國立中央大學

高等微積分共同筆記（**ebook** 版）

Advanced Calculus - Lecture Note

Contents

Chapter 0

Introduction - Sets and Functions

0.1 Sets

Definition 0.1. A ***set*** is a collection of objects called ***elements*** or ***members*** of the set. A set A is said to be a ***subset*** of S if every member of A is also a member of S. We write $x \in A$ if x is a member of A, and write $A \subseteq S$ if A is a subset of S. The empty set, denoted $\varnothing$, is the set with no member.

Definition 0.2. Let S be a given set, and $A \subseteq S$, $B \subseteq S$.
The set $A \cup B$, called the ***union*** of A and B, consists of members belonging to set A or set B. Let $A_1, A_2, \cdots$ be sets. The set $\bigcup\limits_{i=1}^{\infty} A_i = \{x \,|\, x \in A_i \text{ for some } i\}$ is the union of $A_1, A_2, \cdots$.
The set $A \cap B$, called the ***intersection*** of A and B, consists of members belonging to both set A and set B. Let $A_1, A_2, \cdots$ be sets. The set $\bigcap\limits_{i=1}^{\infty} A_i = \{x \,|\, x \in A_i \text{ for all } i\}$ is the intersection of $A_1, A_2, \cdots$.

Remark 0.3. Let $\mathscr{F}$ be the collection of some subsets in S. Sometimes we also write the union of sets in $\mathscr{F}$ as $\bigcup\limits_{A \in \mathscr{F}} A$; that is,

$$\bigcup_{A \in \mathscr{F}} A = \left\{x \in S \,\middle|\, \exists\, A \in \mathscr{F} \ni x \in A\right\}$$

Similarly, $\bigcap\limits_{A \in \mathscr{F}} A = \left\{x \in S \,\middle|\, \forall A \in \mathscr{F} \ni x \in A\right\}$ is the intersection of sets in $\mathscr{F}$.

Example 0.4. Let $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 3, 7\}$, $S = \{1, 2, 3, \cdots\}$, and $\mathscr{F} = \{A, B\}$. Then $\bigcup\limits_{E \in \mathscr{F}} E \equiv A \cup B = \{1, 2, 3, 4, 5, 7\}$, and $\bigcap\limits_{E \in \mathscr{F}} E \equiv A \cap B = \{1, 3\}$.

Definition 0.5. Let S be a given set, and $A \subseteq S$, $B \subseteq S$. The ***complement*** of A relative to B, denoted $B \backslash A$, is the set consisting of members of B that are not members of A. When the universal set S under consideration is fixed, the complement of A relative to S or simply the complement of A, is denoted by $A^{\complement}$, or $S \backslash A$.

Theorem 0.6. (De Morgan's Law)

1. $B \backslash \bigcup_{i=1}^{\infty} A_i = \bigcap_{i=1}^{\infty} (B \backslash A_i)$ *or* $B \backslash \bigcup_{A \in \mathscr{F}} A = \bigcap_{A \in \mathscr{F}} (B \backslash A)$.

2. $B \backslash \bigcap_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} (B \backslash A_i)$ *or* $B \backslash \bigcap_{A \in \mathscr{F}} A = \bigcup_{A \in \mathscr{F}} (B \backslash A)$.

Proof. By definition,

$$
\begin{aligned}
x \in B \backslash \bigcup_{i=1}^{\infty} A_i \Leftrightarrow x \in B \text{ but } x \notin \bigcup_{i=1}^{\infty} A_i \Leftrightarrow x \in B \text{ and } x \notin A_i \text{ for all } i \\
\Leftrightarrow x \in B \backslash A_i \text{ for all } i \Leftrightarrow x \in \bigcap_{i=1}^{\infty} (B \backslash A_i)
\end{aligned}
$$

The proof of the second identity is similar, and is left as an exercise. □

Definition 0.7. Given sets A and B, the ***Cartesian product*** $A \times B$ of A and B is the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$, $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$.

Example 0.8. Let $A = \{1, 3, 5\}$ and $B = \{\star, \diamond\}$. Then

$$A \times B = \{(1, \star), (3, \star), (5, \star), (1, \diamond), (3, \diamond), (5, \diamond)\}.$$

Example 0.9. Let $A = [2, 7]$ and $B = [1, 4]$. The Cartesian product of A and B is the square plotted below:

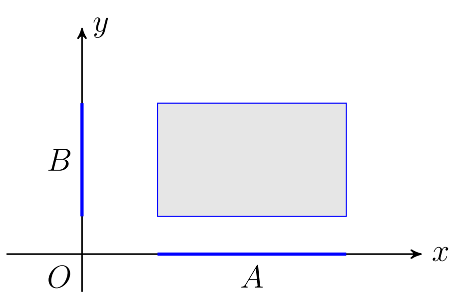
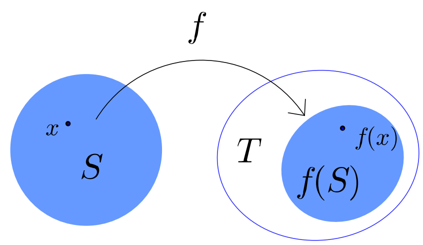


Figure 1: The Cartesian product $[2, 7] \times [1, 4]$

0.2 Functions

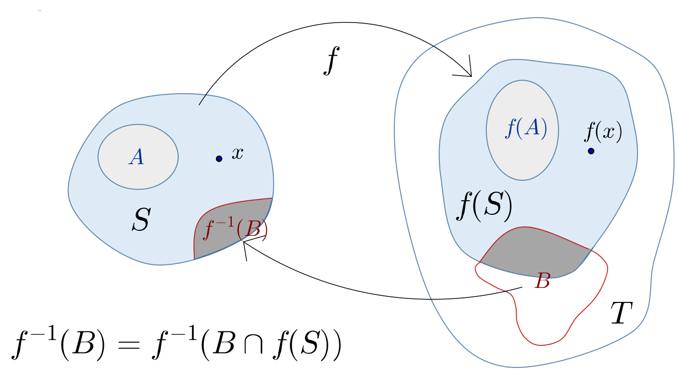
Definition 0.10. Let S and T be given sets. A ***function*** $f : S \to T$ consists of two sets S and T together with a "rule" that assigns to each $x \in S$ a special element of T denoted by $f(x)$. One writes $x \mapsto f(x)$ to denote that x is mapped to the element $f(x)$. S is called the ***domain*** (定義域) of f, and T is called the ***target*** or ***co-domain*** of f. The ***range*** (值域) of f or the ***image*** of f, is the subset of T defined by $f(S) = \{f(x) \,|\, x \in S\}$.



Definition 0.11. A function $f : S \to T$ is called ***one-to-one***（一對一）, ***injective*** or an ***injection*** if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ (which is equivalent to that $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$). A function $f : S \to T$ is called ***onto***（映成）, ***surjective*** or an ***surjection*** if $\forall\, y \in T$, $\exists\, x \in S$, $\ni f(x) = y$ (that is, $f(S) = T$). A function $f : S \to T$ is called an ***bijection*** if it is one-to-one and onto.

Remark 0.12 (映成函數的反敘述)**.** If $f : S \to T$ is **not** onto, then $\exists\, y \in T$, $\ni \forall\, x \in S$, $f(x) \neq y$. 一般來說，若有一個的數學的敘述 $\forall$statement A, $\exists$statement B $\ni$ statement C 成立，那麼它的相反敘述的寫法為: $\exists$statement A, $\ni \forall$statement B, statement C 不成立。簡單的記法：1. $\forall \leftrightarrow \exists$ 2. $\exists\, P \ni Q \leftrightarrow \ni \forall\, P \sim Q$.

Definition 0.13. For $f : S \to T$, $A \subseteq S$, we call $f(A) = \{f(x) \,|\, x \in A\}$ the ***image*** of A under f. For $B \subseteq T$, we call $f^{-1}(B) = \{x \in S \,|\, f(x) \in B\}$ the ***pre-image*** of B under f.



Example 0.14. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$, $B = [-1, 4] \subseteq T$, $f^{-1}(B) = [-2, 2]$.

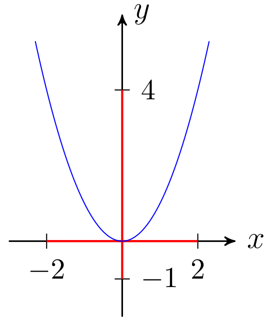


Figure 2: The preimage $f^{-1}([-1, 4])$ is $[-2, 2]$ if $f(x) = x^2$

Proposition 0.15. *Let $f : S \to T$ be a function, C_1 ,$C_2 \subseteq T$ and D_1, $D_2 \subseteq S$.*

(a) $f^{-1}(C_1 \cup C_2) = f^{-1}(C_1) \cup f^{-1}(C_2)$.

(b) $f(D_1 \cup D_2) = f(D_1) \cup f(D_2)$.

(c) $f^{-1}(C_1 \cap C_2) = f^{-1}(C_1) \cap f^{-1}(C_2)$.

(d) $f(D_1 \cap D_2) \subseteq f(D_1) \cap f(D_2)$.

(e) $f^{-1}(f(D_1)) \supseteq D_1$ *("=" if f is one-to-one).*

(f) $f(f^{-1}(C_1)) \subseteq C_1$ *("=" if $C_1 \subseteq f(S)$).*

Proof. We only prove (c) and (d), and the proof of the other statements are left as an exercise.

(c) We first show that $f^{-1}(C_1 \cap C_2) \subseteq f^{-1}(C_1) \cap f^{-1}(C_2)$. Suppose that $x \in f^{-1}(C_1 \cap C_2)$. Then $f(x) \in C_1 \cap C_2$. Therefore, $f(x) \in C_1$ and $f(x) \in C_2$, or equivalently, $x \in f^{-1}(C_1)$ and $x \in f^{-1}(C_2)$; thus $x \in f^{-1}(C_1) \cap f^{-1}(C_2)$.

Next, we show that $f^{-1}(C_1) \cap f^{-1}(C_2) \subseteq f^{-1}(C_1 \cap C_2)$. Suppose that $x \in f^{-1}(C_1) \cap f^{-1}(C_2)$. Then $x \in f^{-1}(C_1)$ and $x \in f^{-1}(C_2)$ which suggests that $f(x) \in C_1$ and $f(x) \in C_2$; thus $f(x) \in C_1 \cap C_2$ or equivalently, $x \in f^{-1}(C_1 \cap C_2)$.

(d) Suppose that $y \in f(D_1 \cap D_2)$. Then $\exists\, x \in D_1 \cap D_2$ such that $y = f(x)$. As a consequence, $y \in f(D_1)$ and $y \in f(D_2)$ which implies that $y \in f(D_1) \cap f(D_2)$. $\square$

Example 0.16. We note it might happen that $f(D_1 \cap D_2) \subsetneq f(D_1) \cap f(D_2)$. Take $D_1 = [-1, 0]$ and $D_2 = [0, 1]$, and define $f : S = \mathbb{R} \to T = \mathbb{R}$ to be $f(x) = x^2$. Then $f(D_1) = f([-1, 0]) = [0, 1]$ and $f(D_2) = f([0, 1]) = [0, 1]$. However,

$$f(D_1 \cap D_2) = f(\{0\}) = \{0\} \subsetneq [0, 1] = f(D_1) \cap f(D_2)\,.$$

Chapter 1

The Real Line and Euclidean Space

1.1 Ordered Fields and the Number Systems

1.1.1 Fields and partial orders

Definition 1.1. A set $\mathcal{F}$ is said to be a ***field*** (體) if there are two operations $+$ and $\cdot$ such that

1. $x + y \in \mathcal{F}$, $x \cdot y \in \mathcal{F}$ if $x, y \in \mathcal{F}$. (封閉性)

2. $x + y = y + x$ for all $x, y \in \mathcal{F}$. (commutativity, 加法的交換性)

3. $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathcal{F}$. (associativity, 加法的結合性)

4. There exists $0 \in \mathcal{F}$, called 加法單位元素, such that $x + 0 = x$ for all $x \in \mathcal{F}$. (the existence of zero)

5. For every $x \in \mathcal{F}$, there exists $y \in \mathcal{F}$ (usually y is denoted by $-x$ and is called x 的加法反元素) such that $x + y = 0$. One writes $x - y \equiv x + (-y)$.

6. $x \cdot y = y \cdot x$ for all $x, y \in \mathcal{F}$. (乘法的交換性)

7. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in \mathcal{F}$. (乘法的結合性)

8. There exists $1 \in \mathcal{F}$, called 乘法單位元素, such that $x \cdot 1 = x$ for all $x \in \mathcal{F}$. (the existence of unity)

9. For every $x \in \mathcal{F}$, $x \neq 0$, there exists $y \in \mathcal{F}$ (usually y is denoted by x^{-1} and is called x 的乘法反元素) such that $x \cdot y = 1$. One writes $x \cdot y \equiv x \cdot x^{-1} = 1$.

10. $x \cdot (y+z) = x \cdot y + x \cdot z$ for all $x, y, z \in \mathcal{F}$. (distributive law, 分配律)

11. $0 \neq 1$.

Remark 1.2. Let x and y be both multiplicative inverse (乘法反元素) of a number a in $(\mathcal{F}, +, \cdot)$. Then

$$x \cdot a = 1 \quad \Rightarrow \quad (x \cdot a) \cdot y = 1 \cdot y = y \quad \Rightarrow \quad x \cdot 1 = x \cdot (a \cdot y) = y\,;$$

thus $x = y$. In other words, the multiplicative inverse of a number is unique.

Remark 1.3. A set $\mathcal{F}$ satisfying properties 1 to 10 with $0 = 1$ consists of only one member: By distributive law, $x \cdot 0 = x \cdot (0+0) = x \cdot 0 + x \cdot 0$; thus $-(x \cdot 0) + (x \cdot 0) = -(x \cdot 0) + (x \cdot 0) + (x \cdot 0)$ which implies that $x \cdot 0 = 0$. Therefore, if $0 = 1$, then $x = x \cdot 1 = x \cdot 0 = 0$ for all $x \in \mathcal{F}$. Hence, the set $\mathcal{F}$ consists only one element 0.

Remark 1.4. If $x \in \mathcal{F}$, then $\big((1 + (-1)\big) \cdot x = 0$ which implies that $x + (-1) \cdot x = 0$. Therefore, $(-1) \cdot x = -x + x + (-1) \cdot x = -x + 0 = -x$.

Example 1.5. Let $\mathbb{Q} = \Big\{ \dfrac{q}{p} \,\Big|\, p \neq 0, p, q \in \mathbb{Z} : \text{integers} \Big\}$. Then $\mathbb{Q}$ is a field. (Check all the properties from 1 to 11).

Example 1.6. Let $\mathbb{N} = \big\{ n \in \mathbb{Z} \,\big|\, n > 0 \big\}$. Then $\mathbb{N}$ is not a field because there is no zero.

Example 1.7. Let $\mathcal{F} = \{a, b, c\}$ with the operations $+$ and $\cdot$ defined by

$$\begin{array}{c|ccc} + & a & b & c \\ \hline a & a & b & c \\ b & b & c & a \\ c & c & a & b \end{array} \qquad \begin{array}{c|ccc} \cdot & a & b & c \\ \hline a & a & a & a \\ b & a & b & c \\ c & a & c & b \end{array}\,.$$

Then $\mathcal{F}$ is a field because of the following: Properties 1, 2, 3, 6, 7 are obvious.
Property 4: $\exists$ "0" $\ni x +$ "0" $= x$ for all $x \in \mathcal{F}$. In fact, "0" $= a$.
Property 5: $\forall\, x \in \mathcal{F}$, $\exists\, y \in \mathcal{F} \ni x + y = 0$, here $b = -c$, $c = -b$.
Property 8: $\exists$ "1" $\ni x \cdot$ "1" $= x$ for all $x \in \mathcal{F}$. In fact, "1" $= b$ (so Property 11 holds since $a \neq b$).
Property 9: $\forall\, x \neq 0, \in \mathcal{F}$, $\exists\, z \in \mathcal{F} \ni x \cdot z = 1$, here $z = x$.
The validity of Property 10 is left as an exercise.

Example 1.8. Let $(\mathcal{F}, +, \cdot)$ be a field. Then $(x-y)(x+y) = x^2 - y^2$ for all $x, y \in \mathcal{F}$. In fact,

$$\begin{aligned}
(x-y)(x+y) &= (x-y)\cdot x + (x-y)\cdot y && \text{(by 分配律)}\\
&= x\cdot(x-y) + y\cdot(x-y) && \text{(by 乘法交換律)}\\
&= x\cdot x + x\cdot(-y) + y\cdot x + y\cdot(-y) && \text{(by 分配律)}\\
&= x^2 - x\cdot y + x\cdot y - y^2 && \text{(by Remark 1.4 and 乘法交換律)}\\
&= x^2 + 0 - y^2 && \text{(by Property 5)}\\
&= x^2 - y^2 && \text{(by Property 4).}
\end{aligned}$$

Definition 1.9. A ***partial order*** over a set P is a binary relation $\preccurlyeq$ which is reflexive, anti-symmetric and transitive (滿足遞移律), in the sense that

1. $x \preccurlyeq x$ for all $x \in P$ (reflexivity).
2. $x \preccurlyeq y$ and $y \preccurlyeq x \Rightarrow x = y$ (anti-symmetry).
3. $x \preccurlyeq y$ and $y \preccurlyeq z \Rightarrow x \preccurlyeq z$ (transitivity).

A set with a partial order is called a ***partially ordered set***.

Example 1.10. Let S be a given set, and 2^S be the ***power set*** of S; that is,

$$P = 2^S = \{A \,|\, A \subseteq S\} = \text{the collection of all subsets of } S\,.$$

We define $\preccurlyeq$ as $\supseteq$. Then

1. $A \supseteq A$ (reflexivity).
2. $A \supseteq B$ and $B \supseteq A \Rightarrow A = B$ (anti-symmetry).
3. $A \supseteq B$ and $B \supseteq C \Rightarrow A \supseteq C$ (transitivity).

Hence, $\supseteq$ is a partial order over 2^S (or equivalently, $(2^S, \supseteq)$ is a partially ordered set). Similarly, $\subseteq$ on 2^S is also a partial order.

Definition 1.11. Let $(P, \preccurlyeq)$ be a partially ordered set. Two elements $x, y \in P$ are said to be ***comparable*** if either $x \preccurlyeq y$ or $y \preccurlyeq x$.

Definition 1.12. A partial order under which every pair of elements is comparable is called a ***total order*** or ***linear order***.

Definition 1.13. An ***ordered field*** is a totally ordered field $(\mathcal{F},+,\cdot,\preccurlyeq)$ satisfying that

1. If $x \preccurlyeq y$, then $x+z \preccurlyeq y+z$ for all $z \in \mathcal{F}$ (compatibility of $\preccurlyeq$ and $+$).

2. If $0 \preccurlyeq x$ and $0 \preccurlyeq y$, then $0 \preccurlyeq x \cdot y$ (compatibility of $\preccurlyeq$ and $\cdot$).

Example 1.14. $(\mathbb{Q},+,\cdot,\geqslant)$ is a totally ordered field, but is **not** an ordered field (since Property 2 in Definition 1.13 is violated). On the other hand, $(\mathbb{Q},+,\cdot,\leqslant)$ is an ordered field.

From now on, the total order $\preccurlyeq$ of an ordered field will be denoted by $\leqslant$.

Definition 1.15. In an ordered field $(\mathcal{F},+,\cdot,\leqslant)$, the binary relations $<$, $\geqslant$ and $>$ are defined by:

1. $x < y$ if $x \leqslant y$ and $x \neq y$.

2. $x \geqslant y$ if $y \leqslant x$.

3. $x > y$ if $y < x$.

Adopting the definition above, it is not immediately clear that $x \nleqslant y \Leftrightarrow x > y$. However, this is indeed the case, and to be more precise we have the following

Proposition 1.16. (Law of Trichotomy, 三一律) *If x and y are elements of an ordered field $(\mathcal{F},+,\cdot,\leqslant)$, then exactly one of the relations $x<y$, $x=y$ or $y<x$ holds.*

Proof. Since $\mathcal{F}$ is a totally ordered field, x and y are comparable. Therefore, either $x \leqslant y$ or $y \leqslant x$. Assume that $x \leqslant y$.

1. If $x = y$, then $x \nless y$ and $x \ngtr y$.

2. If $x \neq y$, then $x < y$. If it also holds that $x > y$, then $x \geqslant y$; thus by the property of anit-symmetry of an order, we must have $x = y$, a contradiction. Therefore, it can only be that $x < y$.

The proof for the case $y \leqslant x$ is similar, and is left as an exercise. □

Proposition 1.17. *Let $(\mathcal{F},+,\cdot,\leqslant)$ be an ordered field, and $a,b,x,y,z \in \mathcal{F}$.*

1. *If $a + x = a$, then $x = 0$.*
   If $a \cdot x = a$ and $a \neq 0$, then $x = 1$.

2. *If* $a + x = 0$*, then* $x = -a$*.*
   If $a \cdot x = 1$ *and* $a \neq 0$*, then* $x = a^{-1}$*.*

3. *If* $x \cdot y = 0$*, then* $x = 0$ *or* $y = 0$*.*

4. *If* $x \leqslant y < z$ *or* $x < y \leqslant z$*, then* $x < z$ *(the transitivity of* $<$*).*

5. *If* $a < b$*, then* $a + x < b + x$ *(the compatibility of* $<$ *and* $+$*).*
   If $0 < a$ *and* $0 < b$*, then* $0 < a \cdot b$ *(the compatibility of* $<$ *and* $\cdot$*).*

6. *If* $a + x = b + x$*, then* $a = b$*.*
   If $a + x \leqslant (<)\, b + x$*, then* $a \leqslant (<)\, b$*.*
   If $a \cdot x = b \cdot x$ *and* $x \neq 0$*, then* $a = b$*.*
   If $a \cdot x \leqslant (<)\, b \cdot x$ *and* $x > 0$*, then* $a \leqslant (<)\, b$*.*

7. $0 \cdot x = 0$.

8. $-(-x) = x$.

9. $-x = (-1) \cdot x$.

10. *If* $x \neq 0$*, then* $x^{-1} \neq 0$ *and* $(x^{-1})^{-1} = x$*.*

11. *If* $x \neq 0$ *and* $y \neq 0$*, then* $x \cdot y \neq 0$ *and* $(x \cdot y)^{-1} = x^{-1} \cdot y^{-1}$*.*

12. *If* $x \leqslant (<)\, y$ *and* $0 \leqslant (<)\, z$*, then* $x \cdot z \leqslant (<)\, y \cdot z$*.*
    If $x \leqslant (<)\, y$ *and* $0 \geqslant (>)\, z$*, then* $x \cdot z \geqslant (>)\, y \cdot z$*.*

13. *If* $x \leqslant (<)\, 0$ *and* $y \leqslant (<)\, 0$*, then* $x \cdot y \geqslant (>)\, 0$*.*
    If $x \leqslant (<)\, 0$ *and* $y \geqslant (>)\, 0$*, then* $x \cdot y \leqslant (<)\, 0$*.*

14. $0 < 1$ *and* $-1 < 0$*.*

15. $x \cdot x \equiv x^2 \geqslant 0$*.*

16. *If* $x > 0$*, then* $x^{-1} > 0$*. If* $x < 0$*, then* $x^{-1} < 0$*.*

Proof. 1. $(-a) + a + x = (-a) + a = 0 \Rightarrow x = 0$.
$(a^{-1}) \cdot a \cdot x = (a^{-1}) \cdot a = 1 \Rightarrow x = 1$.

2. $(-a) + a + x = (-a) + 0 = -a \Rightarrow x = -a$.
   $(a^{-1}) \cdot a \cdot x = (a^{-1}) \cdot 1 = a^{-1} \Rightarrow x = a^{-1}$.

3. Assume that $x \neq 0$, then $x^{-1} \cdot x \cdot y = x^{-1} \cdot 0 = 0 \Rightarrow y = 0$.
   Assume that $y \neq 0$, then $x \cdot y \cdot y^{-1} = 0 \cdot y^{-1} = 0 \Rightarrow x = 0$.

4 and 5 are Left as an exercise.

6. $a + 0 = a + x + (-x) = b + x + (-x) = b + 0 \Rightarrow a = b$.
   $a + 0 = a + x + (-x) \leqslant b + x + (-x) = b + 0 \Rightarrow a \leqslant b$ (compatibility of $\leqslant$ and $+$).
   $a \cdot x \cdot x^{-1} = b \cdot x \cdot x^{-1} \Rightarrow a = b$.
   Suppose the contrary that $b < a$. Then $0 = b + (-b) \leqslant a + (-b)$. Since $x > 0$, $x \geqslant 0$; thus
   $$0 \leqslant \big(a + (-b)\big) \cdot x = a \cdot x + (-b) \cdot x\,.$$
   As a consequence, $b \cdot x = 0 + b \cdot x \leqslant a \cdot x + (-b) \cdot x + b \cdot x = a \cdot x$. By assumption, we must have $a \cdot x = b \cdot x$ or $(a - b) \cdot x = 0$. Using 3, $x = 0$ (since $a \neq b$), a contradiction.

7. See Remark 1.3.

8. $(-x) + (-(-x)) = 0 = (-x) + x \Rightarrow x = -(-x)$.

9. See Remark 1.4.

10. Assume $x^{-1} = 0$, $1 = x \cdot x^{-1} = x \cdot 0 = 0$, a contradiction. Therefore, $x^{-1} \neq 0$; thus $(x^{-1})^{-1} \cdot x^{-1} = 1 = x \cdot x^{-1} \Rightarrow (x^{-1})^{-1} = x$ (by 4).

11. That $x \cdot y = 0$ cannot be true since it is against Property 3, so $x \cdot y \neq 0$. Moreover,
    $$(x \cdot y)^{-1}(x \cdot y) = 1 = 1 \cdot 1 = (x \cdot x^{-1}) \cdot (y \cdot y^{-1}) = (x^{-1} \cdot y^{-1}) \cdot (x \cdot y)\,;$$
    thus $(x \cdot y)^{-1} = x^{-1} \cdot y^{-1}$ (by 4).

12. If $x \leqslant (<)\, y$, then $0 = x + (-x) \leqslant (<)\, y + (-x)$. Since $0 \leqslant (<)\, z$, by the compatibility of $\leqslant (<)$ and $\cdot$ we must have $0 \leqslant (<)\, (y + (-x)) \cdot z = y \cdot z + (-x) \cdot z$. Therefore, by the compatibility of $\leqslant (<)$ and $+$, $x \cdot z = 0 + x \cdot z \leqslant (<)\, y \cdot z + (-x) \cdot z + x \cdot z = y \cdot z$. The second statement can be proved in a similar fashion.

13. Left as an exercise.

14. If $1 \leqslant 0$, then compatibility of $\leqslant$ and $+$ implies that $0 \leqslant -1$. By the compatibility of $\leqslant$ and $\cdot$, using 6 and 7 we find that $0 \leqslant (-1) \cdot (-1) = -(-1) = 1$; thus we conclude that $1 = 0$, a contradiction. As a consequence, $0 < 1$; thus the compatibility of $<$ and $+$ implies that $-1 < 0$.

15. Left as an exercise.

16. If $x > 0$ but $x^{-1} \leqslant 0$, then $1 = x \cdot x^{-1} \leqslant x \cdot 0 = 0$, a contradiction. □

Proposition 1.18. *Let $(\mathcal{F}, +, \cdot, \leqslant)$ be an ordered field, and $x, y \in \mathcal{F}$.*

1. *If $0 \leqslant x < y$, then $x^2 < y^2$.*

2. *If $0 \leqslant x, y$ and $x^2 < y^2$, then $x < y$.*

Proof. 1. By definition of "$<$", $0 \leqslant x \leqslant y$ and $x \neq y$. Using 12 of Proposition 1.17,

$$x^2 \leqslant y \cdot x < y \cdot y = y^2 .$$

By the transitivity of $<$, we conclude that $x^2 < y^2$.

2. Note that $x \neq y$, for if not, then $x^2 - y^2 = 0$ which contradicts to the assumption $x^2 < y^2$. Assume that $y < x$, then 1 implies that $y^2 < x^2$, a contradiction. □

Remark 1.19. Proposition 1.18 can be summarized as follows: if $x, y \geqslant 0$, then

$$x < y \Leftrightarrow x^2 < y^2 .$$

Moreover, Example 1.8, Proposition 1.17 and Proposition 1.18 together suggest that if $x, y \geqslant 0$, then $x \leqslant y$ if and only if $x^2 \leqslant y^2$.

Definition 1.20. The ***magnitude*** or the ***absolute value*** of x, denoted $|x|$, is defined as

$$|x| = \begin{cases} x & \text{if } x \geqslant 0\,, \\ -x & \text{if } x < 0\,. \end{cases}$$

Proposition 1.21. *Let $(\mathcal{F}, +, \cdot, \leqslant)$ be an ordered field. Then*

1. *$|x| \geqslant 0$ for all $x \in \mathcal{F}$.*

2. *$|x| = 0$ if and only if $x = 0$.*

3. $-|x| \leqslant x \leqslant |x|$ *for all* $x \in \mathcal{F}$.

4. $|x \cdot y| = |x| \cdot |y|$ *for all* $x, y \in \mathcal{F}$.

5. $|x + y| \leqslant |x| + |y|$ *for all* $x, y \in \mathcal{F}$ (***triangle inequality***, 三角不等式).

6. $\big||x| - |y|\big| \leqslant |x - y|$ *for all* $x, y \in \mathcal{F}$.

Proof. Left as an exercise. □

Proposition 1.22. *Define* $d(x, y) = |x - y|$. *Then*

1. $d(x, y) \geqslant 0$ *for all* $x, y \in \mathcal{F}$.

2. $d(x, y) = 0$ *if and only if* $x = y$.

3. $d(x, y) = d(y, x)$ *for all* $x, y \in \mathcal{F}$.

4. $d(x, y) \leqslant d(x, z) + d(z, y)$ *for all* $x, y, z \in \mathcal{F}$ (***triangle inequality***, 三角不等式).

Proof. Left as an exercise. □

Remark 1.23. $d(x, y)$ is the "distance" of two elements $x, y \in \mathcal{F}$.

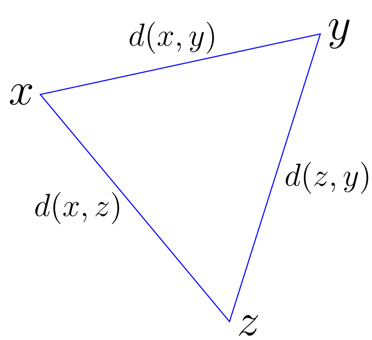


Figure 1.1: An illustration of why 4 of Proposition 1.22 is called the triangle inequality.

1.1.2 The nature numbers, the integers, and the rational numbers

Definition 1.24. Let $(\mathcal{F}, +, \cdot, \leqslant)$ be an ordered field. The ***natural number system***, denoted by $\mathbb{N}$, is the collection of all the numbers 1, 1+1, 1+1+1, $1+1+\cdots+1$ and etc. in $\mathcal{F}$. We write $2 \equiv 1+1$, $3 \equiv 2+1$, and $n \equiv \underbrace{1 + 1 + \cdots + 1}_{\text{(n times)}}$. In other words, $\mathbb{N} = \{1, 2, 3, \cdots\}$. The ***integer number system***, denoted by $\mathbb{Z}$, is the set $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$.

Principle of mathematical induction (Peano axiom, 皮亞諾公設):
If S is a subset of $\mathbb{N} \cup \{0\}$ (or $\mathbb{N}$) such that $0 \in S$ (or $1 \in S$) and $k+1 \in S$ if $k \in S$, then $S = \mathbb{N} \cup \{0\}$ (or $S = \mathbb{N}$).

Example 1.25. Prove $\sum_{k=1}^{n} k = \dfrac{n(n+1)}{2}$. $(\star)$

Proof. Let $S = \left\{ n \in \mathbb{N} \,\middle|\, \sum_{k=1}^{n} k = \dfrac{n(n+1)}{2} \right\}$（把所有滿足 $(\star)$ 的 n 收集起來）. Then

1. If $n = 1$, $\sum_{k=1}^{1} k = \dfrac{1 \times 2}{2} = 1$.
2. Assume that $m \in S$, then
$$\sum_{k=1}^{m+1} k = \sum_{k=1}^{m} k + (m+1) = \frac{m(m+1)}{2} + (m+1) = \frac{(m+1)(m+2)}{2}$$
which implies that $m+1 \in S$.

By mathematical induction, we have $S = \mathbb{N}$. □

Example 1.26. Prove that $\dfrac{1}{2^n} < \dfrac{1}{n}$ for all $n \in \mathbb{N}$.

Proof. Let $S = \left\{ n \in \mathbb{N} \,\middle|\, \dfrac{1}{2^n} < \dfrac{1}{n} \right\}$. We show $S = \mathbb{N}$ by mathematical induction as follows:

(i) $1 \in S \Leftrightarrow \dfrac{1}{2} < \dfrac{1}{1}$.

(ii) If $n \in S$, then
$$\frac{1}{2^{n+1}} = \frac{1}{2^n} \cdot \frac{1}{2} < \frac{1}{n} \cdot \frac{1}{2} = \frac{1}{n+n} \leqslant \frac{1}{n+1}.$$
which implies that $n+1 \in S$.

By mathematical induction, we have $S = \mathbb{N}$. □

Let $(\mathcal{F}, +, \cdot, \leqslant)$ be an ordered field. By the property of being a field, for any non-zero $n \in \mathbb{N}$, there exists a unique multiplicative inverse n^{-1}. This inverse is usually denoted by $\dfrac{1}{n}$. We also use $\dfrac{m}{n}$ to denote $m \cdot n^{-1}$. Giving this notation, we have the following

Definition 1.27. Let $(\mathcal{F}, +, \cdot, \leqslant)$ be an order field. The ***rational number system***, denoted by $\mathbb{Q}$, is the collection of all numbers of the form $\dfrac{q}{p}$ with $p, q \in \mathbb{Z}$ and $p \neq 0$; that is,
$$\mathbb{Q} = \left\{ x \in \mathcal{F} \,\middle|\, x = \frac{q}{p}, p, q \in \mathbb{Z}, p \neq 0 \right\}.$$

Definition 1.28. An order field $(\mathcal{F}, +, \cdot, \leqslant)$ is said to have the ***Archimedean property*** if $\forall\, x \in \mathcal{F}$, $\exists\, n \in \mathbb{Z} \ni x < n$.

Theorem 1.29. *$\mathbb{Q}$ has the Archimedean property.*

Proof. If $x \leqslant 0$, we take $n = 1$. Otherwise if $0 < x = \dfrac{q}{p}$ with $p, q \in \mathbb{N}$, we take $n = q + 1$ and it is obvious that $\dfrac{q}{p} \leqslant q < q + 1 = n$. □

Definition 1.30. A ***well-ordered*** relation on a set S is a total order on S with the property that every non-empty subset of S has a least (smallest) element in this ordering.

Proposition 1.31. *If $S \subseteq \mathbb{N}$ and $S \neq \varnothing$, then S has a smallest element; that is, $\exists\, s_0 \in S \ni \forall\, x \in S$, $s_0 \leqslant x$.*

Proof. Assume the contrary that there exists a non-empty set $S \subseteq \mathbb{N}$ such that S does not have the smallest element. Define $T = \mathbb{N} \backslash S$, and $T_0 = \big\{ n \in \mathbb{N} \,\big|\, \{1, 2, \cdots, n\} \subseteq T \big\}$. Then we have $T_0 \subseteq T$. Also note that $1 \notin S$ for otherwise 1 is the smallest element in S, so $1 \in T$ (thus $1 \in T_0$).

Assume $k \in T_0$. Since $\{1, 2, \cdots, k\} \subseteq T$, $1, 2, \cdots k \notin S$. If $k + 1 \in S$, then $k + 1$ is the smallest element in S. Since we assume that S does not have the smallest element, $k+1 \notin S$; thus $k + 1 \in T \Rightarrow k + 1 \in T_0$.

Therefore, by mathematical induction we conclude that $T_0 = \mathbb{N}$; thus $T = \mathbb{N}$ (since $T_0 \subseteq T$) which further implies that $S = \varnothing$ (since $T = \mathbb{N} \backslash S$). This contradicts to the assumption $S \neq \varnothing$. □

1.1.3 Countability

Definition 1.32. A set S is called ***denumerable*** or ***countably infinite***（無窮可數的）if S can be put into one-to-one correspondence with $\mathbb{N}$; that is, S is denumerable if and only if $\exists\, f : \mathbb{N} \to S$ which is one-to-one and onto. A set is called ***countable***（可數的）if S is either finite or denumerable.

Remark 1.33. If $f : \mathbb{N} \xrightarrow[\textit{onto}]{1-1} S$, then $f^{-1} : S \xrightarrow[\textit{onto}]{1-1} \mathbb{N}$. Therefore,

$$S \text{ is denumerable} \Leftrightarrow \exists\, f : \mathbb{N} \xrightarrow[\textit{onto}]{1-1} S \Leftrightarrow \exists\, g = f^{-1} : S \xrightarrow[\textit{onto}]{1-1} \mathbb{N}.$$

f can be thought as a rule of counting/labeling elements in S since $S = \big\{ f(1), f(2), \cdots \big\}$.

Example 1.34. $\mathbb{N}$ is countable since $f : \mathbb{N} \xrightarrow[onto]{1-1} \mathbb{N}$ with $f(x) = x$, $\forall\, n \in \mathbb{N}$.

Example 1.35. $\mathbb{Z}$ is countable. $f : \mathbb{Z} \to \mathbb{N}$ with $f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 2x & \text{if } x > 0 \\ -2x+1 & \text{if } x < 0 \end{cases}$.

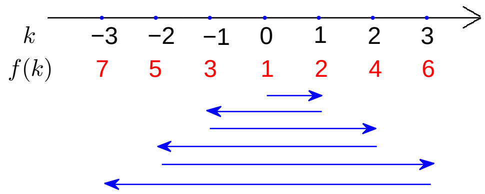


Figure 1.2: An illustration of how elements in $\mathbb{Z}$ are labeled

Example 1.36. The set $\mathbb{N} \times \mathbb{N} = \{(a, b) \,|\, a, b \in \mathbb{N}\}$ is countable. In fact, two ways of mapping are shown in the figures below.

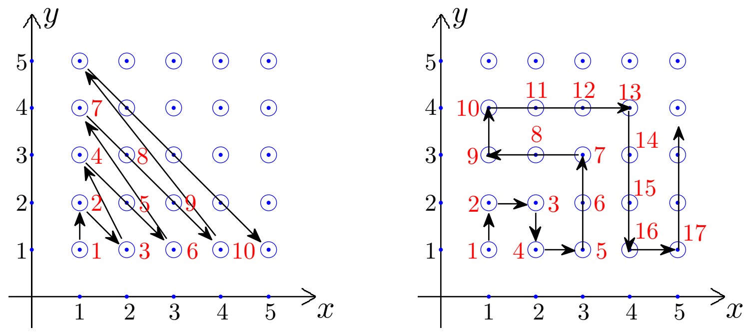


Figure 1.3: The illustration of two ways of labeling elements in $\mathbb{N} \times \mathbb{N}$

Proposition 1.37. *A non-empty set S is countable if and only if there exists a surjection $f : \mathbb{N} \to S$.*

Proof. "$\Rightarrow$" First suppose that $S = \{x_1, \cdots, x_n\}$ is finite. Define $f : \mathbb{N} \to S$ by

$$f(k) = \begin{cases} x_k & \text{if } k < n\,, \\ x_n & \text{if } k \geqslant n\,. \end{cases}$$

Then $f : \mathbb{N} \to S$ is a surjection. Now suppose that S is denumerable. Then by definition of countability, there exists $f : \mathbb{N} \xrightarrow[onto]{1-1} S$.

"$\Leftarrow$" W.L.O.G. (without loss of generality, 不失一般性) we assume that S is an infinite set. Let $k_1 = 1$. Since $\#(S) = \infty$, $S_1 \equiv S \backslash \{f(k_1)\} \neq \varnothing$; thus $N_1 \equiv f^{-1}(S_1)$ is a non-empty subset of $\mathbb{N}$. By the well-ordered property of $\mathbb{N}$ (Proposition 1.31), N_1 has a smallest element denoted by k_2. Since $\#(S) = \infty$, $S_2 = S\backslash\{f(k_1), f(k_2)\} \neq \varnothing$; thus $N_2 \equiv f^{-1}(S_2)$ is a non-empty subset of $\mathbb{N}$ and possesses a smallest element denoted by k_3. We continue this process and obtain a set $\{k_1, k_2, \cdots\} \subseteq \mathbb{N}$, where $k_1 < k_2 < \cdots$, and k_j is the smallest element of $N_{j-1} \equiv f^{-1}(\mathbb{N}\backslash\{f(k_1), f(k_2), \cdots, f(k_{j-1})\})$.

Claim: $f : \{k_1, k_2, \cdots\} \to S$ is one-to-one and onto.

Proof of claim: The injectivity of f is easy to see since by construction $f(k_j) \notin \big\{f(k_1), f(k_2), \cdots, f(k_{j-1})\big\}$ for all $j \geqslant 2$. For surjectivity, assume that there is $s \in S$ such that $s \notin f(\{k_1, k_2, \cdots\})$. Since $f : \mathbb{N} \to \mathbb{S}$ is onto, $f^{-1}(\{s\})$ is a non-empty subset of $\mathbb{N}$; thus possesses a smallest element k. Since $s \notin f(\{k_1, k_2, \cdots\})$, there exists $\ell \in \mathbb{N}$ such that $k_\ell < k < k_{\ell+1}$. As a consequence, there exists $k \in N_\ell$ such that $k < k_{\ell+1}$ which contradicts to the fact that $k_{\ell+1}$ is the smallest element of N_ℓ.

Define $g : \mathbb{N} \to \{k_1, k_2, \cdots\}$ by $g(j) = k_j$. Then $g : \mathbb{N} \to \{k_1, k_2, \cdots\}$ is one-to-one and onto; thus $h = f \circ g : \mathbb{N} \xrightarrow[onto]{1-1} S$. □

Lemma 1.38. *Let S be a set, and $A \subseteq S$ is a non-empty subset of S. Then there exists a surjection $f : S \to A$.*

Proof. Since A is a non-empty subset of S, $\exists\, a \in A$. Define

$$f(x) = \begin{cases} x & \text{if } x \in A\,, \\ a & \text{if } x \notin A\,. \end{cases}$$

Then $f : S \to A$ is clearly a surjection. □

Theorem 1.39. *Any non-empty subset of a countable set is countable.*

Proof. Let S be a countable set, and A be a non-empty subset of S. Since S is countable, by definition there exists a $f : \mathbb{N} \xrightarrow[onto]{1-1} S$. On the other hand, by Lemma 1.38 there exists a surjection $g : S \to A$. Then $h = g \circ f : \mathbb{N} \to A$ is a surjection, and Proposition 1.37 suggests that A is countable. □

Corollary 1.40. *A non-empty set S is countable if and only if there exists an injection $f : S \to \mathbb{N}$.*

Proof. "$\Rightarrow$" If $S = \{x_1, \cdots, x_n\}$ is finite, we simply let $f : S \to \mathbb{N}$ be $f(x_n) = n$. Then f is clearly an injection. If S is denumerable, by definition there exists $g : \mathbb{N} \xrightarrow[onto]{1-1} S$ which suggests that $f = g^{-1} : S \to \mathbb{N}$ is an injection.

"$\Leftarrow$" Suppose that $f : S \to \mathbb{N}$ is an injection. W.L.O.G., we can assume that $\#\big(f(S)\big) = \infty$ for otherwise $\#(S) < \infty$ which trivially implies that S is countable. Since $f(S)$ is a subset of $\mathbb{N}$, by Theorem 1.39 $f(S)$ is countably infinite. By definition, there exists $g : f(S) \xrightarrow[onto]{1-1} \mathbb{N}$; thus $h = g \circ f : S \xrightarrow[onto]{1-1} \mathbb{N}$. □

Example 1.41. The set $\mathbb{N} \times \mathbb{N}$ is countable since the map $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $f((m,n)) = 2^m 3^n$ is an injection.

Theorem 1.42. *The union of countable countable sets is countable.* (可數個可數集的聯集是可數的)

Proof. Let A_i be countable, and define $A = \bigcup\limits_{i=1}^{\infty} A_i$. Write $A_i = \{x_{i1}, x_{i2}, x_{i3}, \cdots\}$. Then $A = \big\{x_{ij} \,\big|\, i = 1, 2, \cdots,\ j < \#(A_i) + 1\big\}$, where $\#(A_i) = \infty$ if A_i is countably infinite. Let $S = \big\{(i,j) \,\big|\, i = 1, 2, \cdots, j < \#(A_i) + 1\big\}$, and define $f : S \to A$ by $f((i,j)) = x_{ij}$. Then $f : S \to A$ is a surjection. On the other hand, since S is a subset of $\mathbb{N} \times \mathbb{N}$, Theorem 1.39 implies that S is countable; thus Proposition 1.37 guarantees the existence of a surjection $g : \mathbb{N} \to S$. Then $h = f \circ g : \mathbb{N} \to A$ is a surjection which, by Proposition 1.37 again, implies that A is countable. □

Example 1.43. $\mathbb{Z} \times \mathbb{Z}$ is countable.

Proof. For $i \in \mathbb{Z}$, let $A_i = \big\{(i,j) \,\big|\, j \in \mathbb{Z}\big\}$. By Example 1.35, A_i is countable for all $i \in \mathbb{Z}$. Since $\mathbb{Z} \times \mathbb{Z} = \bigcup_{i \in \mathbb{Z}} A_i$ which is countable union of countable sets, Theorem 1.42 implies that $\mathbb{Z} \times \mathbb{Z}$ is countable. □

Theorem 1.44. $\mathbb{Q}$ *is countable.*

Proof. Define

$$f(x) = \begin{cases} (p,q), & \text{if } x > 0, \quad x = \dfrac{q}{p}, \quad \gcd(p,q) = 1,\ p > 0. \\ (0,0), & \text{if } x = 0. \\ (p,-q), & \text{if } x < 0, \quad x = -\dfrac{q}{p}, \quad \gcd(p,q) = 1,\ p > 0. \end{cases}$$

Then $f : \mathbb{Q} \to \mathbb{Z} \times \mathbb{Z}$ is one-to-one; thus $f : \mathbb{Q} \xrightarrow[onto]{1-1} f(\mathbb{Q})$. Since $\mathbb{Z} \times \mathbb{Z}$ is countable, its non-empty subset $f(\mathbb{Q})$ is also countable. As a consequence, there exists $g : f(\mathbb{Q}) \xrightarrow[onto]{1-1} \mathbb{N}$; thus $h = g \circ f : \mathbb{Q} \xrightarrow[onto]{1-1} \mathbb{N}$. □

1.2 Completeness and the Real Number System

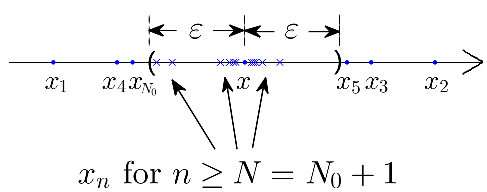
1.2.1 Sequences

Definition 1.45. A ***sequence*** in a set S is a function $f : \mathbb{N} \to S$ (not necessary one-to-one or onto). The values of f are called the ***terms*** of the sequence.

Remark 1.46. A sequence in S is a countable list of elements in S arranged in a particular order, and is usually denoted by $\big\{f(n)\big\}_{n=1}^{\infty}$ or $\{x_n\}_{n=1}^{\infty}$ with $x_n = f(n)$.

