Exercise Problems for Advanced Calculus

MA2045, National Central University, Fall Semester 2013

§5.1 Pointwise and Uniform Convergence, §5.2 The Weierstrass M-Test, §5.3 Integration and Differentiation of Series

Problem 1. Let (M,d) be a metric space, $A \subseteq M$, and $f_k : A \to \mathbb{R}$ be a sequence of functions (not necessary continuous) which converges uniformly on A. Suppose that $a \in \text{cl}(A)$ and

$$\lim_{x\to a} f_k(x) = A_k$$

exists for all $k \in \mathbb{N}$. Show that $\{A_k\}_{k=1}^{\infty}$ converges, and

$$\lim_{x\to a}\lim_{k\to\infty} f_k(x) = \lim_{k\to\infty}\lim_{x\to a} f_k(x)\,.$$

Problem 2. Let (M,d) and (N,ρ) be metric spaces, $A \subseteq M$, and $f_k : A \to N$ be uniformly continuous functions, and $\{f_k\}_{k=1}^{\infty}$ converges uniformly to $f : A \to N$ on A. Show that f is uniformly continuous on A.

Problem 3. Determine which of the following real series $\sum\limits_{k=1}^{\infty} g_k$ converge (pointwise or uniformly). Check the continuity of the limit in each case.

1. $g_k(x) = \begin{cases} 0 & \text{if } x \leqslant k\,, \\ (-1)^k & \text{if } x > k\,. \end{cases}$

2. $g_k(x) = \begin{cases} \dfrac{1}{k^2} & \text{if } |x| \leqslant k\,, \\ \dfrac{1}{x^2} & \text{if } |x| > k\,. \end{cases}$

3. $g_k(x) = \Big(\dfrac{(-1)^k}{\sqrt{k}}\Big)\cos(kx)$ on $\mathbb{R}$.

4. $g_k(x) = x^k$ on $(0,1)$.

Problem 4. Complete the following.

(a) Suppose that $f_k, f, g : [0,\infty) \to \mathbb{R}$ are functions such that

1. $\forall\, R > 0$, f_k and g are Riemann integrable on $[0,R]$;
2. $|f_k(x)| \leqslant g(x)$ for all $k \in \mathbb{N}$ and $x \in [0,\infty)$;
3. $\forall\, R > 0$, $\{f_k\}_{k=1}^{\infty}$ converges to f uniformly on $[0,R]$;
4. $\displaystyle\int_0^{\infty} g(x)dx \equiv \lim_{R\to\infty}\int_0^R g(x)dx < \infty$.

Show that $\displaystyle\lim_{k\to\infty}\int_0^{\infty} f_k(x)dx = \int_0^{\infty} f(x)dx$; that is,

$$\lim_{k\to\infty}\lim_{R\to\infty}\int_0^R f_k(x)dx = \lim_{R\to\infty}\lim_{k\to\infty}\int_0^R f_k(x)dx\,.$$

(b) Let $f_k(x)$ be given by $f_k(x) = \begin{cases} 1 & \text{if } k-1 \leqslant x < k\,, \\ 0 & \text{otherwise.} \end{cases}$ Find the (pointwise) limit f of the sequence $\{f_k\}_{k=1}^{\infty}$, and check whether $\lim\limits_{k\to\infty} \int_0^\infty f_k(x)dx = \int_0^\infty f(x)dx$ or not. Briefly explain why one can or cannot apply (a).

(c) Let $f_k : [0,\infty) \to \mathbb{R}$ be given by $f_k(x) = \dfrac{x}{1+kx^4}$. Find $\lim\limits_{k\to\infty} \int_0^\infty f_k(x)dx$.

Problem 5. Construct the function $g(x)$ by letting $g(x) = |x|$ if $x \in \big[-\frac{1}{2}, \frac{1}{2}\big]$ and extending g so that it becomes periodic (with period 1). Define

$$f(x) = \sum_{k=1}^{\infty} \frac{g(4^{k-1}x)}{4^{k-1}}\,.$$

1. Use the Weierstrass M-test to show that f is continuous on $\mathbb{R}$.

2. Prove that f is differentiable at no point.

(So there exists a continuous which is nowhere differentiable!)
Hint: Google Blancmange function!

