Exercise Problems for Advanced Calculus

MA2045, National Central University, Spring Semester 2015

§5.1 Pointwise and Uniform Convergence

Problem 1. Let (M, d) be a metric space, $A \subseteq M$, and $f_k : A \to \mathbb{R}$ be a sequence of functions (not necessary continuous) which converges uniformly on A. Suppose that $a \in \text{cl}(A)$ and

$$\lim_{x \to a} f_k(x) = A_k$$

exists for all $k \in \mathbb{N}$. Show that $\{A_k\}_{k=1}^{\infty}$ converges, and

$$\lim_{x \to a} \lim_{k \to \infty} f_k(x) = \lim_{k \to \infty} \lim_{x \to a} f_k(x) .$$

Problem 2. Let (M, d) and (N, ρ) be metric spaces, $A \subseteq M$, and $f_k : A \to N$ be uniformly continuous functions, and $\{f_k\}_{k=1}^{\infty}$ converges uniformly to $f : A \to N$ on A. Show that f is uniformly continuous on A.

Problem 3. Complete the following.

(a) Suppose that $f_k, f, g : [0, \infty) \to \mathbb{R}$ are functions such that

1. $\forall\, R > 0$, f_k and g are Riemann integrable on $[0, R]$;
2. $|f_k(x)| \leqslant g(x)$ for all $k \in \mathbb{N}$ and $x \in [0, \infty)$;
3. $\forall\, R > 0$, $\{f_k\}_{k=1}^{\infty}$ converges to f uniformly on $[0, R]$;
4. $\displaystyle\int_0^{\infty} g(x)dx \equiv \lim_{R \to \infty} \int_0^R g(x)dx < \infty$.

Show that $\displaystyle\lim_{k \to \infty} \int_0^{\infty} f_k(x)dx = \int_0^{\infty} f(x)dx$; that is,

$$\lim_{k \to \infty} \lim_{R \to \infty} \int_0^R f_k(x)dx = \lim_{R \to \infty} \lim_{k \to \infty} \int_0^R f_k(x)dx .$$

(b) Let $f_k(x)$ be given by $f_k(x) = \begin{cases} 1 & \text{if } k-1 \leqslant x < k\,, \\ 0 & \text{otherwise.} \end{cases}$ Find the (pointwise) limit f of the sequence $\{f_k\}_{k=1}^{\infty}$, and check whether $\displaystyle\lim_{k \to \infty} \int_0^{\infty} f_k(x)dx = \int_0^{\infty} f(x)dx$ or not. Briefly explain why one can or cannot apply (a).

(c) Let $f_k : [0, \infty) \to \mathbb{R}$ be given by $f_k(x) = \dfrac{x}{1 + kx^4}$. Find $\displaystyle\lim_{k \to \infty} \int_0^{\infty} f_k(x)dx$.

§5.2 The Weierstrass M-Test

Problem 4. Show that the series

$$\sum_{k=1}^{\infty} (-1)^k \frac{x^2 + k}{k^2}$$

converges uniformly on every bounded interval.

Problem 5. Consider the function

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{1+k^2 x}\,.$$

On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges? If f bounded?

Problem 6. Determine which of the following real series $\sum_{k=1}^{\infty} g_k$ converge (pointwise or uniformly). Check the continuity of the limit in each case.

1. $g_k(x) = \begin{cases} 0 & \text{if } x \leqslant k\,, \\ (-1)^k & \text{if } x > k\,. \end{cases}$

2. $g_k(x) = \begin{cases} \dfrac{1}{k^2} & \text{if } |x| \leqslant k\,, \\ \dfrac{1}{x^2} & \text{if } |x| > k\,. \end{cases}$

3. $g_k(x) = \Big(\dfrac{(-1)^k}{\sqrt{k}}\Big)\cos(kx)$ on $\mathbb{R}$.

4. $g_k(x) = x^k$ on $(0,1)$.

