^k$, where $\{a_k\}_{k=1}^{\infty}$ is defined by the recursive relation $a_k = 3a_{k-1} - 2a_{k-2}$ for $k \geqslant 2$, and $a_0 = 1$, $a_1 = 2$.

Also find the sum of the series in (5).

Problem 60. In this problem we investigate the differentiability of a complex power series. This requires another proof of $\dfrac{d}{dx}\sum\limits_{k=0}^{\infty} a_k x^k = \sum\limits_{k=1}^{\infty} k a_k x^{k-1}$ instead of making use of Theorem 6.10 in the lecture note.

Let $\{a_k\}_{k=0}^{\infty} \subseteq \mathbb{R}$ be a real sequence, and $f(x) = \sum\limits_{k=0}^{\infty} a_k x^k$ be a (real) power series with radius of convergence $R > 0$. Let $s_n(x) = \sum\limits_{k=0}^{n} a_k x^k$ be the n-th partial sum, $R_n(x) = f(x) - s_n(x)$, and $g(x) = \sum\limits_{k=1}^{\infty} k a_k x^{k-1}$. For $x, x_0 \in (-\rho, \rho)$, where $0 < \rho < R$ and $x \neq x_0$, consider

$$\frac{f(x)-f(x_0)}{x-x_0} - g(x) = \Big(\frac{s_n(x)-s_n(x_0)}{x-x_0} - s_n'(x_0)\Big) + \big(s_n'(x_0) - g(x_0)\big) + \Big(\frac{R_n(x)-R_n(x_0)}{x-x_0}\Big).$$

(1) Show that
$$\Big|\frac{R_n(x)-R_n(x_0)}{x-x_0}\Big| \leqslant \sum_{k=n+1}^{\infty} k|a_k|\rho^{k-1}.$$

(2) Use (1) to show that $\lim\limits_{x\to x_0} \dfrac{f(x)-f(x_0)}{x-x_0} = g(x_0)$.

(3) Generalize the conclusion to complex power series; that is, show that if $\{a_k\}_{k=0}^{\infty} \subseteq \mathbb{C}$ and the power series $\sum\limits_{k=0}^{\infty} a_k z^k$ has radius of convergence $R > 0$, then
$$\frac{d}{dz}\sum_{k=0}^{\infty} a_k z^k = \sum_{k=1}^{\infty} k a_k z^{k-1} \qquad \forall\, |z| < R.$$

Assume that you have known $\dfrac{d}{dz}\sum\limits_{k=0}^{n} a_k z^k = \sum\limits_{k=1}^{n} k a_k z^{k-1}$ for all $n \in \mathbb{N} \cup \{0\}$ (this can be proved using the definition of differentiability of functions with values in normed vector spaces provided in Chapter 4).

Problem 61. Suppose that the series $\sum_{n=0}^{\infty} a_n = 0$, and $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $-1 < x \leqslant 1$. Show that f is continuous at $x = 1$ by complete the following.

1. Write $s_n = a_0 + a_1 + \cdots + a_n$ and $s_n(x) = a_0 + a_1 x + \cdots + a_n x^n$. Show that

$$s_n(x) = (1 - x)(s_0 + s_1 x + \cdots + s_{n-1} x^{n-1}) + s_n x^n$$

   and $f(x) = (1 - x) \sum_{n=0}^{\infty} s_n x^n$.

2. Using the representation of f from above to conclude that $\lim_{x \to 1^-} f(x) = 0$.

3. What if $\sum_{n=0}^{\infty} a_n$ is convergent but not zero?

Problem 62. Let (M, d) be a metric space, and $K \subseteq M$ be a compact subset.

1. Show that the set $U = \{f \in \mathscr{C}(K; \mathbb{R}) \,|\, a < f(x) < b \text{ for all } x \in K\}$ is open in $(\mathscr{C}(K; \mathbb{R}), \|\cdot\|_\infty)$ for all $a, b \in \mathbb{R}$.

2. Show that the set $F = \{f \in \mathscr{C}(K; \mathbb{R}) \,|\, a \leqslant f(x) \leqslant b \text{ for all } x \in K\}$ is closed in $(\mathscr{C}(K; \mathbb{R}), \|\cdot\|_\infty)$ for all $a, b \in \mathbb{R}$.

3. Let $A \subseteq M$ be a subset, not necessarily compact. Prove or disprove that the set $B = \{f \in \mathscr{C}_b(A; \mathbb{R}) \,|\, f(x) > 0 \text{ for all } x \in A\}$ is open in $(\mathscr{C}_b(A; \mathbb{R}), \|\cdot\|_\infty)$.

Weak 15 (May 21 - May 27):

Problem 63. Which of the following set B of continuous functions are equi-continuous in the metric space M?