§5.4 The Space of Continuous Functions §5.5 The Arzela-Ascoli Theorem

Problem 6. Let (M, d) be a metric space, and $K \subseteq M$ be a compact subset.

1. Show that the set $U = \big\{f \in \mathscr{C}(K;\mathbb{R}) \,\big|\, a < f(x) < b \text{ for all } x \in K\big\}$ is open in $\big(\mathscr{C}(K;\mathbb{R}), \|\cdot\|_\infty\big)$ for all $a, b \in \mathbb{R}$.

2. Show that the set $F = \big\{f \in \mathscr{C}(K;\mathbb{R}) \,\big|\, a \leqslant f(x) \leqslant b \text{ for all } x \in K\big\}$ is closed in $\big(\mathscr{C}(K;\mathbb{R}), \|\cdot\|_\infty\big)$ for all $a, b \in \mathbb{R}$.

3. Let $A \subseteq M$ be a subset, not necessarily compact. Prove or disprove that the set $B = \big\{f \in \mathscr{C}_b(A;\mathbb{R}) \,\big|\, f(x) > 0 \text{ for all } x \in A\big\}$ is open in $\big(\mathscr{C}_b(A;\mathbb{R}), \|\cdot\|_\infty\big)$.

Problem 7. Let $\delta : \mathscr{C}([0,1];\mathbb{R}) \to \mathbb{R}$ be defined by $\delta(f) = f(0)$. Show that δ is linear and continuous.

Problem 8. Let (M, d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a normed space, and $A \subseteq M$ be a subset. Suppose that $B \subseteq \mathscr{C}_b(A;\mathcal{V})$ be equi-continuous. Prove or disprove that $\mathrm{cl}(B)$ is equi-continuous.

Problem 9. Let $\mathscr{C}^{0,\alpha}([0,1];\mathbb{R})$ denote the "space"

$$\mathscr{C}^{0,\alpha}([0,1];\mathbb{R}) \equiv \Big\{f \in \mathscr{C}([0,1];\mathbb{R}) \,\Big|\, \sup_{x,y\in[0,1]} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty\Big\}\,,$$

where $\alpha \in (0,1]$. For each $f \in \mathscr{C}^{0,\alpha}([0,1];\mathbb{R})$, define

$$\|f\|_{\mathscr{C}^{0,\alpha}} = \sup_{x\in[0,1]} |f(x)| + \sup_{\substack{x,y\in[0,1]\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha}\,.$$

1. Show that $\big(\mathscr{C}^{0,\alpha}([0,1];\mathbb{R}), \|\cdot\|_{\mathscr{C}^{0,\alpha}}\big)$ is a complete normed space.

2. Show that the set $B = \big\{ f \in \mathscr{C}([0,1];\mathbb{R}) \,\big|\, \|f\|_{\mathscr{C}^{0,\alpha}} < 1 \big\}$ is equi-continuous.

3. Show that $\text{cl}(B)$ is compact in $\big(\mathscr{C}([0,1];\mathbb{R}), \|\cdot\|_\infty\big)$.

Problem 10. Assume that $\{f_k\}_{k=1}^\infty$ is a sequence of monotone increasing functions on $\mathbb{R}$ with $0 \leqslant f_k(x) \leqslant 1$ for all x and $k \in \mathbb{N}$.

1. Show that there is a subsequence $\big\{f_{k_j}\big\}_{j=1}^\infty$ which converges **pointwise** to a function f (This is usually called the Helly selection theorem).

2. If the limit f is continuous, show that $\big\{f_{k_j}\big\}_{j=1}^\infty$ converges uniformly to f on any compact set of $\mathbb{R}$.