Definition 1.47. A sequence $\{x_n\}_{n=1}^{\infty}$ is said to ***converge*** to a limit x if $\forall\, \varepsilon > 0$, $\exists\, N > 0$, $\ni |x_n - x| < \varepsilon$ whenever $n \geqslant N$ (落在 $(x-\varepsilon, x+\varepsilon)$ 之外的 x_n 最多只有 $N-1$ 個). One writes $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$ to denote that the sequence $\{x_n\}_{n=1}^{\infty}$ converges to x.

Intuition: If $\{x_n\}_{n=1}^{\infty}$ converges to x, we expect that $\forall\, \varepsilon > 0$, $\#\{n \in \mathbb{N} \,|\, x_n \notin (x-\varepsilon, x+\varepsilon)\} < \infty$. Let $N_0 = \max\big\{n \in \mathbb{N} \,\big|\, x_n \notin (x-\varepsilon, x+\varepsilon)\big\}$ (把落在 $(x-\varepsilon, x+\varepsilon)$ 外面所有的 x_n 中最大的指標定為 N_0), and let $N = N_0 + 1$. Then $x_n \in (x-\varepsilon, x+\varepsilon)$ whenever $n \geqslant N$.



This intuition sometimes is very useful for proving the convergence of a sequence. For example, let $\pi : \mathbb{N} \to \mathbb{N}$ be a rearrangement of $\mathbb{N}$ (that is, π is bijective), and $\{x_n\}_{n=1}^{\infty}$ be a convergent sequence. Then $\big\{x_{\pi(n)}\big\}_{n=1}^{\infty}$ is also convergent since if x is the limit of $\{x_n\}_{n=1}^{\infty}$ and $\varepsilon > 0$,

$$\big\{n \in \mathbb{N} \,\big|\, x_{\pi(n)} \notin (x-\varepsilon, x+\varepsilon)\big\} = \big\{n \in \mathbb{N} \,\big|\, x_n \notin (x-\varepsilon, x+\varepsilon)\big\} < \infty\,.$$

One should try to prove the convergence of $\big\{x_{\pi(n)}\big\}_{n=1}^{\infty}$ using the ε-N argument, and it will be immediately seen that the approach above is much easier.

Remark 1.48. The number N may depend on ε, and smaller ε usually requires larger N.

Remark 1.49. If a sequence $\{x_n\}_{n=1}^{\infty}$ does not converge to x, then $\exists\, \{x_{n_1}, x_{n_2}, \cdots\} \subseteq \{x_n\}_{n=1}^{\infty}$, $n_i \neq n_j$ if $i \neq j$, such that $x_{n_j} \notin (x-\varepsilon, x+\varepsilon)$ for all $j \in \mathbb{N}$. In other words, $x_n \not\twoheadrightarrow x$ as $n \to \infty$ if and only if $\exists\, \varepsilon > 0$, $\ni \forall\, N > 0$, $\exists\, n > N$ such that $|x_n - x| \geqslant \varepsilon$.

Lemma 1.50 (Sandwich). *If* $\lim\limits_{n\to\infty} x_n = L$, $\lim\limits_{n\to\infty} y_n = L$, $\{z_n\}_{n=1}^{\infty}$ *is a sequence such that* $x_n \leqslant z_n \leqslant y_n$, *then* $\lim\limits_{n\to\infty} z_n = L$.

Proof. Let $\varepsilon > 0$ be given. Since $\lim\limits_{n\to\infty} x_n = L$ and $\lim\limits_{n\to\infty} y_n = L$, by definition

$$\exists\, N_1 > 0 \ni L - \varepsilon < x_n < L + \varepsilon,\ \forall\, n \geqslant N_1$$

and

$$\exists\, N_2 > 0 \ni L - \varepsilon < y_n < L + \varepsilon,\ \forall\, n \geqslant N_2\,.$$

Let $N = \max\{N_1, N_2\}$. Then for $n \geqslant N$, $L - \varepsilon < x_n \leqslant z_n \leqslant y_n < L + \varepsilon$; thus $\lim\limits_{n\to\infty} z_n = L$. □

Proposition 1.51. *If* $a \leqslant x_n \leqslant b$ *and* $\lim\limits_{n\to\infty} x_n = x$, *then* $a \leqslant x \leqslant b$.

Proof. Assume the contrary that $x \notin [a, b]$. If $x < a$, let $\varepsilon = a - x > 0$. Since $\lim\limits_{n\to\infty} x_n = x$, $\exists\, N > 0 \ni x_n \in (x - \varepsilon, x + \varepsilon)$ whenever $n \geqslant N$. Therefore, $x_n < a$ for all $n \geqslant N$, a contradiction. So $a \leqslant x$.

We can prove $x \leqslant b$ in a similar way, and the proof is left as an exercise. □

Corollary 1.52. *If* $a < x_n < b$ *and* $\lim\limits_{n\to\infty} x_n = x$, *then* $a \leqslant x \leqslant b$.

Proposition 1.53. *If* $\{x_n\}_{n=1}^{\infty}$ *is a sequence in an ordered field, and* $x_n \to x$ *and* $x_n \to y$ *as* $n \to \infty$, *then* $x = y$. *(The uniqueness of the limit).*

Proof. Assume the contrary that $x \neq y$. W.L.O.G. we may assume that $x < y$, and let $\varepsilon = \dfrac{y - x}{2} > 0$ $(x + \varepsilon = y - \varepsilon)$. Since $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} x_n = y$,

$$\exists\, N_1 > 0 \ni |x_n - x| < \varepsilon,\ \forall\, n \geqslant N_1$$

and

$$\exists\, N_2 > 0 \ni |x_n - y| < \varepsilon,\ \forall\, n \geqslant N_2\,.$$

Then if $n \geqslant N \equiv \max\{N_1, N_2\}$, we have both $|x_n - x| < \varepsilon$ and $|x_n - y| < \varepsilon$ for all $n \geqslant N$. As a consequence, $x_n < x + \varepsilon$ and $x_n > y - \varepsilon$ for all $n \geqslant N$, a contradiction. So $x = y$. □

Definition 1.54. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in an order field $\mathcal{F}$.

1. $\{x_n\}_{n=1}^{\infty}$ is said to be ***bounded*** (有界的) if there exists $M > 0$ such that $|x_n| \leqslant M$ for all $n \in \mathbb{N}$.

2. $\{x_n\}_{n=1}^{\infty}$ is said to be ***bounded from above*** (有上界) if there exists $B \in \mathcal{F}$, called an ***upper bound*** of the sequence, such that $x_n \leqslant B$ for all $n \in \mathbb{N}$.

3. $\{x_n\}_{n=1}^{\infty}$ is said to be ***bounded from below*** (有下界) if there exists $A \in \mathcal{F}$, called a ***lower bound*** of the sequence, such that $A \leqslant x_n$ for all $n \in \mathbb{N}$.

Proposition 1.55. *A convergent sequence is bounded* (數列收斂必有界).

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a convergent sequence with limit x. Then there exists $N > 0$ such that

$$x_n \in (x-1, x+1) \qquad \forall\, n \geqslant N.$$

Let $M = \max\big\{|x_1|, |x_2|, \cdots, |x_{N-1}|, |x|+1\big\}$. Then $|x_n| \leqslant M$ for all $n \in \mathbb{N}$. ▫

Theorem 1.56. *Suppose that $x_n \to x$ and $y_n \to y$ as $n \to \infty$, λ is a constant. Then*

1. $x_n \pm y_n \to x \pm y$ *as* $n \to \infty$.

2. $\lambda \cdot x_n \to \lambda \cdot x$ *as* $n \to \infty$.

3. $x_n \cdot y_n \to x \cdot y$ *as* $n \to \infty$.

4. *If* $y_n, y \neq 0$, *then* $\dfrac{x_n}{y_n} \to \dfrac{x}{y}$ *as* $n \to \infty$.

Proof. The proof of 1 and 2 are left as an exercise.

3. Since $x_n \to x$ and $y_n \to y$ as $n \to \infty$, by Proposition 1.55 $\exists\, M > 0 \ni |x_n| \leqslant M$ and $|y_n| \leqslant M$. Let $\varepsilon > 0$ be given. Moreover,

$$\exists\, N_1 > 0 \ni |x_n - x| < \frac{\varepsilon}{2M}\ \forall\, n \geqslant N_1$$

and

$$\exists\, N_2 > 0 \ni |y_n - y| < \frac{\varepsilon}{2M}\ \forall\, n \geqslant N_2\,.$$

Define $N = \max\{N_1, N_2\}$. Then for all $n \geqslant N$,

$$\begin{aligned}|x_n \cdot y_n - x \cdot y| &= |x_n \cdot y_n - x_n \cdot y + x_n \cdot y - x \cdot y| \leqslant |x_n \cdot (y_n - y)| + |y \cdot (x_n - x)| \\ &\leqslant M \cdot |y_n - y| + M \cdot |x_n - x| < M \cdot \frac{\varepsilon}{2M} + M \cdot \frac{\varepsilon}{2M} = \varepsilon.\end{aligned}$$

4. It suffices to show that $\lim\limits_{n\to\infty} \frac{1}{y_n} = \frac{1}{y}$ if $y_n, y \neq 0$ (because of 3). Since $\lim\limits_{n\to\infty} y_n = y$, $\exists\, N_1 > 0 \ni |y_n - y| < \frac{|y|}{2}$ for all $n \geqslant N_1$. Therefore, $|y| - |y_n| < \frac{|y|}{2}$ for all $n \geqslant N_1$ which further implies that $|y_n| > \frac{|y|}{2}$ for all $n \geqslant N_1$.

   Let $\varepsilon > 0$ be given. Since $\lim\limits_{n\to\infty} y_n = y$, $\exists\, N_2 > 0 \ni |y_n - y| < \frac{|y|^2}{2}\varepsilon$ for all $n \geqslant N_2$. Define $N = \max\{N_1, N_2\}$. Then for all $n \geqslant N$,

$$\Big|\frac{1}{y_n} - \frac{1}{y}\Big| = \frac{|y_n - y|}{|y_n||y|} < \frac{|y|^2}{2}\varepsilon \cdot \frac{1}{|y|}\frac{2}{|y|} = \varepsilon\,. \qquad \square$$

1.2.2 Monotone sequence property and completeness

Definition 1.57. A sequence $\{x_n\}_{n=1}^{\infty}$ is said to be ***increasing/non-decreasing***, ***decreasing/non-increasing***, ***strictly increasing*** and ***strictly decreasing*** if $x_n \leqslant x_{n+1}$, $x_n \geqslant x_{n+1}$, $x_n < x_{n+1}$ and $x_n > x_{n+1}$ $\forall\, n \in \mathbb{N}$, respectively. A sequence is called (strictly) ***monotone*** if it is either (strictly) increasing or (strictly) decreasing.

Definition 1.58. An ordered field $\mathcal{F}$ is said to satisfy the **(*strictly*) *monotone sequence property*** if every bounded (strictly) monotone sequence converges to a limit in $\mathcal{F}$.

Remark 1.59. An equivalent definition of the monotone sequence property is that every monotone *increasing* sequence *bounded above* converges; that is, if each sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$ satisfying

(i) $x_n \leqslant x_{n+1}$ for all $n \in \mathbb{N}$,

(ii) $\exists\, M \in \mathcal{F} \ni \forall\, n \in \mathbb{N} : x_n \leqslant M$,

is convergent, then we say $\mathcal{F}$ satisfies the monotone sequence property.

Example 1.60. $(\mathbb{Q}, +, \cdot, \leqslant)$ is an ordered field.
Question: Is there any bounded monotone sequence in $\mathbb{Q}$ that does not converge to a limit in $\mathbb{Q}$?
Answer: Yes! Consider the sequence

$$x_1 = \frac{1}{2}, \quad x_2 = \frac{1}{2 + \frac{1}{2}}, \quad x_3 = \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}, \quad \cdots, \quad x_{n+1} = \frac{1}{2 + x_n}\,.$$

Then $\{x_n\}_{n=1}^{\infty}$ is a monotone decreasing sequence in $\mathbb{Q}$. If $\lim\limits_{n\to\infty} x_n = x$, then Theorem 1.56 implies that $x = \dfrac{1}{2+x}$ from which we conclude that $x = -1+\sqrt{2}$. Since $x \notin \mathbb{Q}$, $\{x_n\}_{n=1}^{\infty}$ does not converge (to a limit) in $\mathbb{Q}$. In other words, $\mathbb{Q}$ does not have the monotone sequence property.

Proposition 1.61. *An ordered field satisfying the monotone sequence property has the Archimedean property; that is, if $\mathcal{F}$ is an ordered field satisfying the monotone sequence property, then $\forall\, x \in \mathcal{F}$, $\exists\, n \in \mathbb{N} \ni x < n$.*

Proof. Assume the contrary that there exists $x \in \mathcal{F}$ such that $n \leqslant x$ for all $n \in \mathbb{N}$. Let $x_n = n$. Then $\{x_n\}_{n=1}^{\infty}$ is increasing and bounded above. By the monotone sequence property of $\mathcal{F}$, $\exists\, \widehat{x} \in \mathcal{F} \ni x_n \to \widehat{x}$ as $n \to \infty$; thus $\exists\, N > 0$ such that

$$|x_n - \widehat{x}| < \frac{1}{4} \qquad \forall\, n \geqslant N\,.$$

In particular, $|N - \widehat{x}| < \dfrac{1}{4}$, $|N+1-\widehat{x}| < \dfrac{1}{4}$; thus

$$1 = |N+1-N| \leqslant |N+1-\widehat{x}| + |\widehat{x} - N| < \frac{1}{4} + \frac{1}{4} = \frac{1}{2}\,,$$

a contradiction. □

Example 1.62. Let $(\mathcal{F}, +, \cdot, \leqslant)$ be an ordered field satisfying the monotone sequence property, and $y \in \mathcal{F}$ be a given positive number (that is, $y > 0$). Define $x_n = \dfrac{N_n}{2^n}$, where N_n is the largest integer such that $x_n^2 \leqslant y$; that is, $\big(\dfrac{N_n}{2^n}\big)^2 \leqslant y$ but $\big(\dfrac{N_n+1}{2^n}\big)^2 > y$ (for example, if $y = 2$, then $x_1 = \dfrac{2}{2^1}$, $x_2 = \dfrac{5}{2^2}$, $x_3 = \dfrac{11}{2^3}$, $\cdots$). Then

1. x_n is bounded above: since $x_n^2 \leqslant y \leqslant 2y + y^2 + 1 = (y+1)^2$, by the non-negativity of x_n and y and Remark 1.19 we must have $0 \leqslant x_n \leqslant y+1$.

2. x_n is increasing: by the definition of N_n,

$$N_n^2 \leqslant 2^{2n} \cdot y \Rightarrow 4 \cdot N_n^2 \leqslant 2^{2n+2} \cdot y = 2^{2(n+1)} \cdot y \Rightarrow \Big(\frac{2N_n}{2^{n+1}}\Big)^2 \leqslant y \Rightarrow 2N_n \leqslant N_{n+1}\,.$$

   Therefore, $x_n = \dfrac{N_n}{2^n} = \dfrac{2N_n}{2^{n+1}} \leqslant \dfrac{N_{n+1}}{2^{n+1}} = x_{n+1}$. Since $\mathcal{F}$ satisfies the monotone sequence property, $\exists\, x \in \mathcal{F} \ni x_n \to x$ as $n \to \infty$. By Theorem 1.56, $x_n^2 \to x^2$, and by Proposition 1.51, $x^2 \leqslant y$.

Now we show $x^2 = y$. To this end observe that

$$\left(x_n + \frac{1}{2^n}\right)^2 = \left(\frac{N_n}{2^n} + \frac{1}{2^n}\right) = \left(\frac{N_n + 1}{2^n}\right)^2 > y;$$

thus $x_n^2 \leqslant y \leqslant \left(x_n + \frac{1}{2^n}\right)^2$. By the Archimedean propery of $\mathcal{F}$ (Proposition 1.61), $\lim\limits_{n\to\infty} \frac{1}{2^n} = 0$; thus Theorem 1.56 implies that $x^2 = \lim\limits_{n\to\infty} x_n^2 = \lim\limits_{n\to\infty} \left(x_n + \frac{1}{2^n}\right)^2 = y$. Note that Proposition 1.18 implies that such an x is unique if $x > 0$.

In general, one can define the n-th root of non-negative number y in an ordered field satisfying the monotone sequence property. The construction of the n-th root of $y \in \mathcal{F}$ is left as an exercise.

Definition 1.63. For $n \in \mathbb{N}$, the ***n-th root*** of a non-negative number y in an ordered field satisfying the monotone sequence property is the unique non-negative number x satisfying $x^n = y$. One writes $y^{1/n}$ or $\sqrt[n]{y}$ to denote n-th root of y.

Definition 1.64. An ordered field $\mathcal{F}$ is said to be ***complete*** (完備) (have the completeness property, 具備完備性) if it satisfies the monotone sequence property.

Remark 1.65. In an ordered field, completeness $\Leftrightarrow$ monotone sequence property (在 ordered field 裡，完備性 = 數列單調有界必收斂 = 數列遞增有上界必收斂). Moreover,

1. A complete ordered field is "Archimedean" (Proposition 1.61).

2. For $n \in \mathbb{N}$, the n-th root of a non-negative number in a complete ordered field is well-defined (Definition 1.63).

Proposition 1.66. *Let $(\mathcal{F}, +, \cdot, \leqslant)$ be an ordered field. Then $\mathcal{F}$ satisfies the monotone sequence property if and only if $\mathcal{F}$ satisfies the strictly monotone sequence property.*

Proof. The "only if" part is trivial, so we only prove the "if" part. Let $\{x_n\}_{n=1}^\infty$ be a bounded increasing sequence in $\mathcal{F}$. If $\{x_n\}_{n=1}^\infty$ has finite number of values; that is,

$$\#\left\{n \in \mathbb{N} \,\middle|\, x_n < x_{n+1}\right\} < \infty\,,$$

then there exists $N \in \mathbb{N}$ such that $x_n = x_N$ for all $n \geqslant N$ which implies that $\{x_n\}_{n=1}^\infty$ converges to x_N. Now suppose that

$$\#\left\{n \in \mathbb{N} \,\middle|\, x_n < x_{n+1}\right\} = \infty\,.$$

Then there exists an infinite set $\{n_1, n_2, \cdots\} \subseteq \mathbb{N}$ such that $x_{n_k} \neq x_{n_{k+1}}$ for all $k \in \mathbb{N}$. Let $y_k = x_{n_k}$. Since $\mathcal{F}$ satisfies the strictly monotone sequence property, $y_k \to y$ as $k \to \infty$ for some $x \in \mathcal{F}$. However, it is easy to see that the sequence $\{x_n\}_{n=1}^\infty$ also converges to y since $\{x_n\}_{n=1}^\infty$ is monotone increasing. $\square$

Theorem 1.67. *There is a "unique" complete ordered field, called the real number system* $\mathbb{R}$.

Remark 1.68. Uniqueness means if $\mathcal{F}$ is any other complete ordered field $(\mathcal{F}, \oplus, \odot, \preccurlyeq)$, then there exists an field isomorphism $\phi : \mathbb{R} \to \mathcal{F}$; that is, $\phi : \mathbb{R} \to \mathcal{F}$ is one-to-one and onto, and satisfies that

1. $\phi(x + y) = \phi(x) \oplus \phi(y)$ for all $x, y \in \mathbb{R}$.
2. $\phi(x \cdot y) = \phi(x) \odot \phi(y)$ for all $x, y \in \mathbb{R}$.
3. $x \leqslant y \Rightarrow \phi(x) \preccurlyeq \phi(y)$ for all $x, y \in \mathbb{R}$.

Sketch of proof of Theorem 1.67. Let S be the collection of all bounded increasing sequences in $\mathbb{Q}$ in which all terms in every sequence have the same sign; that is,

$$S = \Big\{ \{x_n\}_{n=1}^\infty \,\Big|\, x_n \in \mathbb{Q} \text{ for all } n \in \mathbb{N},\ x_j \cdot x_k \geqslant 0 \text{ for all } k, j \in \mathbb{N}, \text{ and } \{x_n\}_{n=1}^\infty \text{ is increasing and bounded above} \Big\}.$$

Define on S an equivalence relation $\sim$: $\{x_n\}_{n=1}^\infty \sim \{y_n\}_{n=1}^\infty$ if every upper bound of $\{x_n\}_{n=1}^\infty$ is also an upper bound of $\{y_n\}_{n=1}^\infty$, and vice versa. Let $\mathbb{R} = \big\{ \big[\{x_n\}_{n=1}^\infty\big] \,\big|\, \{x_n\}_{n=1}^\infty \in S \big\}$ be the set of equivalence class of S (the existence of such a set relies on the *axiom of choice*). We define on $\mathbb{R}$, $+$, $\cdot$, $\leqslant$ as follows: if $r = \big[\{x_n\}_{n=1}^\infty\big]$ and $s = \big[\{y_n\}_{n=1}^\infty\big]$ (where $\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty \in S$), then

1. $r + s = \big[\{x_n + y_n\}_{n=1}^\infty\big]$; 2. $r \cdot s = \begin{cases} \big[\{x_n \cdot y_n\}_{n=1}^\infty\big] & \text{if } r, s \geqslant 0\,, \\ -\big((-r) \cdot s\big) & \text{if } r < 0 \text{ and } s > 0\,, \\ -\big(r \cdot (-s)\big) & \text{if } r > 0 \text{ and } s < 0\,, \\ (-r) \cdot (-s) & \text{if } r, s < 0\,; \end{cases}$

3. $r \leqslant s$ if every upper bound of $\{y_n\}_{n=1}^\infty$ is also an upper bound for $\{x_n\}_{n=1}^\infty$.

One needs to verify that $\mathbb{R}$ is an ordered field, and this part is left as an exercise.

Claim 1: If $\{x_{n_k}\}_{k=1}^\infty$ is a subsequence of $\{x_n\}_{n=1}^\infty$, then $\big[\{x_{n_k}\}_{k=1}^\infty\big] = \big[\{x_n\}_{n=1}^\infty\big]$.

Claim 2: If $\left[\{x_n\}_{n=1}^\infty\right] < \left[\{y_n\}_{n=1}^\infty\right]$, then for some $N \in \mathbb{N}$, $x_n < y_N$ for all $n \geqslant N$.

The proofs of the claims above are not difficult and are left as an exercise.

Now we show the completeness of $\mathbb{R}$ by showing that $\mathbb{R}$ satisfies the strictly monotone sequence property (Proposition 1.66). Let $\{r_k\}_{k=1}^\infty$ be a bounded, strictly increasing sequence in $\mathbb{R}^+$. Write $r_k = \left[\{x_{k,n}\}_{n=1}^\infty\right]$, where $x_{k,n} \leqslant x_{k,n+1}$ for all $k, n \in \mathbb{N}$. Since $\{r_k\}_{k=1}^\infty$ is bounded in $\mathbb{R}$, there is $M \in \mathbb{Q}$ such that $x_{k,n} \leqslant M$ for all $k, n \in \mathbb{N}$. Moreover, since $r_k < r_{k+1}$ for all $k \in \mathbb{N}$, by claims above we can assume that $x_{k,n} < x_{k+1,1}$ for all $k, n \in \mathbb{N}$; thus

$$x_{k,n} < x_{\ell,m} \qquad \forall\, \ell > k \text{ and } n, m \in \mathbb{N}. \tag{$\star$}$$

Therefore, $\{x_{n,n}\}_{n=1}^\infty$ is bounded and monotone increasing, so $\{x_{n,n}\}_{n=1}^\infty \in S$. Define $r = \left[\{x_{n,n}\}_{n=1}^\infty\right]$. Then $r \in \mathbb{R}$, and

(i) r **is an upper bound of** $\{r_k\}_{k=1}^\infty$: Suppose the contrary that there exists $M \in \mathbb{Q}$ such that $x_{n,n} \leqslant M$ for all $n \in \mathbb{N}$ but $x_{k,\ell} > M$ for some $k, \ell \in \mathbb{N}$.

  (a) If $k \geqslant \ell$, then $x_{k,k} \geqslant x_{k,\ell} > M$ since $\{x_{k,\ell}\}_{\ell=1}^\infty$ is increasing.

  (b) If $k < \ell$, then $x_{\ell,\ell} > x_{k,\ell} > M$ because of $(\star)$.

  In either case we conclude that M cannot be an upper bound of r, a contradiction.

(ii) $r - \varepsilon$ **is not an upper bound of** $\{r_k\}_{k=1}^\infty$ **for all** $\varepsilon > 0$: Suppose the contrary that $r - \varepsilon$ is an upper bound of $\{r_k\}_{k=1}^\infty$. Write $\varepsilon = \{\varepsilon_k\}_{k=1}^\infty$, and W.L.O.G. we can assume that there exists $\delta \in \mathbb{Q}$ such that $\varepsilon_k \geqslant 2\delta > 0$ for all $k \in \mathbb{N}$. Then for all (fixed) $k \in \mathbb{N}$,

$$\left[\{x_{k,\ell} + \delta\}_{\ell=1}^\infty\right] < \left[\{x_{k,\ell} + 2\delta\}_{\ell=1}^\infty\right] \leqslant \left[\{x_{k,\ell} + \varepsilon_k\}_{\ell=1}^\infty\right] \leqslant \left[\{x_{\ell,\ell}\}_{\ell=1}^\infty\right].$$

  Let $N_1 = 1$. By claim 2, for each $k \in \mathbb{N}$ there exists $N_{k+1} \in \mathbb{N}$ such that $N_{k+1} \geqslant N_k$ and $x_{N_k,\ell} + \delta < x_{N_{k+1},N_{k+1}}$ for all $\ell \geqslant N_{k+1}$. On the other hand,

$$x_{N_{k+1},N_{k+1}} \geqslant x_{N_k,N_{k+1}} + \delta \geqslant x_{N_k,N_k} + \delta \geqslant \cdots \geqslant x_{1,1} + k\delta$$

  which implies that $\{x_{\ell,\ell}\}_{\ell=1}^\infty$ is not bounded, a contradiction.

As a consequence, r is the least upper bound of $\{r_k\}_{k=1}^\infty$. □

From now on $\mathbb{R}$ is the complete ordered field containing $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$.

Example 1.69. In $\mathbb{R}$, define x_n inductively by $x_1 = 0$, $x_2 = \sqrt{2}$, $x_3 = \sqrt{2+\sqrt{2}}$, $\cdots$, $x_{n+1} = \sqrt{2 + x_n}$. It is easy to see that $\{x_n\}_{n=1}^\infty$ satisfies $x_n \geqslant 0$ for all $n \in \mathbb{N}$.

1. $x_n \leqslant 2$ for all $n \in \mathbb{N}$ (boundedness): First of all, $x_1 \leqslant 2$. Assume that $x_n \leqslant 2$. Then $x_{n+1} = \sqrt{2 + x_n} \leqslant \sqrt{2+2} = 2$. By mathematical induction, $x_n \leqslant 2$ for all $n \in \mathbb{N}$.

2. $x_n \leqslant x_{n+1}$ (monotonicity): Since $x_n - 2 \leqslant 0$ and $x_n + 1 \geqslant 0$, $(x_n - 2) \cdot (x_n + 1) \leqslant 0$. Expanding the product, we obtain that $x_n^2 \leqslant x_n + 2 = x_{n+1}^2$ which implies that $x_n \leqslant x_{n+1}$.

3. $x_n \to 2$ as $n \to \infty$ (convergence): Since $\{x_n\}_{n=1}^{\infty}$ is a bounded monotone sequence in $\mathbb{R}$, $\lim\limits_{n\to\infty} x_n = x$ for some $x \in \mathbb{R}$. Note that then $x_{n+1} \to x$ as $n \to \infty$. Since $x_{n+1}^2 = x_n + 2$, by Theorem 1.56 we must have $x^2 = x + 2$. Then $(x-2)(x+1) = 0$ which implies $x = 2$ or $x = -1$ (failed). Therefore, $\{x_n\}_{n=1}^{\infty}$ converges to 2.

Theorem 1.70. *The interval* $(0, 1)$ *in* $\mathbb{R}$ *is uncountable* (不可數).

Proof. Assume the contrary that there exists $f : \mathbb{N} \to (0, 1)$ which is one-to-one and onto. Write $f(k)$ in decimal expansion (十進位展開); that is,

$$
\begin{aligned}
f(1) &= 0.d_{11}d_{21}d_{31}\cdots \\
f(2) &= 0.d_{12}d_{22}d_{32}\cdots \\
\vdots \qquad & \qquad \vdots \\
f(k) &= 0.d_{1k}d_{2k}d_{3k}\cdots \\
\vdots \qquad & \qquad \vdots
\end{aligned}
$$

Here we note that repeated 9's are chosen by preference over terminating decimals; that is, for example, we write $\dfrac{1}{4} = 0.249999\cdots$ instead of $\dfrac{1}{4} = 0.250000\cdots$.

Let $x \in (0, 1)$ be such that $x = 0.d_1 d_2 \cdots$, where

$$
d_k = \begin{cases} 5 & \text{if} \quad d_{kk} \neq 5\,, \\ 7 & \text{if} \quad d_{kk} = 5\,. \end{cases}
$$

(建構一個 x 使其小數點下第 k 位數與 $f(k)$ 的小數點下第 k 位數不相等). Then $x \neq f(k)$ for all $k \in \mathbb{N}$, a contradiction; thus $(0, 1)$ is uncountable. □

Corollary 1.71. $\mathbb{R}$ *is uncountable.*

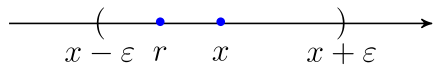
Proposition 1.72. $\mathbb{Q}$ *is dense* (稠密) *in* $\mathbb{R}$*; that is, if* $x, y \in \mathbb{R}$ *and* $x < y$*, then* $\exists\, r \in \mathbb{Q} \ni x < r < y$.

Proof. Since $\frac{1}{n} \to 0$ as $n \to \infty$ (by the Archimedean property of $\mathbb{R}$, Proposition 1.61), $\exists\, N > 0 \ni \left|\frac{1}{n} - 0\right| < y - x$ for all $n \geqslant N$.

Claim: $\left\{\frac{k}{N} \,\middle|\, k \in \mathbb{Z}\right\} \cap (x, y) \neq \varnothing$.

Proof of claim: Suppose the contrary that $\left\{\frac{k}{N} \,\middle|\, k \in \mathbb{Z}\right\} \cap (x, y) = \varnothing$. Then $\frac{\ell}{N} \leqslant x$ and $\frac{\ell + 1}{N} \geqslant y$ for some $\ell \in \mathbb{Z}$, while this fact will imply that $y - x \leqslant \frac{1}{N}$, a contradiction. □

Remark 1.73. The denseness of $\mathbb{Q}$ in $\mathbb{R}$ can be rephrased as follows: if $x \in \mathbb{R}$ and $\varepsilon > 0$, then $\exists\, r \in \mathbb{Q} \ni |x - r| < \varepsilon$.



Corollary 1.74. *The collection of irrational numbers $\mathbb{Q}^\complement \equiv \mathbb{R}\backslash\mathbb{Q}$ is dense in $\mathbb{R}$; that is, if $x, y \in \mathbb{R}$ and $x < y$, $\exists\, c \in \mathbb{Q}^\complement \ni x < c < y$.*

Proof. Let $x, y \in \mathbb{R}$ with $x < y$. By Proposition 1.72 there exists $r \in \mathbb{Q}$, $r \neq 0$ such that $\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}$. Let $c = \sqrt{2}r$. Then $c \in \mathbb{Q}^\complement$ and $x < c < y$. □

Example 1.75. The harmonic sequence

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 1 + \frac{1}{2} \\ \vdots \quad & \quad \vdots \\ x_n &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k} \\ \vdots \quad & \quad\quad\quad \vdots \end{aligned}$$

is (monotone) increasing but not bounded above.

Proof. That the sequence is increasing is trivial. For the unboundedness, we observe that

$$\begin{aligned} x_{2^n} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots + \frac{1}{2^n} \\ &\geqslant 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \cdots + \frac{2^{n-1}}{2^n} \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} = 1 + \frac{n}{2} \end{aligned}$$

which is not bounded above (沒有上界). □

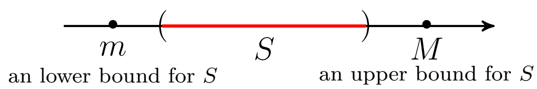
1.3 Least Upper Bounds and Greatest Lower Bounds

Definition 1.76. Let $\varnothing \neq S \subseteq \mathbb{R}$. A number $M \in \mathbb{R}$ is called an ***upper bound*** (上界) for S if $x \leqslant M$ for all $x \in S$, and a number $m \in \mathbb{R}$ is called a ***lower bound*** (下界) for S if $x \geqslant m$ for all $x \in S$. If there is an upper bound for S, then S is said to be ***bounded from above***, while if there is a lower bound for S, then S is said to be ***bounded from below***. A number $b \in \mathbb{R}$ is called a ***least upper bound*** (最小上界) if

1. b is an upper bound for S, and
2. if M is an upper bound for S, then $M \geqslant b$.

A number a is called a ***greatest lower bound*** (最大下界) if

1. a is a lower bound for S, and
2. if m is a lower bound for S, then $m \leqslant a$.



If S is not bounded above, the least upper bound of S is set to be ∞, while if S is not bounded below, the greatest lower bound of S is set to be $-\infty$. The least upper bound of S is also called the ***supremum*** of S and is usually denoted by lub$S$ or $\sup S$, and "the" greatest lower bound of S is also called the ***infimum*** of S, and is usually denoted by glb$S$ or $\inf S$. If $S = \varnothing$, then $\sup S = -\infty$, $\inf S = \infty$.

Example 1.77. Let $S = (0, 1)$. Then $\sup S = 1$, $\inf S = 0$.

Example 1.78. Let $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 - x^2 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Define

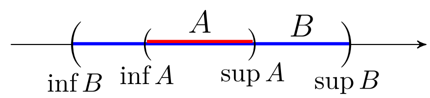
$$S = \{f(x) \mid x \in R\}, \quad T = \Big\{x \in \mathbb{R} \mid f(x) > \frac{1}{4}\Big\}.$$

We can get $S = (-\infty, 1)$, so $\sup(S) = 1$, $\inf(S) = -\infty$.

Solve $1 - x^2 = \dfrac{1}{4} \Rightarrow x = \pm\dfrac{\sqrt{3}}{2}$, then we can get $T = \big(-\dfrac{\sqrt{3}}{2}, 0\big) \cup \big(0, \dfrac{\sqrt{3}}{2}\big)$, so $\sup(T) = \dfrac{\sqrt{3}}{2}$, $\inf(T) = -\dfrac{\sqrt{3}}{2}$.

Remark 1.79. The least upper bound and the greatest lower bound of S need not be a member of S.

Remark 1.80. The reason for defining $\sup\varnothing = -\infty$ and $\inf\varnothing = \infty$ is as follows: if $\varnothing \neq A \subseteq B$, then $\sup A \leqslant \sup B$ and $\inf A \geqslant \inf B$.



Since $\varnothing$ is a subset of any other sets, we shall have $\sup\varnothing$ is smaller then any real number, and $\inf\varnothing$ is greater than any real number. However, this "definition" would destroy the property that $\inf A \leqslant \sup A$.

The "definition" of $\sup\varnothing$ and $\inf\varnothing$ is purely artificial. One can also define $\sup\varnothing = \infty$ and $\inf\varnothing = -\infty$.

Definition 1.81. An ***open interval*** in $\mathbb{R}$ is of the form (a, b) which consists of all $x \in \mathbb{R} \ni a < x < b$. A ***closed interval*** in $\mathbb{R}$ is of the form $[a, b]$ which consists of all $x \in \mathbb{R} \ni a \leqslant x \leqslant b$.

Proposition 1.82. *Let $S \subseteq \mathbb{R}$ be non-empty. Then*

1. $b = \sup S \in \mathbb{R}$ *if and only if*
   (a) *b is an upper bound of S.*
   (b) $\forall\, \varepsilon > 0$, $\exists\, x \in S \ni x > b - \varepsilon$.
2. $a = \inf S \in \mathbb{R}$ *if and only if*
   (a) *a is a lower bound of S.*
   (b) $\forall\, \varepsilon > 0$, $\exists\, x \in S \ni x < a + \varepsilon$.

Proof. "$\Rightarrow$" (a) is part of the definition of being a least upper bound.

(b) If M is an upper bound of S, then we must have $M \geqslant b$; thus $b - \varepsilon$ is not an upper bound of S. Therefore, $\exists\, x \in S \ni x > b - \varepsilon$.

"$\Leftarrow$" We only need to show that if M is an upper bound of S, then $M \geqslant b$. Assume the contrary. Then $\exists\, M$ such that M is an upper bound of S but $M < b$. Let $\varepsilon = b - M$, then there is no $x \in S \ni x > b - \varepsilon$. $\rightarrow\leftarrow$ $\square$

So far it is not clear that whether the least upper bound or the greatest lower bound for a subset $S \subseteq \mathbb{R}$ exists or not. The following theorem provides the existence of the least upper bound or the greatest lower bound of a set S provided that S has certain properties.

Theorem 1.83. *In $\mathbb{R}$, the following two properties hold:*

1. ***Least upper bound property*** (L.U.B.P.)*:*

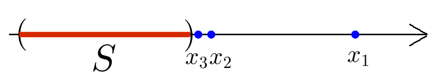
   Let S be a non-empty set in $\mathbb{R}$ that has an upper bound (*or is bounded from above*)*, then S has a least upper bound.* (非空集合有上界，則有最小上界)

2. ***Greatest lower bound property****:*

   Let S be a non-empty set in $\mathbb{R}$ that has a lower bound (*or is bounded from below*)*, then S has a greatest lower bound.* (非空集合有下界，則有最大下界)

Proof. We only prove the least upper bound property since the proof of the greatest lower bound property is similar.

Let $\varnothing \neq S \subseteq \mathbb{R}$ be given. Let x_0 be the smallest integer such that x_0 is an upper bound of S. Let $x_1 = x_0 - \dfrac{N_1}{10}$, where N_1 is the largest integer such that x_2 is still an upper bound of S. We continue this process, and define $x_n = x_{n-1} - \dfrac{N_n}{10^n}$, where N_n is the largest integer such that x_n is an upper bound of S.（事實上，x_n 就是十進位下小數點以下只有 n 位的小數裡面，S 的上界中最小的那個數）



Note that in the process of constructing $\{x_n\}_{n=1}^{\infty}$, N_n is always non-negative which implies that $\{x_n\}_{n=1}^{\infty}$ is decreasing. Moreover, any $a \in S$ is a lower bound of $\{x_n\}_{n=1}^{\infty}$. By completeness of $\mathbb{R}$, $\{x_n\}_{n=1}^{\infty}$ converges. Assume that $x_n \to x$ as $n \to \infty$.
Claim: $x = \sup S$ ($\Leftrightarrow$ 1. x is an upper bound of S. 2. $\forall\, \varepsilon > 0, \exists\, s \in S \ni s > x - \varepsilon$).

1. Assume the contrary that x is not an upper bound of S. Then $\exists\, s \in S \ni s > x$. Since $x_n \to x$ as $n \to \infty$, $\exists\, N > 0 \ni |x_n - x| < s - x$ for all $n \geqslant N$; thus

$$2x - s < x_n < s \qquad n \geqslant N\,.$$

   Therefore, x_n cannot be an upper bound of S for all $n \geqslant N$, a contradiction.

2. Assume the contrary that $\exists\, \varepsilon > 0 \ni \forall\, s \in S,\ s < x - \varepsilon$. Choose $k \in \mathbb{N}$ such that $\varepsilon > \dfrac{1}{10^k}$. Then
$$x_{k-1} - \frac{N_k + 1}{10^k} = x_k - \frac{1}{10^k} \geqslant x - \varepsilon > s$$
which suggests that N_k is not the largest integer such that $x_{k-1} - \dfrac{N_k}{10^k}$ is still an upper bound, a contradiction. $\square$

Proposition 1.84. *Suppose that $\varnothing \neq A \subseteq B \subseteq \mathbb{R}$. Then $\inf B \leqslant \inf A \leqslant \sup A \leqslant \sup B$.*

Proof. We proceed as follows.

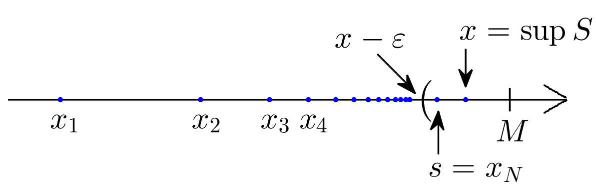
1. $\sup A \leqslant \sup B$: Let $b = \sup B$, then $\forall x \in B$, $x \leqslant b$. Since $A \subseteq B$, then $\forall x \in A$, $x \leqslant b$; hence b is also an upper bound for A. Since $\sup A$ is the least upper bound for A and b is an upper bound for A, then $\sup A \leqslant b = \sup B$.

2. It is similar to prove $\inf B \leqslant \inf A$.

3. It is trivially true that $\inf A \leqslant \sup A$. $\square$

Theorem 1.85. *Let $(\mathcal{F}, +, \cdot, \leqslant)$ be an ordered field such that $\mathcal{F}$ has the least upper bound property, then $\mathcal{F}$ is complete.*

Proof. We would like to show that any increasing bounded sequence converges. Let $\{x_n\}_{n=1}^{\infty}$ be increasing and bounded above (by M).



Define $S = \{x_1, x_2, \cdots, x_n, \cdots\}$. Then S is non-empty and has an upper bound; thus by the assumption that $\mathcal{F}$ satisfies the least upper bound property, $\sup S \equiv x$ exists.

1. x is an upper bound of $S \Rightarrow x_n \leqslant x$ for all $n \in \mathbb{N}$.

2. By Proposition 1.82, $\forall\, \varepsilon > 0$, $\exists\, s \in S \ni s > x - \varepsilon$. Note that $s = x_N$ for some $N \in \mathbb{N}$. Since $\{x_n\}_{n=1}^{\infty}$ is increasing, $x_N \leqslant x_n \leqslant x$ for all $n \geqslant N$. Therefore, if $n \geqslant N$,
$$x - \varepsilon < x_N \leqslant x_n \leqslant x < x + \varepsilon$$

which implies that $|x_n - x| < \varepsilon$ if $n \geqslant N$. □

Example 1.86. $\mathbb{Q}$ is not complete. Let $S = \{x_1 = 3, x_2 = 3.1, x_3 = 3.14, \cdots\}$. Then S has 4 as an upper bound, but S has no least upper bound (in $\mathbb{Q}$).

Remark 1.87. The two theorems above suggest that in an ordered field, completeness $\Leftrightarrow$ the least upper bound property.

1.4 Cauchy Sequences

So far the only criteria that we learn (from previous sections) for the convergence of a sequence in an ordered field is that a bounded monotone sequence in $\mathbb{R}$ converges. Are there any other criteria for the convergence of a sequence in an ordered field? By Proposition 1.53, we know that if a sequence $\{x_n\}_{n=1}^{\infty}$ in an ordered field $\mathcal{F}$ converges, then

$$\exists!\, x \in \mathcal{F} \ni \forall\, \varepsilon > 0, \#\big\{n \in \mathbb{N} \,\big|\, x_n \notin (x-\varepsilon, x+\varepsilon)\big\} < \infty.$$

We would like to investigate if the following much weaker statement

$$\forall\, \varepsilon > 0, \exists\, (\text{a limit candidate})\, y \in \mathcal{F} \ni \#\big\{n \in \mathbb{N} \,\big|\, x_n \notin (y-\varepsilon, y+\varepsilon)\big\} < \infty \tag{$\star$}$$

leads to the convergence of a sequence. Note that statement ($\star$) is equivalent to statement ($\star\star$) in the following

Definition 1.88. A sequence $\{x_n\}_{n=1}^{\infty}$ in an ordered field is said to be ***Cauchy*** if

$$\forall\, \varepsilon > 0, \exists\, N > 0 \ni |x_n - x_m| < \varepsilon \text{ whenever } n, m \geqslant N\,. \tag{$\star\star$}$$

Remark 1.89. ($\star$) 這個敘述的中心思想是：給定一正值 ε, 我們都能找到一個長度是 2ε 的區間使得落在此區間外的 x_n 只有有限個。因為當對每個長度我們都能找到這樣的區間時，才有機會找到 $\{x_n\}_{n=1}^{\infty}$ 的極限（極限若真的存在的話，那麼這個極限一定落在所有這樣的區間之內）；要是連這樣的區間都找不到，就不可能會收斂了。

Example 1.90. In $\mathbb{Q}$, $x_1 = 3, x_2 = 3.1, x_3 = 3.14, x_4 = 3.141, \cdots$. Then $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence, but is not convergent. Therefore, a Cauchy sequence may not converge.

Proposition 1.91. *Every convergent sequence is Cauchy.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a convergent sequence with limit x. For any $\varepsilon > 0$, $\exists\, N > 0 \ni |x_n - x| < \dfrac{\varepsilon}{2}$ if $n \geqslant N$. Then by triangle inequality, if $n, m \geqslant N$,

$$|x_n - x_m| \leqslant |x_n - x| + |x - x_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,;$$

thus $\{x_n\}_{n=1}^{\infty}$ is Cauchy. □

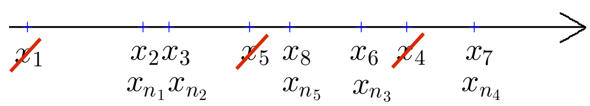
Lemma 1.92. *Every Cauchy sequence is bounded.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be Cauchy. $\exists\, N > 0 \ni |x_n - x_m| < 1$ for all $n, m \geqslant N$. In particular, $|x_n - x_N| < 1$ if $n \geqslant N$ or equivalently,

$$x_N - 1 < x_n < x_N + 1 \qquad \forall\, n \geqslant N\,.$$

Let $M = \max\big\{|x_1|, |x_2|, \cdots, |x_{N-1}|, |x_N| + 1\big\}$. Then $|x_n| \leqslant M$ for all $n \in \mathbb{N}$. □

Definition 1.93. A ***subsequence*** (子數列) is a sequence that can be derived from another sequence by deleting some elements without changing the order of remaining elements. In other words, let $f : \mathbb{N} \to \mathbb{R}$ be a sequence and $x_n = f(n)$. A subsequence of $\{x_n\}_{n=1}^{\infty}$, denoted by $\{x_{n_j}\}_{j=1}^{\infty}$ with $n_{j+1} > n_j$, is the image of an infinite subset $\{n_1, n_2, \cdots\}$ of $\mathbb{N}$ under the map f.

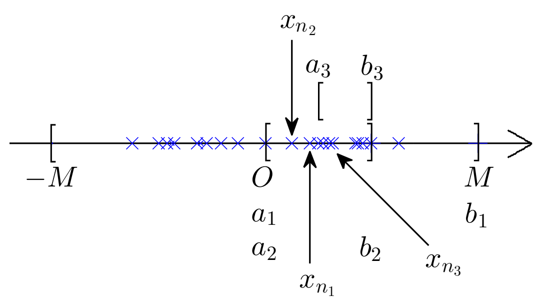


Example 1.94. Let $\{x_n\}_{n=1}^{\infty} = \Big\{1, \dfrac{1}{2}, \dfrac{1}{7}, \dfrac{1}{3}, \dfrac{2}{3}, \dfrac{11}{8}, \cdots\Big\}$, and $\{y_n\}_{n=1}^{\infty} = \Big\{\dfrac{1}{2}, \dfrac{1}{7}, \dfrac{2}{3}, \dfrac{11}{8}, \cdots\Big\}$. Then $\{y_n\}_{n=1}^{\infty}$ can be viewed as a subsequence of $\{x_n\}_{n=1}^{\infty}$ by the relation $y_j = x_{n_j}$; that is, $y_1 = x_2$, $y_2 = x_3$, $y_3 = x_5$, $y_4 = x_6$, and etc. The sequence $\{x_{n_j}\}_{j=1}^{\infty}$ is obtained by deleting x_1 and x_4 (and maybe more) from the original sequence $\{x_n\}_{n=1}^{\infty}$. However, if $\{z_n\}_{n=1}^{\infty} = \Big\{\dfrac{1}{7}, \dfrac{11}{8}, 1, \cdots\Big\}$, then $\{z_n\}_{n=1}^{\infty}$ is not a subsequence of $\{x_n\}_{n=1}^{\infty}$ (but only a subset) of $\{x_n\}_{n=1}^{\infty}$ because the order is changed.