§5.3 Integration and Differentiation of Series

Problem 7. Suppose that the series $\sum_{n=0}^{\infty} a_n = 0$, and $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $-1 < x \leqslant 1$. Show that f is continuous at $x = 1$ by complete the following.

1. Write $s_n = a_0 + a_1 + \cdots + a_n$ and $s_n(x) = a_0 + a_1 x + \cdots + a_n x^n$. Show that

$$s_n(x) = (1-x)(s_0 + s_1 x + \cdots + s_{n-1}x^{n-1}) + s_n x^n$$

   and $f(x) = (1-x)\sum_{n=0}^{\infty} s_n x^n$.

2. Using the representation of f from above to conclude that $\lim_{x\to 1^-} f(x) = 0$.

3. What if $\sum_{n=0}^{\infty} a_n$ is convergent but not zero?

§5.4 The Space of Continuous Functions

Problem 8. Let $\delta : (\mathscr{C}([0,1];\mathbb{R}), \|\cdot\|_\infty) \to \mathbb{R}$ be defined by $\delta(f) = f(0)$. Show that δ is linear and continuous.

Problem 9. Let (M,d) be a metric space, and $K \subseteq M$ be a compact subset.

1. Show that the set $U = \big\{f \in \mathscr{C}(K;\mathbb{R}) \,\big|\, a < f(x) < b \text{ for all } x \in K\big\}$ is open in $\big(\mathscr{C}(K;\mathbb{R}), \|\cdot\|_\infty\big)$ for all $a, b \in \mathbb{R}$.

2. Show that the set $F = \left\{ f \in \mathscr{C}(K;\mathbb{R}) \,\middle|\, a \leqslant f(x) \leqslant b \text{ for all } x \in K \right\}$ is closed in $\big(\mathscr{C}(K;\mathbb{R}), \|\cdot\|_\infty\big)$ for all $a, b \in \mathbb{R}$.

3. Let $A \subseteq M$ be a subset, not necessarily compact. Prove or disprove that the set $B = \left\{ f \in \mathscr{C}_b(A;\mathbb{R}) \,\middle|\, f(x) > 0 \text{ for all } x \in A \right\}$ is open in $\big(\mathscr{C}_b(A;\mathbb{R}), \|\cdot\|_\infty\big)$.

§5.5 The Arzela-Ascoli Theorem

Problem 10. Let (M,d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a normed space, and $A \subseteq M$ be a subset. Suppose that $B \subseteq \mathscr{C}_b(A;\mathcal{V})$ be equi-continuous. Prove or disprove that $\text{cl}(B)$ is equi-continuous.

Problem 11. Let $\mathscr{C}^{0,\alpha}([0,1];\mathbb{R})$ denote the "space"

$$\mathscr{C}^{0,\alpha}([0,1];\mathbb{R}) \equiv \left\{ f \in \mathscr{C}([0,1];\mathbb{R}) \,\middle|\, \sup_{x,y \in [0,1]} \frac{|f(x) - f(y)|}{|x-y|^\alpha} < \infty \right\},$$

where $\alpha \in (0,1]$. For each $f \in \mathscr{C}^{0,\alpha}([0,1];\mathbb{R})$, define

$$\|f\|_{\mathscr{C}^{0,\alpha}} = \sup_{x \in [0,1]} |f(x)| + \sup_{\substack{x,y \in [0,1] \\ x \neq y}} \frac{|f(x) - f(y)|}{|x-y|^\alpha}.$$

1. Show that $\big(\mathscr{C}^{0,\alpha}([0,1];\mathbb{R}), \|\cdot\|_{\mathscr{C}^{0,\alpha}}\big)$ is a complete normed space.

2. Show that the set $B = \left\{ f \in \mathscr{C}([0,1];\mathbb{R}) \,\middle|\, \|f\|_{\mathscr{C}^{0,\alpha}} < 1 \right\}$ is equi-continuous.