§5.6 The Contraction Mapping Principle and its Applications

Problem 11. Suppose that $f : [a,b] \to \mathbb{R}$ is twice continuous differentiable; that is, $f', f'' : [a,b] \to \mathbb{R}$ are continuous, and $f(a) < 0 = f(c) < f(b)$, and $f'(x) \neq 0$ for all $x \in [a,b]$. Consider the function

$$\Phi(x) = x - \frac{f(x)}{f'(x)}\,.$$

1. Show that $\Phi : [a,b] \to \mathbb{R}$ satisfies

$$|\Phi(x) - \Phi(y)| \leqslant k|x-y| \quad \forall\, x, y \in [a,b]$$

   for some $k \in [0,1)$ if $|b-a|$ are small enough.

2. Suppose that $f''(x) > 0$ for all $x \in [a,b]$. Show that there exists $a \leqslant \widetilde{a} < c$ such that $\Phi : [\widetilde{a}, b] \to [\widetilde{a}, b]$.

3. Under the condition of 2, show that if $x_0 \in [\widetilde{a}, b]$, then the iteration

$$x_{k+1} = \Phi(x_k) \quad \forall\, k \in \mathbb{N} \cup \{0\}$$

   provides a convergent sequence $\{x_k\}_{k=1}^\infty$ with limit c.

(The iteration scheme above of finding the zero c of f is called the Newton method.)

Problem 12. Let (M,d) be a metric space, $K \subseteq M$ be a compact subset, and $\Phi : K \to K$ be such that $d\big(\Phi(x), \Phi(y)\big) < d(x,y)$ for all $x, y \in K$, $x \neq y$.

1. Show that Φ has a unique fixed-point.

2. Show that 1 is false if K is not compact.

§5.7 The Stone-Weierstrass Theorem

Problem 13. Suppose that f is continuous on $[0,1]$ and

$$\int_0^1 f(x)x^n dx = 0 \qquad \forall\, n \in \mathbb{N} \cup \{0\}\,.$$

Show that $f = 0$ on $[0,1]$.

Problem 14. Put $p_0 = 0$ and define

$$p_{k+1}(x) = p_k(x) + \frac{x^2 - p_k^2(x)}{2} \qquad \forall\, k \in \mathbb{N} \cup \{0\}\,.$$

Show that $\{p_k\}_{k=1}^{\infty}$ converges uniformly to $|x|$ on $[-1,1]$.
Hint: Use the identity

$$|x| - p_{k+1}(x) = \big[|x| - p_k(x)\big]\Big[1 - \frac{|x| + p_k(x)}{2}\Big]$$

to prove that $0 \leqslant p_k(x) \leqslant p_{k+1}(x) \leqslant |x|$ if $|x| \leqslant 1$, and that

$$|x| - p_k(x) \leqslant |x|\Big(1 - \frac{|x|}{2}\Big)^k < \frac{2}{k+1}$$

if $|x| \leqslant 1$.

Problem 15. A function $g : [0,1] \to \mathbb{R}$ is called simple if we can divide up $[0,1]$ into sub-intervals on which g is constant, except perhaps at the endpoints (see Definition 5.88 in the lecture note). Let $f : [0,1] \to \mathbb{R}$ be continuous and $\varepsilon > 0$. Prove that there is a simple function g such that $\|f - g\|_\infty < \varepsilon$.

Problem 16. （挑戰自我之期中考不考題）Suppose that p_n is a sequence of polynomials converging uniformly to f on $[0,1]$ and f is not a polynomial. Prove that the degrees of p_n are not bounded.
Hint: An Nth-degree polynomial p is uniquely determined by its values at $N+1$ points $x_0, \cdots, x_N$ via Lagrange's interpolation formula

$$p(x) = \sum_{k=0}^{N} \pi_k(x) \frac{p(x_k)}{\pi_k(x_k)}\,,$$

where $\pi_k(x) = (x - x_0)(x - x_1)\cdots(x - x_N)/(x - x_k) = \prod\limits_{\substack{1 \leqslant j \leqslant N \\ j \neq k}} (x - x_j)$.

Problem 17. （挑戰自我之期中考不考題）Consider the set of all functions on $[0,1]$ of the form

$$h(x) = \sum_{j=1}^{n} a_j e^{b_j x}\,,$$

where a_j, $b_j \in \mathbb{R}$. Is this set dense in $\mathscr{C}([0,1];\mathbb{R})$?