Theorem 1.95 (Bolzano-Weierstrass property)**.** *Every bounded sequence in* $\mathbb{R}$ *has a convergent subsequence; that is, every bounded sequence in* $\mathbb{R}$ *has a subsequence that converges to a limit in* $\mathbb{R}$*.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence satisfying $|x_n| \leqslant M$ for all $n \in \mathbb{N}$. Divide $[-M, M]$ into two intervals $[-M, 0]$, $[0, M]$, and denote one of the two intervals containing infinitely many x_n as $[a_1, b_1]$; that is, $\#\big\{n \in \mathbb{N} \,\big|\, x_n \in [a_1, b_1]\big\} = \infty$. Divide $[a_1, b_1]$ into two intervals $\big[a_1, \dfrac{a_1 + b_1}{2}\big], \big[\dfrac{a_1 + b_1}{2}, b_1\big]$, and denote one of the two intervals containing infinitely many x_n as $[a_2, b_2]$. We continue this process, and obtain a sequence of intervals $[a_k, b_k]$ such that $\#\{n \in \mathbb{N} \,|\, x_n \in [a_k, b_k]\} = \infty$.

Let x_{n_1} be an element belonging to $[a_1, b_1]$. Since $\#\{n \in \mathbb{N} \,|\, x_n \in [a_1, b_1]\} = \infty$, we can choose $n_2 > n_1$ such that $x_{n_2} \in [a_2, b_2]$, and for the same reason we can choose $n_3 > n_2$ such that $x_{n_3} \in [a_3, b_3]$. We continue this process and obtain $x_{n_k} \in [a_k, b_k]$ with $n_k > n_{k-1}$.



Since $[a_k, b_k] \supseteq [a_{k+1}, b_{k+1}]$ for all $k \in \mathbb{N}$, we find that $\{a_k\}_{k=1}^{\infty}$ is increasing and $\{b_k\}_{k=1}^{\infty}$ is decreasing. Moreover, $a_k \leqslant M$, $b_k \geqslant -M$. As a consequence, by the monotone sequence property, a_k converges to a and b_k converges to b.

On the other hand, we observe that $b_k - a_k = \dfrac{M}{2^{k-1}}$. Then $b - a = \lim\limits_{k\to\infty} \dfrac{M}{2^{k-1}} = 0$; thus $a = b$. Since $a_k \leqslant x_{n_k} \leqslant b_k$, by Sandwich lemma $\lim\limits_{k\to\infty} x_{n_k} = a = b \in \mathbb{R}$. □

Lemma 1.96. *If a subsequence of a Cauchy sequence is convergent, then this Cauchy sequence also converges.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence with a convergent subsequence $\{x_{n_j}\}_{j=1}^{\infty}$. Assume $\lim\limits_{j\to\infty} x_{n_j} = x$. Then $\forall\, \varepsilon > 0$,

$$\begin{aligned} &\exists\, K > 0 \ni |x_{n_j} - x| < \frac{\varepsilon}{2} \qquad \text{if} \quad j \geqslant K, \ \text{and} \\ &\exists\, N > 0 \ni |x_n - x_m| < \frac{\varepsilon}{2} \qquad \text{if} \quad n, m \geqslant N. \end{aligned}$$

Choose $j \geqslant \max\{K, N\}$. Then $n_j \geqslant N$; thus if $n \geqslant N$,

$$|x_n - x| \leqslant |x_n - x_{n_j}| + |x_{n_j} - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.$$ □

Theorem 1.97. *Every Cauchy sequence in $\mathbb{R}$ is convergent.*

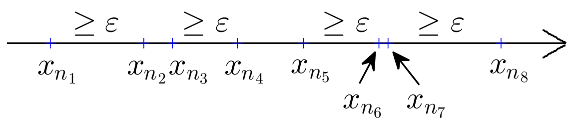
Theorem 1.98. *Suppose that $\mathcal{F}$ is an ordered field with Archimedean property and every Cauchy sequence converges. Then $\mathcal{F}$ is complete.*

Proof. Suppose the contrary that there is a bounded increasing sequence $\{x_n\}_{n=1}^{\infty}$ that does not converge to a limit in $\mathcal{F}$. By assumption, $\{x_n\}_{n=1}^{\infty}$ cannot be Cauchy; thus

$$\exists\, \varepsilon > 0 \ni \forall\, N > 0\ \exists\, n, m \geqslant N \ni |x_n - x_m| \geqslant \varepsilon\,.$$

Let $N = 1$, $\exists\, n_2 > n_1 \geqslant 1 \ni |x_{n_1} - x_{n_2}| \geqslant \varepsilon$. Let $N = n_2 + 1$, $\exists\, n_4 > n_3 \geqslant n_2 + 1 \ni |x_{n_3} - x_{n_4}| \geqslant \varepsilon$. We continue this process and obtain a sequence $\{x_{n_j}\}_{j=1}^{\infty}$ satisfying

$$\left|x_{n_{2k-1}} - x_{n_{2k}}\right| \geqslant \varepsilon \quad \forall\, k \in \mathbb{N}.$$



Claim: $\left\{x_{n_j}\right\}_{j=1}^{\infty}$ is unbounded (thus a contradiction to the boundedness of $\{x_n\}_{n=1}^{\infty}$).
Proof of claim: Assume the contrary that there exists $M \in \mathcal{F}$ such that $x_{n_j} \leqslant M$ for all $j \in \mathbb{N}$. Since $x_{n_{2k}} \geqslant x_1 + k\varepsilon$ for all $k \in \mathbb{N}$, we must have

$$k \leqslant \frac{M - x_1}{\varepsilon} \qquad \forall\, k \in \mathbb{N}$$

which violates the Archimedean property, a contradiction. □

Remark 1.99. In an ordered field with Archimedean property, Completeness ⇔ Cauchy completeness (Every Cauchy sequence converges).

Example 1.100. $x_n \in \mathbb{R}$, $|x_n - x_{n+1}| < \dfrac{1}{2^{n+1}}\ \forall\, n \in \mathbb{N}$.
Claim: $\{x_n\}_{n=1}^{\infty}$ is Cauchy. Given $\varepsilon > 0$, choose $N > 0 \ni \dfrac{1}{2^N} < \varepsilon$. Then if $N \leqslant n < m$,

$$\begin{aligned}
|x_n - x_m| &\leqslant |x_n - x_{n+1}| + |x_{n+1} - x_m| \\
&\leqslant |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + |x_{n+2} - x_m| \\
&\leqslant \cdots \\
&\leqslant |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} - x_m| \\
&\leqslant \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \cdots + \frac{1}{2^m} \\
&\leqslant \frac{1}{2^n} \leqslant \frac{1}{2^N} < \varepsilon\,;
\end{aligned}$$

thus $\{x_n\}_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}$. This implies that the sequence is convergent.

1.5 Cluster Points and Limit Inferior, Limit Superior

Definition 1.101. A point x is called a ***cluster point*** of a sequence $\{x_n\}_{n=1}^{\infty}$ if

$$\forall\, \varepsilon > 0,\ \#\big\{n \in \mathbb{N} \,\big|\, x_n \in (x-\varepsilon, x+\varepsilon)\big\} = \infty\,.$$

Example 1.102. Let $x_n = (-1)^n$. Then 1 and -1 are the only two cluster points of $\{x_n\}_{n=1}^{\infty}$.

Example 1.103. Let $x_n = (-1)^n + \dfrac{1}{n}$.
Claim: 1 and -1 are cluster points of $\{x_n\}_{n=1}^{\infty}$.
Let $\varepsilon > 0$ be given. We observe that

$$\big\{n \in \mathbb{N} \,\big|\, x_n \in (1-\varepsilon, 1+\varepsilon)\big\} \supseteq \Big\{n \in \mathbb{N} \,\Big|\, n \text{ is even}, \frac{1}{n} < \varepsilon\Big\}\,;$$

thus $\#\big\{n \in \mathbb{N} \,\big|\, x_n \in (1-\varepsilon, 1+\varepsilon)\big\} = \infty$. Similarly, -1 is a cluster point.
Claim: $\forall\, a \neq \pm 1$, a is not a cluster point of $\{x_n\}_{n=1}^{\infty}$ (reasoning in the following proposition).

Proposition 1.104. *Let $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ and $x \in \mathbb{R}$.*

1. *x is a cluster point of $\{x_n\}_{n=1}^{\infty}$ if and only if $\forall\, \varepsilon > 0$, $N > 0$, $\exists\, n \geqslant N \ni |x_n - x| < \varepsilon$.*

2. *x is a cluster point of $\{x_n\}_{n=1}^{\infty}$ if and only if there exists a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ converges to x.*

3. *$x_n \to x$ as $n \to \infty$ if and only if every proper subsequence of $\{x_n\}_{n=1}^{\infty}$ converges to x.*

4. *$x_n \to x$ as $n \to \infty$ if and only if $\{x_n\}_{n=1}^{\infty}$ is bounded and x is the only cluster point of $\{x_n\}_{n=1}^{\infty}$.*

5. *$x_n \to x$ as $n \to \infty$ if and only if every proper subsequence of $\{x_n\}_{n=1}^{\infty}$ has a further subsequence that converges to x.*

Proof. We only prove 1-4, and the proof of 5 is left as an exercise.

1. ($\Rightarrow$) Let $\varepsilon > 0$ be given. Since there are infinitely many $n's$ with $|x_n - x| < \varepsilon$, for any fixed $N \in \mathbb{N}$, there are only finite number of the indices that are smaller than N. So there must be some $n \geqslant N$ with $|x_n - x| < \varepsilon$.

   ($\Leftarrow$) Let $\varepsilon > 0$ be given. Pick $n_1 \geqslant 1 \ni |x_{n_1} - x| < \varepsilon$, then pick $n_2 \geqslant n_1 + 1 \ni |x_{n_2} - x| < \varepsilon$. We continue this process and obtain a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ satisfying $|x_{n_j} - x| < \varepsilon$ for all $j \in \mathbb{N}$. Then $\big\{n \in \mathbb{N} \,\big|\, x_n \in (x-\varepsilon, x+\varepsilon)\big\} \supseteq \{n_1, n_2, \cdots\}$.

2. ($\Rightarrow$) By 1, we can pick $n_1 \geqslant 1 \ni |x_{n_1} - x| < 1$ and pick $n_2 \geqslant n_1 + 1 \ni |x_{n_2} - x| < \frac{1}{2}$. In general, we can pick $n_k \geqslant n_{k-1} + 1 \ni |x_{n_k} - x| < \frac{1}{k}$ for all $k \geqslant 2$. Then
$$x - \frac{1}{k} < x_{n_k} < x + \frac{1}{k} \quad \forall\, k \in \mathbb{N}.$$
By Sandwich lemma, $\lim\limits_{k\to\infty} x_{n_k} = x$.

($\Leftarrow$) $\forall\, \varepsilon > 0$, $\exists\, J > 0 \ni |x_{n_j} - x| < \varepsilon$ if $j \geqslant J$. Then $\big\{n \in \mathbb{N} \,\big|\, x_n \in (x-\varepsilon, x+\varepsilon)\big\} \supseteq \{n_J, n_{J+1}, \cdots\}$.

3. ($\Rightarrow$) Let $\{x_{n_j}\}_{j=1}^{\infty}$ be a subsequence of a convergent sequence $\{x_n\}_{n=1}^{\infty}$ and $\lim\limits_{n\to\infty} x_n = x$. Then $\forall\, \varepsilon > 0$, $\exists\, N > 0 \ni |x_n - x| < \varepsilon$ for all $n \geqslant N$. Since $n_j \to \infty$ as $j \to \infty$, $\exists\, J > 0$ $\ni n_j \geqslant N$; thus $|x_{n_j} - x| < \varepsilon$ whenever $j \geqslant J$.

($\Leftarrow$) Assume the contrary that $x_n \nrightarrow x$ as $n \to \infty$. Then
$$\exists\, \varepsilon > 0 \ni \forall\, N > 0,\, \exists\, n \geqslant N \ni |x_n - x| \geqslant \varepsilon.$$
Let $n_1 \geqslant 1$ such that $|x_{n_1} - x| \geqslant \varepsilon$, and $n_2 \geqslant n_1 + 1$ such that $|x_{n_2} - x| \geqslant \varepsilon$. In general, we can chose $n_k \geqslant n_{k-1}$ such that $|x_{n_k} - x| \geqslant \varepsilon$ for all $k \geqslant 2$. The subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ clearly does not converge to x, a contradiction.

4. ($\Rightarrow$) This direction is a direct consequence of Proposition 1.53 and 1.55.

($\Leftarrow$) Suppose that $\{x_n\}_{n=1}$ is a bounded sequence in $\mathbb{R}$ and has x as the only cluster point but $\{x_n\}_{n=1}^{\infty}$ does not converge to x. Then
$$\exists\, \varepsilon > 0 \ni \#\big\{n \in \mathbb{N} \,\big|\, x_n \notin (x-\varepsilon, x+\varepsilon)\big\} = \infty\,.$$
Write $\big\{n \in \mathbb{N} \,\big|\, x_n \notin (x-\varepsilon, x+\varepsilon)\big\} = \{n_1, n_2, \cdots, n_k, \cdots\}$. Then we find a subsequence $\big\{x_{n_k}\big\}_{k=1}^{\infty}$ lying outside $(x-\varepsilon, x+\varepsilon)$. Since $\big\{x_{n_k}\big\}_{k=1}^{\infty}$ is bounded, the Bolzano-Weierstrass property (Theorem 1.95) suggests that there exists a convergent subsequence $\big\{x_{n_{k_j}}\big\}_{j=1}^{\infty}$ with limit y. Since $x_{n_{k_j}} \notin (x-\varepsilon, x+\varepsilon)$, $y \notin [x-\varepsilon, x+\varepsilon]$; thus $y \neq x$. On the other hand, 2 suggests that y is a cluster point of $\{x_n\}_{n=1}^{\infty}$, a contradiction to the assumption that x is the only cluster point of $\{x_n\}_{n=1}^{\infty}$. □

Definition 1.105. A sequence $\{x_n\}_{n=1}^{\infty}$ is said to ***diverge to infinity*** if $\forall\, M > 0$, $\exists\, N > 0$ $\ni x_n > M$ whenever $n \geqslant N$. It is said to ***diverge to negative infinity*** if $\{-x_n\}_{n=1}^{\infty}$ diverge to infinity. We use $\lim\limits_{n\to\infty} x_n = \infty$ or $-\infty$ to denote that $\{x_n\}_{n=1}^{\infty}$ diverges to infinity or negative infinity, and call ∞ or $-\infty$ the limit of $\{x_n\}_{n=1}^{\infty}$.

Definition 1.106. The ***extended real number system***, denoted by $\mathbb{R}^*$, is the number system $\mathbb{R} \cup \{\infty, -\infty\}$, where ∞ and $-\infty$ are two symbols satisfying $-\infty < x < \infty$ for all $x \in \mathbb{R}$.

Remark 1.107. 1. $\mathbb{R}^*$ is not a field since ∞ and $-\infty$ do not have multiplicative inverse.

2. The definition of the least upper bound of a set can be simplified as follows: Let $S \subseteq \mathbb{R}^*$ be a set (not necessary non-empty set). A number $b \in \mathbb{R}^*$ is said to be the least upper bound of S if

   (a) b is an upper bound of S (that is, $s \leqslant b$ for all $s \in S$);

   (b) If $M \in \mathbb{R}^*$ is an upper bound of S, then $b \leqslant M$.

   No further discussion (such as $S = \varnothing$ or S is not bounded above) has to be made. The greatest lower bound can be defined in a similar fashion.

3. Any sets in $\mathbb{R}^*$ has a least upper bound and a greatest lower bound in $\mathbb{R}^*$, even the empty set and unbounded set.

4. Proposition 1.82 can be rephrased as follows: Let $S \subseteq \mathbb{R}^*$. Then $b = \sup S \in \mathbb{R}$ if and only if

   (a) b is an upper bound of S;

   (b) $\forall\, \varepsilon > 0$, $\exists\, s \in S \ni s > b - \varepsilon$.

   Note that $b \in \mathbb{R}$ is crucial since there is no $s \in \mathbb{R}^*$ such that $s > \infty - \varepsilon = \infty$. The greatest lower bound counterpart can be made in a similar fashion.

5. In light of Proposition 1.104 and Definition 1.105, we can redefine cluster points of a real sequence as follows: A number $x \in \mathbb{R}^*$ is said to be a cluster point of a sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ if there exists a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ such that $\lim\limits_{j\to\infty} x_{n_j} = x$. Note that now we can talk about if ∞ or $-\infty$ is a cluster points of a real sequence.

In the rest of the section, one is allowed to find the least upper bound and the greatest lower bound of a subset in $\mathbb{R}^*$.

Definition 1.108. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$.

1. The ***limit superior*** of $\{x_n\}_{n=1}^\infty$, denoted by $\limsup\limits_{n\to\infty} x_n$ or $\overline{\lim\limits_{n\to\infty}}\, x_n$, is the infimum of the sequence $\Big\{\sup\big\{x_n \,\big|\, n \geqslant k\big\}\Big\}_{k=1}^\infty$.
2. The ***limit inferior*** of $\{x_n\}_{n=1}^\infty$, denoted by $\liminf\limits_{n\to\infty} x_n$ or $\underline{\lim\limits_{n\to\infty}}\, x_n$, is the supremum of the sequence $\Big\{\inf\big\{x_n \,\big|\, n \geqslant k\big\}\Big\}_{k=1}^\infty$.

Remark 1.109. Let $\sup\limits_{n\geqslant k} x_n$ denote the number $\sup\big\{x_n \,\big|\, n \geqslant k\big\}$ and $\inf\limits_{n\geqslant k} x_n$ denote the number $\inf\big\{x_n \,\big|\, n \geqslant k\big\}$. Then the limit superior and the limit inferior can be written as

$$\limsup_{n\to\infty} x_n = \inf_{k\geqslant 1}\sup_{n\geqslant k} x_n \quad \text{and} \quad \liminf_{n\to\infty} x_n = \sup_{k\geqslant 1}\inf_{n\geqslant k} x_n\,.$$

Remark 1.110. Let $\{x_n\}_{n=1}^\infty$ be a sequence in $\mathbb{R}$, and $y_k = \sup\limits_{n\geqslant k} x_n$ and $z_k = \inf\limits_{n\geqslant k} x_n$. Then $\{y_k\}_{k=1}^\infty$ is a decreasing sequence, and $\{z_k\}_{k=1}^\infty$ is an increasing sequence. Therefore, the limit of $\{y_k\}_{k=1}^\infty$ and the limit of $\{z_k\}_{k=1}^\infty$ both "exist" in the sense of Definition 1.47 and 1.105. In fact, the limit of $\{y_k\}_{k=1}^\infty$ is the infimum of $\{y_k\}_{k=1}^\infty$, and the limit of $\{z_k\}_{k=1}^\infty$ is the supremum of $\{z_k\}_{k=1}^\infty$. In other words,

$$\lim_{k\to\infty}\sup_{n\geqslant k} x_n = \inf_{k\geqslant 1}\sup_{n\geqslant k} x_n \quad \text{and} \quad \lim_{k\to\infty}\inf_{n\geqslant k} x_n = \sup_{k\geqslant 1}\inf_{n\geqslant k} x_n\,;$$

thus

$$\limsup_{n\to\infty} x_n = \lim_{k\to\infty}\sup_{n\geqslant k} x_n \quad \text{and} \quad \liminf_{n\to\infty} x_n = \lim_{k\to\infty}\inf_{n\geqslant k} x_n\,.$$

Example 1.111. Let $\{x_n\}_{n=1}^\infty = \{1, 0, -1, 1, 0, -1, 1, 0, -1, \cdots\}$. Then

$$\begin{aligned} y_k &= \sup_{n\geqslant k} x_n = 1 \quad \Rightarrow \limsup_{n\to\infty} x_n = 1. \\ z_k &= \inf_{n\geqslant k} x_n = -1 \Rightarrow \liminf_{n\to\infty} x_n = -1. \end{aligned}$$

Example 1.112. Let $x_n = \dfrac{1}{n}$. Then

$$\begin{aligned} y_k &= \sup_{n\geqslant k} x_n = \frac{1}{k} \Rightarrow \limsup_{n\to\infty} x_n = 0. \\ z_k &= \inf_{n\geqslant k} x_n = 0 \ \Rightarrow \liminf_{n\to\infty} x_n = 0. \end{aligned}$$

Example 1.113. Let $x_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$; that is, $\{x_n\}_{n=1}^\infty = \{1, 0, 3, 0, 5, \cdots\}$. Then

$$\begin{aligned} y_k &= \sup_{n\geqslant k} x_n = \infty \Rightarrow \limsup_{n\to\infty} x_n = \infty. \\ z_k &= \inf_{n\geqslant k} x_n = 0 \ \Rightarrow \liminf_{n\to\infty} x_n = 0. \end{aligned}$$

Example 1.114. Let $x_n = \begin{cases} 1+\dfrac{1}{n} & \text{if } n = 4k+1, \\ -1-\dfrac{1}{n} & \text{if } n = 4k+2, \\ 1-\dfrac{1}{n} & \text{if } n = 4k+3, \\ -1+\dfrac{1}{n} & \text{if } n = 4k. \end{cases}$

$y_k = \sup\limits_{n \geqslant k} x_n = 1 + \dfrac{1}{\bigcirc}$, $z_k = \inf\limits_{n \geqslant k} x_n = -1 - \dfrac{1}{\bigcirc}$. $\limsup\limits_{n\to\infty} x_n = 1$. $\liminf\limits_{n\to\infty} x_n = -1$.

Proposition 1.115. *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. Then*

$$\limsup_{n\to\infty} -x_n = -\liminf_{n\to\infty} x_n \quad \textit{and} \quad \liminf_{n\to\infty} -x_n = -\limsup_{n\to\infty} x_n\,.$$

Proof. By the fact that $\sup\limits_{n\geqslant k} -x_n = -\inf\limits_{n\geqslant k} x_n$,

$$\limsup_{n\to\infty} -x_n = \lim_{k\to\infty}\sup_{n\geqslant k}(-x_n) = \lim_{k\to\infty}\big(-\inf_{n\geqslant k} x_n\big) = -\lim_{k\to\infty}\inf_{n\geqslant k} x_n = -\liminf_{n\to\infty} x_n\,.$$

The second identity holds simply by replacing x_n by $-x_n$ in the first identity. □

Proposition 1.116. *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. Then*

1. $a = \liminf\limits_{n\to\infty} x_n \in \mathbb{R}$ *if and only if*

   (a) $\forall\, \varepsilon > 0$, $\exists\, N > 0$ *such that* $a - \varepsilon < x_n$ *whenever* $n \geqslant N$*; that is,*

   $$\forall\, \varepsilon > 0,\ \#\big\{n \in \mathbb{N} \,\big|\, x_n \leqslant a - \varepsilon\big\} < \infty\,,$$

   and

   (b) $\forall\, \varepsilon > 0$ *and* $N > 0$, $\exists\, n \geqslant N$ *such that* $x_n < a + \varepsilon$*; that is,*

   $$\forall\, \varepsilon > 0,\ \#\big\{n \in \mathbb{N} \,\big|\, x_n < a + \varepsilon\big\} = \infty\,.$$

2. $b = \limsup\limits_{n\to\infty} x_n \in \mathbb{R}$ *if and only if*

   (a) $\forall\, \varepsilon > 0$, $\exists\, N > 0$ *such that* $b + \varepsilon > x_n$ *whenever* $n \geqslant N$*; that is,*

   $$\forall\, \varepsilon > 0,\ \#\big\{n \in \mathbb{N} \,\big|\, x_n \geqslant b + \varepsilon\big\} < \infty\,,$$

   and

(b) $\forall\, \varepsilon > 0$ *and* $N > 0$, $\exists\, n \geqslant N$ *such that* $x_n > b - \varepsilon$; *that is,*

$$\forall\, \varepsilon > 0,\ \#\big\{n \in \mathbb{N} \,\big|\, x_n > b - \varepsilon\big\} = \infty\,.$$

Proof. We only prove 1 since the proof of 2 is similar. Let $z_k = \inf\limits_{n \geqslant k} x_n$, and

$$\sup_{k \geqslant 1} z_k = \lim_{k \to \infty} z_k = a \in \mathbb{R}^*\,.$$

We show that $a \in \mathbb{R}$ if and only if 1-(a) and 1-(b). Nevertheless, by Proposition 1.82 (or Remark 1.107), $a \in \mathbb{R}$ if and only if

(i) a is an upper bound of $\{z_k\}_{k=1}^{\infty}$.

(ii) $\forall\, \varepsilon > 0$, $\exists\, N \in \mathbb{N} \ni z_N > a - \varepsilon$.

We justify the equivalency between 1-(a) and (ii), as well as the equivalency between 1-(b) and (i) as follows:

(i) a is an upper bound of $\{z_k\}_{k=1}^{\infty}$ $\Leftrightarrow$ $a \geqslant z_k$ for all $k \in \mathbb{N}$ $\Leftrightarrow$ $\forall\, \varepsilon > 0, a + \varepsilon > z_k$ for all $k \in \mathbb{N}$ $\Leftrightarrow$ $\forall\, \varepsilon > 0$ and $k \in \mathbb{N}$, $a + \varepsilon > \inf\limits_{n \geqslant k} x_n$ $\Leftrightarrow$ $\forall\, \varepsilon > 0$ and $k \in \mathbb{N}$, $a + \varepsilon$ is not a lower bound of $\{x_n\}_{n \geqslant k}^{\infty}$ $\Leftrightarrow$ $\forall\, \varepsilon > 0$ and $k \in \mathbb{N}$, $\exists\, n \geqslant k \ni a + \varepsilon > x_n$ $\Leftrightarrow$ 1-(b).

(ii) $\forall\, \varepsilon > 0$, $\exists\, N \in \mathbb{N} \ni z_N > a - \varepsilon$ $\Leftrightarrow$ $\forall\, \varepsilon > 0$, $\exists\, N > 0 \ni \inf\limits_{n \geqslant N} x_n > a - \varepsilon$ $\Leftrightarrow$ $\forall\, \varepsilon > 0$, $\exists\, N > 0$ such that $a - \varepsilon$ is a lower bound of $\{x_N, x_{N+1}, \cdots\}$ $\Leftrightarrow$ $\forall\, \varepsilon > 0$, $\exists\, N > 0$ such that $a - \varepsilon \leqslant x_n$ for all $n \geqslant N$ $\Leftrightarrow$ $\forall\, \varepsilon > 0$, $\exists\, N > 0$ such that $a - \varepsilon < x_n$ for all $n \geqslant N$ $\Leftrightarrow$ 1-(a). □

Remark 1.117. By Proposition 1.116, if $a = \liminf\limits_{n \to \infty} x_n \in \mathbb{R}$, then

$$\forall\, \varepsilon > 0,\ \#\big\{n \in \mathbb{N} \,\big|\, x_n \in (a - \varepsilon, a + \varepsilon)\big\} = \infty$$

which suggests that a is a cluster point of $\{x_n\}_{n=1}^{\infty}$. Moreover, 1-(a) of Proposition 1.116 implies that no other cluster points can be smaller than a. In other words, if $a = \liminf\limits_{n \to \infty} x_n \in \mathbb{R}$, then a is the smallest cluster point of $\{x_n\}_{n=1}^{\infty}$. Similarly, b is the largest cluster point of $\{x_n\}_{n=1}^{\infty}$ if $b = \limsup\limits_{n \to \infty} x_n \in \mathbb{R}$.

Theorem 1.118. *Let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence in* $\mathbb{R}$. *Then*

1. $\liminf\limits_{n \to \infty} x_n \leqslant \limsup\limits_{n \to \infty} x_n$.

2. *If* $\{x_n\}_{n=1}^{\infty}$ *is bounded above by* M*, then* $\limsup_{n\to\infty} x_n \leqslant M$.

3. *If* $\{x_n\}_{n=1}^{\infty}$ *is bounded below by* m*, then* $\liminf_{n\to\infty} x_n \geqslant m$.

4. $\limsup_{n\to\infty} x_n = \infty$ *if and only if* $\{x_n\}_{n=1}^{\infty}$ *is not bounded above.*

5. $\liminf_{n\to\infty} x_n = -\infty$ *if and only if* $\{x_n\}_{n=1}^{\infty}$ *is not bounded below.*

6. *If* x *is a cluster point of* $\{x_n\}_{n=1}^{\infty}$*, then* $\liminf_{n\to\infty} x_n \leqslant x \leqslant \limsup_{n\to\infty} x_n$.

7. *If* $a = \liminf_{n\to\infty} x_n$ *is finite, then* a *is a cluster point.*

8. *If* $b = \limsup_{n\to\infty} x_n$ *is finite, then* b *is a cluster point.*

9. *If* $\{x_n\}_{n=1}^{\infty}$ *converges to* x *in* $\mathbb{R}$ *if and only if* $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = x \in \mathbb{R}$.

Proof. Left as an exercise. □

Remark 1.119. Using the definition of cluster points of a sequence in Remark 1.107, Remark 1.117 and Theorem 1.118 together imply that the limit superior/inferior of a sequence is the largest/smallest cluster point of that sequence.

Example 1.120. Let $S = \mathbb{Q} \cap [0,1]$. Then S is countable since it is a subset of a countable set $\mathbb{Q}$. Therefore, $\exists\, f : \mathbb{N} \xrightarrow[onto]{1-1} S$ or equivalently $S = \{q_1, q_2, \cdots, q_n, \cdots\}$. The collection of all cluster points of $\{q_n\}_{n=1}^{\infty}$ is $[0,1]$ since $\mathbb{Q} \cap [0,1]$ is dense in $[0,1]$.

1.6 Euclidean Spaces and Vector Spaces

Definition 1.121. ***Euclidean*** $\boldsymbol{n}$***-space***, denoted by $\mathbb{R}^n$, consists of all ordered n-tuples of real numbers. Symbolically,

$$\mathbb{R}^n = \left\{x \,\middle|\, x = (x_1, x_2, \cdots, x_n), x_i \in \mathbb{R}\right\}.$$

Elements of $\mathbb{R}^n$ are generally denoted by single letters that stand for n-tuples such as $x = (x_1, x_2, \cdots, x_n)$, and speak of x as a "point" in $\mathbb{R}^n$.

Definition 1.122. A ***real vector space*** $\mathcal{V}$ is a set of elements called vectors, with given operations of vector addition $+ : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ and scalar multiplication $\cdot : \mathbb{R} \times \mathcal{V} \to \mathcal{V}$ such that

1. $v + w = w + v$ for all $v, w \in \mathcal{V}$.
2. $(v + w) + u = v + (u + w)$ for all $u, v, w \in \mathcal{V}$.
3. $\exists\, 0$, the zero vector, $\ni v + 0 = v$ for all $v \in \mathcal{V}$.
4. $\forall\, v \in \mathcal{V},\ \exists\, w \in \mathcal{V}\ \ni v + w = 0$.
5. $\lambda \cdot (v + w) = \lambda \cdot v + \lambda \cdot w$ for all $\lambda \in \mathbb{R}$ and $v, w \in \mathcal{V}$.
6. $(\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v$ for all $\lambda, \mu \in \mathbb{R}$ and $v \in \mathcal{V}$.
7. $(\lambda \cdot \mu) \cdot v = \lambda \cdot (\mu \cdot v)$ for all $\lambda, \mu \in \mathbb{R}$ and $v \in \mathcal{V}$.
8. $1 \cdot v = v$ for all $v \in \mathcal{V}$.

Example 1.123. Let the vector addition and scalar multiplication on $\mathbb{R}^n$ be defined by

$$x + y = (x_1 + y_1, \cdots, x_n + y_n) \quad \text{if} \quad x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n)$$

and

$$\lambda \cdot x = (\lambda x_1, \cdots, \lambda x_n) \quad \text{if} \quad \lambda \in \mathbb{R}, x = (x_1, \cdots, x_n)\,.$$

Then $\mathbb{R}^n$ is a real vector space.

Example 1.124. Let $\mathcal{M} \equiv \big\{n \times m$ matrix with entries in $\mathbb{R}\big\}$. Define

$$A + B \equiv [a_{ij} + b_{ij}], \quad \lambda \cdot A \equiv [\lambda \cdot a_{ij}] \quad \text{if} \quad \lambda \in \mathbb{R}, A = [a_{ij}], B = [b_{ij}] \in \mathcal{M}\,.$$

Then $\mathcal{M}$ is a real vector space.

Definition 1.125. $\mathcal{W}$ is called a ***subspace*** of a real vector space $\mathcal{V}$ if

1. $\mathcal{W}$ is a subset of $\mathcal{V}$.
2. $(\mathcal{W}, +, \cdot)$, with vector addition and scalar multiplication in $\mathcal{V}$, is a real vector space.

Example 1.126. $\mathcal{V} = \mathbb{R}^3$, $W = \mathbb{R}^2 \times \{0\} \equiv \{(x, y, 0) | x, y \in \mathbb{R}\}$. $\mathcal{W}$ is a subspace of $\mathcal{V}$.

Lemma 1.127. *If $\mathcal{W}$ is a subset of a real vector space $\mathcal{V}$, then $\mathcal{W}$ is a subspace if and only if $\lambda \cdot v + \mu \cdot w \in \mathcal{W},\ \forall\, \lambda, \mu \in \mathbb{R},\ v, w \in \mathcal{W}$.*

Remark 1.128. "n" is called the ***dimension*** of $\mathbb{R}^n$.
There are n linearly independent vectors $e_1 = (1, 0, \cdots, 0), e_2 = (0, 1, 0, \cdots, 0), \cdots, e_n = (0, 0, \cdots, 0, 1)$, but if $v_1, v_2, \cdots, v_{n+1}$ are $(n+1)$ vectors in $\mathbb{R}^n$, $\exists\, \lambda_1, \cdots, \lambda_{n+1} \in \mathbb{R}$, $\ni \lambda_1 v_1 + \cdots + \lambda_{n+1} v_{n+1} = 0$, $(\lambda_1, \cdots, \lambda_{n+1}) \neq (0, \cdots, 0)$.

Definition 1.129. A subset $H \subseteq \mathbb{R}^n$ is called a ***hyperplane*** if H is $(n-1)$-dimensional subspace of $\mathbb{R}^n$. An ***affine hyperplane*** is a set $x + H \equiv \{x + y \mid y \in H\}$ for some hyperplane H.

1.7 Normed Vector Spaces, Inner Product Spaces and Metric Spaces

Definition 1.130. A ***nomed vector space*** $(\mathcal{V}, \|\cdot\|)$ is a real vector space $\mathcal{V}$ associated with a function $\|\cdot\| : \mathcal{V} \to \mathbb{R}$ such that

(a) $\|x\| \geqslant 0$ for all $x \in \mathcal{V}$.

(b) $\|x\| = 0$ if and only if $x = 0$.

(c) $\|\lambda \cdot x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{R}$ and $x \in \mathcal{V}$.

(d) $\|x + y\| \leqslant \|x\| + \|y\|$ for all $x, y \in \mathcal{V}$.

A function $\|\cdot\|$ satisfies (a)-(d) is called a ***norm*** on $\mathcal{V}$.

Example 1.131. Let $\mathcal{V} = \mathbb{R}^n$, and $\|x\|_2 \equiv \Big(\sum_{i=1}^n x_i^2\Big)^{\frac{1}{2}}$ if $x = (x_1, x_2, \cdots, x_n)$. Then $\|\cdot\|_2$ is a norm, called 2-norm, on $\mathbb{R}^n$. It suffices to show that (d) in Definition 1.130 holds. Let $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$. Then

$$\begin{aligned}
\left(\|x+y\|_2\right)^2 &= \sum_{i=1}^n (x_i + y_i)^2 = \sum_{i=1}^n (x_i^2 + 2x_i y_i + y_i^2) = \sum_{i=1}^n x_i^2 + \sum_{i=1}^n y_i^2 + 2\sum_{i=1}^n x_i y_i \\
&\leqslant \|x\|_2^2 + \|y\|_2^2 + 2\|x\|_2 \|y\|_2 \qquad \text{(By Cauchy's inequality)} \\
&= \left(\|x\|_2 + \|y\|_2\right)^2 ;
\end{aligned}$$

thus $\|x + y\|_2 \leqslant \|x\|_2 + \|y\|_2$.

Example 1.132. Let $\mathcal{V} = \mathbb{R}^n$, and define

$$\|x\|_p \equiv \begin{cases} \Big(\sum\limits_{i=1}^{n} |x_i|^p\Big)^{\frac{1}{p}} & \text{if } 1 \leqslant p < \infty, \\ \max\big\{|x_1|, \cdots, |x_n|\big\} & \text{if } p = \infty, \end{cases} \quad \text{for all } x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n.$$

Then $\|\cdot\|_p$ is a norm, called p-norm, on $\mathbb{R}^n$. Property (d) in Definition 1.130; that is, $\|x+y\|_p \leqslant \|x\|_p + \|y\|_p$, is left as an exercise.

Example 1.133. Let $\mathcal{M}_{n\times m} \equiv \big\{n \times m \text{ matrix with entries in } \mathbb{R}\big\}$, and we remind the readers that if $A \in \mathcal{M}_{n\times m}$, then $A : \begin{cases} \mathbb{R}^m \to \mathbb{R}^n \\ \;x \;\mapsto Ax \end{cases}$. Define

$$\|A\|_p = \sup_{\|x\|_p=1} \|Ax\|_p = \sup_{x\neq 0} \frac{\|Ax\|_p}{\|x\|_p} \quad \forall\, A \in \mathcal{M}_{n\times m}\,;$$

that is, $\|A\|_p$ is the least upper bound of the set $\Big\{\dfrac{\|Ax\|_p}{\|x\|_p} \,\Big|\, x \neq 0, x \in \mathbb{R}^m\Big\}$. Therefore, $\dfrac{\|Ax\|_p}{\|x\|_p} \leqslant \|A\|_p \;\forall\, x \neq 0$; thus

$$\|Ax\|_p \leqslant \|A\|_p \|x\|_p \quad \forall\, x \in \mathbb{R}^m\,.$$

Consider the case $p = 1, p = 2$ and $p = \infty$ respectively.

1. $p = 2$: Let $(\cdot,\cdot)_{\mathbb{R}^k}$ denote the inner product in Euclidean space $\mathbb{R}^k$. Then

$$\|Ax\|_2^2 = (Ax, Ax)_{\mathbb{R}^n} = (x, A^{\mathrm{T}}Ax)_{\mathbb{R}^m} = (x, P\Lambda P^{\mathrm{T}}x)_{\mathbb{R}^m} = (P^{\mathrm{T}}x, \Lambda P^{\mathrm{T}}x)_{\mathbb{R}^n}\,,$$

   in which we use the fact that $A^{\mathrm{T}}A$ is symmetric; thus diagonalizable by an orthonormal matrix P (that is, $A^{\mathrm{T}}A = P\Lambda P^{\mathrm{T}}$, $P^{\mathrm{T}}P = I$, Λ is a diagonal matrix). Therefore,

$$\begin{aligned} \sup_{\|x\|_2=1} \|Ax\|_2^2 &= \sup_{\|x\|_2=1} (P^{\mathrm{T}}x, \Lambda P^{\mathrm{T}}x) = \sup_{\|y\|_2=1} (y, \Lambda y) \quad (\text{Let } y = P^{\mathrm{T}}x, \text{ then } \|y\|_2 = 1) \\ &= \sup_{\|y\|_2=1} (\lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2) \\ &= \max\big\{\lambda_1, \cdots, \lambda_n\big\} = \text{maximum eigenvalue of } A^{\mathrm{T}}A \end{aligned}$$

   which implies that $\|A\|_2 = \sqrt{\text{maximum eigenvalue of } A^{\mathrm{T}}A}$.

2. $p = \infty$: $\|A\|_\infty = \sup\limits_{\|x\|_\infty=1} \|Ax\|_\infty = \max\Big\{\sum\limits_{j=1}^{m} |a_{1j}|, \sum\limits_{j=1}^{m} |a_{2j}|, \cdots \sum\limits_{j=1}^{m} |a_{nj}|\Big\}$.

Reason: Let $x = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}$ and $A = \big[a_{ij}\big]_{n\times m}$. Then

$$Ax = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1m}x_m \\ a_{21}x_1 + \cdots + a_{2m}x_m \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nm}x_m \end{bmatrix}$$

Assume $\max\limits_{1\leqslant i\leqslant n} \sum\limits_{j=1}^{m} |a_{ij}| = \sum\limits_{j=1}^{m} |a_{kj}|$ for some $1 \leqslant k \leqslant n$. Let

$$x = \big(\operatorname{sgn}(a_{k1}), \operatorname{sgn}(a_{k2}), \cdots, \operatorname{sgn}(a_{kn})\big) .$$

Then $\|x\|_\infty = 1$, and $\|Ax\|_\infty = \sum\limits_{j=1}^{m} |a_{kj}|$.

On the other hand, if $\|x\|_\infty = 1$, then

$$|a_{i1}x_1 + a_{i2}x_2 + \cdots a_{im}x_m| \leqslant \sum_{j=1}^{m} |a_{ij}| \leqslant \max\Big\{\sum_{j=1}^{m} |a_{1j}|, \sum_{j=1}^{m} |a_{2j}|, \cdots \sum_{j=1}^{m} |a_{nj}|\Big\};$$

thus $\|A\|_\infty = \max\Big\{\sum\limits_{j=1}^{m} |a_{1j}|, \sum\limits_{j=1}^{m} |a_{2j}|, \cdots \sum\limits_{j=1}^{m} |a_{nj}|\Big\}$. In other words, $\|A\|_\infty$ is the largest sum of the absolute value of row entries.

3. $p = 1$: $\|A\|_1 = \max\Big\{\sum\limits_{i=1}^{n} |a_{i1}|, \sum\limits_{i=1}^{n} |a_{i2}|, \cdots, \sum\limits_{i=1}^{n} |a_{im}|\Big\}$.

Example 1.134. Let $\mathscr{C}$ be the collection of all continuous real-valued functions on the interval $[0, 1]$; that is,

$$\mathscr{C} = \big\{f : [0, 1] \to \mathbb{R} \,\big|\, f \text{ is continuous on } [0, 1]\big\} .$$

For each $f \in \mathscr{C}$, we define

$$\|f\|_p = \begin{cases} \Big[\displaystyle\int_0^1 |f(x)|^p dx\Big]^{\frac{1}{p}} & \text{if } 1 \leqslant p < \infty\,, \\ \max\limits_{x\in[0,1]} |f(x)| & \text{if } p = \infty\,. \end{cases}$$

The function $\|\cdot\|_p : \mathscr{C} \to \mathbb{R}$ is a norm on $\mathscr{C}$ (Minkowski's inequality).

Definition 1.135. An ***inner product space*** $\big(\mathcal{V}, \langle\cdot,\cdot\rangle\big)$ is a real vector space $\mathcal{V}$ associated with a function $\langle\cdot,\cdot\rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ such that

(1) $\langle x, x\rangle \geqslant 0, \ \forall\, x \in \mathcal{V}$.

(2) $\langle x, x\rangle = 0$ if and only if $x = 0$.

(3) $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$ for all $x, y, z \in \mathcal{V}$.

(4) $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$ for all $\lambda \in \mathbb{R}$ and $x, y \in \mathcal{V}$.

(5) $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{V}$.

A symmetric bilinear form $\langle \cdot, \cdot\rangle$ satisfies (1)-(5) is called an ***inner product*** on $\mathcal{V}$.

Example 1.136. Let $(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by

$$(x, y) = \sum_{i=1}^{n} x_i y_i \quad \forall\, x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n)\,.$$

Then $(\cdot, \cdot)$ is an inner product on $\mathbb{R}^n$.

Example 1.137. Let $\mathscr{C}$ be defined as in Example 1.134. Define

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx\,.$$

Then $\langle \cdot, \cdot\rangle : \mathscr{C} \times \mathscr{C} \to \mathbb{R}$ satisfies all the properties that an inner product has. Note that $\langle f, f\rangle = \|f\|_2^2$.

Proposition 1.138. *If $\langle \cdot, \cdot\rangle$ is an inner product on a real vector space $\mathcal{V}$. Then*

1. $\langle \lambda v + \mu w, u\rangle = \lambda\langle v, u\rangle + \mu\langle w, u\rangle$ *for all* $u, v, w \in \mathcal{V}$.

2. $\langle u, \lambda v + \mu w\rangle = \lambda\langle u, v\rangle + \mu\langle u, w\rangle$ *for all* $u, v, w \in \mathcal{V}$.

3. $\langle v, \lambda w\rangle = \lambda\langle v, w\rangle$ *for all* $v, w \in \mathcal{V}$.

4. $\langle 0, w\rangle = \langle w, 0\rangle = 0$ *for all* $w \in \mathcal{V}$.

Theorem 1.139. *The inner product $\langle \cdot, \cdot\rangle$ on a real vector space induces a norm $\|\cdot\|$ given by $\|x\| = \sqrt{\langle x, x\rangle}$ and satisfies the **Cauchy-Schwartz inequality***

$$\big|\langle x, y\rangle\big| \leqslant \|x\| \cdot \|y\| \qquad \forall\, x, y \in \mathcal{V}\,. \tag{1.7.1}$$

Proof. First, we observe that for all $x, y \in \mathcal{V}$ fixed, we must have

$$0 \leqslant \langle \lambda x + y, \lambda x + y \rangle = \|x\|^2 \lambda^2 + 2\langle x, y \rangle \lambda + \|y\|^2$$

for all $\lambda \in \mathbb{R}$. Therefore,

$$\langle x, y \rangle^2 - \|x\|^2 \cdot \|y\|^2 \leqslant 0$$

which implies (1.7.1).

It should be clear that (a)-(c) in Definition 1.130 are satisfied. To show that $\|\cdot\|$ satisfies the triangle inequality, by (1.7.1) we find that

$$\begin{aligned}\big(\|x\| + \|y\|\big)^2 - \|x + y\|^2 &= \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 - \langle x + y, x + y \rangle \\ &= 2\big(\|x\|\|y\| - \langle x, y \rangle\big) \geqslant 0\,;\end{aligned}$$

thus the triangle inequality is also valid. □

Corollary 1.140. *Let $f, g : [0, 1] \to \mathbb{R}$ be continuous. Then*

$$\left|\int_0^1 f(x)g(x)dx\right| \leqslant \left(\int_0^1 |f(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_0^1 |g(x)|^2 dx\right)^{\frac{1}{2}}.$$

Definition 1.141. A ***metric space*** (M, d) is a set M associated with a function $d : M \times M \to \mathbb{R}$ such that

(i) $d(x, y) \geqslant 0$ for all $x, y \in M$.

(ii) $d(x, y) = 0$ if and only if $x = y$.

(iii) $d(x, y) = d(y, x)$ for all $x, y \in M$.

(iv) $d(x, y) \leqslant d(x, z) + d(z, y)$ for all $x, y, z \in M$.

A function d satisfies (i)-(iv) is called a ***metric*** on M.

Example 1.142 (Discrete metric)**.** Let M be a non-empty set, and $d_0 : M \times M \to \mathbb{R}$ be defined by

$$d_0(x, y) = \begin{cases} 0 & \text{if } x = y\,, \\ 1 & \text{if } x \neq y\,. \end{cases}$$

Then d_0 is a metric on M, and we call d_0 the discrete metric.