3. Show that $\text{cl}(B)$ is compact in $\big(\mathscr{C}([0,1];\mathbb{R}), \|\cdot\|_\infty\big)$.

Problem 12. Assume that $\{f_k\}_{k=1}^\infty$ is a sequence of monotone increasing functions on $\mathbb{R}$ with $0 \leqslant f_k(x) \leqslant 1$ for all x and $k \in \mathbb{N}$.

1. Show that there is a subsequence $\{f_{k_j}\}_{j=1}^\infty$ which converges **pointwise** to a function f (This is usually called the Helly selection theorem).

2. If the limit f is continuous, show that $\{f_{k_j}\}_{j=1}^\infty$ converges uniformly to f on any compact set of $\mathbb{R}$.

§5.6 The Contraction Mapping Principle and its Applications

Problem 13. Let (M,d) be a complete metric space, and $f : M \to M$. Define $f_k = f \circ f \circ \cdots \circ f$, here the composition was taken for $k-1$ times. Assume that there exists a sequence $\{\alpha_k\}_{k=1}^\infty \subseteq \mathbb{R}$ such that

1. $\alpha_k \to 0$ as $k \to \infty$.

2. $d\big(f_k(x), f_k(y)\big) \leqslant \alpha_k d(x,y)$ for all $k \in \mathbb{N}$, $x, y \in M$.

Show that f has a unique fixed-point.

Problem 14. Let (M, d) be a metric space, and $f : M \to M$ be such that $d\big(f(x), f(y)\big) < d(x, y)$ for all $x, y \in M$, $x \neq y$.

1. Fix $x_0 \in M$. Let $x_{n+1} = f(x_n)$, and $c_n = d(x_n, x_{n+1})$. Show that $\{c_n\}_{n=1}^{\infty}$ is a decreasing sequence; thus $c = \lim\limits_{n\to\infty} c_n$ exists.

2. Assume that there is a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $x_{n_j} \to x$ as $j \to \infty$. Show that
$$c = d\big(x, f(x)\big) = d\big(f(x), f(f(x))\big) .$$
and deduce that x is a fixed-point of f.

3. Suppose further that M is compact. Show that the sequence $\{x_n\}_{n=1}^{\infty}$ itself converges to x.

Problem 15. Let (M, d) be a metric space, $K \subseteq M$ be a compact subset, and $\Phi : K \to K$ be such that $d\big(\Phi(x), \Phi(y)\big) < d(x, y)$ for all $x, y \in K$, $x \neq y$.

1. Show that Φ has a unique fixed-point.

2. Show that 1 is false if K is not compact.

§5.7 The Stone-Weierstrass Theorem

Problem 16. Suppose that f is continuous on $[0, 1]$ and
$$\int_0^1 f(x)x^n dx = 0 \qquad \forall\, n \in \mathbb{N} \cup \{0\} .$$
Show that $f = 0$ on $[0, 1]$.

Problem 17. Put $p_0 = 0$ and define
$$p_{k+1}(x) = p_k(x) + \frac{x^2 - p_k^2(x)}{2} \qquad \forall\, k \in \mathbb{N} \cup \{0\} .$$
Show that $\{p_k\}_{k=1}^{\infty}$ converges uniformly to $|x|$ on $[-1, 1]$.
Hint: Use the identity
$$|x| - p_{k+1}(x) = \big[|x| - p_k(x)\big]\Big[1 - \frac{|x| + p_k(x)}{2}\Big]$$
to prove that $0 \leqslant p_k(x) \leqslant p_{k+1}(x) \leqslant |x|$ if $|x| \leqslant 1$, and that
$$|x| - p_k(x) \leqslant |x|\Big(1 - \frac{|x|}{2}\Big)^k < \frac{2}{k+1}$$
if $|x| \leqslant 1$.