§6.1 Bounded Linear Maps

Problem 18. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces.

1. Show that $\big(\mathscr{B}(X,Y), \|\cdot\|_{\mathscr{B}(X,Y)}\big)$ is a normed space.
2. Show that $\big(\mathscr{B}(X,Y), \|\cdot\|_{\mathscr{B}(X,Y)}\big)$ is complete if $(Y, \|\cdot\|_Y)$ is complete.

Problem 19. Let $\mathscr{P}((0,1)) \subseteq \mathscr{C}_b((0,1);\mathbb{R})$ be the collection of all polynomials defined on $(0,1)$.

1. Show that the operator $\dfrac{d}{dx} : \mathscr{P}((0,1)) \to \mathscr{C}_b((0,1))$ is linear.
2. Show that $\dfrac{d}{dx} : \big(\mathscr{P}((0,1)), \|\cdot\|_\infty\big) \to \big(\mathscr{C}_b((0,1)), \|\cdot\|_\infty\big)$ is unbounded; that is, show that
$$\sup_{\|p\|_\infty=1} \|p'\|_\infty = \infty\,.$$

§6.2 Definition of Derivatives and the Matrix Representation of Derivatives

Problem 20. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces, $\mathcal{U} \subseteq X$ be open, and $f : \mathcal{U} \subseteq X \to Y$ be a map. Show that f is differentiable at $a \in \mathcal{U}$ if and only if there exists $L \in \mathscr{B}(X,Y)$ such that

$$\forall\, \varepsilon > 0, \exists\, \delta > 0 \ni \|f(x) - f(a) - L(x-a)\|_Y \leqslant \varepsilon \|x-a\|_X \text{ whenever } x \in D(a,\delta)\,.$$

Problem 21. Let $f : \mathrm{GL}(n) \to \mathrm{GL}(n)$ be given by $f(L) = L^{-1}$. In class we have shown that f is continuous on $\mathrm{GL}(n)$. Show that f is differentiable at each "point" (or more precisely, linear map) of $\mathrm{GL}(n)$.
Hint: In order to show the differentiability of f at $L \in \mathrm{GL}(n)$, we need to figure out what $(Df)(L)$ is. So we need to compute $f(L+h) - f(L)$, where $h \in \mathscr{B}(\mathbb{R}^n, \mathbb{R}^n)$ is a "small" linear map. Compute $(L+h)^{-1} - L^{-1}$ and make a conjecture what $(Df)(L)$ should be.

Problem 22. Let $I : \mathscr{C}([0,1];\mathbb{R}) \to \mathbb{R}$ be defined by

$$I(f) = \int_0^1 f(x)^2 dx\,.$$

Show that I is differentiable at every "point" $f \in \mathscr{C}([0,1];\mathbb{R})$.
Hint: Figure out what $(DI)(f)$ is by computing $I(f+h) - I(f)$, where $h \in \mathscr{C}([0,1];\mathbb{R})$ is a "small" continuous function.
Remark. A map from a space of functions such as $\mathscr{C}([0,1];\mathbb{R})$ to a scalar field such as $\mathbb{R}$ or $\mathbb{C}$ is usually called a ***functional***. The derivative of a functional I is usually denoted by δI instead of DI.

Problem 23. Let $\mathcal{U} = \mathbb{R}^2 \backslash \{(x,0) \in \mathbb{R}^2 \,|\, x \geqslant 0\}$. Check the differentiability of the function $f : \mathcal{U} \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} \cos^{-1} \dfrac{x}{\sqrt{x^2+y^2}} & \text{if } y > 0\,, \\ \pi & \text{if } y = 0\,, \\ 2\pi - \cos^{-1} \dfrac{x}{\sqrt{x^2+y^2}} & \text{if } y < 0\,, \end{cases}$$

at point $(-1,0)$ by the definition of differentiability of a function.