Example 1.143 (Bounded metric)**.** Let (M, d) be a metric space. Define $\rho : M \times M \to \mathbb{R}$ by

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}\,.$$

Then ρ is also a metric on M.

Proposition 1.144. *If $(\mathcal{V}, \|\cdot\|)$ is a normed vector space, then the function $d : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ defined by $d(x, y) = \|x - y\|$ is a metric on $\mathcal{V}$. In other words, (V, d) is a metric space, and we usually write $(\mathcal{V}, \|\cdot\|)$ as the metric space.*

Chapter 2

Point-Set Topology of Metric spaces

2.1 Open Sets and the Interior of Sets

Definition 2.1. Let (M, d) be a metric space. For each $x \in M$ and $\varepsilon > 0$, the set

$$D(x, \varepsilon) = \left\{ y \in M \,\middle|\, d(x, y) < \varepsilon \right\}$$

is called the ε***-disk*** (ε***-ball***) about x or the disk/ball centered at x with radius ε.

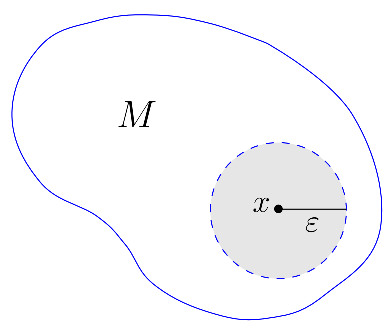


Figure 2.1: The ε-ball about x in a metric space

Example 2.2. $(\mathbb{R}^2, \|\cdot\|_p)$ is a normed vector space. Consider $x = 0$, $\varepsilon = 1$ and $p = 1,\ p = 2$ and $p = \infty$ respectively.

1. $p = 1$: $\|x\|_1 = |x_1| + |x_2|$, $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$.

2. $p = 2$: $\|x\|_2 = \sqrt{x_1^2 + x_2^2}$, $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.

3. $p = \infty$: $\|x\|_\infty = \max\left\{|x_1|, |x_2|\right\}$, $d(x, y) = \max\left\{|x_1 - y_1|, |x_2 - y_2|\right\}$.

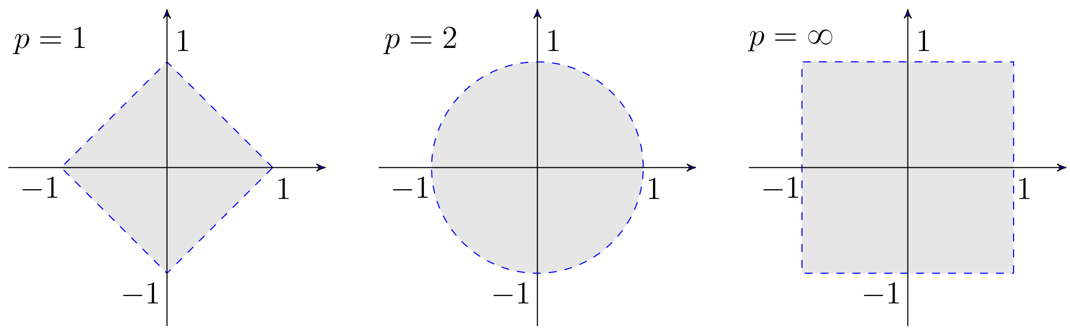


Figure 2.2: The 1-ball about 0 in $\mathbb{R}^2$ with different p

Example 2.3. Let (M, d) be a metric space with discrete metric; that is,

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Then $D(x, \varepsilon) = \begin{cases} \{x\} & \text{if } 0 < \varepsilon \leqslant 1, \\ M & \text{if } \varepsilon > 1. \end{cases}$

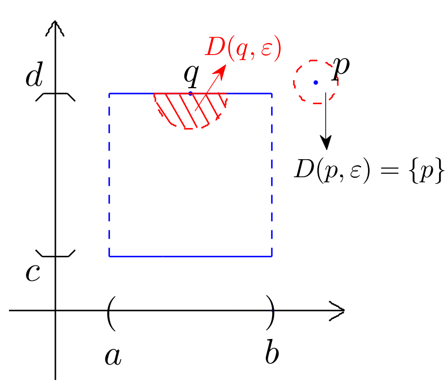
Definition 2.4. Let (M, d) be a metric space. A set $\mathcal{U} \subseteq M$ is said to be ***open*** (in M) if

$$\forall\, x \in \mathcal{U},\ \exists\, \varepsilon > 0 \ni D(x, \varepsilon) \subseteq \mathcal{U}\,.$$

Example 2.5. The set $A = \big\{(x, y) \in \mathbb{R}^2 \,\big|\, 0 < x < 1\big\}$ is open: given $(x, y) \in A$, take $\varepsilon = \min\{1 - x, x\}$, then $D(x, \varepsilon) \subseteq A$.

Example 2.6. $A = \big\{(x, y) \in \mathbb{R}^2 \,\big|\, 0 < x \leqslant 1\big\}$ is not open: let $u = (1, 0)$, then $\forall\, \varepsilon > 0 \ni D(u, \varepsilon) \nsubseteq A$ (since $\big(1 + \frac{\varepsilon}{2}, 0\big) \in D(u, \varepsilon)$ but $\big(1 + \frac{\varepsilon}{2}, 0\big) \notin A$).

Example 2.7. $M \equiv (a, b) \times [c, d] \cup \{p\}$, $p \notin [a, b] \times [c, d]$, $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. Then $D(p, \varepsilon) = \{p\}$ if $\varepsilon \ll 1$. Let $q = \big(\frac{a + b}{2}, d\big)$, $\varepsilon < \min\big\{\frac{b - a}{2}, d - c\big\}$. Then $D(q, \varepsilon)$ is the shaded region in red color shown in the figure below.



Proposition 2.8. *Let (M,d) be a metric space. Then every ε-disk is open.*

Proof. Let $D(x,\varepsilon)$ be an ε-disk. We would like to show that $\forall\, y \in D(x,\varepsilon)$, $\exists\, \delta > 0 \ni D(y,\delta) \subseteq D(x,\varepsilon)$. Let $\delta = \varepsilon - d(x,y) > 0$. Then if $z \in D(y,\delta)$, we have

$$d(z,x) \leqslant d(z,y) + d(x,y) < \delta + d(x,y) = \varepsilon\,;$$

thus $z \in D(x,\varepsilon)$. □

Proposition 2.9. *Let (M,d) be a metric space.*

1. *The intersection of finitely many open sets is open.*
2. *The union of arbitrary family of open sets is open.*
3. *The empty set $\varnothing$ and the universal set M are open.*

Proof. 1. Let $U_1, U_2, \cdots, U_k$ be open sets in M, and $U \equiv \bigcap\limits_{i=1}^{k} U_i$. If $y \in U$, then $y \in U_i$ for all $1 \leqslant i \leqslant k$. Since U_i is open, $\exists\, \delta_i > 0 \ni D(y,\delta_i) \subseteq U_i$. Let $\delta = \min\{\delta_1, \cdots, \delta_k\}$.

   Claim: $D(y,\delta) \subseteq U$.

   Proof of claim: Let $z \in D(y,\delta)$. Then $d(y,z) < \delta \leqslant \delta_i$ if $i = 1, 2, \cdots, k$.

   $\Rightarrow z \in D(y,\delta_i)\ \forall\, i = 1, 2, \cdots, k \Rightarrow z \in U_i$ if $i = 1, 2, \cdots, k \Rightarrow z \in \bigcap\limits_{i=1}^{k} U_i \equiv U$.

2. Let $\mathscr{F} = \big\{U_\alpha \,\big|\, U_\alpha \text{ open in } M,\ \alpha \in I\big\}$ be a family of open sets, and $U \equiv \bigcup\limits_{\alpha \in I} U_\alpha$. If $y \in U$, then $y \in U_\beta$ for some $\beta \in I$. Since U_β is open, $\exists\, \delta > 0 \ni D(y,\delta) \subseteq U_\beta$; thus $D(y,\delta) \subseteq \bigcup\limits_{\alpha \in I} U_\alpha \equiv U$.
3. $\varnothing$ is trivially an open set. Moreover, if $y \in M$, then $D(y,1) \subseteq M$ (by definition). □

Corollary 2.10. *Let (M,d_0) be a metric space with discrete metric. Then every subset of M is open.*

Proof. $\forall\, y \in M, \{y\} = D\big(y, \frac{1}{2}\big)$ is an open set in M. If $A \subseteq M$, $A \neq \varnothing$, then $A = \bigcup\limits_{y \in A} D\big(y, \frac{1}{2}\big)$ which suggests that A is open since it is an arbitrary union of open sets. □

Remark 2.11. Infinite intersection of open sets need not be open:

1. Take $A_n = \big(-\frac{1}{n}, \frac{1}{n}\big)$, then $\bigcap\limits_{n=1}^{\infty} A_n = \{0\}$ which is not open.

2. Let $U_k = (-2 - \frac{1}{k}, 2 + \frac{1}{k}) \subseteq \mathbb{R}$. Then $A = \bigcap\limits_{k=1}^{\infty} U_k \supseteq [-2, 2]$. Moreover, if $x \notin [-2, 2]$, then $\exists\, k \in \mathbb{N} \ni x \notin U_k$ $\big($If $x > 2$, $\frac{1}{k} < \frac{x-2}{2}$. If $x < -2$, $\frac{1}{k} < \frac{-x-2}{2}\big)$. Therefore, $\bigcap\limits_{k=1}^{\infty} U_k = [-2, 2]$.

Example 2.12. Let $A \subseteq \mathbb{R}^n$ be open, and $B \subseteq \mathbb{R}^n$. Then $A + B = \{a + b \,|\, a \in A, b \in B\}$ is open.

Proof. Let $y \in A + B$. Then $y = a + b$ for some $a \in A, b \in B$. Since A is open, $\exists\, \delta > 0 \ni D(a, \delta) \subseteq A$.
Claim: $D(y, \delta) \subseteq A + B$.
Proof of claim: Let $z \in D(y, \delta)$. Then $\|z - y\|_2 < \delta$. Since $z = b + (z - b)$, if we can show that $z - b \in A$, then $z \in A + B$. Nevertheless, we have

$$\|(z - b) - a\|_2 = \|z - a - b\|_2 = \|z - y\|_x < \delta$$

which implies that $z - b \in D(a, \delta) \subseteq A$. ▫

Definition 2.13. Let (M, d) be a metric space, and $A \subseteq M$ be a subset of M. A point $x \in A$ is called an ***interior point*** of A if $\exists\, \varepsilon > 0 \ni D(x, \varepsilon) \subseteq A$. The ***interior*** of A is the collection of all interior points of A, and is denoted by $\text{int}(A)$ or $\mathring{A}$.

Example 2.14. Let $M = \mathbb{R}$ with $d(x, y) = |x - y|$, and $A = [0, 1)$, $B = \big\{1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots\big\} \cup \{0\} = \big\{\frac{1}{n}\big\}_{n=1}^{\infty} \cup \{0\}$. Then $\mathring{A} = (0.1)$ and $\mathring{B} = \varnothing$ since

1. If $x \in (0, 1)$, then $\exists\, \varepsilon > 0 \ni D(x, \varepsilon) \subseteq (0, 1) \subseteq A$.

2. 0 is not an interior point since $(-\varepsilon, \varepsilon) \cap [0, 1)^{\complement} \neq \phi \ \forall\, \varepsilon > 0$.

Remark 2.15. $\mathring{A}$ might be empty.

Theorem 2.16. *Let (M, d) be a metric space, and $A \subseteq M$ be a subset of M. The interior of A is the largest open set contained in A. In other words, if $U \subseteq A$ is open, then $U \subseteq \text{int}(A)$.*

Proof. Let $z \in U$. Since U is open, $\exists\, \delta > 0 \ni D(z, \delta) \subseteq U \subseteq A \Rightarrow z \in \mathring{A} \Rightarrow U \subseteq \mathring{A}$.
To show that $\mathring{A}$ is open, we observe that $\mathring{A} = \bigcup\limits_{x \in \mathring{A}} D(x, \varepsilon_x)$, where $\varepsilon_x > 0$ is chosen so that $D(x, \varepsilon_x) \subseteq \mathring{A}$ if $x \in \mathring{A}$, for the following reason:

1. "$\subseteq$": trivial.

2. "$\supseteq$": Let $y \in \bigcup_{x \in \mathring{A}} D(x, \varepsilon_x) \Rightarrow \exists\, x \in \mathring{A} \ni y \in D(x, \varepsilon_x)$. Then if $\delta = \varepsilon_x - d(x, y)$,
$$D(y, \delta) \subseteq D(x, \varepsilon_x) \subseteq A \Rightarrow y \in \mathring{A}.$$ □

Theorem 2.17. *Let (M, d) be a metric space. A set $A \subseteq M$ is open if and only if $A = \mathring{A}$.*

Example 2.18. Let (M, d) be a metric space, and A and B be two subsets of M.

1. $\operatorname{int}(A) \cup \operatorname{int}(B) \subseteq \operatorname{int}(A \cup B)$.

   Proof. Let $x \in \operatorname{int}(A) \cup \operatorname{int}(B)$. W.L.O.G. Assume $x \in \operatorname{int}(A)$, then $\exists\, r > 0$ such that $D(x, r) \subseteq A$. Therefore, $x \in D(x, r) \subseteq A \cup B$, so $x \in \operatorname{int}(A \cup B)$. □

2. Strict containment might happen because of the following example:

   Take $A = [0, 1]$, $B = [1, 2]$, then $\operatorname{int}(A) = (0, 1)$, $\operatorname{int}(B) = (1, 2)$.

   Sine $A \cup B = [0, 2]$, $\operatorname{int}(A \cup B) = (0, 2)$; however, $\operatorname{int}(A) \cup \operatorname{int}(B) = (0, 2)\backslash\{1\}$.

   Hence, $\operatorname{int}(A) \cup \operatorname{int}(B) \neq \operatorname{int}(A \cup B)$.

   Another example is stated as follows: Let $A = \mathbb{Q} \cap [0, 1]$ and $B = \mathbb{Q}^{\complement} \cap [0, 1]$. Then
$$(0, 1) = \operatorname{int}([0, 1]) = \operatorname{int}(A \cup B) \supsetneq \operatorname{int}(A) \cup \operatorname{int}(B) = \varnothing.$$

Example 2.19. In a metric space (M, d), it is not always true that $\operatorname{int}\big(\{y \in M \,\big|\, d(x, y) \leqslant R\}\big) = \{y \in M \,\big|\, d(x, y) < R\}$. To see this, we consider the discrete metric
$$d_0(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = 0. \end{cases}$$

Let $R = 1$, and fix $x_0 \in M \neq \varnothing$. Then
$$\{y \in M \,\big|\, d_0(y, x_0) \leqslant 1\} = M \Rightarrow \operatorname{int}\big(\{y \in M \,\big|\, d_0(y, x_0) \leqslant 1\}\big) = \operatorname{int}(M) = M\,.$$

Now $\{y \in M \,\big|\, d_0(y, x_0) < 1\} = \{x_0\}$. As long as M has more than one point, we have $\operatorname{int}\big(\{y \in M \,\big|\, d_0(y, x_0) \leqslant 1\}\big) = M \neq \{x_0\} = D(x_0, 1)$.

2.2 Closed Sets, the Closure of Sets, and the Boundary of Sets

Definition 2.20. Let (M,d) be a metric space. A set $F \subseteq M$ is said to be ***closed*** if $F^{\complement} = M\backslash F$ is open. In other words,

$$F \text{ is closed} \Leftrightarrow \forall\, x \in F^{\complement}, \exists\, \varepsilon > 0 \ni D(x,\varepsilon) \subseteq F^{\complement}\,.$$

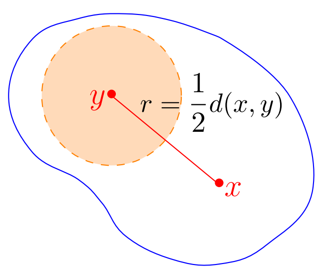
Example 2.21. The set $[0,1] \subseteq \mathbb{R}$ is closed, and the set $(0,1] \subseteq \mathbb{R}$ is not open and not closed.

Example 2.22. Let $S = \{(x,y) \mid x^2+y^2 \leqslant 1\}$. Take $z \in \mathbb{R}^2\backslash S$, then $D(z, \|z\|_2 - 1) \subseteq \mathbb{R}^2\backslash S$. As a consequence, $\mathbb{R}^2\backslash S$ is open; thus S is closed.

Example 2.23. Let $S = \big\{(x,y)\,\big|\,0 < x \leqslant 1, 0 \leqslant y \leqslant 1\big\}$. Since $\mathbb{R}^2\backslash S$ is not open, S is not closed.

Proposition 2.24. *Any point in a metric space is closed; that is, if (M,d) is a metric space and $A = \{x\}$ for some $x \in M$, then A is closed.*

Proof. We show that $M\backslash\{x\}$ is open. Let $y \in M\backslash\{x\}$. Pick $r = \dfrac{1}{2}d(x,y) > 0$.
Claim: $D(y,r) \subseteq M\backslash\{x\}$.



Proof of claim: Let $z \in D(y,r)$. Then $d(z,y) < r = \dfrac{1}{2}d(x,y)$. Then

$$d(z,x) \geqslant d(x,y) - d(y,z) \geqslant d(x,y) - \frac{1}{2}d(x,y) = \frac{1}{2}d(x,y) > 0 \Rightarrow z \neq x\,. \qquad \square$$

Proposition 2.25. *Let (M,d) be a metric space.*

1. *The union of finitely many closed sets is closed.*

2. *The intersection of arbitrary family of closed sets is closed.*

3. *The universal set M and the empty set $\varnothing$ are closed.*

Proof. 1. Let $F_1, \cdots, F_k$ be closed sets, and $F = \bigcup_{j=1}^{k} F_j$. Then by De Morgan's law,

$$F^{\complement} = M \backslash F = M \backslash \bigcup_{j=1}^{k} F_j = \bigcap_{j=1}^{k} (M \backslash F_j) = \bigcap_{j=1}^{k} F_j^{\complement}.$$

Since F_j is closed, $F_j^{\complement}$ is open. By Proposition 2.9, $\bigcap_{j=1}^{k} F_j^{\complement}$ is open.

2. Let $\mathscr{F} = \{F_\alpha \,|\, F_\alpha \text{ closed in } M, \alpha \in I\}$ be a family of closed sets, and $F \equiv \bigcap_{\alpha \in I} F_\alpha$. Then by De Morgan's law,

$$F^{\complement} = M \backslash \bigcap_{\alpha \in I} F_\alpha = \bigcup_{\alpha \in I} (M \backslash F_\alpha) = \bigcup_{\alpha \in I} F_\alpha^{\complement}$$

which suggests that $F^{\complement}$ is the union of open sets $\{F_\alpha^{\complement}\}_{\alpha \in I}$. By Proposition 2.9 we conclude that $F^{\complement}$ is open or equivalently, F is closed.

3. $M^{\complement} = \varnothing, \varnothing^{\complement} = M$ are both closed. □

Corollary 2.26. *Any set consists of finitely many points of a metric space is closed.*

Example 2.27. Let $F_k = \left[-2 + \frac{1}{k}, 2 - \frac{1}{k}\right] \subseteq \mathbb{R}$. Then $B = \bigcup_{k=1}^{\infty} F_k \subseteq (-2, 2)$. Moreover, if $x \in (-2, 2)$, then $\exists\, k > 0$, $\ni x \in F_k$ $\left(\text{If } x \leqslant 0, \frac{1}{k} < \frac{x+2}{2}. \text{ If } x > 0, \frac{1}{k} < \frac{2-x}{2}\right)$. Therefore, $\bigcup_{k=1}^{\infty} F_k = (-2, 2)$. This example suggests that an arbitrary union of closed sets might not be closed.

Example 2.28. Let (M, d) be a metric space, and $A = \{y_1, y_2, \ldots, y_k\} \subseteq M$. Define $B = \{x \in M \,|\, d(x, y_i) \leqslant 1 \text{ for some } y_i \in A\} = \bigcup_{i=1}^{k} \{x \in M \,|\, d(x, y_i) \leqslant 1\}$. Then B is closed.

Proof. It suffices to show $B_i = \{x \in M \,|\, d(x, y_i) \leqslant 1\}$ is closed for $i = 1, 2, \ldots, k$ since $B = \bigcup_{i=1}^{n} F_i$.. Take $z \in M \backslash B_i$ (if $M \backslash B_i = \varnothing$, then $B_i = M$ and B_i is closed). Let $N = \{u \in M \,|\, d(u, z) < d(z, y_i) - 1\}$.
Claim: $N \subseteq M \backslash B_i$; that is, $M \backslash B_i$ is open.
Proof of claim: Take $u \in N$ and compute $d(u, y_i) \geqslant d(y_i, z) - d(u, z) > d(z, y_i) - (d(z, y_i) - 1) = 1$. Hence $u \notin B_i \Rightarrow u \in M \backslash B_i$. So $N \subseteq M \backslash B_i$. □

Example 2.29. Let (M, d) be a metric space, $A \subseteq M$ be closed, and $B \subseteq M$ be finite ($\#(B) < \infty$). Then $A + B$ is closed.

Proof. Left as an exercise. $\square$

Definition 2.30. Let (M, d) be a metric space, and $A \subseteq M$.

1. A point $x \in M$ is called an ***accumulation point*** of A if $\forall\, \varepsilon > 0, D(x, \varepsilon)$ contains points in A other than x; that is, $\forall\, \varepsilon > 0, D(x, \varepsilon) \cap (A \backslash \{x\}) = (D(x, \varepsilon) \backslash \{x\}) \cap A \neq \varnothing$.

2. A point $x \in M$ is called a ***limit point*** of A if $\forall\, \varepsilon > 0, D(x, \varepsilon)$ contains points in A; that is, $\forall\, \varepsilon > 0, D(x, \varepsilon) \cap A \neq \varnothing$.

3. A point $x \in A$ is called an ***isolated point*** (孤立點) if $\exists\, \varepsilon > 0 \ni D(x, \varepsilon) \cap A = \{x\}$.

4. The ***derived set*** of A is the collection of all accumulation points of A, and is denoted by A'.

5. The collection of all limit points of A is denoted by $\bar{A}$.

Remark 2.31. 1. An accumulation point of A needs not to be in A.

2. If $A = \{x\}$ (that is, a single point), then A has no accumulation point; that is, $A' = \varnothing$.

3. Accumulation points are called cluster points in some books.

4. If $x \in A'$, then x is a limit point of A. In other words, $A' \subseteq \bar{A}$.

5. If $x \in A$, then x is a limit point of A. In other words, $A \subseteq \bar{A}$.

Example 2.32. Let $A = (0, 1) \subseteq \mathbb{R}$, then $A' = [0, 1]$ and $\bar{A} = [0, 1]$.

Example 2.33. Let $A = (0, 1) \cup \{2\} \subseteq \mathbb{R}$. Then

1. for any $x \in [0, 1], x \in A'$;

2. $2 \notin A'$, but 2 is a limit point of A;

3. if $x \notin [0, 1] \cup \{2\}$ then $x \notin A'$.

Therefore, $A' = [0, 1]$. Note that $\sup A = 2$; thus $\sup A$ might not belong to A'.

Example 2.34. Let $A = \{x_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ consists of a bounded sequence of distinct points. Then $A' \neq \varnothing$.

Proof. By Bolzano-Weierstrass property (Theorem 1.95), A has a convergent subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ converging to $x \in \mathbb{R}$.
Claim: $x \in A'$.
Proof of claim: $\forall\, \varepsilon > 0, \exists\, K \in \mathbb{N} \ni \left|x_{n_j} - x\right| < \varepsilon$ for $j \geqslant K$. Moreover, $x_{n_j} \in A$. ▫

Example 2.35. In a metric space (M, d), let $B(x,r) = \{y \in M \,|\, d(x,y) \leqslant r\}$. Is it true that $B(x,r) \subseteq D(x,r)'$; that is, every point of $B(x,r)$ is an accumulation point of $D(x,r)$? Answer: No, take a metric space with discrete metric

$$d_0(x,y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = 0. \end{cases}$$

and M has more than one point. We have $D(x,1) = \{x\}$, then $D(x,1)' = \varnothing$. Also, $B(x,1) = M \nsubseteq \varnothing = D(x,1)'$.

Proposition 2.36. *If $A \subseteq B$, then $A' \subseteq B'$.*

Proof. Let $x \in A'$. Then $\forall\, \varepsilon > 0, \exists\, y \in A, y \neq x \ni y \in D(x,\varepsilon) \cap A$. Since $A \subseteq B$, $y \in B$. Therefore, $\forall\, \varepsilon > 0, \exists\, y \in B, y \neq x \ni y \in D(x,\varepsilon) \cap B \Leftrightarrow x \in B'$. ▫

Example 2.37. Let A be a subset of $\mathbb{R}^n$. An interior point of A is an accumulation point of A ($\mathring{A} \subseteq A'$ if $A \subseteq \mathbb{R}^n$).

Proof. If $x \in \mathring{A}$, then $\exists\, r > 0, \ni D(x,r) \subseteq A$. Let $\varepsilon > 0$ be given.

1. $\varepsilon \geqslant r, D(x,\varepsilon) \cap (A\backslash\{x\}) \supseteq D(x,r) \cap (A\backslash\{x\}) \neq \varnothing$.

2. $\varepsilon < r, D(x,\varepsilon) \subseteq D(x,r) \subseteq A \Rightarrow D(x,\varepsilon) \cap (A\backslash\{x\}) \neq \varnothing$.

Then for all $\varepsilon > 0$, $D(x,\varepsilon) \cap (A\backslash\{x\}) \neq \varnothing \Rightarrow x \in A'$. ▫

Theorem 2.38. *Let (M, d) be a metric space and $A \subseteq M$, then A is closed if and only if $A = \bar{A}$.*（一個集合是閉集合若且唯若該集合包含了它所有的 *limit points*）

Proof. A is closed $\Leftrightarrow \forall\, y \in A^{\complement}, \exists\, r > 0 \ni D(y,r) \subseteq A^{\complement}$ (or $D(y,r) \cap A = \varnothing$).
$\Leftrightarrow \forall\, y \in A^{\complement}, y \notin \bar{A}$ (or $y \in \bar{A}^{\complement}$).
$\Leftrightarrow$ if $y \in \bar{A}$, then $y \in A$. ▫

Theorem 2.39. *Let (M,d) be a metric space and $A \subseteq M$. Then $\bar{A} = A \cup A'$ $\big(= (A\backslash A') \cup A'\big)$.*

Proof. By definition, $x \in \bar{A} \Leftrightarrow \forall\, \varepsilon > 0, D(x,\varepsilon) \cap A \neq \varnothing$.

$\Rightarrow$ If $x \in \bar{A}\backslash A$, then $\forall\, \varepsilon > 0, D(x,\varepsilon) \cap (A\backslash\{x\}) \neq \varnothing$.

$\Rightarrow$ If $x \in \bar{A}\backslash A$, then $x \in A'$.

Therefore, $\bar{A}\backslash A \subseteq A' \Rightarrow \bar{A} \subseteq A \cup A'$. On the other hand, we also have (1) $A \subseteq \bar{A}$ and (2) $A' \subseteq \bar{A}$; thus $A \cup A' \subseteq \bar{A}$. □

Corollary 2.40. *Let (M,d) be a metric space, and $A \subseteq B \subseteq M$. Then $\bar{A} \subseteq \bar{B}$. In particular, if $A \subseteq B$ and B is closed, then $\bar{A} \subseteq B$.*

Proposition 2.41. *Let (M,d) be a metric space, and $A \subseteq M$. Then $A\backslash A'$ is the collection of all isolated points of A.*

Proof. Let $x \in A\backslash A'$. Then $x \in A$, but $\exists\, \varepsilon > 0 \ni D(x,\varepsilon) \cap (A\backslash\{x\}) = \varnothing$. Therefore, $D(x,\varepsilon) \cap A = \{x\}$ which implies that x is an isolated point. □

Theorem 2.42. *Let (M,d) be a metric space, and $A \subseteq M$. Then A' is closed; that is, $\forall\, y \notin A', \exists\, r > 0 \ni D(y,r) \cap A' = \varnothing$.*

Proof. Let $y \notin A'$. Then $\exists\, \varepsilon > 0 \ni D(y,\varepsilon) \cap (A\backslash\{y\}) = (D(y,\varepsilon)\backslash\{y\}) \cap A = \varnothing$. Then

$$A \subseteq \big(D(y,\varepsilon)\backslash\{y\}\big)^\complement .$$

Since $D(y,\varepsilon)\backslash\{y\} = D(y,\varepsilon) \cap \{y\}^\complement$ is open, $\big(D(y,\varepsilon)\backslash\{y\}\big)^\complement$ is closed; thus Corollary 2.40 implies that

$$\bar{A} \subseteq \big(D(y,\varepsilon)\backslash\{y\}\big)^\complement \quad \text{or equivalently,} \quad \bar{A} \cap D(y,\varepsilon)\backslash\{y\} = \varnothing .$$

Since $\bar{A} = A \cup A'$, the equality above suggests that

$$A' \cap D(y,\varepsilon)\backslash\{y\} = \varnothing ;$$

thus the fact that $y \notin A'$ implies that $D(y,\varepsilon) \cap A' = \varnothing$. □

Definition 2.43. Let (M,d) be a metric space and $A \subseteq M$. The ***closure*** of A is the intersection of closed sets containing A, and is denoted by $\text{cl}(A)$. In other word, $\text{cl}(A) = \bigcap\limits_{\substack{F \text{ closed.} \\ A \subseteq F}} F$ (thus $\text{cl}(A)$ is the smallest closed set containing A).

Proposition 2.44. *Let (M,d) be a metric space, and $A \subseteq M$.*

1. $A \subseteq \mathrm{cl}(A)$ *($x \in A \Rightarrow$ if $F \supseteq A$ is closed, then $x \in F$).*

2. *A is closed if and only if $A = \mathrm{cl}(A)$.*

Proposition 2.45. *Let (M,d) be a metric space, and $A \subseteq M$. Then* $\mathrm{cl}(A) = \bar{A}(= A \cup A')$.

Proof. Since $A \subseteq \mathrm{cl}(A)$ and $\mathrm{cl}(A)$ is closed, Corollary 2.40 implies that $\bar{A} \subseteq \mathrm{cl}(A)$.

On the other hand, if $x \notin A \cup A' = \bar{A}$, then $\exists\, r > 0 \ni D(x,r) \cap A = \varnothing$ or in other words, $A \subseteq D(x,r)^{\complement}$. By the definition of the closure of sets, $\mathrm{cl}(A) \subseteq D(x,r)^{\complement}$ or equivalently, $D(x,r) \subseteq \mathrm{cl}(A)^{\complement}$; thus $x \notin \mathrm{cl}(A)$. Therefore, $\mathrm{cl}(A) \subseteq \bar{A}$. ▫

Example 2.46. Let $A = [0.1) \cup \{2\} \subseteq \mathbb{R}$. Find $\mathrm{cl}(A)$.
Answer: $A' = [0.1]$, $\mathrm{cl}(A) = A \cup A' = [0,1] \cup \{2\}$.

Example 2.47. $\mathrm{cl}(A \cap B) \stackrel{?}{=} \mathrm{cl}(A) \cap \mathrm{cl}(B)$.
Answer: No. Take $A = [0,1], B = (1,2]$. Since A is closed, then $\mathrm{cl}(A) = A$. Since $\mathrm{cl}(B) = [1,2]$, $A \cap B = \varnothing$. So $\mathrm{cl}(A \cap B) = \varnothing \neq \{1\} = \mathrm{cl}(A) \cap \mathrm{cl}(B)$; thus $\mathrm{cl}(A \cap B) \subsetneq \mathrm{cl}(A) \cap \mathrm{cl}(B)$.

Example 2.48. In a metric space (M,d),

$$x \in \mathrm{cl}(A) \text{ if and only if } d(x,A) \equiv \inf\big\{d(x,y) \,\big|\, y \in A\big\} = 0.$$

Proof. "$\Leftarrow$" Suppose $d(x,A) = 0$. If $x \in A$, then $x \in A \cup A' = \mathrm{cl}(A)$. If $x \notin A$, since $d(x,A) = 0$, $\forall\, \varepsilon > 0\ \exists\, y \in A \ni d(x,y) < d(x,A) + \varepsilon = \varepsilon$. In other words, $(D(x,\varepsilon)\backslash\{x\}) \cap A \neq \varnothing$. Therefore, $x \in A'$; thus $x \in A \cup A' = \mathrm{cl}(A)$.

"$\Rightarrow$" Suppose $x \in \mathrm{cl}(A)$. Since $\bar{A} = \mathrm{cl}(A)$, $\forall\, \varepsilon > 0$, $D(x,\varepsilon) \cap A \neq \varnothing$. In other words,

$$\forall\, \varepsilon > 0,\ \exists\, y \in A \ni d(x,y) < \varepsilon.$$

Therefore, $d(x,A) < \varepsilon$ for all $\varepsilon > 0$ which implies that $d(x,A) = 0$. ▫

Example 2.49. $A = \big\{\frac{1}{n} \,\big|\, n = 1,2,\cdots\big\}$. Find $\mathrm{cl}(A)$.
Answer: $A' = \{0\} \Rightarrow \mathrm{cl}(A) = A \cup A' = \big\{\frac{1}{n} \,\big|\, n = 1,2,\cdots\big\} \cup \{0\}$.

Example 2.50. $A = \big\{(x,y) \,\big|\, x \in \mathbb{Q}\big\}$. Find $\mathrm{cl}(A)$.
Answer: $A' = \mathbb{R}^2 \Rightarrow \mathrm{cl}(A) = \mathbb{R}^2$.

Definition 2.51. Let (M, d) be a metric space. A subset $A \subseteq M$ is said to be ***dense***（稠密）in another subset $B \subseteq M$ if $A \subseteq B \subseteq \mathrm{cl}(A)$.

Example 2.52. The rational numbers $\mathbb{Q}$ is dense in the real number system $\mathbb{R}$.

Definition 2.53. Let (M, d) be a metric space, and $A \subseteq M$. The ***boundary*** of A, denoted by $\mathrm{bd}(A)$ or ∂A, is the intersection of $\bar{A}$ and $\overline{A^{\complement}}$ ($\partial A = \bar{A} \cap \overline{A^{\complement}}$).

Remark 2.54. 1. ∂A is closed since the closure of a set is closed.

2. By the definition of limit points of a set, we find that $x \in \partial A \Leftrightarrow \forall\, \varepsilon > 0, D(x, \varepsilon) \cap A \neq \varnothing$ and $D(x, \varepsilon) \cap A^{\complement} \neq \varnothing$.

3. $\partial A = \partial(A^{\complement})$.

Proposition 2.55. *Let (M, d) be a metric space, and $A \subseteq M$. Then $\partial A = \bar{A} \backslash \mathring{A}$.*

Proof. If $x \in \partial A$, then $\forall\, \varepsilon > 0, D(x, \varepsilon) \cap A^{\complement} \neq \varnothing$. Therefore, $x \notin \mathring{A}$ which implies that $\partial A \subseteq \bar{A} \backslash \mathring{A}$.

On the other hand, if $x \in \bar{A} \backslash \mathring{A}$, then $\forall\, \varepsilon > 0, D(x, \varepsilon) \nsubseteq A$. As a consequence, $\forall\, \varepsilon > 0, D(x, \varepsilon) \cap A^{\complement} \neq \varnothing$; thus $x \in \overline{A^{\complement}}$ and this further implies that $x \in \bar{A} \cap \overline{A^{\complement}} = \partial A$. □

Example 2.56. Let $M = \mathbb{R}$, $d(x, y) = |x - y|$, and $A = [0, 1] \cap \mathbb{Q}$. Then

1. $A' = [0, 1]$.
   $\big(r \in A, r + \dfrac{1}{n} \in A, r + \dfrac{1}{n} \to r \Rightarrow r \in A'$.
   If $s \in [0, 1] \cap \mathbb{Q}^{\complement}$. $\exists\, s_n \in A, s_n \to s \Rightarrow s \in A'$.
   If $t \notin [0, 1]$, $\exists \varepsilon > 0 \ni D(t, \varepsilon) \cap [0, 1] = \varnothing \Rightarrow t \notin A'\big)$.

2. $\bar{A} = [0, 1] (= A \cup A')$. 3. $\mathring{A} = \varnothing$. 4. $\partial A = [0, 1]$.

Example 2.57. Let (M, d) be a metric space with discrete metric, and $A \subseteq M$. Recall that every point is an open set.

1. A is open. 2. A is also closed since $A^{\complement}$ is open. 3. $\mathring{A} = A$. 4. $A' = \varnothing$.

5. $\mathrm{cl}(A) = \bar{A} = A$. 6. $\partial A = \varnothing$.

Remark 2.58. If $A \subseteq B$, then $\partial A \nsubseteq \partial B$. For example, let $A = \mathbb{Q} \cap [0, 1]$ and $B = [0, 1]$. Then $A \subseteq B$ but $\partial A = [0, 1]$, $\partial B = \{0, 1\}$.

Example 2.59. $\partial A \nsubseteq A'$: take $A = \{0\}$, then $A' = \varnothing$, $\partial A = \{0\}$.

Example 2.60. It is not always true that $\partial A = \partial(\operatorname{int}(A))$. For example, take $A = [0,1] \cup \{2\}$, then $\partial A = \{0,1,2\}, \operatorname{int}(A) = (0,1), \partial(\operatorname{int}(A)) = \{0,1\}$, so $\partial A \neq \partial(\operatorname{int}(A))$.

Example 2.61. Let (M, d) be a metric space, and $A, B \subseteq M$. Then

$$\partial(A \cup B) \subseteq \partial A \cup \partial B \qquad \text{and} \qquad \partial(A \cap B) \subseteq \partial A \cup \partial B$$

since

$$\begin{aligned}
x \in \partial(A \cup B) &\Leftrightarrow \forall\, r > 0, D(x,r) \cap (A \cup B) \neq \varnothing \text{ and } D(x,r) \cap (A^{\complement} \cap B^{\complement}) \neq \varnothing \\
&\Rightarrow \forall\, r > 0, D(x,r) \cap A^{\complement} \neq \varnothing, D(x,r) \cap B^{\complement} \neq \varnothing, \text{ and one of the following} \\
&\qquad \text{holds: } D(x,r) \cap A \neq \varnothing \text{ or } D(x,r) \cap B \neq \varnothing \\
&\Rightarrow x \in \bar{A} \cap \overline{A^{\complement}} \text{ or } x \in \bar{B} \cap \overline{B^{\complement}}\,,
\end{aligned}$$

and with $A^{\complement}, B^{\complement}$ replacing A, B in the inclusion we just arrive,

$$\partial(A \cap B) = \partial(A \cap B)^{\complement} = \partial(A^{\complement} \cup B^{\complement}) \subseteq \partial A^{\complement} \cup \partial B^{\complement} = \partial A \cup \partial B\,.$$

2.3 Sequences and Completeness（完備性）

Definition 2.62. Let (M, d) be a metric space. A sequence in (M, d) is a function $f : \mathbb{N} \to M$, and is denoted by $\big\{f(n)\big\}_{n=1}^{\infty}$. Write x_n for $f(n)$. A sequence $\{x_n\}_{n=1}^{\infty}$ in M is said to converge to x if

$$\begin{aligned}
&\forall\, \varepsilon > 0,\ \exists\, N > 0\ \ni d(x_n, x) < \varepsilon \text{ whenever } n \geqslant N. \\
\Leftrightarrow\ &\forall\, \varepsilon > 0,\ \#\big\{n \in \mathbb{N} \,\big|\, d(x_n, x) \geqslant \varepsilon\big\} < \infty. \\
\Leftrightarrow\ &\forall\, \varepsilon > 0,\ \#\big\{n \in \mathbb{N} \,\big|\, x_n \notin D(x, \varepsilon)\big\} < \infty.
\end{aligned}$$

As Definition 1.47, one writes $\lim\limits_{n \to \infty} x_n = x$ or $x_n \to x$ as $n \to \infty$ to denote that the sequence $\{x_n\}_{n=1}^{\infty}$ converges to x.

Remark 2.63. Let (M, d) be a metric space, $A \subseteq M$ be a subset.

$$\begin{aligned}
x \text{ is a limit point of } A &\Leftrightarrow \forall\, \varepsilon > 0,\ D(x, \varepsilon) \cap A \neq \varnothing. \\
&\Leftrightarrow \forall\, n > 0,\ \exists\, x_n \in A,\ x_n \in D(x, \frac{1}{n}). \\
&\Leftrightarrow \forall\, n > 0,\ \exists\, x_n \in A, d(x_n, x) < \frac{1}{n}. \\
&\Leftrightarrow \exists\, \{x_n\}_{n=1}^{\infty} \subseteq A \ni x_n \to x \text{ as } n \to \infty
\end{aligned}$$

and

$$
\begin{aligned}
y \text{ is an accumulation point of } A &\Leftrightarrow \forall\, \varepsilon > 0,\ D(y,\varepsilon) \cap (A\backslash\{y\}) \neq \varnothing. \\
&\Leftrightarrow \forall\, n > 0,\ \exists\, y_n \neq y,\ y_n \in A,\ y_n \in D(y, \frac{1}{n}). \\
&\Leftrightarrow \forall\, n > 0,\ \exists\, y_n \neq y,\ y_n \in A,\ d(y_n, y) < \frac{1}{n}. \\
&\Leftrightarrow \exists\, \{y_n\}_{n=1}^{\infty} \subseteq A\backslash\{y\} \ni y_n \to y \text{ as } n \to \infty.
\end{aligned}
$$

Remark 2.64. A is closed $\Leftrightarrow A = \mathrm{cl}(A) = \bar{A} \Leftrightarrow$ If $\{x_n\}_{n=1}^{\infty} \subseteq A$ and $x_n \to x$ as $n \to \infty$, then $x \in A$.

Remark 2.65. The sequence $\{x_k\}_{k=1}^{\infty}$ does not converge to x as $k \to \infty$ if

$$
\begin{aligned}
\exists\, \varepsilon > 0 \ni \forall\, N > 0,\ \exists\, k \geqslant N \ni d(x_k, x) \geqslant \varepsilon &\Leftrightarrow \exists\, \varepsilon > 0 \ni \#\big\{n \in \mathbb{N} \,\big|\, d(x_n, x) \geqslant \varepsilon\big\} = \infty \\
&\Leftrightarrow \exists\, \varepsilon > 0 \ni \#\big\{n \in \mathbb{N} \,\big|\, x_n \notin D(x,\varepsilon)\big\} = \infty.
\end{aligned}
$$

Proposition 2.66. *In $\mathbb{R}^n$, a sequence of vectors converges if and only if every component of the vectors converges. In other words, in $\mathbb{R}^n$*

$$\textit{Componentwise convergence} \ \Leftrightarrow\ \textit{Convergence.}$$

Proof. Let $\{v_k\}_{k=1}^{\infty}$, $v_k = (v_k^{(1)}, v_k^{(2)}, \cdots, v_k^{(n)})$, be a sequence of vectors in $\mathbb{R}^n$.

"$\Rightarrow$" Suppose $v_k \to v = (v^{(1)}, \cdots, v^{(n)})$ as $k \to \infty$. Then

$$\forall\, \varepsilon > 0,\ \exists\, N > 0 \ni \|v_k - v\|_2 < \varepsilon \text{ whenever } k \geqslant N;$$

thus if $k \geqslant N$,

$$|v_k^{(i)} - v^{(i)}| \leqslant \|v_k - v\|_2 = \sqrt{(v_k^{(1)} - v^{(1)})^2 + \cdots + (v_k^{(n)} - v^{(n)})^2} < \varepsilon.$$

"$\Leftarrow$" Assume that $v_k^{(i)} \to u_i$ as $k \to \infty$ for $i = 1, 2, \cdots, n$. Then

$$\forall\, \varepsilon > 0, \exists\, N_i > 0,\ \ni |v_k^{(i)} - u_i| < \frac{\varepsilon}{\sqrt{n}} \text{ whenever } k \geqslant N_i.$$

Let $N = \max\{N_1, N_2, \cdots, N_n\}$. Then if $k \geqslant N$,

$$\|v_k - u\|_2 = \sqrt{(v_k^{(1)} - u_1)^2 + \cdots + (v_k^{(n)} - u_n)^2} < \sqrt{\frac{\varepsilon^2}{n} + \cdots + \frac{\varepsilon^2}{n}} = \varepsilon. \qquad \square$$

Example 2.67. Let $v_k = \big(\frac{1}{k}, \frac{1}{k^2}\big) \in \mathbb{R}^2$. Then $v_k \to (0,0)$ as $k \to \infty$ since

$$\sqrt{(\frac{1}{k}-0)^2 + (\frac{1}{k^2}-0)^2} = \frac{1}{k^2}\sqrt{k^2+1} \to 0 \text{ as } k \to \infty\,.$$

Proposition 2.68. *Suppose that $\{v_k\}_{k=1}^\infty$ and $\{w_k\}_{k=1}^\infty$ are sequences of vectors in a normed space $(\mathcal{V}, \|\cdot\|)$, λ_k is a sequence in $\mathbb{R}$, and $v_k \to v,\ w_k \to w$ in $\mathcal{V}$, $\lambda_k \to \lambda$ in $\mathbb{R}$ as $k \to \infty$. Then*

1. $v_k + w_k \to v + w$ *as* $k \to \infty$.

2. $\lambda_k v_k \to \lambda v$ *as* $k \to \infty$.

3. $\dfrac{1}{\lambda_k} v_k \to \dfrac{1}{\lambda} v$ *as* $k \to \infty$ *if* $\lambda_k \neq 0,\ \lambda \neq 0$.

Proposition 2.69. *Let (M, d) be a metric space.*

1. *A set $A \subseteq M$ is closed if and only if every convergent sequence $\{x_k\}_{k=1}^\infty \subseteq A$ converges to a limit in A.*

2. *$x \in \bar{A}$ if and only if there is a sequence $\{x_k\}_{k=1}^\infty \subseteq A,\ \ni x_k \to x$ as $k \to \infty$*

Proof. "$\Rightarrow$" Since A is closed, Theorem 2.38 implies that $A = \bar{A}$. Let $\{x_k\}_{k=1}^\infty \subseteq A$ be a convergent sequence with limit x. Then

$$\forall\, \varepsilon > 0,\ \exists\, N > 0 \ni d(x_k, x) < \varepsilon \text{ whenever } k \geqslant N.$$

Therefore,

$$\forall\, \varepsilon > 0,\ D(x,\varepsilon) \cap A \supseteq \{x_k\}_{k=N}^\infty \neq \varnothing$$

which implies that $x \in \bar{A} (= A)$.

"$\Leftarrow$" Assume the contrary that A is not closed. Then

$$\exists\, x \in A^\complement \ \ni \forall\, \varepsilon > 0,\ D(x,\varepsilon) \nsubseteq A^\complement.$$

Let $\varepsilon = \dfrac{1}{n},\ x_n \in D\big(x, \dfrac{1}{n}\big) \cap A$. Then $\{x_n\}_{n=1}^\infty \subseteq A$ and $x_n \to x$ as $n \to \infty$; thus we obtain a sequence $\{x_n\}_{n=1}^\infty$ which converges to a point $x \notin A$, a contradiction. □

Example 2.70. Suppose $\{x_k\} \subseteq \mathbb{R}^n$ is such that (i) $\|x_k\| \leqslant 1$ (ii) $x_k \to x$ as $k \to \infty$.
Question 1: $\|x\| \leqslant 1$?
Question 2: Can $\leqslant$ be replaced by $<$; that is, is it true that $\|x_k\| < 1$, $x_k \to x$ as $k \to \infty$, then $\|x\| < 1$?
Answer to Question 1: Yes, consider $B(0,1) = \{x \in \mathbb{R}^n \,|\, \|x\| \leqslant 1\}$. Then B is closed since if $x \in B^{\complement}$, $\exists\, \varepsilon = \|x\|_2 - 1 > 0 \ni D(x,\varepsilon) \subseteq B^{\complement}$. Since $\{x_k\}_{k=1}^{\infty} \subseteq B$ and $x_k \to x$ as $k \to \infty$, by Proposition 2.69 $x \in B$; thus $\|x\| \leqslant 1$.