Problem 18. A function $g : [0, 1] \to \mathbb{R}$ is called simple if we can divide up $[0, 1]$ into sub-intervals on which g is constant, except perhaps at the endpoints (see Definition 5.88 in the lecture note). Let $f : [0, 1] \to \mathbb{R}$ be continuous and $\varepsilon > 0$. Prove that there is a simple function g such that $\|f - g\|_\infty < \varepsilon$.

§6.1 Bounded Linear Maps

Problem 19. Let $\mathscr{P}((0,1)) \subseteq \mathscr{C}_b((0,1);\mathbb{R})$ be the collection of all polynomials defined on $(0,1)$.

1. Show that the operator $\dfrac{d}{dx} : \mathscr{P}((0,1)) \to \mathscr{C}_b((0,1))$ is linear.

2. Show that $\dfrac{d}{dx} : \big(\mathscr{P}((0,1)), \|\cdot\|_\infty\big) \to \big(\mathscr{C}_b((0,1)), \|\cdot\|_\infty\big)$ is unbounded; that is, show that
$$\sup_{\|p\|_\infty=1} \|p'\|_\infty = \infty\,.$$

§6.2 Definition of Derivatives and the Matrix Representation of Derivatives

Problem 20. Consider the map δ defined in Problem 8; that is, $\delta : \mathscr{C}([0,1];\mathbb{R}) \to \mathbb{R}$ be defined by $\delta(f) = f(0)$. Show that δ is differentiable. Find $(D\delta)(f)$ (for $f \in \mathscr{C}([0,1];\mathbb{R})$.

Problem 21. Let $f : \mathrm{GL}(n) \to \mathrm{GL}(n)$ be given by $f(L) = L^{-1}$. In class we have shown that f is continuous on $\mathrm{GL}(n)$. Show that f is differentiable at each "point" (or more precisely, linear map) of $\mathrm{GL}(n)$.
Hint: In order to show the differentiability of f at $L \in \mathrm{GL}(n)$, we need to figure out what $(Df)(L)$ is. So we need to compute $f(L+h) - f(L)$, where $h \in \mathscr{B}(\mathbb{R}^n, \mathbb{R}^n)$ is a "small" linear map. Compute $(L+h)^{-1} - L^{-1}$ and make a conjecture what $(Df)(L)$ should be.

Problem 22. Let $I : \mathscr{C}([0,1];\mathbb{R}) \to \mathbb{R}$ be defined by
$$I(f) = \int_0^1 f(x)^2 dx\,.$$
Show that I is differentiable at every "point" $f \in \mathscr{C}([0,1];\mathbb{R})$.
Hint: Figure out what $(DI)(f)$ is by computing $I(f+h) - I(f)$, where $h \in \mathscr{C}([0,1];\mathbb{R})$ is a "small" continuous function.
Remark. A map from a space of functions such as $\mathscr{C}([0,1];\mathbb{R})$ to a scalar field such as $\mathbb{R}$ or $\mathbb{C}$ is usually called a ***functional***. The derivative of a functional I is usually denoted by δI instead of DI.

§6.3 Continuity of Differentiable Mappings, §6.4 Conditions for Differentiability

Problem 23. Investigate the differentiability of
$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

Problem 24. Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, and $f : \mathcal{U} \to \mathbb{R}$. Suppose that the partial derivatives $\dfrac{\partial f}{\partial x_1}, \cdots, \dfrac{\partial f}{\partial x_n}$ are bounded on $\mathcal{U}$; that is, there exists a real number $M > 0$ such that
$$\Big|\frac{\partial f}{\partial x_j}(x)\Big| \leqslant M \qquad \forall\, x \in \mathcal{U} \text{ and } j = 1, \cdots, n\,.$$

Show that f is continuous on $\mathcal{U}$.
Hint: Mimic the proof of Theorem 6.32 in 共筆。