§6.3 Continuity of Differentiable Mappings, §6.4 Conditions for Differentiability

Problem 24. Investigate the differentiability of

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

Problem 25. Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, and $f : \mathcal{U} \to \mathbb{R}$. Suppose that the partial derivatives $\dfrac{\partial f}{\partial x_1}, \cdots, \dfrac{\partial f}{\partial x_n}$ are bounded on $\mathcal{U}$; that is, there exists a real number $M > 0$ such that

$$\Big|\frac{\partial f}{\partial x_j}(x)\Big| \leqslant M \qquad \forall\, x \in \mathcal{U} \text{ and } j = 1, \cdots, n\,.$$

Show that f is continuous on $\mathcal{U}$.
Hint: Mimic the proof of Theorem 6.31 in 共筆。

Problem 26. Investigate the differentiability of

$$f(x,y) = \begin{cases} \dfrac{xy}{x+y^2} & \text{if } x+y^2 \neq 0\,, \\ 0 & \text{if } x+y^2 = 0\,. \end{cases}$$

Problem 27. (True or false) Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open. Then $f : \mathcal{U} \to \mathbb{R}$ is differentiable at $a \in \mathcal{U}$ if and only if each directional derivative $(D_u f)(a)$ exists and

$$(D_u f)(a) = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(a) u_j = \Big(\frac{\partial f}{\partial x_1}(a), \cdots, \frac{\partial f}{\partial x_n}(a)\Big) \cdot u$$

where $u = (u_1, \cdots, u_n)$ is a unit vector.
Hint: Consider the function

$$f(x,y) = \begin{cases} \dfrac{x^3 y}{x^4+y^2} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

§6.5 The Chain Rule

Problem 28. Verify the chain rule for

$$u(x,y,z) = xe^y, \quad v(x,y,z) = yz\sin x$$

and

$$f(u,v) = u^2 + v\sin u$$

with $h(x,y,z) = f\big(u(x,y,z), v(x,y,z)\big)$.

Problem 29. Let (r,θ,φ) be the spherical coordinate of $\mathbb{R}^3$ so that

$$x = r\cos\theta\sin\varphi, y = r\sin\theta\sin\varphi, z = r\cos\varphi\,.$$

1. Find the Jacobian matrices of the map $(x,y,z) \mapsto (r,\theta,\varphi)$ and the map $(r,\theta,\varphi) \mapsto (x,y,z)$.
2. Suppose that $f(x,y,z)$ is a differential function in $\mathbb{R}^3$. Express $|\nabla f|^2$ in terms of the spherical coordinate.

§6.6 The Product Rules and Gradients, §6.7 The Mean Value Theorem

Problem 30. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Assume that for all $x \in \mathbb{R}$, $0 \leqslant f'(x) \leqslant f(x)$. Show that $g(x) = e^{-x}f(x)$ is decreasing. If f vanishes at some point, conclude that f is zero.

§6.8 Higher Derivatives and Taylor's Theorem

Problem 31. Let $f(x,y,z) = (x^2+1)\cos(yz)$, and $a = (0, \frac{\pi}{2}, 1)$, $u = (1,0,0)$, $v = (0,1,0)$ and $w = (2,0,1)$.

1. Compute $(Df)(a)(u)$.
2. Compute $(D^2f)(a)(v)(u)$.
3. Compute $(D^3f)(a)(w)(v)(u)$.

Problem 32. 1. If $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $g : B \subseteq \mathbb{R}^m \to \mathbb{R}^\ell$ are twice differentiable and $f(A) \subseteq B$, then for $x_0 \in A$, $u, v \in \mathbb{R}^n$, show that

$$\begin{aligned} &D^2(g\circ f)(x_0)(u,v) \\ &\quad = (D^2g)(f(x_0))\big((Df)(x_0)(u), Df(x_0)(v)\big) + (Dg)(f(x_0))\big((D^2f)(x_0)(u,v)\big)\,. \end{aligned}$$