On the other hand, we can obtain the inequality above by the triangle inequality:

$$\|x\|_2 \leqslant \|x_k - x\|_2 + \|x_k\|_2 \leqslant \|x_k - x\|_2 + 1 \;\forall\, k > 0 \Rightarrow \|x\|_2 \leqslant \lim_{k\to\infty} \|x_k - x\|_2 + 1 = 1.$$

Answer to Question 2: No. For example, consider the case $n = 1$, and take $x_k = 1 - \dfrac{1}{n}$. Then $|x_k| < 1$ and $x_k \to x = 1$ as $k \to \infty$. However, $|x| = 1 \nless 1$.

Definition 2.71. A point x in a metric space is said to be a ***cluster point*** of a sequence $\{x_n\}_{n=1}^{\infty}$ if

$$\forall\, \varepsilon > 0, \#\{n \in \mathbb{N} \,|\, x_n \in D(x,\varepsilon)\} = \infty\,.$$

Proposition 2.72. *If $\{x_n\}_{n=1}^{\infty}$ is a sequence in a metric space (M,d), then*

1. *x is a cluster point of $\{x_n\}_{n=1}^{\infty}$ if and only if $\forall\, \varepsilon > 0$ and $N > 0, \exists\, n \geqslant N \ni d(x_n,x) < \varepsilon$.*
2. *x is a cluster point of $\{x_n\}_{n=1}^{\infty}$ if and only if $\exists\, \{x_{n_j}\}_{j=1}^{\infty} \ni x_{n_j} \to x$ as $j \to \infty$.*
3. *$x_n \to x$ as $n \to \infty$ if and only if every subsequence of $\{x_n\}_{n=1}^{\infty}$ converges to x.*
4. *$x_n \to x$ as $n \to \infty$ if and only if every proper subsequence of $\{x_n\}_{n=1}^{\infty}$ has a further subsequence that converges to x.*

Proof. See the proof of Proposition 1.104 by changing $|\cdot - \cdot|$ to $d(\cdot,\cdot)$. □

Theorem 2.73. *The collection of cluster points of a sequence is closed.*

Proof. Let $\{x_k\}_{k=1}^{\infty} \subseteq M$ be a sequence, and A be the collection of cluster points of $\{x_k\}_{k=1}^{\infty}$. If $y \in A^{\complement}$, then y is not a cluster point of $\{x_k\}_{k=1}^{\infty}$; thus

$$\exists\, \varepsilon > 0 \ni \#\{n \in \mathbb{N} \,|\, x_n \in D(y,\varepsilon)\} < \infty\,.$$

If $z \in D(y,\varepsilon)$, let $r = \varepsilon - d(y,z) > 0$, then $D(z,r) \subseteq D(y,\varepsilon)$ **(Check!)**. As a consequence, $\#\{n \in \mathbb{N} \,|\, x_n \in D(z,r)\} < \infty$.

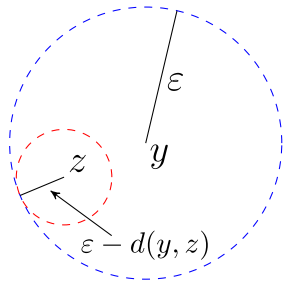


Figure 2.3: $D(z, \varepsilon - d(y,z)) \subseteq D(y,\varepsilon)$ if $z \in D(y,\varepsilon)$

Therefore, $z \in A^{\complement}$ which implies that $D(y,\varepsilon) \subseteq A^{\complement}$; thus A is closed. □

Next we talk about the completeness of a metric space. Recall that the completeness of an order field is defined by the monotone sequence property (or the least upper bound property) which relies on the concept of order, so we cannot define the completeness of a metric space via these two properties. On the other hand, Theorem 1.98 suggests that when the concept of order is out of scope, the convergence of all Cauchy sequences seems a good replacement for completeness. This is in fact how we define the completeness of general metric spaces. To be more precise, we start with the following

Definition 2.74. Let (M,d) be a metric space. A sequence $\{x_k\}_{k=1}^{\infty} \subseteq M$ is said to be ***Cauchy*** if

$$\forall\, \varepsilon > 0,\ \exists\, N > 0 \ \ni d(x_n, x_m) < \varepsilon \text{ whenever } n, m \geqslant N.$$

Definition 2.75. A metric space (M,d) is said to be ***complete*** if every Cauchy sequence in M converges to a limit in M.

Definition 2.76. A sequence $\{x_k\}_{k=1}^{\infty}$ in a normed space $(\mathcal{V}, \|\cdot\|)$ is said to be ***bounded*** if

$$\exists\, B > 0 \ni \|x_k\| \leqslant B \ \forall\, k \in \mathbb{N}.$$

Definition 2.77. A sequence $\{x_k\}_{k=1}^{\infty}$ in a metric space (M,d) is said to be ***bounded*** if

$$\exists\, x_0 \in M \text{ and } B > 0 \ni d(x_k, x_0) \leqslant B \ \forall\, k \in \mathbb{N}.$$

Remark 2.78. Adopting the definition of boundedness in a metric space, a sequence $\{x_k\}_{k=1}^{\infty}$ in a nomed space $(\mathcal{V}, \|\cdot\|)$ is bounded if

$$\exists\, x_0 \in \mathcal{V} \text{ and } B > 0 \ni \|x_k - x_0\| \leqslant B \ \forall\, k \in \mathbb{N};$$

thus $\|x_k\| \leqslant \|x_0\| + B \equiv \widetilde{B}$. Therefore, Definition 2.77 implies Definition 2.76.

Proposition 2.79. *A convergent sequence in (M,d) is bounded.*

Proof. Let $\{x_k\}_{k=1}^{\infty}$ be a convergent sequence in M with limit x_0. Then

$$\forall\,\varepsilon > 0,\ \exists\,N > 0 \ni d(x_k, x_0) < \varepsilon \text{ whenever } k \geqslant N.$$

Let $C = \max\big\{d(x_1,x_0), d(x_2,x_0), \cdots, d(x_{N-1},x_0), \varepsilon\big\} + 1$. Then $d(x_k,x_0) \leqslant C\ \forall\,k \in \mathbb{N}$. □

Proposition 2.80.

1. *Every convergent sequence in (M,d) is Cauchy.*
2. *If a subsequence of Cauchy sequence converges, then this Cauchy sequence also converges.*

Proof. See the proof of Proposition 1.91 and Lemma 1.96 by changing $|\cdot|$ to $d(\cdot,\cdot)$. □

Theorem 2.81. *A sequence in $\mathbb{R}^n$ converges if and only if the sequence is Cauchy* $\Big($*because of that* $\max\limits_{1\leqslant i\leqslant n} |v_k^{(i)} - u_i| \leqslant \|v_k - u\|_2 \leqslant \sqrt{n}\max\limits_{1\leqslant i\leqslant n} |v_k^{(i)} - u_i|\Big)$.

Theorem 2.82. *Let (M,d) be a complete metric space, and $N \subseteq M$ be a closed subset. Then (N,d) is complete* (完備空間中之閉集合是完備的).

Proof. Let $\{x_k\}_{k=1}^{\infty} \subseteq N$ be Cauchy sequence. Then

$$\forall\,\varepsilon > 0, \exists\,N_0 > 0 \ni d(x_n, x_m) < \varepsilon \text{ if } n, m \geqslant N_0.$$

Therefore, $\{x_k\}_{k=1}^{\infty}$ is Cauchy in (M,d). By completeness of (M,d), $\exists\,x \in M \ni x_k \to x$ as $k \to \infty$. Note that $x \in N$ since N is closed. □

2.4 Series of Real Numbers and Vectors

Definition 2.83. Let $(\mathcal{V}, \|\cdot\|)$ be a normed space. A series $\sum\limits_{k=1}^{\infty} x_k$, where $\{x_k\}_{k=1}^{\infty} \subseteq \mathcal{V}$, is said to ***converge*** to $S \in \mathcal{V}$ if the partial sum $S_n = \sum\limits_{k=1}^{n} x_k$ converges to S, and one writes $S = \sum\limits_{k=1}^{\infty} x_k$ if this is the case.

Theorem 2.84. *Let $(\mathcal{V}, \|\cdot\|)$ be a complete normed space* (*called* ***Banach space***)*. A series $\sum\limits_{k=1}^{\infty} x_k$ converges if and only if*

$$\forall\,\varepsilon > 0, \exists\,N > 0 \ni \|x_k + x_{k+1} + \cdots + x_{k+p}\| < \varepsilon \quad \textit{if } k \geqslant N, p \geqslant 0.$$

Proof. Let $S_n = \sum\limits_{k=1}^{n} x_k$ be partial sum of $\sum\limits_{k=1}^{\infty} x_k$. Then

$$\begin{aligned}\{S_n\}_{n=1}^{\infty} \text{ converges in } \mathcal{V} \Leftrightarrow \{S_n\}_{n=1}^{\infty} &\text{ is Cauchy}\\ &\Leftrightarrow \forall\,\varepsilon > 0, \exists\, N > 0 \ni \|S_n - S_m\| < \varepsilon \text{ if } n, m \geqslant N\\ &\Leftrightarrow \forall\,\varepsilon > 0, \exists\, N > 0 \ni \|x_{n+1} + x_{n+2} + \cdots + x_m\| < \varepsilon \text{ if } m > n \geqslant N\\ &\Leftrightarrow \forall\,\varepsilon > 0, \exists\, N > 0 \ni \|x_k + x_{k+1} + \cdots + x_{k+p}\| < \varepsilon \text{ if } k \geqslant N+1, p \geqslant 0.\end{aligned}$$

□

Corollary 2.85. *If* $\sum\limits_{k=1}^{\infty} x_k$ *converges, then* $\|x_k\| \to 0$ *as* $k \to \infty$*, and if* $\|x_k\| \nrightarrow 0$ *as* $k \to \infty$*, then* $\sum\limits_{k=1}^{\infty} x_k$ *diverges.*

Proof. Take $p = 0$ in Theorem 2.84. □

Definition 2.86. A series $\sum\limits_{k=1}^{\infty} x_k$ is said to ***converge absolutely*** if $\sum\limits_{k=1}^{\infty} \|x_k\|$ converges in $\mathbb{R}$. A series that is convergent but not absolutely convergent is said to be ***conditionally convergent***.

Example 2.87. $\sum\limits_{k=1}^{\infty} \dfrac{(-1)^k}{k}$ is conditionally convergent.

Theorem 2.88. *In a complete normed space, if* $\sum\limits_{k=1}^{\infty} x_k$ *converges absolutely, then* $\sum\limits_{k=1}^{\infty} x_k$ *converges.*

Proof. If $\sum\limits_{k=1}^{\infty} x_k$ converges absolutely, then $S_n = \sum\limits_{k=1}^{n} \|x_k\|$ converges in $\mathbb{R}$. Then

$$\forall\,\varepsilon > 0, \exists\, N > 0 \ni \big|\|x_k\| + \|x_{k+1}\| + \cdots + \|x_{k+p}\|\big| < \varepsilon \text{ if } k \geqslant N, p \geqslant 0.$$

Therefore, if $k \geqslant N, p \geqslant 0$,

$$\|x_k + x_{k+1} + \cdots + x_{k+p}\| \leqslant \|x_k\| + \cdots + \|x_{k+p}\| < \varepsilon\,.$$

□

Theorem 2.89.

1. ***Geometric series:***

   (a) *If* $|r| < 1$*, then* $\sum\limits_{k=1}^{\infty} r^k$ *converges absolutely to* $\dfrac{r}{1-r}$.

   (b) *If* $|r| > 1$*, then* $\sum\limits_{k=1}^{\infty} r^k$ *does not converge (diverge).*

2. ***Comparison test**:*

   (a) *If* $\sum_{k=1}^{\infty} a_k$ *converges,* $a_k \geqslant 0$*, and* $0 \leqslant b_k \leqslant a_k$*, then* $\sum_{k=1}^{\infty} b_k$ *converges.*

   (b) *If* $\sum_{k=1}^{\infty} a_k$ *diverges,* $a_k \geqslant 0$*, and* $a_k \leqslant b_k$*, then* $\sum_{k=1}^{\infty} b_k$ *diverges.*

3. ***p-series**:*

   $\sum_{k=1}^{\infty} \frac{1}{k^p}$ *converges if* $p > 1$ *and diverges if* $p \leqslant 1$*.*

4. ***Root test**:*

   (a) *If* $\limsup_{k\to\infty} \sqrt[k]{|x_k|} < 1$*, then* $\sum_{k=1}^{\infty} x_k$ *converges absolutely.*

   (b) *If* $\limsup_{k\to\infty} \sqrt[k]{|x_k|} > 1$*, then* $\sum_{k=1}^{\infty} x_k$ *diverges.*

5. ***Ratio and comparison test**:*

   Let $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} b_k$ *be series, and* $b_k > 0$ *for all* $k \in \mathbb{N}$*.*

   (a) $\limsup_{k\to\infty} \frac{|a_k|}{b_k} < \infty$, $\sum_{k=1}^{\infty} b_k$ *is convergent, then* $\sum_{k=1}^{\infty} a_k$ *converges absolutely.*

   (b) $\liminf_{k\to\infty} \frac{a_k}{b_k} > 0$, $\sum_{k=1}^{\infty} b_k$ *is divergent, then* $\sum_{k=1}^{\infty} a_k$ *diverges.*

6. ***Integral test**:*

   If f *is continuous, non-negative, and monotone decreasing on* $[1, \infty)$*, then* $\sum_{k=1}^{\infty} f(k)$ *converges if and only if the improper integral* $\int_1^{\infty} f(x)dx < \infty$*.*

7. ***Alternative series**:*

   $$\sum_{k=1}^{\infty} (-1)^k a_k \text{ is convergent if } a_k \geqslant 0,\ a_k \searrow 0 \ (\text{that is}, a_k \geqslant a_{k+1}, a_k \to 0 \text{ as } k \to \infty).$$

Remark 2.90. From the exercise problem, we have

$$\liminf_{k\to\infty} \frac{|x_{k+1}|}{|x_k|} \leqslant \liminf_{k\to\infty} \sqrt[k]{|x_k|} \leqslant \limsup_{k\to\infty} \sqrt[k]{|x_k|} \leqslant \limsup_{k\to\infty} \frac{|x_{k+1}|}{|x_k|}\,.$$

As a consequence, by the root test we obtain

1. if $\limsup\limits_{k\to\infty} \dfrac{|x_{k+1}|}{|x_k|} < 1$, the series $\sum\limits_{k=1}^{\infty} x_k$ converges absolutely, and

2. if $\liminf\limits_{k\to\infty} \dfrac{|x_{k+1}|}{|x_k|} > 1$, the series $\sum\limits_{k=1}^{\infty} x_k$ diverges.

This is called the ***ratio test***.

Example 2.91. Let

$$x_k = \begin{cases} \dfrac{1}{2^{\frac{k+1}{2}}} & \text{if } k \text{ is odd,} \\ \dfrac{1}{3^{\frac{k}{2}}} & \text{if } k \text{ is even,} \end{cases}$$

that is, $\{x_k\}_{k=1}^{\infty} = \left\{\dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}, \dfrac{1}{9}, \dfrac{1}{8}, \dfrac{1}{27}, \cdots\right\}$, be a sequence in $\mathbb{R}$. Then

1. $\liminf\limits_{k\to\infty} \dfrac{|x_{k+1}|}{|x_k|} = 0$;

2. $\liminf\limits_{k\to\infty} \sqrt[k]{|x_k|} = \dfrac{1}{\sqrt{3}}$;

3. $\limsup\limits_{k\to\infty} \sqrt[k]{|x_k|} = \dfrac{1}{\sqrt{2}}$;

4. $\limsup\limits_{k\to\infty} \dfrac{|x_{k+1}|}{|x_k|} = \infty$.

Therefore, $\sum\limits_{k=1}^{\infty} x_k$ converges absolutely.

Chapter 3

Compact and Connected Sets

3.1 Compactness（緊緻性）

Definition 3.1. Let (M, d) be a metric space. A subset $K \subseteq M$ is called ***sequentially compact*** if every sequence in K has a subsequence that converges to a point in K.

Example 3.2. Any closed and bounded set in $(\mathbb{R}, |\cdot|)$ is sequentially compact.

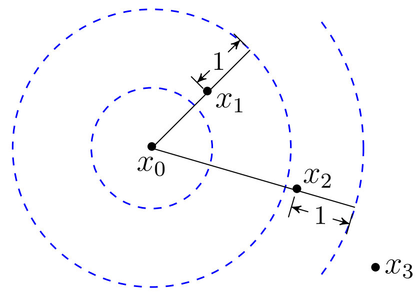
Proof. Let $\{x_k\}_{k=1}^\infty$ be a sequence in a closed and bounded set S. Then $\{x_k\}_{k=1}^\infty$ is also bounded; thus by Bolzano-Weierstrass property of $\mathbb{R}$, there exists a subsequence $\{x_{k_j}\}_{j=1}^\infty$ converging to a point $x \in \mathbb{R}$. Since S is closed, $x \in S$; thus S is sequentially compact. □

Proposition 3.3. *Let (M, d) be a metric space, and $K \subseteq M$ be sequentially compact. Then K is closed and bounded.*

Proof. For closedness, assume that $\{x_k\}_{k=1}^\infty \subseteq K$ and $x_k \to x$ as $k \to \infty$. By the definition of sequential compactness, there exists $\{x_{k_j}\}_{j=1}^\infty$ converging to a point $y \in K$. By Proposition 2.72, $x = y$; thus $x \in K$.

For boundedness, assume the contrary that $\forall\,(x_0, B) \in M \times \mathbb{R}^+$, there exists $y \in K$ such that $d(x_0, y) > B$. In particular, there exists

$$x_k \in K, d(x_k, x_0) > 1 + d(x_{k-1}, x_0) \quad \forall\, k \in \mathbb{N}.$$



Then any subsequence of $\{x_k\}_{k=1}^{\infty}$ cannot be Cauchy since $d(x_k, x_\ell) > 1$ for all $k, \ell \in \mathbb{N}$; thus $\{x_k\}_{k=1}^{\infty}$ has no convergent subsequence, a contradiction. □

Remark 3.4. Example 3.2 and Proposition 3.3 together suggest that in $(\mathbb{R}, |\cdot|)$,

$$\text{sequentially compact} \Leftrightarrow \text{closed and bounded}\,.$$

Corollary 3.5. *If $K \subseteq \mathbb{R}$ is sequentially compact, then $\inf K \in K$ and $\sup K \in K$.*

Proof. By Proposition 3.3, K must be closed and bounded. Therefore, $\inf K \in \mathbb{R}$. Then for each $n \in \mathbb{N}$, there exists $x_n \in K$ such that $\inf K \leqslant x_n < \inf K + \dfrac{1}{n}$. Since $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence in $\mathbb{R}$, the Bolzano-Weierstrass theorem (Theorem 1.95) implies that there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ and $x \in \mathbb{R}$ such that $\lim\limits_{k\to\infty} x_{n_k} = x$. Note that $x = \inf K$, and by the closedness of K, $x \in K$. The proof of $\sup K \in K$ is similar. □

Definition 3.6. Let (M, d) be a metric space, and $A \subseteq M$. A ***cover*** of A is a collection of sets $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ whose union contains A; that is,

$$A \subseteq \bigcup_{\alpha\in I} \mathcal{U}_\alpha\,.$$

It is an ***open cover*** of A if $\mathcal{U}_\alpha$ is open for all $\alpha \in I$. A ***subcover*** of a given cover is a sub-collection $\{\mathcal{U}_\alpha\}_{\alpha\in J}$ of $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ whose union also contains A; that is,

$$A \subseteq \bigcup_{\alpha\in J} \mathcal{U}_\alpha\,, \quad J \subseteq I\,.$$

It is a ***finite subcover*** if $\#J < \infty$.

Definition 3.7. Let (M, d) be a metric space. A subset $K \subseteq M$ is called ***compact*** if every open cover of K possesses a finite subcover; that is, $K \subseteq M$ is compact if

$$\forall\,\text{open cover } \{\mathcal{U}_\alpha\}_{\alpha\in I} \text{ of } K, \exists\, J \subseteq I, \#J < \infty \ni K \subseteq \bigcup_{\alpha\in J} \mathcal{U}_\alpha\,.$$

Example 3.8. Consider $\mathbb{R} \times \{0\}$ in the normed space $(\mathbb{R}^2, \|\cdot\|_2)$. For $x \in \mathbb{R}$, then $\big\{D\big((x,0),1\big)\big\}_{x\in\mathbb{R}}$ is an open cover of $\mathbb{R}\times\{0\}$; that is,

$$\mathbb{R}\times\{0\} \subseteq \bigcup_{x\in\mathbb{R}} D\big((x,0),1\big)\,.$$

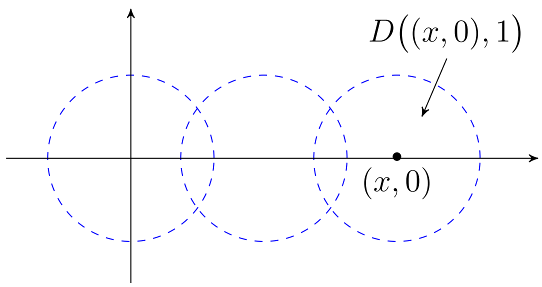


Figure 3.1: An open cover of the x-axis

However, there is no finite subcover; thus $\mathbb{R}\times\{0\}$ is not compact.

Example 3.9. Consider $(0,1]$ in the normed space $(\mathbb{R}, |\cdot|)$. Let $I_k = \big(\frac{1}{k}, 2\big)$. Then $\{I_k\}_{k=1}^\infty$ is an open cover of $(0,1]$; that is,

$$(0,1] \subseteq \bigcup_{k=1}^{\infty} \big(\frac{1}{k}, 2\big)\,.$$

However, there is no finite subcover since

$$\frac{1}{N+1} \notin \bigcup_{k=1}^{N} \big(\frac{1}{k}, 2\big)\,.$$

Therefore, $(0,1]$ is not compact.

Lemma 3.10. *Let (M,d) be a metric space, and $K \subseteq M$ be compact. Then K is closed. In other words, compact subsets of metric spaces are closed.*

Proof. Suppose the contrary that $\exists\,\{x_k\}_{k=1}^\infty \subseteq K$, $x_k \to x$ as $k\to\infty$, but $x \notin K$. For $y \in K$, define the open ball $\mathcal{U}_y$ by

$$\mathcal{U}_y = D\big(y, \frac{1}{2}d(x,y)\big).$$

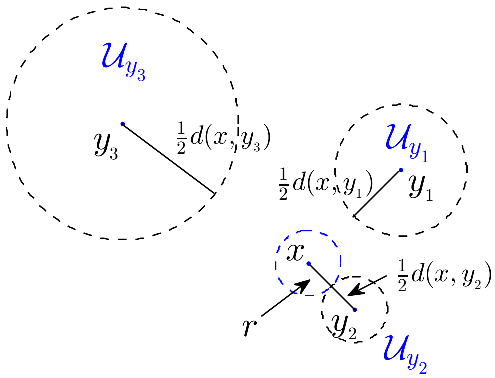
Then $\{\mathcal{U}_y\}_{y\in K}$ is an open cover of K; that is, $K \subseteq \bigcup\limits_{y\in K}\mathcal{U}_y$. Since K is compact, there exist $\{y_1,\cdots,y_n\} \subseteq K$ such that

$$K \subseteq \bigcup_{i=1}^{n} \mathcal{U}_{y_i} = \bigcup_{i=1}^{n} D\big(y_i, \frac{1}{2}d(x,y_i)\big)\,.$$

Let $r = \dfrac{1}{2}\min\big\{d(x,y_1),\cdots,d(x,y_n)\big\} > 0$. Then if $d(x,z) < r$,

$$d(z,y_i) \geqslant d(x,y_i) - d(x,z) > d(x,y_i) - r > d(x,y_i) - \frac{1}{2}d(x,y_i) = \frac{1}{2}d(x,y_i)$$

which implies that $D(x,r) \cap \mathcal{U}_{y_i} = \varnothing$ for all $i = 1,\cdots,n$.



On the other hand, since $x_k \to x$ as $k \to \infty$, $\exists\, N > 0$ such that

$$d(x_k,x) < r \quad \forall\, k \geqslant N\,.$$

In particular, $x_N \in D(x,r) \cap K$; thus $x_N \notin \mathcal{U}_{y_i}$ for all $i = 1,\cdots,n$, which contradicts to that $\{\mathcal{U}_{y_i}\}_{i=1}^n$ is a cover of K. $\square$

Lemma 3.11. *Let (M,d) be a metric space, and $K \subseteq M$ be compact. If $F \subseteq K$ is closed, then F is compact. In other words, closed subsets of compact sets are compact.*

Proof. Let $\big\{\mathcal{U}_\alpha\big\}_{\alpha\in I}$ be an open cover of F. Then $\big\{\mathcal{U}_\alpha\big\}_{\alpha\in I} \cup \{F^\complement\}$ is an open cover of K; thus possessing a finite subcover of K. Therefore, we must have

$$K \subseteq \bigcup_{i=1}^{n} \mathcal{U}_{\alpha_i} \cup F^\complement$$

for some $\alpha_i \in I$. In particular, $F \subseteq \bigcup\limits_{i=1}^{n} \mathcal{U}_{\alpha_i} \cup F^\complement$, so $F \subseteq \bigcup\limits_{i=1}^{n} \mathcal{U}_{\alpha_i}$. $\square$

Definition 3.12. Let (M,d) be a metric space. A subset $A \subseteq M$ is called ***totally bounded*** if for each $r > 0$, there exists $\{x_1,\cdots,x_N\} \subseteq M$ such that

$$A \subseteq \bigcup_{i=1}^{N} D(x_i,r)\,.$$

Proposition 3.13. *Let (M,d) be a metric space, and $A \subseteq M$ be totally bounded. Then A is bounded. In other words, totally bounded sets are bounded.*

Proof. By total boundedness, there exists $\{y_1, \cdots, y_N\} \subseteq M$ such that $A \subseteq \bigcup_{i=1}^{N} D(y_i, 1)$. Let $x_0 = y_1$ and $R = \max\big\{d(x_0, y_2), \cdots, d(x_0, y_N)\big\} + 1$. Then if $z \in A$, $z \in D(y_j, 1)$ for some $j = 1, \cdots, N$, and

$$d(z, x_0) \leqslant d(z, y_j) + d(y_j, x_0) < 1 + d(x_0, y_j) \leqslant R$$

which implies that $A \subseteq D(x_0, R)$. Therefore, A is bounded. □

Example 3.14. In a general metric space (M,d), a bounded set might not be totally bounded. For example, consider the metric space (M,d) with the discrete metric, and $A \subseteq M$ be a set having infinitely many points. Then A is bounded since $A \subseteq D(x, 2)$ for any $x \in M$; however, A is not totally bounded since A cannot be covered by finitely many balls with radius $\frac{1}{2}$.

Example 3.15. Every bounded set in $(\mathbb{R}^n, \|\cdot\|_2)$ is totally bounded **(Check!)**. In particular, the set $\{1\} \times [1,2]$ in $(\mathbb{R}^2, \|\cdot\|)$ is totally bounded.

On the other hand, let $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$d(x,y) = \begin{cases} |x_1 - y_1| & \text{if } x_2 = y_2\,, \\ |x_1 - y_1| + |x_2 - y_2| + 1 & \text{if } x_2 \neq y_2\,. \end{cases} \quad \text{where } x = (x_1, x_2) \text{ and } y = (y_1, y_2).$$

Then $(\mathbb{R}^2, d)$ is also a metric space (exercise). The set $\{1\} \times [1,2]$ is not totally bounded. In fact, consider open ball with radius $\frac{1}{2}$:

$$\begin{aligned} y \in D\big(x, \frac{1}{2}\big) &\Leftrightarrow d(x,y) < \frac{1}{2} \Leftrightarrow |x_1 - y_1| < \frac{1}{2} \text{ and } x_2 = y_2 \\ &\Leftrightarrow y_1 \in \big(x_1 - \frac{1}{2}, x_1 + \frac{1}{2}\big) \text{ and } x_2 = y_2\,. \end{aligned}$$

In other words,

$$D\big(x, \frac{1}{2}\big) = \big(x_1 - \frac{1}{2}, x_1 + \frac{1}{2}\big) \times \{x_2\}\,;$$

thus one cannot cover $\{1\} \times [1,2]$ by the union of finitely many balls with radius $\frac{1}{2}$.

Proposition 3.16. *Let (M,d) be a metric space, and $T \subseteq M$ be totally bounded. If $S \subseteq T$, then S is totally bounded. In other words, subsets of totally bounded sets are totally bounded.*

Proof. Let $r > 0$ be given. By the total boundedness of T, there exists $\{x_1, \cdots, x_N\} \subseteq M$ such that

$$S \subseteq T \subseteq \bigcup_{i=1}^{N} D(x_i, r)\,. \qquad \square$$

Proposition 3.17. *Let (M, d) be a metric space, and $A \subseteq M$. Then A is totally bounded if and only if $\forall\, r > 0$, $\exists \{y_1, \cdots, y_N\} \subseteq A$ such that $A \subseteq \bigcup\limits_{i=1}^{N} D(y_i, r)$.*

Proof. It suffices to show the "only if" part. Let $r > 0$ be given. Since A is totally bounded,

$$\exists\, \{y_1, \cdots, y_N\} \subseteq M \ni A \subseteq \bigcup_{i=1}^{N} D\big(y_i, \frac{r}{2}\big)\,.$$

W.L.O.G., we may assume that for each $i = 1, \cdots, N$, $D\big(y_i, \frac{r}{2}\big) \cap A \neq \varnothing$. Then for each $i = 1, \cdots, N$, there exists $x_i \in D\big(y_i, \frac{r}{2}\big) \cap A$ which suggests that

$$A \subseteq \bigcup_{i=1}^{N} D\big(y_i, \frac{r}{2}\big) \subseteq \bigcup_{i=1}^{N} D(x_i, r)$$

since $D\big(y_i, \frac{r}{2}\big) \subseteq D(x_i, r)$ for all $i = 1, \cdots, N$. $\square$

Lemma 3.18. *Let (M, d) be a metric space, and $K \subseteq M$. If K is either compact or sequentially compact, then K is totally bounded..*

Proof. Suppose first that K is compact. Let $r > 0$ be given, then $\big\{D(x, r)\big\}_{x \in K}$ is an open cover of K. Since K is compact, there exists a finite subcover; thus $\exists\, \{x_1, \cdots, x_N\} \subseteq K$ such that

$$K \subseteq \bigcup_{i=1}^{N} D(x_i, r)\,.$$

Therefore, K is totally bounded.

Now we assume that K is sequentially compact. Suppose the contrary that there is an $r > 0$ such that any finite set $\{y_1, \cdots, y_n\} \subseteq K$, $K \nsubseteq \bigcup\limits_{i=1}^{n} D(y_i, r)$. This implies that we can choose a sequence $\{x_k\}_{k=1}^{\infty} \subseteq K$ such that

$$x_{k+1} \in K \backslash \bigcup_{i=1}^{k} D(x_i, r)\,.$$

Then $\{x_k\}_{k=1}^{\infty}$ is a sequence in K without convergent subsequence since $d(x_k, x_\ell) > r$ for all $k, \ell \in \mathbb{N}$. $\square$

Theorem 3.19. *Let (M, d) be a metric space, and $K \subseteq M$. Then the following three statements are equivalent:*

1. *K is compact.*

2. *K is sequentially compact.*

3. *K is totally bounded and (K, d) is complete.*

Proof. We show that $1 \Rightarrow 3 \Rightarrow 2 \Rightarrow 1$ to conclude the theorem.

"$1 \Rightarrow 3$": By Lemma 3.18, it suffices to show the completeness of (K, d). Let $\{x_k\}_{k=1}^{\infty}$ be a Cauchy sequence in K. Suppose that $\{x_k\}_{k=1}^{\infty}$ does not converge in K. Then

$$\forall\, y \in K, \exists\, \delta_y > 0 \ni \#\big\{k \in \mathbb{N} \,\big|\, x_k \in D(y, \delta_y)\big\} < \infty \tag{3.1.1}$$

for otherwise there is a subsequence of $\{x_k\}_{k=1}^{\infty}$ that converges to x which will suggests the convergence of the Cauchy sequence. The collection $\big\{D(y, \delta_y)\big\}_{y \in K}$ then is an open cover of K; thus possesses a finite subcover $\big\{D\big(y_i, \delta_{y_i}\big)\big\}_{i=1}^{N}$. In particular, $\{x_k\}_{k=1}^{\infty} \subseteq \bigcup_{i=1}^{N} D\big(y_i, \delta_{x_i}\big)$ or

$$\#\Big\{k \in \mathbb{N} \,\Big|\, x_k \in \bigcup_{i=1}^{N} D(y_i, \delta_{y_i})\Big\} = \infty$$

which contradicts to (3.1.1).

"$3 \Rightarrow 2$": The proof of this step is similar to the proof of the Bolzano-Weierstrass Theorem in $\mathbb{R}$ (Theorem 1.95) that we proceed as follows. Let $\{x_k\}_{k=1}^{\infty}$ be a sequence in $T_0 \equiv K$. Since K is totally bound, there exist $\big\{y_1^{(1)}, \cdots y_{N_1}^{(1)}\big\} \subseteq K$ such that

$$T_0 \equiv K \subseteq \bigcup_{i=1}^{N_1} D(y_i^{(1)}, 1)\,.$$

One of these $D(y_i^{(1)}, 1)$'s must contain infinitely many x_k's; that is, $\exists\, 1 \leqslant \ell_1 \leqslant N_1$ such that $\#\big\{k \in \mathbb{N} \,\big|\, x_k \in D(y_{\ell_1}^{(1)}, 1)\big\} = \infty$. Define $T_1 = K \cap D(y_{\ell_1}^{(1)}, 1)$. Then T_1 is also totally bounded by Proposition 3.16, so there exist $\big\{y_1^{(2)}, \cdots, y_{N_2}^{(2)}\big\} \subseteq T_1$ such that

$$T_1 \subseteq \bigcup_{i=1}^{N_2} D\big(y_i^{(2)}, \frac{1}{2}\big)\,.$$

Suppose that $\#\big\{k \in \mathbb{N} \,\big|\, x_k \in D\big(y^{(2)}_{\ell_2}, \frac{1}{2}\big)\big\} = \infty$ for some $1 \leqslant \ell_2 \leqslant N_2$. Define $T_2 = T_1 \cap D\big(y^{(2)}_{\ell_2}, \frac{1}{2}\big)$. We continue this process, and obtain that for all $n \in \mathbb{N}$,

(1) $\exists \big\{y^{(n)}_1, \cdots, y^{(n)}_{N_n}\big\} \subseteq T_{n-1}$ such that

$$T_{n-1} \subseteq \bigcup_{i=1}^{N_n} D\big(y^{(n)}_i, \frac{1}{n}\big)\,.$$

(2) $T_n = T_{n-1} \cap D(y^{(n)}_{\ell_n}, \frac{1}{n})$, where $1 \leqslant \ell_n \leqslant N_n$ is chosen so that

$$\#\big\{k \in \mathbb{N} \,\big|\, x_k \in D\big(y^{(n)}_{\ell_n}, \frac{1}{n}\big)\big\} = \infty\,. \tag{3.1.2}$$

Pick an $k_1 \in \big\{k \in \mathbb{N} \,\big|\, x_k \in D\big(y^{(1)}_{\ell_1}, 1\big)\big\}$, and $k_j \in \big\{k \in \mathbb{N} \,\big|\, x_k \in D\big(y^{(j)}_{\ell_j}, \frac{1}{j}\big)\big\}$ such that $k_j > k_{j-1}$ for all $j \geqslant 2$. We note such k_j always exists because of (3.1.2). Then $\big\{x_{k_j}\big\}_{j=1}^{\infty}$ is a subsequence of $\{x_k\}_{k=1}^{\infty}$, and $x_{k_j} \in T_j \subseteq K$ for all $j \in \mathbb{N}$.

Claim: $\big\{x_{k_j}\big\}_{j=1}^{\infty}$ is a Cauchy sequence.

Proof of claim: Let $\varepsilon > 0$ be given, and $N > 0$ be large enough so that $\frac{1}{N} < \frac{\varepsilon}{2}$. Since if $j \geqslant N$, we must have $x_{k_j} \in D\big(y^{(N)}_{\ell_N}, \frac{1}{N}\big)$, we conclude that if $n, m \geqslant N$, by triangle inequality

$$d\big(x_{k_n}, x_{k_m}\big) \leqslant d\big(x_{k_n}, y^{(N)}_{\ell_N}\big) + d\big(x_{k_m}, y^{(N)}_{\ell_N}\big) < \frac{1}{N} + \frac{1}{N} < \varepsilon.$$

Since (K, d) is complete, the Cauchy sequence $\big\{x_{k_j}\big\}_{j=1}^{\infty}$ converges to a point in K.

"$2 \Rightarrow 1$": Let $\big\{\mathcal{U}_\alpha\big\}_{\alpha \in I}$ be an open cover of K.

Claim: there exists $r > 0$ such that for each $x \in K$, $D(x, r) \subseteq \mathcal{U}_\alpha$ for some $\alpha \in I$.

Proof of claim: Suppose the contrary that for all $k > 0$, there exists $x_k \in K$ such that $D\big(x_k, \frac{1}{k}\big) \nsubseteq \mathcal{U}_\alpha$ for all $\alpha \in I$. Then $\{x_k\}_{k=1}^{\infty}$ is a sequence in K; thus by the assumption of sequential compactness, there exists a subsequence $\big\{x_{k_j}\big\}_{j=1}^{\infty}$ converging in K. Suppose that $x_{k_j} \to x$ as $j \to \infty$, and $x \in \mathcal{U}_\beta$ for some $\beta \in I$. Then

(1) there is $r > 0$ such that $D(x, r) \subseteq \mathcal{U}_\beta$ since $\mathcal{U}_\beta$ is open.

(2) there exists $N > 0$ such that $d(x_{k_j}, x) < \frac{r}{2}$ for all $j \geqslant N$.

Choose $j \geqslant N$ such that $\dfrac{1}{k_j} < \dfrac{r}{2}$. Then $D\big(x_{k_j}, \dfrac{1}{k_j}\big) \subseteq D(x,r) \subseteq \mathcal{U}_\beta$, a contradiction.

By Lemma 3.18, there exists $\{x_1, \cdots, x_N\} \subseteq K$ such that $K \subseteq \bigcup\limits_{i=1}^{N} D(x_i, r)$. For each $1 \leqslant i \leqslant N$, the claim above implies that there exists $\alpha_i \in I$ such that $D(x_i, r) \subseteq \mathcal{U}_{\alpha_i}$. Then $\bigcup\limits_{i=1}^{N} D(x_i, r) \subseteq \bigcup\limits_{i=1}^{N} \mathcal{U}_{\alpha_i}$ which suggests that

$$K \subseteq \bigcup_{i=1}^{N} \mathcal{U}_{\alpha_i} . \qquad \square$$

Remark 3.20.

1. The equivalency between 1 and 2 is sometimes called the Bolzano-Weistrass Theorem.
2. A number $r > 0$ satisfying the claim in the step "2 ⇒ 1" is called a Lebesgue number for the cover $\{\mathcal{U}_\alpha\}_{\alpha \in I}$. The supremum of all such r is called the ***Lebesgue number*** for the cover $\{\mathcal{U}_\alpha\}_{\alpha \in I}$.

Alternative Proof of Theorem 3.19. In this proof we show that 1 ⇒ 2 ⇒ 3 ⇒ 1 to conclude the theorem.

"1 ⇒ 2": Assume the contrary that K is not sequentially compact. Then there is a sequence $\{x_k\}_{k=1}^\infty \subseteq K$ that does not have a convergent subsequence with a limit in K. Therefore, for each $x \in K$, there exists $\delta_x > 0$ such that

$$\#\big\{k \in \mathbb{N} \,\big|\, x_k \in D(x, \delta_x)\big\} < \infty$$

for otherwise x is a cluster point of $\{x_k\}_{k=1}^\infty$ so Proposition 2.72 guarantees the existence of a subsequence of $\{x_k\}_{k=1}^\infty$ converging to x. Since $\big\{D(x, \delta_x)\big\}_{x \in K}$ is an open cover of K, by the compactness of K there exists $\{y_1, \cdots, y_N\} \subseteq K$ such that

$$\{x_k\}_{k=1}^\infty \subseteq K \subseteq \bigcup_{i=1}^{N} D\big(y_i, \delta_{y_i}\big)$$

while this is impossible since $\#\big\{k \in \mathbb{N} \,\big|\, x_k \in D\big(y_i, \delta_{y_i}\big)\big\} < \infty$ for all $i = 1, \cdots N$.

"2 ⇒ 3": By Lemma 3.18, it suffices to show that (K, d) is complete. Let $\{x_k\}_{k=1}^\infty \subseteq K$ be a Cauchy sequence. By sequential compactness of K, there is a subsequence $\big\{x_{k_j}\big\}_{j=1}^\infty$ converging to a point $x \in K$. By Proposition 2.80, $\{x_k\}_{k=1}^\infty$ also converges to x; thus every Cauchy sequence in (K, d) converges to a point in K.

"$3 \Rightarrow 1$": We first prove the following

Claim: If $\{\mathcal{V}_\alpha\}_{\alpha\in I}$ is an open cover of a totally bounded set A such that there is no finite subcover, then for all $r > 0$, there exists $x \in A$ such that $A \cap D(x, r)$ does not admit a finite subcover.

Proof of claim: Let $r > 0$ be given. Since A is totally bounded, by Proposition 3.17 there exists $\{a_1, \cdots, a_N\} \subseteq A$ such that $A \subseteq \bigcup_{j=1}^{N} D(a_j, r)$. If for each $j = 1, \cdots, N$, $A \cap D(a_j, r)$ can be covered by finitely many $\mathcal{V}_\alpha$'s, then A itself can be covered by finitely many $\mathcal{V}_\alpha$'s, a contradiction. Therefore, at least one $A \cap D(a_j, r)$ does not admit a finite subcover.

Now assume the contrary that there exists an open cover $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ of K such that there is no finite subcover. Let $\varepsilon_n = 2^{-n}$. Since K is totally bounded, by the claim there exists $x_1 \in K$ such that $K \cap D(x_1, \varepsilon_1)$ which does not admit a finite subcover. By Proposition 3.16, $K \cap D(x_1, \varepsilon_1)$ is totally bounded, so there must be an $x_2 \in K \cap D(x_1, \varepsilon_1)$ such that $K \cap D(x_1, \varepsilon_1) \cap D(x_2, \varepsilon_2)$ cannot be covered by the union of finitely many $\mathcal{U}_\alpha$. We continuous this process, and obtain a sequence $\{x_k\}_{k=1}^{\infty}$ such that

(1) $x_{k+1} \in K \cap \bigcap_{i=1}^{k} D(x_i, \varepsilon_i)$ (which implies that $d(x_{k+1}, x_k) < \varepsilon_k$);

(2) $K \cap \bigcap_{i=1}^{k} D(x_i, \varepsilon_i)$ cannot be covered by the union of finitely many $\mathcal{U}_\alpha$.

Then similar to Example 1.100, we find that $\{x_k\}_{k=1}^{\infty}$ is a Cauchy sequence in (K, d). By the completeness of K, $x_k \to x$ as $k \to \infty$ for some $x \in K$.

Since $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ is an open cover of K, $x \in \mathcal{U}_\beta$ for some $\beta \in I$. Since $\mathcal{U}_\beta$ is open, $\exists\, r > 0$ such that $D(x, r) \subseteq \mathcal{U}_\beta$. For this particular r, there exists $N > 0$ such that $d(x_k, x) < \frac{r}{2}$. Therefore, if $k \geqslant N$ such that $\varepsilon_k < \frac{r}{2}$,

$$D(x_k, \varepsilon_k) \subseteq D(x, r) \subseteq \mathcal{U}_\beta$$

which contradicts to (2). □

Example 3.21. Let (M, d) be a metric space, and $\{x_k\}_{k=1}^{\infty}$ be a convergent sequence with limit x. Let $A = \{x_1, x_2, \cdots\} \cup \{x\}$. Then A is compact.

Definition 3.22. Let (M, d) be a metric space. A subset $A \subseteq M$ is called ***pre-compact*** if $\bar{A}$ is compact. Let $\mathcal{U} \subseteq M$ be an open set, a subset A of $\mathcal{U}$ is said to be ***compactly contained*** in $\mathcal{U}$, denoted by $A \subset\subset \mathcal{U}$, if A is pre-compact and $\bar{A} \subseteq \mathcal{U}$.

Example 3.23. Let (M, d) be a complete metric space, and $A \subseteq M$ be totally bounded. Then $\bar{A}$ is compact. In other words, in a complete metric space, totally bounded sets are pre-compact.
(Hint: Use the total boundedness equivalence to show compactness.)

Definition 3.24. Let (M, d) be a metric space, and $A \subseteq M$. A collection of closed sets $\{F_\alpha\}_{\alpha \in I}$ is said to have the ***finite intersection property*** for the set A if the intersection of any finite number of F_α with A is non-empty; that is, $\{F_\alpha\}_{\alpha \in I}$ has the finite intersection property for A if

$$A \cap \bigcap_{\alpha \in J} F_\alpha \neq \varnothing \text{ for all } J \subseteq I \text{ and } \#J < \infty.$$

Theorem 3.25. *Let (M, d) be a metric space, and $K \subseteq M$. The K is compact if and only if every collection of closed sets with the finite intersection property for K has non-empty intersection with K; that is,*

$$K \cap \bigcap_{\alpha \in I} F_\alpha \neq \varnothing \textit{ for all } \{F_\alpha\}_{\alpha \in I} \textit{ having the finite intersection property for } K.$$

Proof. It can be proved by contradiction, and is left as an exercise. □

Example 3.26. Let $A = (0, 1) \subseteq \mathbb{R}$, and $K_j = \big[-1, \frac{1}{j}\big]$. Take $K_{j_1}, K_{j_2}, \cdots, K_{j_n}$, where $j_1 < j_2 < \cdots < j_n$. Then $\bigcap\limits_{\ell=1}^{n} K_{j_\ell} \cap A = \big[-1, \frac{1}{j_n}\big] \cap (0, 1) \neq \varnothing$. However $x \in \bigcap\limits_{j=1}^{\infty} K_j \Leftrightarrow -1 \leqslant x \leqslant \frac{1}{j}$ for all $j \in \mathbb{N}$. So $\bigcap\limits_{j=1}^{\infty} K_j = [-1, 0]$; thus $\bigcap\limits_{j=1}^{\infty} K_j \cap A = \varnothing$. Therefore, $(0, 1)$ is not compact.