1. $B = \{ \sin kx \,|\, k = 0, 1, 2, \cdots \}$, $M = \mathscr{C}([0, \pi]; \mathbb{R})$.
2. $B = \{ \sin \sqrt{x + 4k^2\pi^2} \,|\, k = 0, 1, 2, \cdots \}$, $M = \mathscr{C}_b([0, \infty); \mathbb{R})$.
3. $B = \left\{ \dfrac{x^2}{x^2 + (1 - kx)^2} \,\middle|\, k = 0, 1, 2, \cdots \right\}$, $M = \mathscr{C}([0, 1]; \mathbb{R})$.
4. $B = \{ (k + 1)x^k(1 - x) \,|\, k \in \mathbb{N} \}$, $M = \mathscr{C}([0, 1]; \mathbb{R})$.

Problem 64. Let (M, d) be a metric space, $(\mathcal{V}, \| \cdot \|)$ be a normed space, and $A \subseteq M$ be a subset. Suppose that $B \subseteq \mathscr{C}_b(A; \mathcal{V})$ be equi-continuous. Prove or disprove that $\mathrm{cl}(B)$ is equi-continuous.

Problem 65. Let $f_k : [a, b] \to \mathbb{R}$ be a sequence of differentiable functions such that $f_k(a)$ is bounded and $|f_k'(x)| \leqslant M$ for all $x \in [a, b]$ and $k \in \mathbb{N}$. Show that $\{f_k\}_{k=1}^{\infty}$ contains an uniformly convergent subsequence. Must the limit function differentiable?

Weak 16 (May 28 - Jun. 3):

Problem 66. Let (M,d) be a metric space, $(\mathcal{V},\|\cdot\|)$ be a Banach space, $K \subseteq M$ be compact, and $B \subseteq \mathscr{C}(K;\mathcal{V})$. Show that if B is pre-compact in $\mathscr{C}(K;\mathcal{V})$, then B is pointwise pre-compact.

Problem 67. Let $\mathscr{C}^{0,\alpha}([0,1];\mathbb{R})$ denote the "space"

$$\mathscr{C}^{0,\alpha}([0,1];\mathbb{R}) \equiv \left\{ f \in \mathscr{C}([0,1];\mathbb{R}) \,\middle|\, \sup_{x,y\in[0,1]} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty \right\},$$

where $\alpha \in (0,1]$. For each $f \in \mathscr{C}^{0,\alpha}([0,1];\mathbb{R})$, define

$$\|f\|_{\mathscr{C}^{0,\alpha}} = \sup_{x\in[0,1]} |f(x)| + \sup_{\substack{x,y\in[0,1]\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

1. Show that $\left(\mathscr{C}^{0,\alpha}([0,1];\mathbb{R}), \|\cdot\|_{\mathscr{C}^{0,\alpha}}\right)$ is a complete normed space.

2. Show that the set $B = \left\{ f \in \mathscr{C}([0,1];\mathbb{R}) \,\middle|\, \|f\|_{\mathscr{C}^{0,\alpha}} < 1 \right\}$ is equi-continuous.

3. Show that $\mathrm{cl}(B)$ is compact in $\left(\mathscr{C}([0,1];\mathbb{R}), \|\cdot\|_\infty\right)$.

Problem 68. Let (M,d) be a metric space, $(\mathcal{V},\|\cdot\|)$ be a Banach space, $K \subseteq M$ be compact, and $\{f_k\}_{k=1}^\infty \subseteq \mathscr{C}(K;\mathcal{V})$ be a sequence of continuous functions. Suppose that for all $x \in A$, if $\{x_k\}_{k=1}^\infty, \{y_k\}_{k=1}^\infty \subseteq A$ and $\lim\limits_{k\to\infty} x_k = \lim\limits_{k\to\infty} y_k = x$, the limits $\lim\limits_{k\to\infty} f_k(x_k)$ and $\lim\limits_{k\to\infty} f_k(y_k)$ exist and are identical. Show that $\{f_k\}_{k=1}^\infty$ converges uniformly on K. How about if K is not compact?

Problem 69. Assume that $\{f_k\}_{k=1}^\infty$ is a sequence of monotone increasing functions on $\mathbb{R}$ with $0 \leqslant f_k(x) \leqslant 1$ for all x and $k \in \mathbb{N}$.

1. Show that there is a subsequence $\left\{f_{k_j}\right\}_{j=1}^\infty$ which converges pointwise to a function f (This is usually called the ***Helly selection theorem***).

2. If the limit f is continuous, show that $\left\{f_{k_j}\right\}_{j=1}^\infty$ converges uniformly to f on any compact set of $\mathbb{R}$.