2. If $p : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map plus some constant; that is, $p(x) = Lx + c$ for some $L \in \mathscr{B}(\mathbb{R}^n, \mathbb{R}^m)$, and $f : A \subseteq \mathbb{R}^m \to \mathbb{R}^s$ is k-times differentiable, prove that

$$D^k(f\circ p)(x_0)(u^{(1)}, \cdots, u^{(k)}) = (D^kf)\big(p(x_0)\big)\big((Dp)(x_0)(u^{(1)}), \cdots, (Dp)(x_0)(u^{(k)}\big)\,.$$

Problem 33. Let $f(x, y)$ be a real-valued function on $\mathbb{R}^2$. Suppose that f is of class $\mathscr{C}^1$ (that is, all first partial derivatives are continuous on $\mathbb{R}^2$) and $\dfrac{\partial^2 f}{\partial x \partial y}$ exists and is continuous. Show that $\dfrac{\partial^2 f}{\partial y \partial x}$ exists and $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$.

Hint: Mimic the proof of Theorem 6.74.

Problem 34. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable, and Df is a constant map in $\mathscr{B}(\mathbb{R}^n, \mathbb{R}^m)$; that is, $(Df)(x_1)(u) = (Df)(x_2)(u)$ for all $x_1, x_2 \in \mathbb{R}^n$ and $u \in \mathbb{R}^n$. Show that f is a linear term plus a constant and that the linear part of f is the constant value of Df.

Problem 35. Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, and $f : \mathcal{U} \to \mathbb{R}$ is of class $\mathscr{C}^k$ and $(D^j f)(x_0) = 0$ for $j = 1, \cdots, k-1$, but $(D^k f)(x_0)(u, u, \cdots, u) < 0$ for all $u \in \mathbb{R}^n$, $u \neq 0$. Show that f has a local maximum at x_0; that is, $\exists\, \delta > 0$ such that

$$f(x) \leqslant f(x_0) \qquad \forall\, x \in D(x_0, \delta)\,.$$

§6.9 Maxima and Minima

Problem 36. Let $f(x, y) = x^3 + x - 4xy + 2y^2$,

1. Find all critical points of f.
2. Find the corresponding quadratic from $Q(x, y, h, k)$ $\big($or $(D^2 f(x, y)\big((h, k), (h, k)\big)\big)$ at these critical points, and determine which of them is positive definite.
3. Find all relative extrema and saddle points.
4. Find the maximal value of f on the set

$$A = \big\{(x, y) \,\big|\, 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1, x + y \leqslant 1\big\}.$$

Problem 37. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = \begin{cases} x^2 + y^2 - 2x^2 y - \dfrac{4x^6 y^2}{(x^4 + y^2)^2} & \text{if } (x, y) \neq (0, 0)\,, \\ 0 & \text{if } (x, y) = (0, 0)\,. \end{cases}$$

1. Show that f is continuous (at $(0, 0)$) by showing that for all $(x, y) \in \mathbb{R}^2$,

$$4x^4 y^2 \leqslant (x^4 + y^2)^2\,.$$

2. For $0 \leqslant \theta \leqslant 2\pi$, $-\infty < t < \infty$, define

$$g_\theta(t) = f(t\cos\theta, t\sin\theta)\,.$$

Show that each g_θ has a strict local minimum at $t = 0$. In other words, the restriction of f to each straight line through $(0, 0)$ has a strict local minimum at $(0, 0)$.

3. Show that $(0,0)$ is not a local minimum for f.

§7.1 The Inverse Function Theorem

Problem 38. Prove Corollary 7.4; that is, show that if $\mathcal{U} \subseteq \mathbb{R}^n$ is open, $f : \mathcal{U} \to \mathbb{R}^n$ is of class $\mathscr{C}^1$, and $(Df)(x)$ is invertible for all $x \in \mathcal{U}$, then $f(\mathcal{W})$ is open for every open set $\mathcal{W} \subseteq \mathcal{U}$.

Problem 39. Let $\mathcal{D} \subseteq \mathbb{R}^n$ be a bounded open convex set, and $f : \mathcal{D} \to \mathbb{R}^n$ be of class $\mathscr{C}^1$ such that

1. f and Df are continuous on $\overline{\mathcal{D}}$;
2. the Jacobian $\det\big([(Df)(x)]\big) \neq 0$ for all $x \in \overline{\mathcal{D}}$;
3. $f : \partial\mathcal{D} \to \mathbb{R}^n$ is one-to-one.