Example 3.27. Let X be the collection of all bounded real sequences; that is,

$$X = \Big\{\{x_k\}_{k=1}^{\infty} \subseteq \mathbb{R} \,\Big|\, \text{for some } M > 0, |x_k| \leqslant M \text{ for all } k\Big\}.$$

The number $\sup\limits_{k \geqslant 1} |x_k| \equiv \sup\{|x_1|, |x_2|, \cdots, |x_k|, \cdots\} < \infty$ is denoted by $\big\|\{x_k\}_{k=1}^{\infty}\big\|$. For example, if $x_k = \dfrac{(-1)^k}{k}$, then $\big\|\{x_k\}_{k=1}^{\infty}\big\| = 1$. Then $(X, \|\cdot\|)$ is a complete normed space (left as an exercise). Define

$$\begin{aligned}
A &= \big\{\{x_k\}_{k=1}^\infty \in X \,\big|\, |x_k| \leqslant \frac{1}{k}\big\},\\
B &= \big\{\{x_k\}_{k=1}^\infty \in X \,\big|\, x_k \to 0 \text{ as } k \to \infty\big\},\\
C &= \big\{\{x_k\}_{k=1}^\infty \in X \,\big|\, \text{the sequence } \{x_k\}_{k=1}^\infty \text{ converges}\big\},\\
D &= \big\{\{x_k\}_{k=1}^\infty \in X \,\big|\, \sup_{k\geqslant 1} |x_k| = 1\big\} \quad (\text{the unit sphere in } (X, \|\cdot\|)).
\end{aligned}$$

The closedness of A (which implies the completeness of $(A, \|\cdot\|)$) is left as an exercise. We show that A is totally bounded.

Let $r > 0$ be given. Then $\exists\, N > 0 \ni \dfrac{1}{N} < r$. Define

$$\begin{aligned}
E = \Big\{\{x_k\}_{k=1}^\infty \,\Big|\, x_1 = \frac{i_1}{N+1}, x_2 = \frac{i_2}{N+1}, \cdots, x_{N-1} = \frac{i_{N-1}}{N+1} \text{ for some}\\
i_1, \cdots, i_{N-1} = -N, -N+1, \cdots, N-1, N, \text{ and } x_k = 0 \text{ if } k \geqslant N+1\Big\}.
\end{aligned}$$

Then

1. $\#E < \infty$. In fact, $\#E = (2N+1)^{N-1} < \infty$.
2. $A \subseteq \bigcup\limits_{\{x_k\}_{k=1}^\infty \in E} D\big(\{x_k\}_{k=1}^\infty, \frac{1}{N}\big) \subseteq \bigcup\limits_{\{x_k\}_{k=1}^\infty \in E} D\big(\{x_k\}_{k=1}^\infty, r\big)$.

Therefore, A is totally bounded.

On the other hand, B and C are not compact since they are not bounded; thus not totally bounded by Proposition 3.13. D is bounded but not totally bounded. In fact, D cannot be covered by the union of finitely many balls with radius $\frac{1}{2}$ since each ball with radius $\frac{1}{2}$ contains at most one of the points from the subset $\Big\{\{x_j^{(k)}\}_{j=1}^\infty\Big\}_{k=1}^\infty \subseteq D$, where for each k

$$\{x_j^{(k)}\}_{j=1}^\infty = \{\, \underbrace{0, \cdots, 0}_{(k-1) \text{ terms}},\, 1, 0, \cdots \};$$

that is, $x_j^{(k)} = \delta_{kj}$, the kronecker delta.

3.1.1 The Heine-Borel theorem

Theorem 3.28. *In the Euclidean space $(\mathbb{R}^n, \|\cdot\|_2)$, a subset K is compact if and only if it is closed and bounded.*

Proof. By Proposition 3.13 and Theorem 3.19, it is clear that K is closed and bounded if K is compact (in any metric space). It remains to show the direction "$\Leftarrow$". Nevertheless, by Theorem 2.82 closed subsets of a complete metric space must be complete, so it suffices to show that a bounded set in $(\mathbb{R}^n, \|\cdot\|_2)$ is totally bounded.

Let $r > 0$ be given. By the boundedness of K, for some $M > 0$ we have $\|x\|_2 \leqslant M$ for all $x \in K$; thus $K \subseteq [-M, M]^n$. Choose $N > 0$ so that $\dfrac{\sqrt{n}M}{N} < r$, and define

$$E = \Big\{ \Big(\frac{Mi_1}{N}, \cdots, \frac{Mi_n}{N}\Big) \,\Big|\, i_1, i_2, \cdots, i_n \in \big\{-N, -N+1, \cdots, N-1, N\big\} \Big\}.$$

Then $\#E = (2N+1)^n < \infty$, and

$$K \subseteq [-M, M]^n \subseteq \bigcup_{x \in E} D(x, r). \qquad \square$$

Alternative Proof of "$\Leftarrow$". Let $\{x_k\}_{k=1}^{\infty} \subseteq K$ be a sequence. Since $K \subseteq \mathbb{R}^n$, we can write $x_k = (x_k^{(1)}, x_k^{(2)}, \cdots, x_k^{(n)}) \in \mathbb{R}^n$. Since K is bounded, then all the sequence $\{x_k^{(j)}\}_{k=1}^{\infty}$, $j = 1, 2, \cdots, n$, are bounded; that is, $-M_j \leqslant x_k^{(j)} \leqslant M_j$ for all $k \in \mathbb{N}$. Applying the Bolzano-Weierstrass property (Theorem 1.95) to the sequence $\{x_k^{(1)}\}_{k=1}^{\infty}$, we obtain a sequence $\{x_{k_j}^{(1)}\}_{j=1}^{\infty}$ with $x_{k_j}^{(1)} \to y^{(1)}$ as $j \to \infty$. Now $\{x_{k_j}^{(2)}\}_{j=1}^{\infty}$ has a subsequence $\{x_{k_{j_\ell}}^{(2)}\}_{\ell=1}^{\infty}$ converges, say $x_{k_{j_\ell}}^{(2)} \to y^{(2)}$ as $\ell \to \infty$.

Continuing in this way, we obtain a subsequence of $\{x_k\}_{k=1}^{\infty}$ that converges to $y = (y^{(1)}, y^{(2)}, \cdots, y^{(n)})$. Since K is close, $y \in K$; thus K is sequentially compact which is equivalent to the compactness of K. $\square$

Corollary 3.29. *A bounded set A in the Euclidean space $(\mathbb{R}^n, \|\cdot\|_2)$ is pre-compact. In particular, if $\{x_k\}_{k=1}^{\infty}$ is a bounded sequence in $\mathbb{R}^n$, there exists a convergent subsequence $\{x_{k_j}\}_{j=1}^{\infty}$ (the sentence in blue color is again called the* ***Bolzano-Weierstrass theorem****).*

Example 3.30. Let $A = \{0\} \cup \big\{1, \frac{1}{2}, \cdots, \frac{1}{n}, \cdots\big\}$. Then A is compact in $(\mathbb{R}, |\cdot|)$.

Example 3.31. Let $A = [0, 1] \cup (2, 3] \subseteq (\mathbb{R}, |\cdot|)$. Since A is not closed, A is not compact.

3.1.2 The nested set property

Theorem 3.32. *Let $\{K_n\}_{n=1}^{\infty}$ be a sequence of non-empty compact sets in a metric space (M, d) such that $K_n \supseteq K_{n+1}$ for all $n \in \mathbb{N}$. Then there is at least one point in $\bigcap_{n=1}^{\infty} K_n$; that is,*

$$\bigcap_{n=1}^{\infty} K_n \neq \varnothing.$$

Proof. Assume the contrary that $\bigcap\limits_{n=1}^{\infty} K_n = \varnothing$. Then $\bigcup\limits_{n=1}^{\infty} K_n^{\complement} = \big(\bigcap\limits_{n=1}^{\infty} K_n\big)^{\complement} = M$. Since $K_n^{\complement}$ is open, $\{K_n^{\complement}\}_{n=1}^{\infty}$ is an open cover of K_1; thus by compactness of K_1, there exists $J \subseteq \mathbb{N}$, $\#J < \infty$ such that

$$K_1 \subseteq \bigcup_{n \in J} K_n^{\complement} = \big(\bigcap_{n \in J} K_n\big)^{\complement}.$$

Therefore, $K_1 \cap \bigcap\limits_{n \in J} K_n = \varnothing$ which implies that $K_{\max J} = \varnothing$, a contradiction. □

Alternative Proof. By assumption, $\{K_n\}_{n=2}^{\infty}$ has the finite intersection property for K_1. Since K_1 is compact, by Theorem 3.25,

$$K_1 \cap \bigcap_{n=2}^{\infty} K_n \neq \varnothing\,.$$ □

Corollary 3.33. *Let $\{\mathcal{U}_k\}_{k=1}^{\infty}$ be a collection of open sets in a metric space (M, d) such that $\mathcal{U}_k \subseteq \mathcal{U}_{k+1}$ for all $k \in \mathbb{N}$ and $\mathcal{U}_k^{\complement}$ is compact. Then $\bigcup\limits_{k=1}^{\infty} \mathcal{U}_k \neq M$.*

Proof. This is proved by letting $K_n = \mathcal{U}_n^{\complement}$, and applying Theorem 3.32. □

Remark 3.34. If the compactness is removed from the condition, then the intersection might be empty. Suppose that the metric space under consideration is $(\mathbb{R}, |\cdot|)$.

1. If the closedness condition is removed, then $\mathcal{U}_k = \big(0, \frac{1}{k}\big)$ has empty intersection.

2. If the boundedness condition is removed, then $F_k = [k, \infty)$ has empty intersection.

3.2 Connectedness（連通性）

Definition 3.35. Let (M, d) be a metric space, and $A \subseteq M$. Two non-empty open sets $\mathcal{U}$ and $\mathcal{V}$ are said to separate A if

1. $A \cap \mathcal{U} \cap \mathcal{V} = \varnothing$; 2. $A \cap \mathcal{U} \neq \varnothing$; 3. $A \cap \mathcal{V} \neq \varnothing$; 4. $A \subseteq \mathcal{U} \cup \mathcal{V}$.

We say that A is ***disconnected*** or ***separated*** if such separation exists, and A is ***connected*** if no such separation exists.

Proposition 3.36. *Let (M, d) be a metric space. A subset $A \subseteq M$ is disconnected if and only if $A = A_1 \cup A_2$ with $A_1 \cap \bar{A}_2 = \bar{A}_1 \cap A_2 = \varnothing$ for some non-empty A_1 and A_2.*

Proof. "$\Rightarrow$" Suppose that there exist $\mathcal{U}$, $\mathcal{V}$ non-empty open sets such that 1-4 in Definition 3.35 hold. Let $A_1 = A \cap \mathcal{U}$ and $A_2 = A \cap \mathcal{V}$. By 1, $A_1 \subseteq \mathcal{V}^{\complement}$; thus by the definition of the closure of sets, $\bar{A}_1 \subseteq \mathcal{V}^{\complement}$. This implies that $\bar{A}_1 \cap A_2 = \varnothing$. Similarly, $\bar{A}_2 \cap A_1 = \varnothing$.

"$\Leftarrow$" Let $\mathcal{U} = \bar{A}_2^{\complement}$ and $\mathcal{V} = \bar{A}_1^{\complement}$ be two open sets. Then $\mathcal{V} \cap A_1 = \mathcal{U} \cap A_2 = \varnothing$; thus

$$A \cap \mathcal{U} \cap \mathcal{V} = (A_1 \cup A_2) \cap \mathcal{U} \cap \mathcal{V} = (A_1 \cap \mathcal{U}) \cap \mathcal{V} = \mathcal{U} \cap (A_1 \cap \mathcal{V}) = \varnothing\,.$$

Moreover, 2-4 in Definition 3.35 also hold since $A_1 \subseteq \mathcal{U}$ and $A_2 \subseteq \mathcal{V}$. □

Corollary 3.37. *Let (M, d) be a metric space. Suppose that a subset $A \subseteq M$ is connected, and $A = A_1 \cup A_2$, where $A_1 \cap \bar{A}_2 = \bar{A}_1 \cap A_2 = \varnothing$. Then A_1 or A_2 is empty.*

Theorem 3.38. *A subset A of the Euclidean space $(\mathbb{R}, |\cdot|)$ is connected if and only if it has the property that if $x, y \in A$ and $x < z < y$, then $z \in A$.*

Proof. "$\Rightarrow$" Suppose that there exist $x, y \in A$, $x < z < y$ but $z \notin A$. Then $A = A_1 \cup A_2$, where

$$A_1 = A \cap (-\infty, z) \quad \text{and} \quad A_2 = A \cap (z, \infty)\,.$$

Since $x \in A_1$ and $y \in A_2$, A_1 and A_2 are non-empty. Moreover, $\bar{A}_1 \cap A_2 = A_1 \cap \bar{A}_2 = \varnothing$; thus by Proposition 3.36, A is disconnected, a contradiction.

"$\Leftarrow$" Suppose that A is not connected. Then there exist non-empty sets A_1 and A_2 such that $A = A_1 \cup A_2$ with $\bar{A}_1 \cap A_2 = A_1 \cap \bar{A}_2 = \varnothing$. Pick $x \in A_1$ and $y \in A_2$. W.L.O.G., we may assume that $x < y$. Define $z = \sup(A_1 \cap [x, y])$.

Claim: $z \in \bar{A}_1$.

Proof of claim: By definition, for any $n > 0$ there exists $x_n \in A_1 \cap [x, y]$ such that $z - \dfrac{1}{n} < x_n \leqslant z$. Therefore, $x_n \to z$ as $n \to \infty$ which implies that $z \in \bar{A}_1$.

Since $z \in \bar{A}_1$, $z \notin A_2$. In particular, $x \leqslant z < y$.

(a) If $z \notin A_1$, then $x < z < y$ and $z \notin A$, a contradiction.

(b) If $z \in A_1$, then $z \notin \bar{A}_2$; thus $\exists\, r > 0$ such that $(z - r, z + r) \subseteq \bar{A}_2^{\complement}$. Then for all $z_1 \in (z, z + r)$, $z < z_1 < y$ and $z_1 \notin A_2$. Then $x < z_1 < y$ and $z_1 \notin A$, a contradiction. □

3.3 Subspace Topology

Let (M,d) be a metric space, and $N \subseteq M$ be a subset. Then (N,d) is a metric space, and the topology of (N,d) is called the ***subspace topology*** of (N,d).

Remark 3.39. The topology of a metric is the collection of all open sets of that metric space.

Proposition 3.40. *Let (M,d) be a metric space, and $N \subseteq M$. A subset $\mathcal{V} \subseteq N$ is open in (N,d) if and only if $\mathcal{V} = \mathcal{U} \cap N$ for some open set $\mathcal{U}$ in (M,d).*

Proof. "$\Rightarrow$" Let $\mathcal{V} \subseteq N$ be open in (N,d). Then $\forall\, x \in \mathcal{V}$, $\exists\, r_x > 0$ such that

$$D_N(x,r_x) \equiv \big\{ y \in N \,\big|\, d(x,y) < r_x \big\} \subseteq \mathcal{V}\,.$$

In particular, $\mathcal{V} = \bigcup\limits_{x\in\mathcal{V}} D_N(x,r_x)$. Note that $D_N(x,r) = D(x,r) \cap N$; thus if $\mathcal{U} = \bigcup\limits_{x\in\mathcal{V}} D(x,r_x)$, then $\mathcal{U}$ is open in (M,d), and

$$\mathcal{V} = \bigcup_{x\in\mathcal{V}} D(x,r_x) \cap N = \mathcal{U} \cap N\,.$$

"$\Leftarrow$" Suppose that $\mathcal{V} = \mathcal{U} \cap N$ for some open set $\mathcal{U}$ in (M,d). Let $x \in \mathcal{V}$. Then $x \in \mathcal{U}$; thus $\exists\, r > 0$ such that $D(x,r) \subseteq \mathcal{U}$. Therefore,

$$D_N(x,r) \equiv \big\{ y \in N \,\big|\, d(x,y) < r \big\} = D(x,r) \cap N \subseteq \mathcal{U} \cap N = \mathcal{V}\,;$$

hence $\mathcal{V}$ is open in (N,d). ◻

Corollary 3.41. *Let (M,d) be a metric space, and $N \subseteq M$. Let (M,d) be a metric space, and $N \subseteq M$. A subset $E \subseteq N$ is closed in (N,d) if and only if $E = F \cap N$ for some closed set F in (M,d).*

Definition 3.42. Let (M,d) be a metric space, and $N \subseteq M$. A subset A is said to be $\begin{array}{c}\text{open}\\ \text{closed}\\ \text{compact}\end{array}$ relative to N if $A \cap N$ is $\begin{array}{c}\text{open}\\ \text{closed}\\ \text{compact}\end{array}$ in the metric space (N,d).

Theorem 3.43. *Let (M,d) be a metric space, and $K \subseteq N \subseteq M$. Then K is compact in (M,d) if and only if K is compact in (N,d).*

Proof. "⇒" Let $\{\mathcal{V}_\alpha\}_{\alpha\in I}$ be an open cover of K in (N,d). By Proposition 3.40, there are open sets $\mathcal{U}_\alpha$ in (M,d) such that $\mathcal{V}_\alpha = \mathcal{U}_\alpha \cap N$ for all $\alpha \in I$. Then $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ is also an open cover of K; thus possesses a finite subcover; that is, $\exists\, J \subseteq I$, $\#J < \infty$ such that $K \subseteq \bigcup\limits_{\alpha\in J} \mathcal{U}_\alpha$ which, together with the fact that $K \subseteq N$, implies that

$$K \subseteq \Big(\bigcup_{\alpha\in J}\mathcal{U}_\alpha\Big) \cap N = \bigcup_{\alpha\in J}(\mathcal{U}_\alpha \cap N) = \bigcup_{\alpha\in J}\mathcal{V}_\alpha\,.$$

"⇐" Let $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ be an open cover of K in (M,d). Letting $\mathcal{V}_\alpha = \mathcal{U}_\alpha \cap N$, by Proposition 3.40 we find that $\{\mathcal{V}_\alpha\}_{\alpha\in I}$ is an open cover of K in (N,d). Since K is compact in (N,d), there exists $J \subseteq I$, $\#J < \infty$ such that $K \subseteq \bigcup\limits_{\alpha\in J} \mathcal{V}_\alpha$; thus

$$K \subseteq \bigcup_{\alpha\in J}\mathcal{U}_\alpha\,.$$ □

Remark 3.44. Another way to look at Theorem 3.43 is using the sequential compactness equivalence. Let $\{x_k\}_{k=1}^\infty \subseteq K$ be a sequence. By sequential compactness of K in either (M,d) or (N,d), there exists $\big\{x_{k_j}\big\}_{j=1}^\infty$ and $x \in K$ such that $x_{k_j} \to x$ as $j \to \infty$. As long as the metric d used in different space are identical, the concept of convergence of a sequence are the same; thus compactness in (M,d) or (N,d) are the same.

Example 3.45. Let (M,d) be $(\mathbb{R},|\cdot|)$, and $N = \mathbb{Q}$. Consider the set $F = [0,1] \cap \mathbb{Q}$. By Corollary 3.41 F is closed in $(\mathbb{Q},|\cdot|)$. However, F is not compact in $(\mathbb{Q},|\cdot|)$ since F is not complete. We can also apply Theorem 3.43 to see this: if $F \subseteq \mathbb{Q}$ is compact in $(\mathbb{Q},|\cdot|)$, then F is compact in $(\mathbb{R},|\cdot|)$ which is clearly not the case since F is not closed in $(\mathbb{R},|\cdot|)$.

Remark 3.46. Let (M,d) be a metric space. By Proposition 3.36 a subset $A \subseteq M$ is disconnected if and only if there exist two subsets $\mathcal{U}_1, \mathcal{U}_2$ of A, open relative to A, such that $A = \mathcal{U}_1 \cup \mathcal{U}_2$ and $\mathcal{U}_1 \cap \mathcal{U}_2 = \varnothing$ (one choice of $(\mathcal{U}_1,\mathcal{U}_2)$ is $\mathcal{U}_1 = A\backslash\bar{A}_1$ and $\mathcal{U}_2 = A\backslash\bar{A}_2$, where A_1 and A_2 are given by Proposition 3.36). Note that $\mathcal{U}_1$ and $\mathcal{U}_2$ are also closed relative to A.

Given the observation above, if A is a connected set and E is a subset of A such that E is closed and open relative to A, then $E = \varnothing$ or $E = A$.

Chapter 4

Continuous Maps

4.1 Continuity

Definition 4.1. Let (M,d) and (N,ρ) be two metric spaces, $A \subseteq M$ and $f : A \to N$ be a map. For a given $x_0 \in A'$, we say that $b \in N$ is the limit of f at x_0, written

$$\lim_{x\to x_0} f(x) = b \quad \text{or} \quad f(x) \to b \text{ as } x \to x_0 \,,$$

if for every sequence $\{x_k\}_{k=1}^{\infty} \subseteq A\backslash\{x_0\}$ converging to x_0, the sequence $\big\{f(x_k)\big\}_{k=1}^{\infty}$ converges to b.

Proposition 4.2. *Let (M,d) and (N,ρ) be two metric spaces, $A \subseteq M$ and $f : A \to N$ be a map. Then $\lim\limits_{x\to x_0} f(x) = b$ if and only if*

$$\forall\,\varepsilon > 0, \exists\,\delta = \delta(x_0,\varepsilon) > 0 \ni \rho(f(x),b) < \varepsilon \text{ whenever } 0 < d(x,x_0) < \delta \text{ and } x \in A\,.$$

Proof. "$\Rightarrow$" Assume the contrary that $\exists\,\varepsilon > 0$ such that for all $\delta > 0$, there exists $x_\delta \in A$ with

$$0 < d(x_\delta, x_0) < \delta \quad \text{and} \quad \rho(f(x_\delta),b) \geqslant \varepsilon\,.$$

In particular, letting $\delta = \dfrac{1}{k}$, we can find $\{x_k\}_{k=1}^{\infty} \subseteq A\backslash\{x_0\}$ such that

$$0 < d(x_k, x_0) < \frac{1}{k} \quad \text{and} \quad \rho(f(x_k),b) \geqslant \varepsilon\,.$$

Then $x_k \to x_0$ as $k \to \infty$ but $f(x_k) \nrightarrow b$ as $k \to \infty$, a contradiction.

"$\Leftarrow$" Let $\{x_k\}_{k=1}^{\infty} \subseteq A\backslash\{x_0\}$ be such that $x_k \to x_0$ as $k \to \infty$, and $\varepsilon > 0$ be given. By assumption,

$$\exists\, \delta = \delta(x_0, \varepsilon) > 0 \ni \rho(f(x), b) < \varepsilon \text{ whenever } 0 < d(x, x_0) < \delta \text{ and } x \in A\,.$$

Since $x_k \to x_0$ as $k \to \infty$, $\exists\, N > 0 \ni d(x_k, x_0) < \delta$ if $k \geqslant N$. Therefore,

$$\rho(f(x_k), b) < \varepsilon \quad \forall\, k \geqslant N$$

which suggests that $\lim\limits_{k\to\infty} f(x_k) = b$. □

Remark 4.3. Let $(M, d) = (N, \rho) = (\mathbb{R}, |\cdot|)$, $A = (a, b)$, and $f : A \to N$. We write $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to b^-} f(x)$ for the limit $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to b} f(x)$, respectively, if the later exist. Following this notation, we have

$$\begin{aligned}
\lim_{x\to a^+} f(x) = L &\Leftrightarrow \forall\, \varepsilon > 0, \exists\, \delta > 0 \ni |f(x) - L| < \varepsilon \text{ if } 0 < x - a < \delta \text{ and } x \in (a, b)\,,\\
\lim_{x\to b^-} f(x) = L &\Leftrightarrow \forall\, \varepsilon > 0, \exists\, \delta > 0 \ni |f(x) - L| < \varepsilon \text{ if } 0 < b - x < \delta \text{ and } x \in (a, b)\,.
\end{aligned}$$

Definition 4.4. Let (M, d) and (N, ρ) be two metric spaces, $A \subseteq M$, and $f : A \to N$ be a map. For a given $x_0 \in A$, f is said to be continuous at x_0 if either $x_0 \in A\backslash A'$ or $\lim\limits_{x\to x_0} f(x) = f(x_0)$.

Example 4.5. The identity map $f : \begin{array}{ccc} \mathbb{R}^n & \to & \mathbb{R}^n \\ x & \mapsto & x \end{array}$ is continuous at each point of $\mathbb{R}^n$.

Example 4.6. The function $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{x}$ is continuous at each point of $(0, \infty)$.

Proposition 4.7. *Let (M, d) and (N, ρ) be two metric spaces, $A \subseteq M$, and $f : A \to N$ be a map. Then f is continuous at $x_0 \in A$ if and only if*

$$\forall\, \varepsilon > 0, \exists\, \delta = \delta(x_0, \varepsilon) > 0 \ni \rho(f(x), f(x_0)) < \varepsilon \textit{ whenever } x \in D(x_0, \delta) \cap A\,.$$

Proof. **Case 1**: If $x_0 \in A'$, then f is continuous at x_0 if and only if

$$\forall\, \varepsilon > 0, \exists\, \delta = \delta(x_0, \varepsilon) > 0 \ni \rho(f(x), f(x_0)) < \varepsilon \text{ whenever } x \in D(x_0, \delta) \cap A\backslash\{x_0\}\,.$$

Since $\rho(f(x_0), f(x_0)) = 0 < \varepsilon$, we find that the statement above is equivalent to that

$$\forall\, \varepsilon > 0, \exists\, \delta = \delta(x_0, \varepsilon) > 0 \ni \rho(f(x), f(x_0)) < \varepsilon \text{ whenever } x \in D(x_0, \delta) \cap A\,.$$

Case 2: Let $x_0 \in A \backslash A'$.

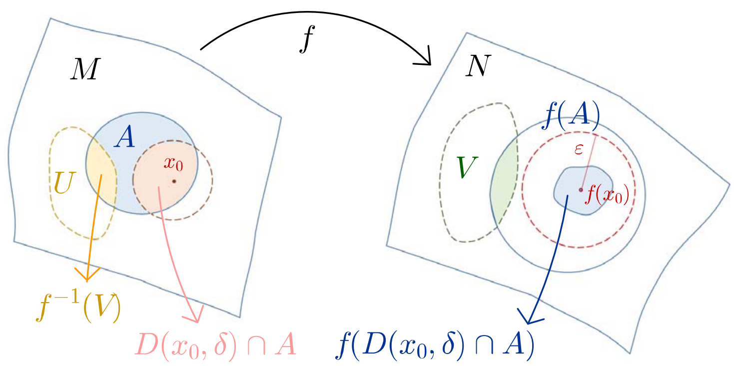
"$\Rightarrow$" then $\exists\, \delta > 0$ such that $D(x_0, \delta) \cap A = \{x_0\}$. Therefore, for this particular δ, we must have

$$\rho(f(x), f(x_0)) = 0 < \varepsilon \quad \text{whenever } x \in D(x_0, \delta) \cap A\,.$$

"$\Leftarrow$" We note that if $x_0 \in A \backslash A'$, f is defined to be continuous at x_0. In other words, f is continuous at each isolated point. $\square$

Remark 4.8. We remark here that Proposition 4.7 suggests that f is continuous at $x_0 \in A$ if and only if

$$\forall\, \varepsilon > 0, \exists\, \delta > 0 \ni f(D(x_0, \delta) \cap A) \subseteq D(f(x_0), \varepsilon)\,.$$



Remark 4.9. In general the number δ in Proposition 4.7 also depends on the function f. For a function $f : A \to \mathbb{R}$ which is continuous at $x_0 \in A$, let $\delta(f, x_0, \varepsilon)$ denote the largest $\delta > 0$ such that if $x \in D(x_0, \delta) \cap A$, then $\rho\big(f(x), f(x_0)\big) < \varepsilon$. In other words,

$$\delta(f, x_0, \varepsilon) = \sup\big\{\delta > 0 \,\big|\, \rho\big(f(x), f(x_0)\big) < \varepsilon \text{ if } x \in D(x_0, \delta) \cap A\big\}\,.$$

This number provides another way for the understanding of the uniform continuity (in Section 4.5) and the equi-continuity (in Section 5.5). See Remark 4.51 and Remark 5.54 for further details.

Definition 4.10. Let (M, d) and (N, ρ) be metric spaces, and $A \subseteq M$. A map $f : A \to N$ is said to be continuous on the set $B \subseteq A$ if f is continuous at each point of B.

Theorem 4.11. *Let (M, d) and (N, ρ) be metric spaces, $A \subseteq M$, and $f : A \to N$ be a map. Then the following assertions are equivalent:*

1. *f is continuous on A.*

2. *For each open set $\mathcal{V} \subseteq N$, $f^{-1}(\mathcal{V}) \subseteq A$ is open relative to A; that is, $f^{-1}(\mathcal{V}) = \mathcal{U} \cap A$ for some $\mathcal{U}$ open in M.*

3. *For each closed set $E \subseteq N$, $f^{-1}(E) \subseteq A$ is closed relative to A; that is, $f^{-1}(E) = F \cap A$ for some F closed in M.*

Proof. It should be clear that $2 \Leftrightarrow 3$ (left as an exercise); thus we show that $1 \Leftrightarrow 2$. Before proceeding, we recall that $B \subseteq f^{-1}(f(B))$ for all $B \subseteq A$ and $f(f^{-1}(B)) \subseteq B$ for all $B \subseteq N$.

"$1 \Rightarrow 2$" Let $a \in f^{-1}(\mathcal{V})$. Then $f(a) \in \mathcal{V}$. Since $\mathcal{V}$ is open in (N, ρ), $\exists\, \varepsilon_{f(a)} > 0$ such that $D(f(a), \varepsilon_{f(a)}) \subseteq \mathcal{V}$. By continuity of f (and Remark 4.8), there exists $\delta_a > 0$ such that

$$f(D(a, \delta_a) \cap A) \subseteq D\big(f(a), \varepsilon_{f(a)}\big)\,.$$

Therefore, by Proposition 0.15, for each $a \in f^{-1}(\mathcal{V})$, $\exists\, \delta_a > 0$ such that

$$D(a, \delta_a) \cap A \subseteq f^{-1}\big(f(D(a, \delta_a) \cap A)\big) \subseteq f^{-1}\big(D\big(f(a), \varepsilon_{f(a)}\big)\big) \subseteq f^{-1}(\mathcal{V})\,. \tag{4.1.1}$$

Let $\mathcal{U} = \bigcup\limits_{a \in f^{-1}(\mathcal{V})} D(a, \delta_a)$. Then $\mathcal{U}$ is open (since it is the union of arbitrarily many open balls), and

(a) $\mathcal{U} \supseteq f^{-1}(\mathcal{V})$ since $\mathcal{U}$ contains every center of balls whose union forms $\mathcal{U}$;

(b) $\mathcal{U} \cap A \subseteq f^{-1}(\mathcal{V})$ by (4.1.1).

Therefore, $\mathcal{U} \cap A = f^{-1}(\mathcal{V})$.

"$2 \Rightarrow 1$" Let $a \in A$ and $\varepsilon > 0$ be given. Define $\mathcal{V} = D(f(a), \varepsilon)$. By assumption there exists $\mathcal{U}$ open in (M, d) such that $f^{-1}(\mathcal{V}) = \mathcal{U} \cap A$. Since $a \in f^{-1}(\mathcal{V})$, $a \in \mathcal{U}$; thus by the openness of $\mathcal{U}$, $\exists\, \delta > 0$ such that $D(a, \delta) \subseteq \mathcal{U}$. Therefore, by Proposition 0.15 we have

$$f(D(a, \delta) \cap A) \subseteq f(\mathcal{U} \cap A) = f(f^{-1}(\mathcal{V})) \subseteq \mathcal{V} = D(f(a), \varepsilon)$$

which suggests that f is continuous at a for all $a \in A$; thus f is continuous on A. ▫

Example 4.12. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous. Then $\big\{x \in \mathbb{R}^n \,\big|\, \|f(x)\|_2 < 1\big\}$ is open since

$$\big\{x \in \mathbb{R}^n \,\big|\, \|f(x)\|_2 < 1\big\} = f^{-1}\big(D(0, 1)\big)\,.$$

Remark 4.13. For a function f of two variable or more, it is important to distinguish the continuity of f and the continuity in each variable (by holding all other variables fixed). For example, let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 1 & \text{if either } x = 0 \text{ or } y = 0, \\ 0 & \text{if } x \neq 0 \text{ and } y \neq 0. \end{cases}$$

Observe that $f(0,0) = 1$, but f is not continuous at $(0,0)$. In fact, for any $\delta > 0$, $f(x,y) = 0$ for infinitely many values of $(x,y) \in D((0,0),\delta)$; that is, $|f(x,y) - f(0,0)| = 1$ for such values. However if we consider the function $g(x) = f(x,0) = 1$ or the function $h(y) = f(0,y) = 1$, then g, h are continuous. Note also that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exists but $\lim\limits_{x\to 0}(\lim\limits_{y\to 0} f(x,y)) = \lim\limits_{x\to 0} 0 = 0$.

4.2 Operations on Continuous Maps

Definition 4.14. Let (M,d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a (real) normed space, $A \subseteq M$, and $f, g : A \to \mathcal{V}$ be maps, $h : A \to \mathbb{R}$ be a function. The maps $f+g$, $f-g$ and hf, mapping from A to $\mathcal{V}$, are defined by

$$\begin{aligned} (f+g)(x) &= f(x) + g(x) \qquad \forall\, x \in A\,, \\ (f-g)(x) &= f(x) - g(x) \qquad \forall\, x \in A\,, \\ (hf)(x) &= h(x)f(x) \qquad \forall\, x \in A\,. \end{aligned}$$

The map $\dfrac{f}{h} : A\backslash\{x \in A \,|\, h(x) = 0\} \to \mathcal{V}$ is defined by

$$\Big(\frac{f}{h}\Big)(x) = \frac{f(x)}{h(x)} \qquad \forall\, x \in A\backslash\{x \in A \,|\, h(x) = 0\}\,.$$

Proposition 4.15. *Let (M,d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a (real) normed space, $A \subseteq M$, and $f, g : A \to \mathcal{V}$ be maps, $h : A \to \mathbb{R}$ be a function. Suppose that $x_0 \in A'$, and $\lim\limits_{x\to x_0} f(x) = a$, $\lim\limits_{x\to x_0} g(x) = b$, $\lim\limits_{x\to x_0} h(x) = c$. Then*

$$\begin{aligned} \lim_{x\to x_0} (f+g)(x) &= a + b\,, \\ \lim_{x\to x_0} (f-g)(x) &= a - b\,, \\ \lim_{x\to x_0} (hf)(x) &= ca\,, \\ \lim_{x\to x_0} \Big(\frac{f}{h}\Big) &= \frac{a}{c} \quad \textit{if } c \neq 0\,. \end{aligned}$$

Corollary 4.16. *Let (M, d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a (real) normed space, $A \subseteq M$, and $f, g : A \to \mathcal{V}$ be maps, $h : A \to \mathbb{R}$ be a function. Suppose that f, g, h are continuous at $x_0 \in A$. Then the maps $f + g$, $f - g$ and hf are continuous at x_0, and $\dfrac{f}{h}$ is continuous at x_0 if $h(x_0) \neq 0$.*

Corollary 4.17. *Let (M, d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a (real) normed space, $A \subseteq M$, and $f, g : A \to \mathcal{V}$ be continuous maps, $h : A \to \mathbb{R}$ be a continuous function. Then the maps $f + g$, $f - g$ and hf are continuous on A, and $\dfrac{f}{h}$ is continuous on $A \backslash \{x \in A \,|\, h(x) = 0\}$.*

Definition 4.18. Let (M, d), (N, ρ) and (P, r) be metric space, $A \subseteq M$, $B \subseteq N$, and $f : A \to N$, $g : B \to P$ be maps such that $f(A) \subseteq B$. The composite function $g \circ f : A \to P$ is the map defined by

$$(g \circ f)(x) = g\big(f(x)\big) \qquad \forall\, x \in A\,.$$

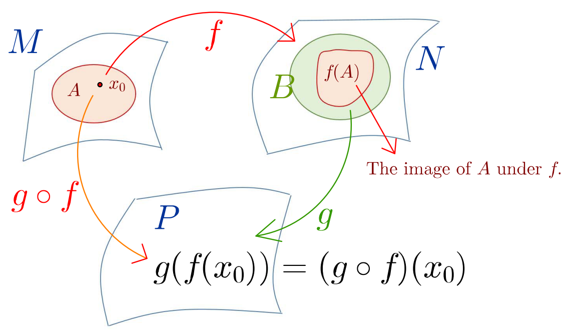


Figure 4.1: The composition of functions

Theorem 4.19. *Let (M, d), (N, ρ) and (P, r) be metric space, $A \subseteq M$, $B \subseteq N$, and $f : A \to N$, $g : B \to P$ be maps such that $f(A) \subseteq B$. Suppose that f is continuous at x_0, and g is continuous at $f(x_0)$. Then the composite function $g \circ f : A \to P$ is continuous at x_0.*

Proof. Let $\varepsilon > 0$ be given. Since g is continuous at $f(x_0)$, $\exists\, r > 0$ such that

$$g(D(f(x_0), r) \cap B) \subseteq D\big((g \circ f)(x_0), \varepsilon\big)\,.$$

Since f is continuous at x_0, $\exists\, \delta > 0$ such that

$$f(D(x_0, \delta) \cap A) \subseteq D\big(f(x_0), r\big)\,.$$

Since $f(A) \subseteq B$, $f(D(x_0, \delta) \cap A) \subseteq D\big(f(x_0), r\big) \cap B$; thus

$$(g \circ f)(D(x_0, \delta) \cap A) \subseteq g(D(f(x_0), r) \cap B) \subseteq D\big((g \circ f)(x_0), \varepsilon\big)\,. \qquad \square$$

Corollary 4.20. *Let (M,d), (N,ρ) and (P,r) be metric space, $A \subseteq M$, $B \subseteq N$, and $f : A \to N$, $g : B \to P$ be continuous maps such that $f(A) \subseteq B$. Then the composite function $g \circ f : A \to P$ is continuous on A.*

Alternative Proof of Corollary 4.20. Let $\mathcal{W}$ be an open set in (P,r). By Theorem 4.11, there exists $\mathcal{V}$ open in (N,ρ) such that $g^{-1}(\mathcal{W}) = \mathcal{V} \cap B$. Since $\mathcal{V}$ is open in (N,ρ), by Theorem 4.11 again there exists $\mathcal{U}$ open in (M,d) such that $f^{-1}(\mathcal{V}) = \mathcal{U} \cap A$. Then

$$(g \circ f)^{-1}(\mathcal{W}) = f^{-1}\big(g^{-1}(\mathcal{W})\big) = f^{-1}(\mathcal{V} \cap B) = f^{-1}(\mathcal{V}) \cap f^{-1}(B) = \mathcal{U} \cap A \cap f^{-1}(B)\,,$$

while the fact that $f(A) \subseteq B$ further suggests that

$$(g \circ f)^{-1}(\mathcal{W}) = \mathcal{U} \cap A\,.$$

Therefore, by Theorem 4.11 we find that $(g \circ f)$ is continuous on A. □

4.3 Images of Compact Sets under Continuous Maps

Theorem 4.21. *Let (M,d) and (N,ρ) be metric spaces, $A \subseteq M$, and $f : A \to N$ be a continuous map.*

1. *If $K \subseteq A$ is compact, then $f(K)$ is compact in (N,ρ).*

2. *Moreover, if $(N,\rho) = (\mathbb{R}, |\cdot|)$, then there exist $x_0, x_1 \in K$ such that*

$$f(x_0) = \inf f(K) = \inf\big\{f(x)\,\big|\,x \in K\big\} \quad \textit{and} \quad f(x_1) = \sup f(K) = \sup\big\{f(x)\,\big|\,x \in K\big\}\,.$$

Proof. 1. Let $\{\mathcal{V}_\alpha\}_{\alpha\in I}$ be an open cover of $f(K)$. Since $\mathcal{V}_\alpha$ is open, by Theorem 4.11 there exists $\mathcal{U}_\alpha$ open in (M,d) such that $f^{-1}(\mathcal{V}_\alpha) = \mathcal{U}_\alpha \cap A$. Since $f(K) \subseteq \bigcup\limits_{\alpha\in I} \mathcal{V}_\alpha$,

$$K \subseteq f^{-1}(f(K)) \subseteq \bigcup_{\alpha\in I} f^{-1}(\mathcal{V}_\alpha) = A \cap \bigcup_{\alpha\in I} \mathcal{U}_\alpha$$

which implies that $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ is an open cover of K. Therefore,

$$\exists\, J \subseteq I, \#J < \infty \ni K \subseteq A \cap \bigcup_{\alpha\in J} \mathcal{U}_\alpha = \bigcup_{\alpha\in J} f^{-1}(\mathcal{V}_\alpha)\,;$$

thus $f(K) \subseteq \bigcup\limits_{\alpha\in J} f(f^{-1}(\mathcal{V}_\alpha)) \subseteq \bigcup\limits_{\alpha\in J} \mathcal{V}_\alpha$.

2. By 1, $f(K)$ is compact; thus sequentially compact. Corollary 3.5 then implies that $\inf f(K) \in f(K)$ and $\sup f(K) \in f(K)$. $\square$

Alternative Proof of Part 1. Let $\{y_n\}_{n=1}^\infty$ be a sequence in $f(K)$. Then there exists $\{x_n\}_{n=1}^\infty \subseteq K$ such that $y_n = f(x_n)$. Since K is sequentially compact, there exists a convergent subsequence $\{x_{n_k}\}_{k=1}^\infty$ with limit $x \in K$. Let $y = f(x) \in f(K)$. By the continuity of f,

$$\lim_{k\to\infty} \rho\big(y_{n_k}, y\big) = \lim_{k\to\infty} \rho\big(f(x_{n_k}), f(x)\big) = 0$$

which implies that the sequence $\big\{y_{n_k}\big\}_{k=1}^\infty$ converges to $y \in f(K)$. Therefore, $f(K)$ is sequentially compact. $\square$

Corollary 4.22 (The Extreme Value Theorem（極值定理）)**.** *Let $f : [a,b] \to \mathbb{R}$ be continuous. Then f attains its maximum and minimum in $[a,b]$; that is, there are $x_0 \in [a,b]$ and $x_1 \in [a,b]$ such that*

$$f(x_0) = \inf\big\{f(x) \,\big|\, x \in [a,b]\big\} \quad \textit{and} \quad f(x_1) = \sup\big\{f(x) \,\big|\, x \in [a,b]\big\}\,. \tag{4.3.1}$$

Proof. The Heine-Borel Theorem suggests that $[a,b]$ is a compact set in $\mathbb{R}$; thus Theorem 4.21 implies that $f([a,b])$ must be compact in $\mathbb{R}$. By the Heine-Borel Theorem again $f([a,b])$ is closed and bounded, so

$$\inf f([a,b]) \in f([a,b]) \quad \text{and} \quad \sup f([a,b]) \in f([a,b])$$

which further imply (4.3.1). $\square$

Remark 4.23. If f attains its maximum (or minimum) on a set B, we use $\max\big\{f(x)\,\big|\,x \in B\big\}$ $\big($or $\min\big\{f(x)\,\big|\,x \in B\big\}\big)$ to denote $\sup\big\{f(x)\,\big|\,x \in B\big\}$ $\big($or $\inf\big\{f(x)\,\big|\,x \in B\big\}\big)$. Therefore, (4.3.1) can be rewritten as

$$f(x_0) = \min\big\{f(x) \,\big|\, x \in [a,b]\big\} \quad \text{and} \quad f(x_1) = \max\big\{f(x) \,\big|\, x \in [a,b]\big\}\,.$$

Example 4.24. Two norms $\|\cdot\|$ and $|||\cdot|||$ on a real vector space $\mathcal{V}$ are called equivalent if there are positive constants C_1 and C_2 such that

$$C_1\|x\| \leqslant |||x||| \leqslant C_2\|x\| \qquad \forall\, x \in \mathcal{V}\,.$$

We note that equivalent norms on a vector space $\mathcal{V}$ induce the same topology; that is, if $\|\cdot\|$ and $|||\cdot|||$ are equivalent norms on $\mathcal{V}$, then $\mathcal{U}$ is open in the normed space $(\mathcal{V}, \|\cdot\|)$ if and only if $\mathcal{U}$ is open in the normed space $(\mathcal{V}, |||\cdot|||)$. In fact, let $\mathcal{U}$ be an open set in $(\mathcal{V}, \|\cdot\|)$. Then for any $x \in \mathcal{U}$, there exists $r > 0$ such that

$$D_{\|\cdot\|}(x,r) \equiv \big\{y \in \mathcal{V} \,\big|\, \|x-y\| < r\big\} \subseteq \mathcal{U}\,.$$

Let $\delta = C_1 r$. Then if $z \in D_{|||\cdot|||}(x,\delta) \equiv \big\{y \in \mathcal{V} \,\big|\, |||x-y||| < \delta\big\}$,

$$\|x-z\| \leqslant \frac{1}{C_1}|||x-z||| < \frac{1}{C_1}\cdot C_1 r = r$$

which implies that $D_{|||\cdot|||}(x,\delta) \subseteq D_{\|\cdot\|}(x,r) \subseteq \mathcal{U}$. Therefore, $\mathcal{U}$ is open in $(\mathcal{V}, |||\cdot|||)$. Similarly, if $\mathcal{U}$ is open in $(\mathcal{V}, |||\cdot|||)$, then the inequality $|||x||| \leqslant C_2\|x\|$ suggests that $\mathcal{U}$ is open in $(\mathcal{V}, \|\cdot\|)$.

Claim: Any two norms on $\mathbb{R}^n$ are equivalent.

Proof of claim: It suffices to show that any norm $\|\cdot\|$ on $\mathbb{R}^n$ is equivalent to the two-norm $\|\cdot\|_2$ (check). Let $\{e_k\}_{k=1}^n$ be the standard basis of $\mathbb{R}^n$; that is,

$$e_k = (\underbrace{0,\cdots,0}_{(k-1)\text{ zeros}},1,0,\cdots,0)\,.$$

Every $x \in \mathbb{R}^n$ can be written as $x = \sum\limits_{i=1}^n x_i e_i$, and $\|x\|_2 = \sqrt{\sum\limits_{i=1}^n |x_i|^2}$. By the definition of norms and the Cauchy-Schwartz inequality,

$$\|x\| \leqslant \sum_{i=1}^n |x_i|\|e_i\| \leqslant \|x\|_2\sqrt{\sum_{i=1}^n \|e_i\|^2}\,; \tag{4.3.2}$$

thus letting $C_2 = \sqrt{\sum\limits_{i=1}^n \|e_i\|^2}$ we have $\|x\| \leqslant C_2\|x\|_2$.

On the other hand, define $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) = \|x\| = \Big\|\sum_{i=1}^n x_i e_i\Big\|\,.$$

Because of (4.3.2), f is continuous on $\mathbb{R}^n$. In fact, for $x, y \in \mathbb{R}^n$,

$$\big|f(x) - f(y)\big| = \big|\|x\| - \|y\|\big| \leqslant \|x-y\| \leqslant C_2\|x-y\|_2$$

which guarantees the continuity of f on $\mathbb{R}^n$. Let $\mathbb{S}^{n-1} = \big\{x \in \mathbb{R}^n \,\big|\, \|x\|_2 = 1\big\}$. Then $\mathbb{S}^{n-1}$ is a compact set in $(\mathbb{R}^n, \|\cdot\|_2)$ (since it is closed and bounded); thus by Theorem 4.21 f attains its minimum on $\mathbb{S}^{n-1}$ at some point $a = (a_1, \cdots, a_n)$. Moreover, $f(a) > 0$ (since if $f(a) = 0$, $a = 0 \notin \mathbb{S}^{n-1}$). Then for all $x \in \mathbb{R}^n$, $\dfrac{x}{\|x\|_2} \in \mathbb{S}^{n-1}$; thus

$$f\Big(\frac{x}{\|x\|_2}\Big) \geqslant f(a)\,.$$

The inequality above further implies that $f(a)\|x\|_2 \leqslant f(x) = \|x\|$; thus letting $C_1 = f(a)$ we have $C_1\|x\|_2 \leqslant \|x\|$.

Remark 4.25.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 0$. Then f is continuous. Note that $\{0\} \subseteq \mathbb{R}$ is compact ($\because$ closed and bounded), but $f^{-1}(\{0\}) = \mathbb{R}$ is not compact.
2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Then f is continuous. Note that $C = \{1\}$ is connected, but $f^{-1}(C) = \{1, -1\}$ is not connected.