Problem 25. (True or false) Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open. Then $f : \mathcal{U} \to \mathbb{R}$ is differentiable at $a \in \mathcal{U}$ if and only if each directional derivative $(D_u f)(a)$ exists and

$$(D_u f)(a) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(a) u_j = \Big(\frac{\partial f}{\partial x_1}(a), \cdots, \frac{\partial f}{\partial x_n}(a)\Big) \cdot u$$

where $u = (u_1, \cdots, u_n)$ is a unit vector.
Hint: Consider the function

$$f(x,y) = \begin{cases} \dfrac{x^3 y}{x^4 + y^2} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

Problem 26. Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, and $f : \mathcal{U} \to \mathbb{R}$. Show that f is differentiable at $a \in \mathcal{U}$ if and only if there exists a vector-valued function $\varepsilon : \mathcal{U} \to \mathbb{R}^n$ such that

$$f(x) - f(a) - \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(a)(x_j - a_j) = \varepsilon(x) \cdot (x - a)$$

and $\varepsilon(x) \to 0$ as $x \to a$.

§6.5 The Chain Rule

Problem 27. Let (r, θ, φ) be the spherical coordinate of $\mathbb{R}^3$ so that

$$x = r\cos\theta\sin\varphi, y = r\sin\theta\sin\varphi, z = r\cos\varphi\,.$$

1. Find the Jacobian matrices of the map $(x, y, z) \mapsto (r, \theta, \varphi)$ and the map $(r, \theta, \varphi) \mapsto (x, y, z)$.

2. Suppose that $f(x, y, z)$ is a differential function in $\mathbb{R}^3$. Express $|\nabla f|^2$ in terms of the spherical coordinate.

§6.6 The Product Rules and Gradients, §6.7 The Mean Value Theorem

Problem 28. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Assume that for all $x \in \mathbb{R}$, $0 \leqslant f'(x) \leqslant f(x)$. Show that $g(x) = e^{-x} f(x)$ is decreasing. If f vanishes at some point, conclude that f is zero.

§6.8 Higher Derivatives and Taylor's Theorem

Problem 29. Let $f(x, y, z) = (x^2 + 1)\cos(yz)$, and $a = (0, \frac{\pi}{2}, 1)$, $u = (1, 0, 0)$, $v = (0, 1, 0)$ and $w = (2, 0, 1)$.

1. Compute $(Df)(a)(u)$.

2. Compute $(D^2 f)(a)(v)(u)$.

3. Compute $(D^3 f)(a)(w)(v)(u)$.

Problem 30. 1. If $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $g : B \subseteq \mathbb{R}^m \to \mathbb{R}^\ell$ are twice differentiable and $f(A) \subseteq B$, then for $x_0 \in A$, $u, v \in \mathbb{R}^n$, show that

$$D^2(g \circ f)(x_0)(u, v) \\ = (D^2 g)(f(x_0))\big((Df)(x_0)(u), Df(x_0)(v)\big) + (Dg)(f(x_0))\big((D^2 f)(x_0)(u, v)\big) .$$

2. If $p : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map plus some constant; that is, $p(x) = Lx + c$ for some $L \in \mathscr{B}(\mathbb{R}^n, \mathbb{R}^m)$, and $f : A \subseteq \mathbb{R}^m \to \mathbb{R}^s$ is k-times differentiable, prove that

$$D^k(f \circ p)(x_0)(u^{(1)}, \cdots, u^{(k)}) = (D^k f)\big(p(x_0)\big)\big((Dp)(x_0)(u^{(1)}), \cdots, (Dp)(x_0)(u^{(k)}\big) .$$

Problem 31. Let $f(x, y)$ be a real-valued function on $\mathbb{R}^2$. Suppose that f is of class $\mathscr{C}^1$ (that is, all first partial derivatives are continuous on $\mathbb{R}^2$) and $\dfrac{\partial^2 f}{\partial x \partial y}$ exists and is continuous. Show that $\dfrac{\partial^2 f}{\partial y \partial x}$ exists and $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$.