Show that $f : \overline{\mathcal{D}} \to \mathbb{R}^n$ is one-to-one by completing the following:

1. Define $E = \big\{x \in \overline{\mathcal{D}} \,\big|\, \exists\, y \in \overline{\mathcal{D}}, y \neq x \ni f(x) = f(y)\big\}$. Then E is open relative to $\overline{\mathcal{D}}$.
2. Show that E is closed.
3. By the previous step, conclude that $E = \varnothing$ or $E = \mathcal{D}$. Also show that $E \neq \mathcal{D}$ (thus $E = \varnothing$ is the only possibility which suggests that f is injective on $\mathcal{D}$).

Problem 40. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be of class $\mathscr{C}^1$, and for some $(a,b) \in \mathbb{R}^2$, $f(a,b) = 0$ and $f_y(a,b) \neq 0$. Show that there exist open neighborhoods $\mathcal{U} \subseteq \mathbb{R}$ of a and $\mathcal{V} \subseteq \mathbb{R}$ of b such that every $x \in \mathcal{U}$ corresponds to a unique $y \in \mathcal{V}$ such that $f(x,y) = 0$. In other words, there exists a function $y = y(x)$ such that $y(a) = b$ and $f(x, y(x)) = 0$ for all $x \in \mathcal{U}$.

§7.2 The Implicit Function Theorem

Problem 41. Assume that one proves the implicit function theorem without applying the inverse theorem. Show the inverse function using the implicit function theorem.

Problem 42. Suppose that $F(x,y,z) = 0$ is such that the functions $z = f(x,y)$, $x = g(y,z)$, and $y = h(z,x)$ all exist by the implicit function theorem. Show that $f_x \cdot g_y \cdot h_z = -1$.

Problem 43. Suppose that the implicit function theorem applies to $F(x,y) = 0$ so that $y = f(x)$. Find a formula for f'' in terms of F and its partial derivatives. Similarly, suppose that the implicit function theorem applies to $F(x_1,x_2,y) = 0$ so that $y = f(x_1,x_2)$. Find formulas for $f_{x_1x_1}$, $f_{x_1x_2}$ and $f_{x_2x_2}$ in terms of F and its partial derivatives.

§8.1 Integrable Functions

Problem 44. Let $A \subseteq \mathbb{R}^n$ be bounded, and $f : A \to \mathbb{R}$ be Riemann integrable.

1. Let $\mathcal{P}$ be a partition of A, and $m \leqslant f(x) \leqslant M$ for all $x \in A$. Show that $m\nu(A) \leqslant L(f,\mathcal{P}) \leqslant U(f,\mathcal{P}) \leqslant M\nu(A)$.
2. Show that $L(f,\mathcal{P}_1) \leqslant \mathcal{U}(f,\mathcal{P}_2)$ if $\mathcal{P}_1$ and $\mathcal{P}_2$ are two partitions of A.

§8.2 Volume and Sets of Measure Zero

Problem 45. Complete the following.

1. Show that if A is a set of volume zero, then A has measure zero. Is it true that if A has measure zero, then A also has volume zero?
2. Let $a, b \in \mathbb{R}$ and $a < b$. Show that the interval $[a,b]$ does not have measure zero (in $\mathbb{R}$).
3. Let $A \subseteq [a,b]$ be a set of measure zero (in $\mathbb{R}$). Show that $[a,b]\backslash A$ does not have measure zero (in $\mathbb{R}$).
4. Show that the Cantor set (defined in Exercise Problem 34 in fall semester) has volume zero.

§8.3 Lebesgue's Theorem

Problem 46. (True or false) If $A \subseteq \mathbb{R}^n$ is a bounded set, and $f : A \to \mathbb{R}$ be bounded continuous. Then f is Riemann integrable over A.