Remark 4.26.

1. If K is not compact, then Theorem 4.21 is not true. Consider the following counter example: $K = (0, 1), f : K \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{x}$. Then $f(K)$ is unbounded.
2. If f is not continuous, then Theorem 4.21 is not true either.
   (a) Counter example 1: $f : K = [0, 1] \to \mathbb{R}$ defined by
   $$f(x, y) = \begin{cases} \dfrac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$
   Then $f(K)$ is unbounded $\Rightarrow \nexists\, x_1 \in K \ni f(x_1) = \sup f(K)$.
   (b) Counter example 2: $f : [0, 1] \to \mathbb{R}$ by
   $$f(x, y) = \begin{cases} x & \text{if } x \neq 1, \\ 0 & \text{if } x = 1. \end{cases}$$
   Then there is no $x_1 \in [0, 1]$ such that $f(x_1) = \sup\limits_{x \in [0,1]} f(x) = 1$.

Example 4.27 (An example show that x_0, x_1 in Theorem 4.21 are not unique)**.** Let $f : [-2, 2] \to \mathbb{R}$ be defined by $f(x) = (x^2 - 1)^2$.

1. Critical point: $f'(x) = 2(x^2 - 1) \cdot 2x = 0 \Leftrightarrow x = 0, \pm 1$.

2. Comparison: $f(0) = 1$, $f(1) = f(-1) = 0$, $f(2) = f(-2) = 9$. Then

$$f(2) = f(-2) = \sup_{x \in [-2,2]} f(x) \quad \text{and} \quad f(1) = f(-1) = \inf_{x \in [-2,2]} f(x)\,.$$

Corollary 4.28. *Let (M, d) be a metric space, $K \subseteq M$ be a compact set, and $f : K \to \mathbb{R}$ be continuous. Then the set*

$$\{x \in K \,|\, f(x) \text{ is the maximum of } f \text{ on } K\}$$

is a non-empty compact set.

Proof. Let $\mathrm{M} = \sup f(K)$. Then the set defined above is $f^{-1}(\{\mathrm{M}\})$, and

1. $f^{-1}(\{\mathrm{M}\})$ is non-empty by Theorem 4.21;

2. $f^{-1}(\{\mathrm{M}\})$ is closed since $\{\mathrm{M}\}$ is a closed set in $(\mathbb{R}, |\cdot|)$ and f is continuous on K.

Lemma 3.11 suggests that $f^{-1}(\{\mathrm{M}\})$ is compact. □

4.4 Images of Connected and Path Connected Sets under Continuous Maps

Definition 4.29. Let (M, d) be a metric space. A subset $A \subseteq M$ is said to be ***path connected*** if for every $x, y \in A$, there exists a continuous map $\varphi : [0, 1] \to A$ such that $\varphi(0) = x$ and $\varphi(1) = y$.

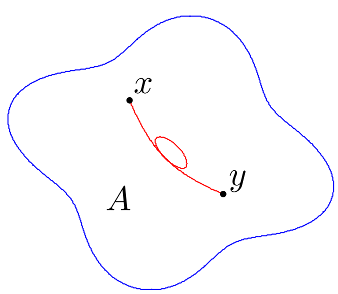


Figure 4.2: Path connected sets

Example 4.30. A set A in a vector space $\mathcal{V}$ is called ***convex*** if for all $x, y \in A$, the line segment joining x and y, denoted by $\overline{xy}$, lies in A. Then a convex set in a normed space is path connected. In fact, for $x, y \in A$, define $\varphi(t) = ty + (1 - t)x$. Then

1. $\varphi : [0,1] \to \overline{xy} \subseteq A$, $\varphi(0) = x$, $\varphi(1) = y$;

2. $\varphi : [0,1] \to A$ is continuous.

$\overline{xy} = \varphi([0,1])$

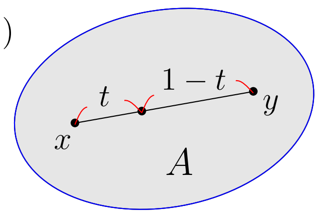


Figure 4.3: Convex sets

Example 4.31. A set S in a vector space $\mathcal{V}$ is called ***star-shaped*** if there exists $p \in S$ such that for any $q \in S$, the line segment joining p and q lies in S. A star-shaped set in a normed space is path connected. In fact, for $x, y \in S$, define

$$\varphi(t) = \begin{cases} 2tp + (1-2t)x & \text{if } t \in \big[0, \frac{1}{2}\big]\,, \\ (2t-1)y + (2-2t)p & \text{if } t \in \big[\frac{1}{2}, 1\big]\,. \end{cases}$$

Then

1. $\varphi : [0,1] \to \overline{xp} \cup \overline{py} \subseteq S$, $\varphi(0) = x$, $\varphi(1) = y$;

2. $\varphi : [0,1] \to A$ is continuous.

Theorem 4.32. *Let (M, d) be a metric space, and $A \subseteq M$. If A is path connected, then A is connected.*

Proof. Assume the contrary that there are two open sets $\mathcal{V}_1$ and $\mathcal{V}_2$ such that

1. $A \cap \mathcal{V}_1 \cap \mathcal{V}_2 = \varnothing$; 2. $A \cap \mathcal{V}_1 \neq \varnothing$; 3. $A \cap \mathcal{V}_2 \neq \varnothing$; 4. $A \subseteq \mathcal{V}_1 \cup \mathcal{V}_2$.

Since A is path connected, for $x \in A \cap \mathcal{V}_1$ and $y \in A \cap \mathcal{V}_2$, there exists $\varphi : [0,1] \to A$ such that $\varphi(0) = x$ and $\varphi(1) = y$. By Theorem 4.11, there exist $\mathcal{U}_1$ and $\mathcal{U}_2$ open in $(\mathbb{R}, |\cdot|)$ such that $\varphi^{-1}(\mathcal{V}_1) = \mathcal{U}_1 \cap [0,1]$ and $\varphi^{-1}(\mathcal{V}_2) = \mathcal{U}_2 \cap [0,1]$. Therefore,

$$[0,1] = \varphi^{-1}(A) \subseteq \varphi^{-1}(\mathcal{V}_1) \cup \varphi^{-1}(\mathcal{V}_2) \subseteq \mathcal{U}_1 \cup \mathcal{U}_2\,.$$

Since $0 \in \mathcal{U}_1$, $1 \in \mathcal{U}_2$, and $[0,1] \cap \mathcal{U}_1 \cap \mathcal{U}_2 = \varphi^{-1}(A \cap \mathcal{V}_1 \cap \mathcal{V}_2) = \varnothing$, we conclude that $[0,1]$ is disconnected, a contradiction. $\square$

Example 4.33. Let $A = \left\{\left(x, \sin\frac{1}{x}\right) \,\middle|\, x \in (0,1]\right\} \cup (\{0\} \times [-1,1])$. Then A is connected in $(\mathbb{R}^2, \|\cdot\|_2)$, but A is not path connected.

Theorem 4.34. *Let (M,d) and (N,ρ) be metric spaces, $A \subseteq M$, and $f : A \to N$ be a continuous map.*

1. *If $C \subseteq A$ is connected, then $f(C)$ is compact in (N,ρ).*

2. *If $C \subseteq A$ is path connected, then $f(C)$ is path connected in (N,ρ).*

Proof. 1. Suppose that there are two open sets $\mathcal{V}_1$ and $\mathcal{V}_2$ in (N,ρ) such that

(a) $f(C) \cap \mathcal{V}_1 \cap \mathcal{V}_2 = \varnothing$; (b) $f(C) \cap \mathcal{V}_1 \neq \varnothing$; (c) $f(C) \cap \mathcal{V}_2 \neq \varnothing$; (d) $f(C) \subseteq \mathcal{V}_1 \cup \mathcal{V}_2$.

By Theorem 4.11, there are $\mathcal{U}_1$ and $\mathcal{U}_2$ open in (M,d) such that $f^{-1}(\mathcal{V}_1) = \mathcal{U}_1 \cap A$ and $f^{-1}(\mathcal{V}_2) = \mathcal{U}_2 \cap A$. By (d),

$$C \subseteq f^{-1}(f(C)) \subseteq f^{-1}(\mathcal{V}_1) \cup f^{-1}(\mathcal{V}_2) = (\mathcal{U}_1 \cup \mathcal{U}_2) \cap A \subseteq \mathcal{U}_1 \cup \mathcal{U}_2 \,.$$

Moreover, by (a) we find that

$$\begin{aligned} C \cap \mathcal{U}_1 \cap \mathcal{U}_2 &= C \cap (\mathcal{U}_1 \cap A) \cap (\mathcal{U}_2 \cap A) = C \cap f^{-1}(\mathcal{V}_1) \cap f^{-1}(\mathcal{V}_2) \\ &\subseteq f^{-1}(f(C) \cap \mathcal{V}_1 \cap \mathcal{V}_2) = \varnothing \end{aligned}$$

which implies $C \cap \mathcal{U}_1 \cap \mathcal{U}_2 = \varnothing$. Finally, (b) implies that for some $x \in C$, $f(x) \in \mathcal{V}_1$. Therefore, $x \in f^{-1}(\mathcal{V}_1) = \mathcal{U}_1 \cap A$ which suggests that $x \in \mathcal{U}_1$; thus $C \cap \mathcal{U}_1 \neq \varnothing$. Similarly, $C \cap \mathcal{U}_2 \neq \varnothing$. Therefore, C is disconnected which is a contradiction.

2. Let $y_1, y_2 \in f(C)$. Then $\exists\, x_1, x_2 \in C$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since C is path connected, $\exists\, r : [0,1] \to C$ such that r is continuous on $[0,1]$ and $r(0) = x_1$ and $r(1) = x_2$. Let $\varphi : [0,1] \to f(C)$ be defined by $\varphi = f \circ r$. By Corollary 4.20 φ is continuous on $[0,1]$, and $\varphi(0) = y_1$ and $\varphi(1) = y_2$. ◻

Corollary 4.35 (The Intermediate Value Theorem（中間值定理）)**.** *Let $f : [a,b] \to \mathbb{R}$ be continuous. If $f(a) \neq f(b)$, then for all d in between $f(a)$ and $f(b)$, there exists $c \in (a,b)$ such that $f(c) = d$.*

Proof. The closed interval $[a,b]$ is connected by Theorem 3.38, so Theorem 4.34 implies that $f([a,b])$ must be connected in $\mathbb{R}$. By Theorem 3.38 again, if d is in between $f(a)$ and $f(b)$, then d belongs to $f([a,b])$. Therefore, for some $c \in (a,b)$ we have $f(c) = d$. ◻

Example 4.36. Let $f : [0,1] \to [0,1]$ be continuous. Then $\exists\, x_0 \in [0,1] \ni f(x_0) = x_0$.

Proof. Let $g(x) = x - f(x)$. Then

1. $g(0) = 0$ or $g(1) = 0 \Rightarrow x_0 = 0$ or 1.

2. $g(0) \neq 0$ or $g(1) \neq 0 \Rightarrow g(0) < 0$ and $g(1) > 0$. Since $g : [0,1] \to \mathbb{R}$ is continuous,

   $$\exists\, x_0 \in [0,1] \ni g(x_0) = 0 \Rightarrow \exists\, x_0 \in (0,1) \ni f(x_0) = x_0. \qquad \square$$

Remark 4.37. Such an x_0 in Example 4.36 is called a ***fixed-point*** of f.

Example 4.38. Let $f : [1,2] \to [0,3]$ be continuous, and $f(1) = 0$ and $f(2) = 3$. Then $\exists\, x_0 \in [1,2] \ni f(x_0) = x_0$.

Proof. Let $g(x) = x - f(x)$. Then $g : [1,2] \to \mathbb{R}$ is continuous. Moreover,

$$g(1) = 1 - f(1) = 1, \quad g(2) = 2 - f(2) = -1\,;$$

thus $\exists\, x_0 \in (1,2) \ni g(x_0) = 0$. $\square$

Example 4.39. Let p be a cubic polynomial; that is, $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ for some $a_0, a_1, a_2 \in \mathbb{R}$ and $a_3 \neq 0$. Then p has a real root x_0 (that is, $\exists\, x_0 \in \mathbb{R}$ such that $p(x_0) = 0$).

Proof. Note that p is obviously continuous and $\mathbb{R}$ is connected. Write

$$p(x) = a_3x^3\Big(1 + \frac{a_2}{a_3x} + \frac{a_1}{a_3x^2} + \frac{a_0}{a_3x^3}\Big)\,.$$

Now $\lim\limits_{x\to\pm\infty} \dfrac{\alpha}{\beta x^n} = 0$ if $n > 0$ and $\beta \neq 0$, so

$$\lim_{x\to\pm\infty} \Big(1 + \frac{a_2}{a_3x} + \frac{a_1}{a_3x^2} + \frac{a_0}{a_3x^3}\Big) = 1\,.$$

Moreover,

$$\lim_{x\to\infty} ax^3 = \begin{cases} \infty & \text{if } a > 0, \\ -\infty & \text{if } a < 0. \end{cases}$$

Suppose that $a > 0$. Then $\lim\limits_{x\to\infty} ax^3 = \infty$ and $\lim\limits_{x\to-\infty} ax^3 = -\infty \Rightarrow \exists\, x, y \in \mathbb{R} \ni p(x) < 0 < p(y)$. By Corollary 4.35 $\exists\, r \in \mathbb{R} \ni p(r) = 0$. The case that $a < 0$ is similar. $\square$

4.5 Uniform Continuity（均勻連續）

Definition 4.40. Let (M,d) and (N,ρ) be metric spaces, $A \subseteq M$, and $f : A \to N$ be a map. For a set $B \subseteq A$, f is said to be ***uniformly continuous on*** B if for any two sequences $\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty \subseteq B$ with the property that $\lim\limits_{n\to\infty} d(x_n,y_n) = 0$, one has $\lim\limits_{n\to\infty} \rho\big(f(x_n), f(y_n)\big) = 0$.

Proposition 4.41. *Let (M,d) and (N,ρ) be metric spaces, $A \subseteq M$, and $f : A \to N$ be a map. If f is uniformly continuous on A, then f is continuous on A.*

Proof. Let $x_0 \in A \cap A'$. Then there exists sequence $\{x_k\}_{k=1}^\infty \subseteq A$ such that $x_k \to x_0$ as $k \to \infty$. Let $\{y_k\}_{k=1}^\infty$ be a constant sequence with value x_0; that is, $y_k = x_0$ for all $k \in \mathbb{N}$. Then $\{y_k\}_{k=1}^\infty \subseteq A$ and $d(x_k,y_k) \to 0$ as $k \to \infty$. By the uniform continuity of f on A,

$$\lim_{k\to\infty} \rho\big(f(x_k), f(x_0)\big) = \lim_{k\to\infty} \rho\big(f(x_k), f(y_k)\big) = 0$$

which implies that f is continuous on x_0. □

Example 4.42. Let $f : [0,1] \to \mathbb{R}$ be the Dirichlet function; that is,

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}\,, \\ 1 & \text{if } x \in \mathbb{Q}^\complement. \end{cases}$$

and $B = \mathbb{Q} \cap [0,1]$. Then f is continuous **nowhere** in $[0,1]$, but f is uniformly continuous on B. However, the proposition above guarantees that if f is uniformly continuous on A, then f must be continuous on A (Check why the proof of Proposition 4.41 does not go through if B is a proper subset of A).

Example 4.43. The function $f(x) = |x|$ is uniformly continuous on $\mathbb{R}$.

Proof. By the triangle inequality,

$$\big|f(x) - f(y)\big| = \big||x| - |y|\big| \leqslant |x-y|\,;$$

thus if $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are sequences in $\mathbb{R}$ and $\lim\limits_{n\to\infty} |x_n - y_n| = 0$, by the Sandwich lemma we must have $\lim\limits_{n\to\infty} \big|f(x_n) - f(y_n)\big| = 0$. □

Example 4.44. The function $f : (0,\infty) \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{x}$ is uniformly continuous on $[a,\infty)$ for all $a > 0$. However, it is not uniformly continuous on $(0,\infty)$.

Proof. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences in $[a,\infty)$ such that $\lim\limits_{n\to\infty}|x_n-y_n|=0$. Then

$$|f(x_n)-f(y_n)|=\Big|\frac{1}{x_n}-\frac{1}{y_n}\Big|=\frac{|x_n-y_n|}{|x_ny_n|}\leqslant\frac{|x_n-y_n|}{a^2}\to 0\quad\text{as}\quad n\to\infty$$

which implies that f is uniformly continuous on $[a,\infty)$ if $a>0$. However, by choosing $x_n=\dfrac{1}{n}$ and $y_n=\dfrac{1}{2n}$, we find that

$$|x_n-y_n|=\frac{1}{2n}\quad\text{but}\quad|f(x_n)-f(y_n)|=n\geqslant 1\,;$$

thus f cannot be uniformly continuous on $(0,\infty)$. □

Remark 4.45. Let (M,d) and (N,ρ) be metric spaces, $A\subseteq M$, and $f:B\subseteq A\to N$ be a map. Then the following four statements are equivalent:

(1) f is **not** uniformly continuous on B.

(2) $\exists\,\{x_n\}_{n=1}^{\infty},\,\{y_n\}_{n=1}^{\infty}\subseteq B\ni\lim\limits_{n\to\infty}d(x_n,y_n)=0$ and $\limsup\limits_{n\to\infty}\rho\big(f(x_n),f(y_n)\big)>0$.

(3) $\exists\,\{x_n\}_{n=1}^{\infty},\,\{y_n\}_{n=1}^{\infty}\subseteq B\ni\lim\limits_{n\to\infty}d(x_n,y_n)=0$ and $\lim\limits_{n\to\infty}\rho\big(f(x_n),f(y_n)\big)>0$.

(4) $\exists\,\varepsilon>0\ni\forall\,n>0,\exists\,x_n,y_n\in B$ and $d(x_n,y_n)<\dfrac{1}{n}\ni\rho\big(f(x_n),f(y_n)\big)\geqslant\varepsilon$.

Example 4.46. Let $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=x^2$. Then f is continuous in $\mathbb{R}$ but not uniformly continuous on $\mathbb{R}$. Let $\varepsilon=1$, $x_n=n$, and $y_n=n+\dfrac{1}{2n}$,

$$\big|f(x_n)-f(y_n)\big|=\big|n^2-(n+\frac{1}{2n})^2\big|=\big|n^2-n^2-1-\frac{1}{4n^2}\big|>1\quad\forall\,n>0\,.$$

Example 4.47. The function $f(x)=\sin(x^2)$ is not uniform continuous on $\mathbb{R}$.

Proof. Let $\varepsilon=1$, $x_n=2n\sqrt{\pi}+\dfrac{\sqrt{\pi}}{8n}$ and $y_n=2n\sqrt{\pi}-\dfrac{\sqrt{\pi}}{8n}$. Then

$$\big|\sin(x_n^2)-\sin(y_n^2)\big|=\Big|\sin\Big(4n^2\pi+\frac{\pi}{2}+\frac{\pi}{64n^2}\Big)-\sin\Big(4n^2\pi-\frac{\pi}{2}+\frac{\pi}{64n^2}\Big)\Big|=2\cos\frac{\pi}{64n^2}\,;$$

thus if n is large enough, $\big|\sin(x_n^2)-\sin(y_n^2)\big|\geqslant 1$. □

Example 4.48. The function $f:(0,1)\to\mathbb{R}$ defined by $f(x)=\sin\dfrac{1}{x}$ is not uniformly continuous.

Proof. Let $\varepsilon = 1$, $x_n = \big(2n\pi + \dfrac{\pi}{2}\big)^{-1}$ and $y_n = \big(2n\pi - \dfrac{\pi}{2}\big)^{-1}$. Then

$$\Big|\sin\frac{1}{x_n} - \sin\frac{1}{y_n}\Big| = 2\,,$$

while $|x_n - y_n| = \dfrac{\pi}{4n^2\pi^2 - \frac{\pi^2}{4}} = \dfrac{1}{(4n^2 - \frac{1}{4})\pi} \leqslant \dfrac{1}{n}$ for all $n \in \mathbb{N}$. ◻

Theorem 4.49. *Let (M,d) and (N,ρ) be metric spaces, $A \subseteq M$, and $f : A \to N$ be a map. For a set $B \subseteq A$, f is uniformly continuous on B if and only if*

$$\forall\,\varepsilon > 0, \exists\,\delta > 0 \ni \rho\big(f(x), f(y)\big) < \varepsilon \textit{ whenever } d(x,y) < \delta \textit{ and } x, y \in B\,.$$

Proof. "$\Leftarrow$" Suppose the contrary that f is not uniformly continuous on B. Then there are two sequences $\{x_n\}_{n=1}^\infty$, $\{y_n\}_{n=1}^\infty$ in B such that

$$\lim_{k\to\infty} d(x_n, y_n) = 0 \quad \text{but} \quad \limsup_{n\to\infty} \rho\big(f(x_n), f(y_n)\big) > 0\,.$$

Let $\varepsilon = \dfrac{1}{2}\limsup\limits_{n\to\infty} \rho\big(f(x_n), f(y_n)\big)$. Then by the definition of the limit and the limit superior (or Proposition 1.116) we conclude that there exist subsequences $\{x_{n_k}\}_{k=1}^\infty$ and $\{y_{n_k}\}_{k=1}^\infty$ such that

$$\rho\big(f(x_{n_k}), f(y_{n_k})\big) \geqslant \limsup_{n\to\infty} \rho\big(f(x_n), f(y_n)\big) - \varepsilon = \varepsilon > 0$$

while $\lim\limits_{k\to\infty} d(x_{n_k}, y_{n_k}) = 0$, a contradiction.

"$\Rightarrow$" Suppose the contrary that there exists $\varepsilon > 0$ such that for all $\delta = \dfrac{1}{n} > 0$, there exist two points x_n and $y_n \in B$ such that

$$d(x_n, y_n) < \frac{1}{n} \quad \text{but} \quad \rho\big(f(x_n), f(y_n)\big) \geqslant \varepsilon\,.$$

These points form two sequences $\{x_n\}_{n=1}^\infty$, $\{y_n\}_{n=1}^\infty$ in B such that $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$, while the limit of $\rho\big(f(x_n), f(y_n)\big)$, if exists, does not converges to zero as $n \to \infty$. As a consequence, f is not uniformly continuous on B, a contradiction. ◻

Remark 4.50. The theorem above provides another way (the blue color part) of defining the uniform continuity of a function over a subset of its domain. Moreover, according to this alternative definition, if $f : A \to N$ is uniformly continuous on $B \subseteq A$, then

$$\forall\,\varepsilon > 0, \exists\,\delta > 0 \ni \forall\, b \in M, f\big(D\big(b, \frac{\delta}{2}\big) \cap B\big) \subseteq D\big(c, \frac{\varepsilon}{2}\big) \text{ for some } c \in N\,;$$

that is, the diameter of the image, under f, of subsets of B whose diameter is not greater than δ is not greater than ε（在 B 中直徑不超過 δ 的子集合被函數 f 映過去之後，在對應域中的直徑不會超過 ϵ）.

Remark 4.51. In terms of the number $\delta(f, x, \varepsilon)$ defined in Remark 4.9, the uniform continuity of a function $f : A \to N$ is equivalent to that

$$\delta_f(\varepsilon) \equiv \inf_{x\in A} \delta(f, x, \varepsilon) > 0 \qquad \forall\, \varepsilon > 0\,.$$

The function $\delta_f(\cdot)$ is called ***the inverse of the modulus of continuity of*** (***a uniform continuous***) ***function*** f.

Theorem 4.52. *Let (M, d) and (N, ρ) be metric spaces, $A \subseteq M$, and $f : A \to N$ be a map. If $K \subseteq A$ is compact and f is continuous on K, then f is uniformly continuous on K.*

Proof. Let $\varepsilon > 0$ be given. Since f is continuous on K,

$$\forall\, a \in K, \exists\, \delta = \delta(a) > 0 \ni \rho\big(f(x), f(a)\big) < \frac{\varepsilon}{2} \text{ whenever } x \in D(a, \delta) \cap A\,.$$

Then $\Big\{D\big(a, \dfrac{\delta(a)}{2}\big)\Big\}_{a\in K}$ is an open cover of K; thus

$$\exists\, \{a_1, \cdots, a_N\} \subseteq K \ni K \subseteq \bigcup_{i=1}^{N} D\big(a_i, \frac{\delta_i}{2}\big)\,,$$

where $\delta_i = \delta(a_i)$. Let $\delta = \dfrac{1}{2}\min\{\delta_1, \cdots, \delta_N\}$. Then $\delta > 0$, and if $x_1, x_2 \in K$ and $d(x_1, x_2) < \delta$, there must be $j = 1, \cdots, N$ such that $x_1, x_2 \in B(a_j, \delta_j)$. In fact, since $x_1 \in D\big(a_j, \dfrac{\delta_j}{2}\big)$ for some $j = 1, \cdots, N$, then

$$d(x_2, a_j) \leqslant d(x_1, x_2) + d(x_1, a_j) < \delta + \frac{\delta_j}{2} < \delta_j\,.$$

Therefore, $x_1, x_2 \in D(a_j, \delta_j) \cap A$ for some $j = 1, \cdots, N$; thus

$$\rho\big(f(x_1), f(x_2)\big) \leqslant \rho\big(f(x_1), f(a_j)\big) + \rho\big(f(x_2), f(a_j)\big) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,. \qquad \square$$

Alternative proof. Assume the contrary that f is not uniformly continuous on K. Then ((3) of Remark 4.45 implies that) there are sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in K such that

$$\lim_{n\to\infty} d(x_n, y_n) = 0 \quad \text{but} \quad \lim_{n\to\infty} \rho\big(f(x_n), f(y_n)\big) > 0\,.$$

Since K is (sequentially) compact, there exist convergent subsequences $\{x_{n_k}\}_{k=1}^{\infty}$ and $\{y_{n_k}\}_{k=1}^{\infty}$ with limits $x, y \in K$. On the other hand, $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$, we must have $x = y$; thus by the continuity of f (on K),

$$0 = \rho\big(f(x), f(x)\big) = \lim_{k\to\infty} \rho\big(f(x_{n_k}), f(y_{n_k})\big) = \lim_{n\to\infty} \rho\big(f(x_n), f(y_n)\big) > 0\,,$$

a contradiction. □

Lemma 4.53. *Let (M, d) and (N, ρ) be metric spaces, $A \subseteq M$, and $f : A \to N$ be uniformly continuous. If $\{x_k\}_{k=1}^{\infty} \subseteq A$ is a Cauchy sequence, so is $\big\{f(x_k)\big\}_{k=1}^{\infty}$.*

Proof. Let $\{x_k\}_{k=1}^{\infty}$ be a Cauchy sequence in (M, d), and $\varepsilon > 0$ be given. Since $f : A \to N$ is uniformly continuous,

$$\exists\,\delta > 0 \ni \rho\big(f(x), f(y)\big) < \varepsilon \text{ whenever } d(x, y) < \delta \text{ and } x, y \in A\,.$$

For this particular δ, $\exists\, N > 0 \ni d(x_k, x_\ell) < \delta$ if $k, \ell \geqslant N$. Therefore,

$$\rho(f(x_k), f(x_\ell)\big) < \varepsilon \text{ if } k, \ell \geqslant N\,.$$ □

Corollary 4.54. *Let (M, d) and (N, ρ) be metric spaces, $A \subseteq M$, and $f : A \to N$ be uniformly continuous. If N is complete, then f has a unique extension to a continuous function on $\bar{A}$; that is, $\exists\, g : \bar{A} \to N$ such that*

(1) *g is uniformly continuous on $\bar{A}$;*

(2) *$g(x) = f(x)$ for all $x \in A$;*

(3) *if $h : \bar{A} \to N$ is a continuous map satisfying* (1) *and* (2)*, then $h = g$.*

Proof. Let $x \in \bar{A}\backslash A$. Then $\exists\,\{x_k\}_{k=1}^{\infty} \subseteq A$ such that $x_k \to x$ as $k \to \infty$. Since $\{x_k\}_{k=1}^{\infty}$ is Cauchy, by Lemma 4.53 $\big\{f(x_k)\big\}_{k=1}^{\infty}$ is a Cauchy sequence in (N, ρ); thus is convergent. Moreover, if $\{z_k\}_{k=1}^{\infty} \subseteq A$ is another sequence converging to x, we must have $d(x_k, z_k) \to 0$ as $k \to \infty$; thus $\rho(f(x_k), f(z_k)) \to 0$ as $k \to \infty$, so the limit of $\big\{f(x_k)\big\}_{k=1}^{\infty}$ and $\big\{f(z_k)\big\}_{k=1}^{\infty}$ must be the same.

Define $g : \bar{A} \to N$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in A\,, \\ \lim\limits_{k\to\infty} f(x_k) & \text{if } x \in \bar{A}\backslash A, \text{ and } \{x_k\}_{k=1}^{\infty} \subseteq A \text{ converging to } x \text{ as } k \to \infty\,. \end{cases}$$

Then the argument above shows that g is well-defined, and (2), (3) hold.

Let $\varepsilon > 0$ be given. Since $f : A \to N$ is uniformly continuous,

$$\exists\,\delta > 0 \ni \rho\big(f(x), f(y)\big) < \frac{\varepsilon}{3} \text{ whenever } d(x,y) < 2\delta \text{ and } x, y \in A\,.$$

Suppose that $x, y \in \bar{A}$ such that $d(x,y) < \delta$. Let $\{x_k\}_{k=1}^\infty, \{y_k\}_{k=1}^\infty \subseteq A$ be sequences converging to x and y, respectively. Then $\exists\, N > 0$ such that

$$d(x_k, x) < \frac{\delta}{2}\,, d(y_k, y) < \frac{\delta}{2} \text{ and } \rho\big(f(x_k), g(x)\big) < \frac{\varepsilon}{3}\,, \rho\big(f(y_k), g(y)\big) < \frac{\varepsilon}{3} \quad \forall\, k \geqslant N\,.$$

In particular, due to the triangle inequality,

$$d(x_N, y_N) \leqslant d(x_N, x) + d(x,y) + d(y, y_N) < \frac{\delta}{2} + \delta + \frac{\delta}{2} = 2\delta\,;$$

thus $\rho\big(f(x_N), f(y_N)\big) < \frac{\varepsilon}{3}$. As a consequence,

$$\rho\big(g(x), g(y)\big) \leqslant \rho\big(g(x), f(x_N)\big) + \rho\big(f(x_N), f(y_N)\big) + \rho\big(f(y_N), f(y)\big) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\,. \ \square$$

4.6 Differentiation of Functions of One Variable

Definition 4.55. A function $f : (a,b) \to \mathbb{R}$ is said to be ***differentiable at*** x_0 if there exists a number m such that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - m(x - x_0)}{x - x_0} = 0\,.$$

The (unique) number m is usually denoted by $f'(x_0)$, and is called the ***derivative*** of f at x_0.

Remark 4.56. The derivative of f at x_0 can be computed by

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}\,.$$

Remark 4.57. By the definition of the limit of functions, $f : (a,b) \to \mathbb{R}$ is differentiable at $x_0 \in (a,b)$ if and only if there exists $m \in \mathbb{R}$, denoted by $f'(x_0)$, such that

$$\forall\, \varepsilon > 0, \exists\, \delta > 0 \ni \big|f(x) - f(x_0) - f'(x_0)(x - x_0)\big| \leqslant \varepsilon|x - x_0| \text{ if } |x - x_0| < \delta\,.$$

Definition 4.58. A function $f : (a,b) \to \mathbb{R}$ is said to be ***differentiable*** (on (a,b)) if f is differentiable at each $x_0 \in (a,b)$.

Proposition 4.59. *Suppose that a function $f : (a,b) \to \mathbb{R}$ is differentiable at x_0. Then f is continuous at x_0.*

Proof. For $x \neq x_0$, $f(x) - f(x_0) = \dfrac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0)$; thus Proposition 4.15 implies that

$$\lim_{x \to x_0} \big(f(x) - f(x_0)\big) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \to x_0} (x - x_0) = f'(x_0) \cdot 0 = 0\,. \qquad \square$$

Theorem 4.60. *Suppose that functions $f, g : (a,b) \to \mathbb{R}$ are differentiable at x_0, and $k \in \mathbb{R}$ is a constant. Then*

1. $(kf)'(x_0) = kf'(x_0)$.

2. $(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$.

3. $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.

4. $\big(\dfrac{f}{g}\big)'(x_0) = \dfrac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$ *if* $g(x_0) \neq 0$.

Theorem 4.61 (Chain Rule)**.** *Suppose that a function $f : (a,b) \to \mathbb{R}$ is differentiable at x_0, and $g : (c,d) \to \mathbb{R}$ is differentiable at $y_0 = f(x_0) \in (c,d)$. Then $g \circ f$ is differentiable at x_0, and*

$$(g \circ f)'(x_0) = g'(f(x_0))f'(x_0)\,.$$

Proof. Let $\varepsilon > 0$ be given. Since $f : (a,b) \to \mathbb{R}$ is differentiable at x_0 and $g : (c,d) \to \mathbb{R}$ is differentiable at $y_0 = f(x_0)$,

$$\exists\, \delta_1 > 0 \ni \big|f(x) - f(x_0) - f'(x_0)(x - x_0)\big| \leqslant \min\Big\{1, \frac{\varepsilon}{2(1 + |g'(y_0)|)}\Big\}|x - x_0| \text{ if } |x - x_0| < \delta_1$$

and

$$\exists\, \delta_2 > 0 \ni \big|g(y) - g(y_0) - g'(y_0)(y - y_0)\big| \leqslant \frac{\varepsilon|y - y_0|}{2(1 + |f'(x_0)|)} \text{ if } |y - y_0| < \delta_2\,.$$

Moreover, by Proposition 4.59 f is continuous at x_0; thus

$$\exists\, \delta_3 > 0 \ni \big|f(x) - f(x_0)\big| < \delta_2 \text{ if } |x - x_0| < \delta_3 \text{ and } x \in (a,b)\,.$$

Let $\delta = \min\{\delta_1, \delta_3\}$, and denote $f(x)$ by y. Then if $|x - x_0| < \delta$, we have $|y - y_0| < \delta_2$ and

$$\begin{aligned}
\big|(g \circ f)(x) - (g \circ f)(x_0) - g'(y_0)f'(x_0)(x - x_0)\big| &= \big|g(y) - g(y_0) - g'(y_0)f'(x_0)(x - x_0)\big| \\
&= \big|g(y) - g(y_0) - g'(y_0)(y - y_0) + g'(y_0)(f(x) - f(x_0) - f'(x_0)(x - x_0)\big| \\
&\leqslant \frac{\varepsilon\big|f(x) - f(x_0)\big|}{2(1 + |f'(x_0)|)} + \big|g'(y_0)\big|\frac{\varepsilon|x - x_0|}{2(1 + |g'(y_0)|)} \\
&\leqslant \frac{\varepsilon}{2(1 + |f'(x_0)|)}\big(|x - x_0| + |f'(x_0)||x - x_0|\big) + \frac{\varepsilon}{2}|x - x_0| = \varepsilon|x - x_0|\,.
\end{aligned}$$

By Remark 4.57, $g \circ f$ is differentiable at x_0 with derivative $g'(f(x_0))f'(x_0)$. □

Proposition 4.62. *If $f : (a, b) \to \mathbb{R}$ is differentiable at $x_0 \in (a, b)$ and f attains a local minimum or maximum at x_0, then $f'(x_0) = 0$.*

Proof. W.L.O.G. we assume that f attains its local minimum at x_0. Then $f(x) - f(x_0) \geqslant 0$ for all $x \in I$, where I is an open interval containing x_0. Therefore,

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} \leqslant 0$$

and

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \geqslant 0 \,.$$

As a consequence, $f'(x_0) = 0$. □

Theorem 4.63 (Rolle)**.** *Suppose that a function $f : [a, b] \to \mathbb{R}$ is continuous, and is differentiable on (a, b). If $f(a) = f(b)$, then $\exists\, c \in (a, b)$ such that $f'(c) = 0$.*

Proof. By the Extreme Value Theorem, there exists x_0 and x_1 in $[a, b]$ such that

$$f(x_0) = \min f([a, b]) \quad \text{and} \quad f(x_1) = \max f([a, b]) \,.$$

Case 1. $f(x_0) = f(x_1)$, then f is constant on $[a, b]$; thus $f'(x) = 0$ for all $x \in (a, b)$.

Case 2. One of $f(x_0)$ and $f(x_1)$ is different from $f(a)$. W.L.O.G. we may assume that $f(x_0) \neq f(a)$. Then $x_0 \in (a, b)$, and f attains its global minimum at x_0. By Proposition 4.62, $f'(x_0) = 0$. □

Theorem 4.64 (Cauchy's Mean Value Theorem)**.** *Suppose that functions $f, g : [a, b] \to \mathbb{R}$ are continuous, and $f, g : (a, b) \to \mathbb{R}$ are differentiable. If $g(a) \neq g(b)$ and $g'(x) \neq 0$ for all $x \in (a, b)$, then $\exists\, c \in (a, b)$ such that*

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)} \,.$$

Proof. Consider the function

$$h(x) \equiv \big(f(x) - f(a)\big)\big(g(b) - g(a)\big) - \big(f(b) - f(a)\big)\big(g(x) - g(a)\big) \,.$$

Then $h : [a, b] \to \mathbb{R}$ is continuous, and is differentiable on (a, b). Moreover, $h(b) = h(a) = 0$. By Rolle's theorem, $\exists\, c \in (a, b)$ such that

$$h'(c) = f'(c)\big(g(b) - g(a)\big) - \big(f(b) - f(a)\big)g'(c) = 0 \,.$$ □

Corollary 4.65 (Mean Value Theorem)**.** *Suppose that a function $f : [a, b] \to \mathbb{R}$ is continuous, and $f : (a, b) \to \mathbb{R}$ is differentiable. Then $\exists\, c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}\,.$$

Proof. Apply the Cauchy Mean Value Theorem for the case that $g(x) = x$. $\square$

Corollary 4.66. *Suppose that a function $f : [a, b] \to \mathbb{R}$ is continuous and $f'(x) = 0$ for all $x \in (a, b)$. Then f is constant.*

Proof. Fix $x \in (a, b)$, by Mean Value Theorem $\exists\, c \in (a, x) \ni f(x) - f(a) = f'(c)(x - a) = 0$. Then $f(x) = f(a) \Rightarrow \forall\, x \in (a, b)$, $f(x) = f(a)$. Now by continuity, $f(b) = \lim\limits_{x \to b^-} f(x) = f(a)$. $\square$

Corollary 4.67 (L'Hôspital's rule)**.** *Let $f, g : (a, b) \to \mathbb{R}$ be differentiable functions. Suppose that for some $x_0 \in (a, b)$, $f(x_0) = g(x_0) = 0$, $g'(x) \neq 0$ for all $x \neq x_0$, and the limit $\lim\limits_{x \to x_0} \dfrac{f'(x)}{g'(x)}$ exists. Then the limit $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)}$ also exists, and*

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}\,.$$

Proof. We first note that $g(x) \neq g(x_0)$ for all $x \neq x_0$ since if not, the Mean Value Theorem implies that $\exists\, c$ in between x and x_0 such that $g'(c) = 0$ which contradicts to that $g'(x) \neq 0$ for all $x \neq x_0$. By Cauchy's Mean Value Theorem, for all $x \in (a, b)$ and $x \neq x_0$, there exists $\xi = \xi(x)$ in between x and x_0 such that

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(\xi)}{g'(\xi)}$$

Since $\xi \to x_0$ as $x \to x_0$, we have

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{\xi \to x_0} \frac{f'(\xi)}{g'(\xi)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}\,.$$ $\square$

Example 4.68. A function $f : [a, b] \to \mathbb{R}$ is said to be ***Lipschitz continuous*** if $\exists\, M > 0$ such that

$$|f(x_1) - f(x_2)| \leqslant M|x_1 - x_2| \qquad \forall\, x_1, x_2 \in [a, b]\,.$$

If the derivative of a differentiable function $f : (a, b) \to \mathbb{R}$ is bounded; that is, $\exists\, M > 0$ $\ni |f'(x)| \leqslant M$ for all $x \in (a, b)$, then the Mean Value Theorem suggests that f is Lipschitz continuous. A Lipschitz continuous function must be uniformly continuous.

Definition 4.69. A function $f : (a, b) \to \mathbb{R}$ is said to be $\begin{array}{c} \textbf{\textit{increasing}} \\ \textbf{\textit{decreasing}} \\ \textbf{\textit{strictly increasing}} \\ \textbf{\textit{strictly decreasing}} \end{array}$ (on (a, b))

if $f(x_1) \begin{array}{c} \leqslant \\ \geqslant \\ < \\ > \end{array} f(x_2)$ if $a < x_1 < x_2 < b$. f is said to be ***monotone*** if f is either increasing or decreasing on (a, b), and ***strictly monotone*** if f is either strictly increasing or strictly decreasing.

Theorem 4.70. *Suppose that $f : (a, b) \to \mathbb{R}$ is differentiable.*

1. *f is increasing on (a, b) if and only if $f'(x) \geqslant 0$ for all $x \in (a, b)$.*

2. *f is decreasing on (a, b) if and only if $f'(x) \leqslant 0$ for all $x \in (a, b)$.*

3. *If $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing.*

4. *If $f'(x) < 0$ for all $x \in (a, b)$, then f is strictly decreasing.*

Theorem 4.71 (Inverse Function Theorem)**.** *Let $f : (a, b) \to \mathbb{R}$ be differentiable, and f' is sign-definite; that is, $f'(x) > 0$ for all $x \in (a, b)$ or $f'(x) < 0$ for all $x \in (a, b)$. Then $f : (a, b) \to f((a, b))$ is a bijection, and f^{-1}, the inverse function of f, is differentiable on $f((a, b))$, and*

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)} \qquad \forall\, x \in (a, b)\,. \tag{4.6.1}$$

Proof. W.L.O.G. we assume that $f'(x) > 0$ for all $x \in (a, b)$. By Theorem 4.70 f is strictly increasing; thus f^{-1} exists.
Claim: $f^{-1} : f((a, b)) \to (a, b)$ is continuous.
Proof of claim: Let $y_0 = f(x_0) \in f((a, b))$, and $\varepsilon > 0$ be given. Then $f((x_0 - \varepsilon, x_0 + \varepsilon)) = \big(f(x_0 - \varepsilon), f(x_0 + \varepsilon)\big)$ since f is continuous on (a, b) and $(x_0 - \varepsilon, x_0 + \varepsilon)$ is connected. Let $\delta = \min\{f(x_0) - f(x_0 - \varepsilon), f(x_0 + \varepsilon) - f(x_0)\big\}$. Then $\delta > 0$, and

$$(y_0 - \delta, y_0 + \delta) = \big(f(x_0) - \delta, f(x_0) + \delta\big) \subseteq f((x_0 - \varepsilon, x_0 + \varepsilon))\,;$$

thus by the injectivity of f,

$$f^{-1}((y_0 - \delta, y_0 + \delta)) \subseteq f^{-1}(f((x_0 - \varepsilon, x_0 + \varepsilon))) = (x_0 - \varepsilon, x_0 + \varepsilon) = (f^{-1}(y_0) - \varepsilon, f^{-1}(y_0) + \varepsilon)\,.$$

The inclusion above implies that f^{-1} is continuous at y_0.

Writing $y = f(x)$ and $x = f^{-1}(y)$. Then if $y_0 = f(x_0) \in f((a,b))$,

$$\frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{x - x_0}{f(x) - f(x_0)} .$$

Since f^{-1} is continuous on $f((a,b))$, $x \to x_0$ as $y \to y_0$; thus

$$\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{f'(x_0)}$$

which implies that f^{-1} is differentiable at y_0. □

4.7 Integration of Functions of One Variable

Definition 4.72. Let $A \subseteq \mathbb{R}$ be a bounded subset. A collection $\mathcal{P}$ of finitely many points $\{x_0, x_1, \cdots, x_n\}$ is called a ***partition*** of A if $\inf A = x_0 < x_1 < \cdots < x_{n-1} < x_n = \sup A$. The ***mesh size*** of the partition $\mathcal{P}$, denoted by $\|\mathcal{P}\|$, is defined by

$$\|\mathcal{P}\| = \max\big\{x_k - x_{k-1} \,\big|\, k = 1, \cdots, n\big\} .$$

Definition 4.73. Let $A \subseteq \mathbb{R}$ be a bounded subset, and $f : A \to \mathbb{R}$ be a bounded function. For any partition $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$ of A, the ***upper sum*** and the ***lower sum*** of f with respect to the partition $\mathcal{P}$, denoted by $U(f,\mathcal{P})$ and $L(f,\mathcal{P})$ respectively, are numbers defined by

$$U(f,\mathcal{P}) = \sum_{k=1}^{n} \sup_{x \in [x_{k-1}, x_k]} \bar{f}(x)(x_k - x_{k-1}) = \sum_{k=0}^{n-1} \sup_{x \in [x_k, x_{k+1}]} \bar{f}(x)(x_{k+1} - x_k) ,$$
$$L(f,\mathcal{P}) = \sum_{k=1}^{n} \inf_{x \in [x_{k-1}, x_k]} \bar{f}(x)(x_k - x_{k-1}) = \sum_{k=0}^{n-1} \inf_{x \in [x_k, x_{k+1}]} \bar{f}(x)(x_{k+1} - x_k) ,$$

where $\bar{f}$ is an extension of f given by

$$\bar{f}(x) = \begin{cases} f(x) & x \in A , \\ 0 & x \notin A . \end{cases} \tag{4.7.1}$$

The two numbers

$$\overline{\int_A} f(x)dx \equiv \inf\big\{U(f,\mathcal{P}) \,\big|\, \mathcal{P} \text{ is a partition of } A\big\} ,$$

and

$$\underline{\int_A} f(x)dx \equiv \sup\big\{L(f,\mathcal{P})\,\big|\,\mathcal{P} \text{ is a partition of } A\big\}$$

are called the ***upper integral*** and ***lower integral*** of f over A, respective. The function f is said to be ***Riemann*** (***Darboux***) ***integrable*** (over A) if $\overline{\int_A} f(x)dx = \underline{\int_A} f(x)dx$, and in this case, we express the upper and lower integral as $\int_A f(x)dx$, called the ***integral*** of f over A. The upper integral, the lower integral, and the integral of f over $[a,b]$ sometimes are also denoted by $\overline{\int_a^b} f(x)dx$, $\underline{\int_a^b} f(x)dx$, and $\int_a^b f(x)dx$.

Example 4.74. $\underline{\int_a^b} f(x)dx$ and $\overline{\int_a^b} f(x)dx$ are not always the same. For example, define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1]\backslash\mathbb{Q}, \\ 0 & \text{if } x \in [0,1] \cap \mathbb{Q}. \end{cases}$$

Let $\mathcal{P} = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ be any partition on $[0,1]$. Then for any $k = 0, 1, \cdots, n-1$, $\sup\limits_{x\in[x_k,x_{k+1}]} f(x) = 1$ and $\inf\limits_{x\in[x_k,x_{k+1}]} f(x) = 0$; thus

$$\begin{aligned} U(f,\mathcal{P}) &= \sum_{k=0}^{n-1} \sup_{x\in[x_k,x_{k+1}]} f(x)(x_k - x_{k-1}) = \sum_{k=0}^{n}(x_k - x_{k-1}) \\ &= (x_1 - x_0) + (x_2 - x_1) + \cdots + (x_n - x_{n-1}) = x_n - x_0 = 1 - 0 = 1 \end{aligned}$$

and

$$L(f,\mathcal{P}) = \sum_{i=1}^{n} 0(x_i - x_{i-1}) = 0\,.$$

As a consequence,

$$\begin{aligned} \overline{\int_0^1} f(x)dx &= \inf\big\{U(f,\mathcal{P})\,\big|\,\mathcal{P} \text{ is a partition on } [0,1]\big\} = 1\,, \\ \underline{\int_0^1} f(x)dx &= \sup\big\{L(f,\mathcal{P})\,\big|\,\mathcal{P} \text{ is a partition on } [0,1]\big\} = 0\,; \end{aligned}$$

hence f is not Riemann integrable over $[0,1]$.