Hint: Mimic the proof of Theorem 6.78.

Problem 32. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable, and Df is a constant map in $\mathscr{B}(\mathbb{R}^n, \mathbb{R}^m)$; that is, $(Df)(x_1)(u) = (Df)(x_2)(u)$ for all $x_1, x_2 \in \mathbb{R}^n$ and $u \in \mathbb{R}^n$. Show that f is a linear term plus a constant and that the linear part of f is the constant value of Df.

Problem 33. Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, and $f : \mathcal{U} \to \mathbb{R}$ is of class $\mathscr{C}^k$ and $(D^j f)(x_0) = 0$ for $j = 1, \cdots, k-1$, but $(D^k f)(x_0)(u, u, \cdots, u) < 0$ for all $u \in \mathbb{R}^n$, $u \neq 0$. Show that f has a local maximum at x_0; that is, $\exists\, \delta > 0$ such that

$$f(x) \leqslant f(x_0) \qquad \forall\, x \in D(x_0, \delta) .$$

§6.9 Maxima and Minima

Problem 34. Let $f(x, y) = x^3 + x - 4xy + 2y^2$,

1. Find all critical points of f.

2. Find the corresponding quadratic from $Q(x, y, h, k)$ $\big($or $(D^2 f(x, y)\big((h, k), (h, k)\big)\big)$ at these critical points, and determine which of them is positive definite.

3. Find all extreme points and saddle points.

4. Find the maximal value of f on the set

$$A = \big\{(x, y) \,\big|\, 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1, x + y \leqslant 1\big\}.$$

Problem 35. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} x^2 + y^2 - 2x^2 y - \dfrac{4x^6 y^2}{(x^4+y^2)^2} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

1. Show that f is continuous (at $(0,0)$) by showing that for all $(x,y) \in \mathbb{R}^2$,

$$4x^4 y^2 \leqslant (x^4+y^2)^2\,.$$

2. For $0 \leqslant \theta \leqslant 2\pi$, $-\infty < t < \infty$, define

$$g_\theta(t) = f(t\cos\theta, t\sin\theta)\,.$$

   Show that each g_θ has a strict local minimum at $t = 0$. In other words, the restriction of f to each straight line through $(0,0)$ has a strict local minimum at $(0,0)$.

3. Show that $(0,0)$ is not a local minimum for f.

§7.1 The Inverse Function Theorem

Problem 36. Prove Corollary 7.4; that is, show that if $\mathcal{U} \subseteq \mathbb{R}^n$ is open, $f : \mathcal{U} \to \mathbb{R}^n$ is of class $\mathscr{C}^1$, and $(Df)(x)$ is invertible for all $x \in \mathcal{U}$, then $f(\mathcal{W})$ is open for every open set $\mathcal{W} \subseteq \mathcal{U}$.

Problem 37. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be of class $\mathscr{C}^1$, and for some $(a,b) \in \mathbb{R}^2$, $f(a,b) = 0$ and $f_y(a,b) \neq 0$. Show that there exist open neighborhoods $\mathcal{U} \subseteq \mathbb{R}$ of a and $\mathcal{V} \subseteq \mathbb{R}$ of b such that every $x \in \mathcal{U}$ corresponds to a unique $y \in \mathcal{V}$ such that $f(x,y) = 0$. In other words, there exists a function $y = y(x)$ such that $y(a) = b$ and $f(x,y(x)) = 0$ for all $x \in \mathcal{U}$.

§7.2 The Implicit Function Theorem

Problem 38. Assume that one proves the implicit function theorem without applying the inverse theorem. Show the inverse function using the implicit function theorem.