Problem 47. Let $A = \bigcup_{k=1}^{\infty} D\big(\frac{1}{k}, \frac{1}{2^k}\big) = \bigcup_{k=1}^{\infty} \big(\frac{1}{k} - \frac{1}{2^k}, \frac{1}{k} + \frac{1}{2^k}\big)$ be a subset of $\mathbb{R}$. Does A have volume?

Problem 48. Prove the following statements.

1. The function $f(x) = \sin\frac{1}{x}$ is Riemann integrable over $(0,1)$.
2. Let $f : [0,1] \to \mathbb{R}$ be given by
$$f(x) = \begin{cases} \frac{1}{p} & \text{if } x = \frac{q}{p} \in \mathbb{Q},\ (p,q) = 1\,, \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$
Then f is Riemann integrable over $[0,1]$. Find $\int_0^1 f(x)dx$ as well.
3. Let $A \subseteq \mathbb{R}^n$ be a bounded set, and $f : A \to \mathbb{R}$ is Riemann integrable. Then f^k (f 的 k 次方) is integrable for all $k \in \mathbb{N}$.

Problem 49. (True or false) Let $A, B \subset \mathbb{R}$ be bounded, and $f : A \to \mathbb{R}$ and $g : f(A) \to \mathbb{R}$ be Riemann integrable. Then $g \circ f$ is Riemann integrable over A.

§8.4 Properties of the Integrals

Problem 50. (True or false) Let $A \subseteq \mathbb{R}^n$ be bounded, $B \subseteq A$, and $f : A \to \mathbb{R}$ be bounded. If $f1_B$ is Riemann integrable over A, then f is Riemann integrable over B. Moreover,

$$\int_A (f1_B)(x)dx = \int_B f(x)dx\,.$$

§8.5 Fubini's Theorem

Problem 51. Let $A = [a,b] \times [c,d]$ be a rectangle in $\mathbb{R}^2$, and $f : A \to \mathbb{R}$ be Riemann integrable. Show that the sets

$$\Big\{x \in [a,b] \,\Big|\, \underline{\int_c^d} f(x,y)dy \neq \overline{\int_c^d} f(x,y)dy\Big\} \quad \text{and} \quad \Big\{y \in [c,d] \,\Big|\, \underline{\int_a^b} f(x,y)dx \neq \overline{\int_a^b} f(x,y)dx\Big\}$$

have measure zero (in $\mathbb{R}^1$).

Problem 52. Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} 0 & \text{if } x = 0 \text{ or if } x \text{ or } y \text{ is irrational}\,, \\ \dfrac{1}{p} & \text{if } x, y \in \mathbb{Q} \text{ and } x = \dfrac{q}{p} \text{ with } (p,q) = 1\,. \end{cases}$$

1. Show that $f(\cdot, y) : [0,1] \to \mathbb{R}$ is Riemann integrable for each $y \in [0,1]$.
2. Show that $f(x, \cdot) : [0,1] \to \mathbb{R}$ is Riemann integrable if $x \notin \mathbb{Q}$.
3. Find $\underline{\int_0^1} f(x,y)dy$ and $\overline{\int_0^1} f(x,y)dy$ if $x = \dfrac{q}{p}$ in reduced form.
4. Show that f is Riemann integrable over $[0,1] \times [0,1]$. Find $\displaystyle\int_{[0,1]\times[0,1]} f(x,y)d\mathbb{A}$.

Problem 53. Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} 1 & \text{if } (x,y) = \big(\dfrac{k}{2^n}, \dfrac{\ell}{2^n}\big),\ 0 < k, \ell < 2^n \text{ odd numbers, } n \in \mathbb{N}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Show that

$$\int_0^1 \int_0^1 f(x,y)dydx = \int_0^1 \int_0^1 f(x,y)dxdy$$

but f is not Riemann integrable.

Problem 54.

1. Draw the region corresponding to the integral $\displaystyle\int_0^1 \Big(\int_1^{e^x} (x+y)dy \Big) dx$ and evaluate.
2. Change the order of integration of the integral in 1 and check if the answer is unaltered.