Problem 39. Suppose that the implicit function theorem applies to $F(x,y) = 0$ so that $y = f(x)$. Find a formula for f'' in terms of F and its partial derivatives. Similarly, suppose that the implicit function theorem applies to $F(x_1,x_2,y) = 0$ so that $y = f(x_1,x_2)$. Find formulas for $f_{x_1x_1}$, $f_{x_1x_2}$ and $f_{x_2x_2}$ in terms of F and its partial derivatives.

§8.1 Integrable Functions

Problem 40. Let $A \subseteq \mathbb{R}^n$ be a bounded rectangle, and $f : A \to \mathbb{R}$ be Riemann integrable.

1. Let $\mathcal{P}$ be a partition of A, and $m \leqslant f(x) \leqslant M$ for all $x \in A$. Show that $m\nu(A) \leqslant L(f,\mathcal{P}) \leqslant U(f,\mathcal{P}) \leqslant M\nu(A)$.

2. Show that $L(f,\mathcal{P}_1) \leqslant \mathcal{U}(f,\mathcal{P}_2)$ if $\mathcal{P}_1$ and $\mathcal{P}_2$ are two partitions of A.

§8.2 Volume and Sets of Measure Zero

Problem 41. Complete the following.

1. Show that if A is a set of volume zero, then A has measure zero. Is it true that if A has measure zero, then A also has volume zero?
2. Let $a, b \in \mathbb{R}$ and $a < b$. Show that the interval $[a,b]$ does not have measure zero (in $\mathbb{R}$).
3. Let $A \subseteq [a,b]$ be a set of measure zero (in $\mathbb{R}$). Show that $[a,b]\backslash A$ does not have measure zero (in $\mathbb{R}$).
4. Show that the Cantor set (defined in Exercise Problem 38 in fall semester) has volume zero.

Problem 42. Let $A = \bigcup_{k=1}^{\infty} D\big(\frac{1}{k}, \frac{1}{2^k}\big) = \bigcup_{k=1}^{\infty} \big(\frac{1}{k} - \frac{1}{2^k}, \frac{1}{k} + \frac{1}{2^k}\big)$ be a subset of $\mathbb{R}$. Does A have volume?

§8.3 Lebesgue's Theorem

Problem 43. (True or false) If $A \subseteq \mathbb{R}^n$ is a bounded set, and $f : A \to \mathbb{R}$ be bounded continuous. Then f is Riemann integrable over A.

Problem 44. Prove the following statements.

1. The function $f(x) = \sin\frac{1}{x}$ is Riemann integrable over $(0,1)$.
2. Let $f : [0,1] \to \mathbb{R}$ be given by
$$f(x) = \begin{cases} \dfrac{1}{p} & \text{if } x = \dfrac{q}{p} \in \mathbb{Q},\ (p,q) = 1\,, \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$
Then f is Riemann integrable over $[0,1]$. Find $\int_0^1 f(x)dx$ as well.
3. Let $A \subseteq \mathbb{R}^n$ be a bounded set, and $f : A \to \mathbb{R}$ is Riemann integrable. Then f^k（f 的 k 次方）is integrable for all $k \in \mathbb{N}$.

Problem 45. (True or false) Let $A, B \subseteq \mathbb{R}$ be bounded, and $f : A \to \mathbb{R}$ and $g : f(A) \to \mathbb{R}$ be Riemann integrable. Then $g \circ f$ is Riemann integrable over A.

§8.4 Properties of the Integrals, §8.5 Fubini's Theorem

Problem 46. Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} 1 & \text{if } y \in \mathbb{Q}\,, \\ x & \text{if } y \notin \mathbb{Q}\,. \end{cases}$$

Justify the integrability of f over $[0,1] \times [0,1]$ using

1. the Lebesgue theorem;

2. the Fubini theorem.

Problem 47.

1. Draw the region corresponding to the integral $\displaystyle\int_0^1 \left(\int_1^{e^x} (x+y)dy \right) dx$ and evaluate.

2. Change the order of integration of the integral in 1 and check if the answer is unaltered.