Example 4.75. Suppose $f : [a,b] \to \mathbb{R}$ is integrable and $f \geqslant 0$ on $[a,b]$, then $\int_a^b f(x)dx \geqslant 0$.

Reason: Since $f \geqslant 0$ on $[a,b] \Rightarrow \sup\limits_{x\in[x_k,x_{k+1}]} f(x) \geqslant 0$ for $k = 0,1,\dots,n-1$. Therefore, $U(f,\mathcal{P}) \geqslant 0$ for all partition $\mathcal{P}$ on $[a,b]$, so

$$\int_a^b f(x)dx = \overline{\int_a^b} f(x)dx = \inf\{U(f,\mathcal{P}) \,\big|\, \mathcal{P} \text{ is a partition on } [a,b]\} \geqslant 0\,.$$

Definition 4.76. A partition $\mathcal{P}'$ of a bounded set $A \subseteq \mathbb{R}$ is said to be a ***refinement*** of another partition $\mathcal{P}$ if $\mathcal{P} \subseteq \mathcal{P}'$.

Proposition 4.77. *Let $A \subseteq \mathbb{R}$ be a bounded subset, and $f : A \to \mathbb{R}$ be a bounded function. If $\mathcal{P}$ and $\mathcal{P}'$ are partitions of A and $\mathcal{P}'$ is a refinement of $\mathcal{P}$, then*

$$L(f,\mathcal{P}) \leqslant L(f,\mathcal{P}') \leqslant U(f,\mathcal{P}') \leqslant U(f,\mathcal{P})\,.$$

Proof. Let $\bar{f}$ be the extension of f given by (4.7.1). Suppose that $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$, $\mathcal{P}' = \{y_0, y_1, \cdots, y_m\}$, and $\mathcal{P} \subseteq \mathcal{P}'$. For any fixed $k = 0,1,\cdots,n-1$, either $\mathcal{P}' \cap (x_k, x_{k+1}) = \varnothing$ or $\mathcal{P}' \cap (x_k, x_{k+1}) \neq \varnothing$.

1. If $\mathcal{P}' \cap (x_k, x_{k+1}) = \varnothing$, then $x_k = y_\ell$ and $x_{k+1} = y_{\ell+1}$ for some ℓ. Therefore,
$$\sup_{x\in[x_k,x_{k+1}]} \bar{f}(x)(x_{k+1} - x_k) = \sup_{x\in[y_\ell,y_{\ell+1}]} \bar{f}(x)(y_{\ell+1} - y_\ell).$$

2. If $\mathcal{P}' \cap (x_k, x_{k+1}) = \{y_{\ell+1}, y_{\ell+2}, \cdots, y_{\ell+p}\}$, then $x_k = y_\ell$ and $x_{k+1} = y_{\ell+p+1}$. Therefore,
$$\begin{aligned}\sum_{i=1}^{p+1} \sup_{x\in[y_{\ell+i-1},y_{\ell+i}]} \bar{f}(x)(y_{\ell+i} - y_{\ell+i-1}) &= \sup_{x\in[y_\ell,y_{\ell+1}]} \bar{f}(x)(y_{\ell+1} - y_\ell) \\ &\quad + \sup_{x\in[y_{\ell+1},y_{\ell+2}]} \bar{f}(x)(y_{\ell+2} - y_{\ell+1}) + \cdots + \sup_{x\in[y_{\ell+p},y_{\ell+p+1}]} \bar{f}(x)(y_{\ell+p+1} - y_{\ell+p}) \\ &\leqslant \sup_{x\in[x_k,x_{k+1}]} \bar{f}(x)(y_{\ell+1} - y_\ell) + \sup_{x\in[x_k,x_{k+1}]} \bar{f}(x)(y_{\ell+2} - y_{\ell+1}) + \cdots \\ &\quad + \sup_{x\in[x_k,x_{k+1}]} \bar{f}(x)(y_{\ell+p+1} - y_{\ell+p}) = \sup_{x\in[x_k,x_{k+1}]} \bar{f}(x)(x_{k+1} - x_k)\,.\end{aligned}$$

In either case,

$$\sum_{[y_{\ell-1},y_\ell]\subseteq[x_k,x_{k+1}]} \sup_{x\in[y_{\ell-1},y_\ell]} \bar{f}(x)(y_\ell - y_{\ell-1}) \leqslant \sup_{x\in[x_k,x_{k+1}]} \bar{f}(x)(x_{k+1} - x_k)\,.$$

As a consequence,

$$\begin{aligned} U(f,\mathcal{P}') &= \sum_{\ell=0}^{m-1} \sup_{x\in[y_\ell,y_{\ell+1}]} \bar{f}(x)(y_{\ell+1}-y_\ell) = \sum_{k=0}^{n-1} \sum_{[y_{\ell-1},y_\ell]\subseteq[x_k,x_{k+1}]} \bar{f}(x)(y_\ell - y_{\ell-1}) \\ &\leqslant \sum_{k=0}^{n-1} \sup_{x\in[x_k,x_{k+1}]} \bar{f}(x)(x_{k+1}-x_k) = U(f,\mathcal{P})\,. \end{aligned}$$

Similarly, $L(f,\mathcal{P}) \leqslant L(f,\mathcal{P}')$; thus the fact that $L(f,\mathcal{P}') \leqslant U(f,\mathcal{P}')$ concludes the proposition. □

Corollary 4.78. *Let $f : [a,b] \to \mathbb{R}$ be a function bounded by M; that is, $|f(x)| \leqslant M$ for all $a \leqslant x \leqslant b$. Then for all partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of $[a,b]$,*

$$-M(b-a) \leqslant L(f,\mathcal{P}_1) \leqslant \underline{\int_a^b} f(x)dx \leqslant \overline{\int_a^b} f(x)dx \leqslant U(f,\mathcal{P}_2) \leqslant M(b-a)\,.$$

Proof. It suffices to show that $\underline{\int_a^b} f(x)dx \leqslant \overline{\int_a^b} f(x)dx$. By the definition of infimum and supremum, for any given $\varepsilon > 0$, $\exists$ partitions $\bar{\mathcal{P}}$ and $\widetilde{\mathcal{P}}$ such that

$$\underline{\int_a^b} f(x)dx - \frac{\varepsilon}{2} < L(f,\bar{\mathcal{P}}) \leqslant \underline{\int_a^b} f(x)dx \quad \text{and} \quad \overline{\int_a^b} f(x)dx \leqslant U(f,\widetilde{\mathcal{P}}) < \overline{\int_a^b} f(x)dx + \frac{\varepsilon}{2}\,.$$

Let $\mathcal{P} = \bar{\mathcal{P}} \cup \widetilde{\mathcal{P}}$. Then $\mathcal{P}$ is a refinement of both $\bar{\mathcal{P}}$ and $\widetilde{\mathcal{P}}$; thus

$$\underline{\int_a^b} f(x)dx - \frac{\varepsilon}{2} < L(f,\bar{\mathcal{P}}) \leqslant L(f,\mathcal{P}) \leqslant U(f,\mathcal{P}) \leqslant U(f,\widetilde{\mathcal{P}}) < \overline{\int_a^b} f(x)dx + \frac{\varepsilon}{2}\,.$$

Since $\varepsilon > 0$ is given arbitrarily, we must have $\underline{\int_a^b} f(x)dx \leqslant \overline{\int_a^b} f(x)dx$. □

Proposition 4.79 (Riemann's condition)**.** *Let $A \subseteq \mathbb{R}$ be a bounded set, and $f : A \to \mathbb{R}$ be a bounded function. Then f is Riemann integrable over A if and only if*

$$\forall\, \varepsilon > 0, \exists \text{ a partition } \mathcal{P} \text{ of } A \ni U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon\,.$$

Proof. "⇒" Let $\varepsilon > 0$ be given. Since f is integrable over A,

$$\inf_{\mathcal{P}:\ \text{Partition of } A} U(f,\mathcal{P}) = \sup_{\mathcal{P}:\ \text{Partition of } A} L(f,\mathcal{P}) = \int_A f(x)dx\,;$$

thus there exist $\mathcal{P}_1$ and $\mathcal{P}_2$, partitions of A, such that

$$\int_A f(x)dx - \frac{\varepsilon}{2} < L(f,\mathcal{P}_1) \leqslant \int_A f(x)dx \leqslant U(f,\mathcal{P}_2) < \int_A f(x)dx + \frac{\varepsilon}{2}.$$

Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Then $\mathcal{P}$ is a refinement of $\mathcal{P}_1$ and $\mathcal{P}_2$; thus

$$\begin{aligned}\int_A f(x)dx - \frac{\varepsilon}{2} < L(f,\mathcal{P}_1) \leqslant L(f,\mathcal{P}) \leqslant \int_A f(x)dx \\ \leqslant U(f,\mathcal{P}) \leqslant U(f,\mathcal{P}_2) < \int_A f(x)dx + \frac{\varepsilon}{2}\end{aligned}$$

which implies that $U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon$.

"$\Leftarrow$" We note that for any partition $\mathcal{P}$ of A,

$$L(f,\mathcal{P}) \leqslant \underline{\int_A} f(x)dx \leqslant \overline{\int_A} f(x)dx \leqslant U(f,\mathcal{P});$$

so we have that for all partition $\mathcal{P}$ of A,

$$\overline{\int_A} f(x)dx - \underline{\int_A} f(x)dx < U(f,\mathcal{P}) - L(f,\mathcal{P}).$$

Let $\varepsilon > 0$ be given. By choosing $\mathcal{P}$ so that $U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon$, we conclude that

$$\overline{\int_A} f(x)dx - \underline{\int_A} f(x)dx < \varepsilon.$$

Since $\varepsilon > 0$ is given arbitrarily, $\overline{\int_A} f(x)dx = \underline{\int_A} f(x)dx$; thus f is Riemann integrable over A. $\square$

Proposition 4.80. *Suppose that $f, g : [a,b] \to \mathbb{R}$ are Riemann integrable, and $k \in \mathbb{R}$. Then*

1. *kf is Riemann integrable, and* $\displaystyle\int_a^b (kf)(x)dx = k\int_a^b f(x)dx$.
2. *$f \pm g$ are Riemann integrable, and* $\displaystyle\int_a^b (f \pm g)(x)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$.
3. *If $f \leqslant g$ for all $x \in [a,b]$, then* $\displaystyle\int_a^b f(x)dx \leqslant \int_a^b g(x)dx$.
4. *If f is also Riemann integrable over $[b,c]$, then f is Riemann integrable over $[a,c]$, and*
$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx. \tag{4.7.2}$$

5. *The function* $|f|$ *is also Riemann integrable, and* $\displaystyle\Big|\int_a^b f(x)dx\Big| \leqslant \int_a^b |f(x)|dx$.

Proof. 1. Case 1. $k \geqslant 0$. We note that

$$\inf_{x\in[x_{i-1},x_i]}(kf)(x) = k\inf_{x\in[x_{i-1},x_i]} f(x) \quad \text{and} \quad \sup_{x\in[x_{i-1},x_i]}(kf)(x) = k\sup_{x\in[x_{i-1},x_i]} f(x)\,.$$

Then

$$\begin{aligned}L(kf,\mathcal{P}) &= \sum_{i=1}^{n}\inf_{x\in[x_{i-1},x_i]}(kf)(x)(x_i - x_{i-1})\\ &= \sum_{i=1}^{n} k\inf_{x\in[x_{i-1},x_i]} f(x)(x_i - x_{i-1}) = kL(f,\mathcal{P})\,.\end{aligned}$$

Similarly, $U(kf,\mathcal{P}) = kU(f,\mathcal{P})$ for every partition $\mathcal{P}$. So

$$\begin{aligned}\underline{\int_a^b}(kf)(x)dx &= \sup_{\mathcal{P}:\text{ Partition of }[a,b]} L(kf,\mathcal{P}) = k\sup_{\mathcal{P}:\text{ Partition of }[a,b]} L(f,\mathcal{P})\\ &= k\underline{\int_a^b} f(x)dx = k\int_a^b f(x)dx\,.\end{aligned}$$

Similarly, $\displaystyle\overline{\int_a^b}(kf)(x)dx = k\int_a^b f(x)dx$. Hence kf is integrable and

$$\int_a^b (kf)(x)dx = \underline{\int_a^b}(kf)(x)dx = k\underline{\int_a^b} f(x)dx = k\int_a^b f(x)dx\,.$$

Case 2. $k < 0$. We have

$$\inf_{x\in[x_{i-1},x_i]}(kf)(x) = k\sup_{x\in[x_{i-1},x_i]} f(x) \quad \text{and} \quad \sup_{x\in[x_{i-1},x_i]}(kf)(x) = k\inf_{x\in[x_{i-1},x_i]} f(x)\,.$$

Then $L(kf,\mathcal{P}) = kU(f,\mathcal{P})$ and $U(kf,\mathcal{P}) = kL(f,\mathcal{P})$; thus

$$\begin{aligned}\underline{\int_a^b}(kf)(x)dx &= \sup_{\mathcal{P}:\text{ Partition of }[a,b]} L(kf,\mathcal{P}) = \sup_{\mathcal{P}:\text{ Partition of }[a,b]} kU(f,\mathcal{P})\\ &= k\inf_{\mathcal{P}:\text{ Partition of }[a,b]} U(f,\mathcal{P}) = k\overline{\int_a^b} f(x)dx = k\int_a^b f(x)dx\,.\end{aligned}$$

Similarly, $\displaystyle\overline{\int_a^b}(kf)(x)dx = k\int_a^b f(x)dx$. Hence kf is Riemann integrable over $[a,b]$ and

$$\int_a^b (kf)(x)dx = \underline{\int_a^b}(kf)(x)dx = k\underline{\int_a^b} f(x)dx = k\int_a^b f(x)dx\,.$$

2. We prove the case of summation. For ant partition $\mathcal{P}$, we have

$$\begin{aligned} L(f+g,\mathcal{P}) &= \sum_{i=1}^{n} \inf_{x\in[x_{i-1},x_i]} (f+g)(x)(x_i - x_{i-1}) \\ &\geqslant \sum_{i=1}^{n} \inf_{x\in[x_{i-1},x_i]} f(x)(x_i - x_{i-1}) + \sum_{i=1}^{n} \inf_{x\in[x_{i-1},x_i]} g(x)(x_i - x_{i-1}) \\ &= L(f,\mathcal{P}) + L(g,\mathcal{P})\,. \end{aligned}$$

Similarly, $U(f+g,\mathcal{P}) \leqslant U(f,\mathcal{P}) + U(g,\mathcal{P})$. Therefore,

$$L(f,\mathcal{P}) + L(g,\mathcal{P}) \leqslant L(f+g,\mathcal{P}) \leqslant U(f+g,\mathcal{P}) \leqslant U(f,\mathcal{P}) + U(g,\mathcal{P})\,. \tag{4.7.3}$$

Let $\varepsilon > 0$ be given. By Proposition 4.79, $\exists\, \mathcal{P}_1, \mathcal{P}_2$ partitions of $[a,b]$ such that

$$U(f,\mathcal{P}_1) - L(f,\mathcal{P}_1) < \frac{\varepsilon}{2} \quad \text{and} \quad U(g,\mathcal{P}_2) - L(g,\mathcal{P}_2) < \frac{\varepsilon}{2}\,.$$

Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. By (4.7.3),

$$\begin{aligned} U(f+g,\mathcal{P}) - L(f+g,\mathcal{P}) &\leqslant \big(U(f,\mathcal{P}) + U(g,\mathcal{P})\big) - \big(L(f,\mathcal{P}) + L(g,\mathcal{P})\big) \\ &= \big(U(f,\mathcal{P}) - L(f,\mathcal{P})\big) + \big(U(g,\mathcal{P}) - L(g,\mathcal{P})\big) \\ &\leqslant \big(U(f,\mathcal{P}_1) - L(f,\mathcal{P}_1)\big) + \big(U(g,\mathcal{P}_2) - L(g,\mathcal{P}_2)\big) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,. \end{aligned}$$

By Proposition 4.79, $f+g$ is Riemann integrable over $[a,b]$.

To see $\int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$, we note that by Proposition 4.77,

$$\begin{aligned} U(f,\mathcal{P}) &\leqslant L(f,\mathcal{P}) + U(f,\mathcal{P}_1) - L(f,\mathcal{P}_1) < L(f,\mathcal{P}) + \frac{\varepsilon}{2} \\ &\leqslant \underline{\int_a^b} f(x)dx + \frac{\varepsilon}{2} = \int_a^b f(x)dx + \frac{\varepsilon}{2} \end{aligned}$$

and similarly, $U(g,\mathcal{P}) < \int_a^b g(x)dx + \frac{\varepsilon}{2}$. Therefore, by (4.7.3),

$$\begin{aligned} \int_a^b (f+g)(x)dx = \overline{\int_a^b} (f+g)(x)dx &\leqslant U(f+g,\mathcal{P}) \\ &\leqslant U(f,\mathcal{P}) + U(g,\mathcal{P}) < \int_a^b f(x)dx + \int_a^b g(x)dx + \varepsilon\,. \end{aligned} \tag{4.7.4}$$

On the other hand,

$$L(f,\mathcal{P}) > U(f,\mathcal{P}) - \frac{\varepsilon}{2} \geqslant \int_a^b f(x)dx - \frac{\varepsilon}{2}$$

and

$$L(g,\mathcal{P}) > U(g,\mathcal{P}) - \frac{\varepsilon}{2} \geqslant \int_a^b g(x)dx - \frac{\varepsilon}{2}\,;$$

hence by (4.7.3),

$$\begin{aligned}\int_a^b (f+g)(x)dx = \underline{\int_a^b}(f+g)(x)dx &\geqslant L(f+g,\mathcal{P}) \geqslant L(f,\mathcal{P}) + L(g,\mathcal{P}) \\ &> \int_a^b f(x)dx + \int_a^b g(x)dx - \varepsilon\,. \end{aligned} \tag{4.7.5}$$

By (4.7.4) and (4.7.5),

$$\int_a^b f(x)dx + \int_a^b g(x)dx - \varepsilon < \int_a^b (f+g)(x)dx < \int_a^b f(x)dx + \int_a^b g(x)dx + \varepsilon\,.$$

Since $\varepsilon > 0$ is arbitrary, $\displaystyle\int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.

3. Let $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a,b]$. Define

$$m_i(f) = \inf_{x\in[x_{i-1},x_i]} f(x) \quad \text{and} \quad m_i(g) = \inf_{x\in[x_{i-1},x_i]} g(x)\,.$$

Since $f(x) \leqslant g(x)$ on $[a,b]$, $m_i(f) \leqslant m_i(g)$. As a consequence, for any partition $\mathcal{P}$,

$$L(f,\mathcal{P}) = \sum_{i=1}^n m_i(f)(x_i - x_{i-1}) \leqslant \sum_{i=1}^n m_i(g)(x_i - x_{i-1}) = L(g,\mathcal{P})\,;$$

thus taking the infimum over all partition $\mathcal{P}$,

$$\int_a^b f(x)dx = \underline{\int_a^b} f(x)dx = \sup_{\mathcal{P}} L(f,\mathcal{P}) \leqslant \sup_{\mathcal{P}} L(g,\mathcal{P}) = \underline{\int_a^b} g(x)dx = \int_a^b g(x)dx\,.$$

4. Let $\varepsilon > 0$ be given. Since f is Riemann integrable of $[a,b]$ and $[b,c]$, there exist a partition $\mathcal{P}_1$ over $[a,b]$ and a partition $\mathcal{P}_2$ of $[b,c]$ such that

$$U(f,\mathcal{P}_1) - L(f,\mathcal{P}_1) < \frac{\varepsilon}{2} \quad \text{and} \quad U(f,\mathcal{P}_2) - L(f,\mathcal{P}_2) < \frac{\varepsilon}{2}\,.$$

Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Then $\mathcal{P}$ is a partition of $[a, c]$ such that

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) = U(f, \mathcal{P}_1) + U(f, \mathcal{P}_2) - L(f, \mathcal{P}_1) - L(f, \mathcal{P}_2) < \varepsilon\,.$$

Therefore, Proposition 4.79 suggests that f is Riemann integrable over $[a, c]$.

Now we show that $\displaystyle\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$. To simplify the notation, we let

$$A = \int_a^c f(x)dx\,, \quad B = \int_a^b f(x)dx\,, \quad C = \int_b^c f(x)dx\,.$$

Let $\varepsilon > 0$ be given. Then $\exists$ partition $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$ of $[a, c]$ such that

$$A \leqslant U(f, \mathcal{P}) < A + \varepsilon\,.$$

Let $\mathcal{P}' = \mathcal{P} \cup \{b\}$. Then $\mathcal{P}'$ is a refinement of $\mathcal{P}$. Moreover,

$$U(f, \mathcal{P}') = U(f, \mathcal{P}_1) + U(f, \mathcal{P}_2)\,,$$

where $\mathcal{P}_1 = \mathcal{P}' \cap [a, b]$ and $\mathcal{P}_2 = \mathcal{P}' \cap [b, c]$ are partitions of $[a, b]$ and $[b, c]$ whose union is $\mathcal{P}$. Therefore,

$$B + C \leqslant U(f, \mathcal{P}_1) + U(f, \mathcal{P}_2) = U(f, \mathcal{P}') \leqslant U(f, \mathcal{P}) < A + \varepsilon\,.$$

On the other hand, $\exists$ partition $\mathcal{P}_1$ of $[a, b]$ and partition $\mathcal{P}_2$ of $[b, c]$ such that

$$B \leqslant U(f, \mathcal{P}_1) < B + \frac{\varepsilon}{2} \quad \text{and} \quad C \leqslant U(f, \mathcal{P}_2) < C + \frac{\varepsilon}{2}\,.$$

Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Then $\mathcal{P}$ is a partition of $[a, c]$. Therefore,

$$A \leqslant U(f, \mathcal{P}) = U(f, \mathcal{P}_1) + U(f, \mathcal{P}_2) < B + C + \varepsilon\,.$$

Therefore, $\forall\, \varepsilon > 0$, $B + C < A + \varepsilon$ and $A < B + C + \varepsilon$; thus $A = B + C$.

5. Note that for any interval $[\alpha, \beta]$,

$$\sup_{x\in[\alpha,\beta]} |f(x)| - \inf_{x\in[\alpha,\beta]} |f(x)| \leqslant \sup_{x\in[\alpha,\beta]} f(x) - \inf_{x\in[\alpha,\beta]} f(x)\,; \quad (\textbf{Check}!)$$

thus for any partition $\mathcal{P}$ of $[a, b]$,

$$U(|f|, \mathcal{P}) - L(|f|, \mathcal{P}) \leqslant U(f, \mathcal{P}) - L(f, \mathcal{P})\,.$$

Therefore, Proposition 4.79 suggests that $|f|$ is Riemann integrable over $[a, b]$. Moreover, since $-|f(x)| \leqslant f(x) \leqslant |f(x)|$ for all $x \in [a, b]$, by 3 we have

$$-\int_a^b |f(x)|dx \leqslant \int_a^b f(x)dx \leqslant \int_a^b |f(x)|dx\,. \qquad \square$$

Remark 4.81. The proof of 4 in Proposition 4.80 in fact also shows that if $a < b < c$, then

$$\overline{\int_a^c} f(x)dx = \overline{\int_a^b} f(x)dx + \overline{\int_b^c} f(x)dx\,.$$

Similar proof also suggests that

$$\underline{\int_a^c} f(x)dx = \underline{\int_a^b} f(x)dx + \underline{\int_b^c} f(x)dx\,.$$

Remark 4.82. If $a < b$, we let the number $\int_b^a f(x)dx$ denote the number $-\int_a^b f(x)dx$. Then (4.7.2) holds for all $a, b, c \in \mathbb{R}$.

Example 4.83. Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ -1 & \text{if } x \in \mathbb{Q}^{\complement}. \end{cases}$$

Then $f(x)$ is not Riemann integrable over $[0,1]$ since $U(f,P) = 1$ and $L(f,P) = -1$. However $|f(x)| \equiv 1$, thus $|f|$ is Riemann integrable. In other words, if $|f|$ is integrable, we cannot know whether f is integrable or not.

Theorem 4.84. *If $f : [a,b] \to \mathbb{R}$ is continuous, then f is Riemann integrable.*

Proof. Let $\varepsilon > 0$ be given. Theorem 4.52 suggests that

$$\exists\, \delta > 0 \ni |f(x) - f(y)| < \frac{\varepsilon}{2(b-a)} \text{ whenever } |x-y| < \delta \text{ and } x, y \in [a,b]\,.$$

Let $\mathcal{P}$ be a partition with mesh size less than δ. Then

$$\begin{aligned} U(f,\mathcal{P}) - L(f,\mathcal{P}) &= \sum_{k=1}^{n} \Big(\sup_{x\in[x_{k-1},x_k]} f(x) - \inf_{x\in[x_{k-1},x_k]} f(x) \Big)(x_k - x_{k-1}) \\ &\leqslant \frac{\varepsilon}{2(b-a)} \sum_{k=1}^{n} (x_k - x_{k-1}) = \frac{\varepsilon}{2(b-a)}(x_n - x_0) < \varepsilon\,; \end{aligned}$$

thus by Proposition 4.79 f is Riemann integrable over $[a,b]$. □

Corollary 4.85. *If $f : (a,b) \to \mathbb{R}$ is continuous and f is bounded on $[a,b]$, then f is Riemann integrable over $[a,b]$.*

Proof. Let $|f(x)| \leqslant M$ for all $x \in [a,b]$, and $\varepsilon > 0$ be given. Since $f : \big[a+\frac{\varepsilon}{8M}, b-\frac{\varepsilon}{8M}\big] \to \mathbb{R}$ is continuous, by Theorem 4.84 f is Riemann integrable; thus

$$\exists\, \mathcal{P}' \text{: partition of } \Big[a+\frac{\varepsilon}{8M}, b-\frac{\varepsilon}{8M}\Big] \ni U(f,\mathcal{P}') - L(f,\mathcal{P}') < \frac{\varepsilon}{2}\,.$$

Let $\mathcal{P} = \mathcal{P}' \cup \{a,b\}$. Then

$$\begin{aligned}
&U(f,\mathcal{P}) - L(f,\mathcal{P}) \\
&\quad < \Big(\sup_{x\in[a,a+\frac{\varepsilon}{8M}]} f(x) - \inf_{x\in[a,a+\frac{\varepsilon}{8M}]} f(x)\Big)\frac{\varepsilon}{8M} + \frac{\varepsilon}{2} + \Big(\sup_{x\in[b-\frac{\varepsilon}{8M},b]} f(x) - \inf_{x\in[b-\frac{\varepsilon}{8M},b]} f(x)\Big)\frac{\varepsilon}{8M} \\
&\quad \leqslant 2M\cdot\frac{\varepsilon}{8M} + \frac{\varepsilon}{2} + 2M\cdot\frac{\varepsilon}{8M} = \varepsilon\,;
\end{aligned}$$

thus Proposition 4.79 suggests that f is Riemann integrable over $[a,b]$. ▫

Corollary 4.86. *If $f : [a,b] \to \mathbb{R}$ is bounded and is continuous at all but finitely many points of $[a,b]$, then f is Riemann integral.*

Proof. Let $\{c_1,\cdots,c_N\}$ be the collection of all discontinuities of f in (a,b) such that $c_1 < c_2 < \cdots < c_N$. Let $a = c_0$ and $b = c_{N+1}$. Then for all $k = 0,1,\cdots,N$, $f : (c_k,c_{k+1})$ is continuous and $f : [c_k,c_{k+1}]$ is bounded; thus f is Riemann integrable by Corollary 4.86. Finally, 4 of Proposition 4.80 suggests that f is Riemann integrable over $[a,b]$. ▫

Theorem 4.87. *Any increasing or decreasing function on $[a,b]$ is Riemann integrable.*

Proof. Let $f : [a,b] \to \mathbb{R}$ be a monotone function, and $\varepsilon > 0$ be given. W.L.O.G. we may assume that $f(b) \neq f(a)$. Let $\mathcal{P} = \{x_0,x_1,\cdots,x_n\}$ be a partition of $[a,b]$ with mesh size less than $\dfrac{\varepsilon}{|f(b)-f(a)|}$. Then

$$\begin{aligned}
U(f,\mathcal{P}) - L(f,\mathcal{P}) &= \sum_{k=1}^{n}\Big(\sup_{x\in[x_{k-1},x_k]} f(x) - \inf_{x\in[x_{k-1},x_k]} f(x)\Big)(x_k - x_{k-1}) \\
&< \sum_{k=1}^{n}\big|f(x_k) - f(x_{k-1})\big|\frac{\varepsilon}{|f(b)-f(a)|} = \big|f(b)-f(a)\big|\frac{\varepsilon}{|f(b)-f(a)|} = \varepsilon\,;
\end{aligned}$$

thus Proposition 4.79 suggests that f is Riemann integrable over $[a,b]$. ▫

Definition 4.88. A continuous function $F : [a.b] \to \mathbb{R}$ is called an ***anti-derivative***（反導函數）of $f : [a,b] \to \mathbb{R}$ if F is differentiable on (a,b) and $F'(x) = f(x)$ for all $x \in (a,b)$.

Theorem 4.89 (Fundamental Theorem of Calculus (微積分基本定理)). *Let $f : [a,b] \to \mathbb{R}$ be continuous. Then f has an anti-derivative F, and*

$$\int_a^b f(x)dx = F(b) - F(a)\,.$$

Moreover, if G is any other anti-derivative of f, we also have $\displaystyle\int_a^b f(x)dx = G(b) - G(a)$.

Proof. Define $F(x) = \displaystyle\int_a^x f(y)dy$, where the integral of f over $[a,x]$ is well-defined because of continuity of f on $[a,x]$. We first show that F is differentiable on (a,b).

Let $x_0 \in (a,b)$ and $\varepsilon > 0$ be given. Since $[a,b]$ is compact,

$$\exists\, \delta_1 > 0 \ni \big|f(x) - f(y)\big| < \frac{\varepsilon}{2} \text{ whenever } |x-y| < \delta_1 \text{ and } x,y \in [a,b]\,.$$

Let $\delta = \min\{\delta_1, x_0 - a, b - x_0\}$. By 4 of Proposition 4.80, if $x, x_0 \in (a,b)$,

$$\int_{x_0}^x f(y)dy = \int_a^x f(y)dy - \int_a^{x_0} f(y)dy = F(x) - F(x_0)\,;$$

thus if $0 < |x - x_0| < \delta$,

$$\begin{aligned}\Big|\frac{F(x)-F(x_0)}{x-x_0} - f(x_0)\Big| &= \Big|\frac{1}{x-x_0}\int_{x_0}^x f(y)dx - f(x_0)\Big| = \Big|\frac{1}{x-x_0}\int_{x_0}^x \big(f(y)-f(x_0)\big)dy\Big| \\ &\leqslant \frac{1}{|x-x_0|}\int_{\min\{x_0,x\}}^{\max\{x_0,x\}} \big|f(y)-f(x_0)\big|dy \leqslant \frac{1}{|x-x_0|}\int_{\min\{x_0,x\}}^{\max\{x_0,x\}} \frac{\varepsilon}{2}dy < \varepsilon\,.\end{aligned}$$

Therefore, $\displaystyle\lim_{x\to x_0}\frac{F(x)-F(x_0)}{x-x_0} = f(x_0)$ for all $x_0 \in (a,b)$, so $F'(x) = f(x)$ for all $x \in (a,b)$.

Next we show that F is continuous at $x = a$ and $x = b$. This is simply because of the boundedness of f on $[a,b]$ which suggests that

$$\limsup_{x\to a^+}\big|F(x)-F(a)\big| = \limsup_{x\to a^+}\Big|\int_a^x f(t)dt\Big| \leqslant \max_{x\in[a,b]}|f(x)| \cdot \limsup_{x\to a^+}\int_a^x 1dt = 0$$

and

$$\limsup_{x\to b^-}\big|F(x)-F(b)\big| = \limsup_{x\to b^-}\Big|\int_x^b f(t)dt\Big| \leqslant \max_{x\in[a,b]}|f(x)| \cdot \limsup_{x\to b^-}\int_x^b 1dt = 0\,.$$

Therefore, F is an anti-derivative of f.

Now suppose that G is another anti-derivative of f. Then $(G-F)'(x) = 0$ for all $x \in (a,b)$. By Corollary 4.66, $(G-F)(x) = (G-F)(a)$ for all $x \in [a,b]$; thus $G(b) - G(a) = F(b) - F(a)$. $\square$

Example 4.90. If f is only integrable but not continuous, then the function

$$F(x) = \int_a^x f(t)dt$$

is not necessarily differentiable. For example, consider

$$f(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant 1, \\ 1 & \text{if } 1 < x \leqslant 2. \end{cases}$$

Then

$$F(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant 1, \\ x-1 & \text{if } 1 \leqslant x \leqslant 2. \end{cases}$$

so F is continuous on $[0,2]$ but not differentiable at $x = 1$.

Theorem 4.91. *Let $f : [a,b] \to \mathbb{R}$ be differentiable and assume that f' is Riemann integrable. Then $\displaystyle\int_a^b f'(x)dx = f(b) - f(a)$.*

Proof. Let $\mathcal{P} = \{x_0, x_1, \cdots x_n\}$ be a partition of $[a,b]$. Since $f : [a,b] \to \mathbb{R}$ is differentiable, by the Mean Value Theorem there exists $\{\xi_1, \cdots, \xi_n\}$ with the property that $x_k < \xi_{k+1} < x_{k+1}$ for all $k = 0, 1 \cdots, n-1$ such that

$$f'(\xi_{k+1})(x_{k+1} - x_k) = f(x_{k+1}) - f(x_k) \quad \forall\, k = 0, 1, \cdots, n-1\,.$$

Therefore,

$$\sum_{k=0}^{n-1} \inf_{x\in[x_k,x_{k+1}]} f'(x)(x_{k+1} - x_k) \leqslant \sum_{k=0}^{n-1} f'(\xi_{k+1})(x_{k+1} - x_k) \leqslant \sum_{k=0}^{n-1} \sup_{x\in[x_k,x_{k+1}]} f'(x)(x_{k+1} - x_k)\,.$$

Since $\sum\limits_{k=0}^{n-1} f'(\xi_{k+1})(x_{k+1} - x_k) = \sum\limits_{k=0}^{n-1} \big(f(x_{k+1}) - f(x_k)\big) = f(b) - f(a)$, the inequality above implies that

$$L(f', \mathcal{P}) \leqslant f(b) - f(a) \leqslant U(f', \mathcal{P}) \text{ for all partitions } \mathcal{P} \text{ of } [a,b]\,;$$

thus by the definition of the upper and the lower integrals,

$$\underline{\int_a^b} f'(x)dx \leqslant f(b) - f(a) \leqslant \overline{\int_a^b} f'(x)dx\,.$$

We then conclude the theorem by the identity

$$\underline{\int_a^b} f'(x)dx = \overline{\int_a^b} f'(x)dx = \int_a^b f'(x)dx$$

since f' is Riemann integrable. □

Definition 4.92. Let $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$ be a partition of a bounded set $A \subseteq \mathbb{R}$. A set of points $\{\xi_1, \cdots, \xi_n\}$ are called ***sample points*** with respect to a partition $\mathcal{P}$ if $\xi_k \in [x_{k-1}, x_k]$ for all $k = 1, \cdots, n$.

Theorem 4.93 (Darboux)**.** *Let $f : A \to \mathbb{R}$ be a bounded function with extension $\bar{f}$ given by (4.7.1). Then f is Riemann integrable if and only if $\exists\, \mathrm{I} \in \mathbb{R}$ such that $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ $\ni$ if $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$ is a partition of A satisfying $\|\mathcal{P}\| < \delta$, then for all sets of sample points $\{\xi_1, \cdots, \xi_n\}$ with respect to $\mathcal{P}$,*

$$\Big|\sum_{k=0}^{n-1} \bar{f}(\xi_{k+1})(x_{k+1} - x_k) - \mathrm{I}\Big| < \varepsilon\,. \tag{4.7.6}$$

Proof. "$\Leftarrow$" Let $\varepsilon > 0$ be given. Then for some $\mathrm{I} \in \mathbb{R}$, $\exists\, \delta > 0$ such that if $\mathcal{P}$ is a partition of A satisfying $\|\mathcal{P}\| < \delta$, then for all sets of sample points $\{\xi_1, \cdots, \xi_n\}$ with respect to $\mathcal{P}$, we must have

$$\Big|\sum_{k=0}^{n-1} \bar{f}(\xi_{k+1})(x_{k+1} - x_k) - \mathrm{I}\Big| < \frac{\varepsilon}{4}\,.$$

Let $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$ be a partition of A with $\|\mathcal{P}\| < \delta$. Choose sets of sample points $\{\xi_1, \cdots, \xi_n\}$ and $\{\eta_1, \cdots, \eta_n\}$ with respect to $\mathcal{P}$ such that

(a) $\displaystyle\sup_{x \in [x_k, x_{k+1}]} \bar{f}(x) - \frac{\varepsilon}{4(x_n - x_0)} < \bar{f}(\xi_{k+1}) \leqslant \sup_{x \in [x_k, x_{k+1}]} \bar{f}(x)$;

(b) $\displaystyle\inf_{x \in [x_k, x_{k+1}]} \bar{f}(x) + \frac{\varepsilon}{4(x_n - x_0)} > \bar{f}(\eta_{k+1}) \geqslant \inf_{x \in [x_k, x_{k+1}]} \bar{f}(x)$.

Then

$$\begin{aligned} U(f, \mathcal{P}) &= \sum_{k=0}^{n-1} \sup_{x \in [x_k, x_{k+1}]} \bar{f}(x)(x_{k+1} - x_k) < \sum_{k=0}^{n-1} \big[\bar{f}(\xi_{k+1}) + \frac{\varepsilon}{4(x_n - x_0)}\big](x_{k+1} - x_k) \\ &= \sum_{k=0}^{n-1} \bar{f}(\xi_{k+1})(x_{k+1} - x_k) + \frac{\varepsilon}{4(x_n - x_0)} \sum_{k=0}^{n-1}(x_{k+1} - x_k) < \mathrm{I} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \mathrm{I} + \frac{\varepsilon}{2} \end{aligned}$$

and

$$\begin{aligned} L(f, \mathcal{P}) &= \sum_{k=0}^{n-1} \inf_{x \in [x_k, x_{k+1}]} \bar{f}(x)(x_{k+1} - x_k) > \sum_{k=0}^{n-1} \big[\bar{f}(\eta_{k+1}) - \frac{\varepsilon}{4(x_n - x_0)}\big](x_{k+1} - x_k) \\ &= \sum_{k=0}^{n-1} \bar{f}(\eta_{k+1})(x_{k+1} - x_k) - \frac{\varepsilon}{4(x_n - x_0)} \sum_{k=0}^{n-1}(x_{k+1} - x_k) > \mathrm{I} - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} = \mathrm{I} - \frac{\varepsilon}{2}\,. \end{aligned}$$

As a consequence, $\mathrm{I} - \frac{\varepsilon}{2} < L(f, \mathcal{P}) \leqslant U(f, \mathcal{P}) < \mathrm{I} + \frac{\varepsilon}{2}$; thus $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon$.

"⇒" Let $\varepsilon > 0$ be given, and $I = \displaystyle\int_A \bar{f}(x)dx$. Since f is Riemann integrable over A, $\exists$ a partition $\mathcal{P}_1 = \{y_0, y_1, \cdots, y_m\}$ of A such that $U(f,\mathcal{P}_1) - L(f,\mathcal{P}_1) < \dfrac{\varepsilon}{2}$. Define

$$\delta = \min\left\{|y_1 - y_0|, |y_2 - y_1|, \cdots, |y_m - y_{m-1}|, \frac{\varepsilon}{4m\big(\sup f(A) - \inf f(A)\big)}\right\}.$$

If $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$ is a partition of A with $\|\mathcal{P}\| < \delta$, then at most $2m$ intervals of the form $[x_k, x_{k+1}]$ contains one of these y_j's, and each such interval $[x_k, x_{k+1}]$ can only contain one of these y_j's. Let $\mathcal{P}' = \mathcal{P} \cup \mathcal{P}_1$.

Claim: $U(f,\mathcal{P}) - U(f,\mathcal{P}') < \dfrac{\varepsilon}{2}$.

Proof of claim: We note that

$$\begin{aligned} U(f,\mathcal{P}) &= \sum_{k=0}^{n-1} \sup_{x\in[x_k,x_{k+1}]} \bar{f}(x)(x_{k+1} - x_k) \\ &= \sum_{\substack{0\leqslant k\leqslant n-1 \text{ with} \\ \mathcal{P}_1\cap[x_k,x_{k+1}]=\varnothing}} \sup_{x\in[x_k,x_{k+1}]} \bar{f}(x)(x_{k+1} - x_k) + \sum_{\substack{0\leqslant k\leqslant n-1 \text{ with} \\ \mathcal{P}_1\cap[x_k,x_{k+1}]\neq\varnothing}} \sup_{x\in[x_k,x_{k+1}]} \bar{f}(x)(x_{k+1} - x_k) \end{aligned}$$

and

$$\begin{aligned} U(f,\mathcal{P}') = &\sum_{\substack{0\leqslant k\leqslant n-1 \text{ with} \\ \mathcal{P}_1\cap[x_k,x_{k+1}]=\varnothing}} \sup_{x\in[x_k,x_{k+1}]} \bar{f}(x)(x_{k+1} - x_k) \\ &+ \sum_{\substack{0\leqslant k\leqslant n-1 \text{ with} \\ \mathcal{P}_1\cap[x_k,x_{k+1}]=y_j}} \Big[\sup_{x\in[x_k,y_j]} \bar{f}(x)(y_j - x_k) + \sup_{x\in[y_j,x_{k+1}]} \bar{f}(x)(x_{k+1} - y_j)\Big]. \end{aligned}$$

Therefore,

$$\begin{aligned} U(f,\mathcal{P}) - U(f,\mathcal{P}') &\leqslant \big(\sup f(A) - \inf f(A)\big) \sum_{\substack{0\leqslant k\leqslant n-1 \text{ with} \\ \mathcal{P}_1\cap[x_k,x_{k+1}]\neq\varnothing}} (x_{k+1} - x_k) \\ &< 2m\big(\sup f(A) - \inf f(A)\big)\delta \leqslant \frac{\varepsilon}{2}. \end{aligned}$$

On the other hand, the inequality $U(f,\mathcal{P}_1) - L(f,\mathcal{P}_1) < \dfrac{\varepsilon}{2}$ suggests that

$$U(f,\mathcal{P}_1) - \mathrm{I} < \frac{\varepsilon}{2}.$$

As a consequence,

$$U(f,\mathcal{P}) - \mathrm{I} \leqslant U(f,\mathcal{P}) - \mathrm{I} + U(f,\mathcal{P}_1) - U(f,\mathcal{P}') < \varepsilon.$$

Therefore, for any sample set $\{\xi_1, \cdots, \xi_n\}$ with respect to $\mathcal{P}$,

$$\sum_{k=0}^{n-1} \bar{f}(\xi_{k+1})(x_{k+1} - x_k) \leqslant U(f, \mathcal{P}) < \mathrm{I} + \varepsilon\,.$$

Similar argument can be used to show that

$$\sum_{k=0}^{n-1} \bar{f}(\xi_{k+1})(x_{k+1} - x_k) \geqslant L(f, \mathcal{P}) > \mathrm{I} - \varepsilon$$

which conclude the Theorem. □

Remark 4.94. The sum $\sum\limits_{k=0}^{n-1} \bar{f}(\xi_{k+1})(x_{k+1} - x_k)$ in (4.7.6) is called a ***Riemann sum*** of f over A.

Theorem 4.95 (Change of Variable Formula)**.** *Let $g : [a, b] \to \mathbb{R}$ be a one-to-one continuously differentiable function, and $f : g([a, b]) \to \mathbb{R}$ be Riemann integrable. Then $(f \circ g)g'$ is also Riemann integrable, and*

$$\int_{g([a,b])} f(y)\, dy = \int_a^b f(g(x))|g'(x)|\, dx\,.$$

Chapter 5

Uniform Convergence and the Space of Continuous Functions

5.1 Pointwise and Uniform Convergence（逐點收斂與均匀收斂）

Definition 5.1. Let (M, d) and (N, ρ) be two metric spaces, $A \subseteq M$ be a set, and $f_k, f : A \to N$ be functions for $k = 1, 2, \cdots$. The sequence of function $\{f_k\}_{k=1}^{\infty}$ is said to ***converge pointwise*** to f if

$$\lim_{k\to\infty} \rho\big(f_k(a), f(a)\big) = 0 \qquad \forall\, a \in A.$$

We often write $f_k \to f$ p.w. if f_k converges pointwise to f.

Let $B \subseteq A$ be a subset. The sequence of functions $\{f_k\}_{k=1}^{\infty}$ is said to ***converge uniformly*** to f on B (or $\{f_k\}_{k=1}^{\infty}$ converges to f uniformly on B) if

$$\lim_{k\to\infty} \sup_{x\in B} \rho\big(f_k(x), f(x)\big) = 0\,.$$

In other words, $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f on B if for every $\varepsilon > 0$, $\exists\, N > 0$ such that

$$\rho\big(f_k(x), f(x)\big) < \varepsilon \qquad \forall\, k \geqslant N \text{ and } x \in B\,.$$

Example 5.2. Let $f_k, f : [0, 1] \to \mathbb{R}$ be given by

$$f_k(x) = \begin{cases} 0 & \text{if } \dfrac{1}{k} \leqslant x \leqslant 1\,, \\ -kx + 1 & \text{if } 0 \leqslant x < \dfrac{1}{k}. \end{cases} \qquad \text{and} \quad f(x) = \begin{cases} 0 & \text{if } x \in (0, 1], \\ 1 & \text{if } x = 0. \end{cases}$$

Then $\{f_k\}_{k=1}^{\infty}$ converges pointwise to f on $[0, 1]$. To see this, fix $x \in [0, 1]$.

1. Case $x \neq 0$: Let $\varepsilon > 0$ be given, take $N > \dfrac{1}{x} \Leftrightarrow \dfrac{1}{N} < x$. If $k \geqslant N$,
$$\big|f_k(x) - f(x)\big| = \big|f_k(x) - 0\big| = |0 - 0| < \varepsilon\,.$$

2. Case $x = 0$: For any $\varepsilon > 0$, $k = 1, 2, 3, \dots$, $\big|f_k(0) - f(0)\big| = |1 - 1| = 0 < \varepsilon$.

However, $\{f_k\}_{k=1}^{\infty}$ does not converge uniformly to f on $[0, 1]$ because
$$\sup_{x\in[0,1]} \big|f_k(x) - f(x)\big| = 1 \Rightarrow \lim_{k\to\infty} \sup_{x\in[0,1]} \big|f_k(x) - f(x)\big| = 1 \neq 0\,.$$

Example 5.3. Let $f_k : [0, 1] \to \mathbb{R}$ be given by $f_k(x) = x^k$. Then for each $a \in [0, 1)$, $f_k(a) \to 0$ as $k \to \infty$, while if $a = 1$, $f_k(a) = 1$ for all k. Therefore, if $f(x) = \begin{cases} 0 & \text{if } x \in [0, 1)\,, \\ 1 & \text{if } x = 1\,, \end{cases}$ then $f_k \to f$ p.w.. However, since
$$\sup_{x\in[0,1]} \big|f_k(x) - f(x)\big| = \sup_{x\in[0,1)} \big|f_k(x)\big| = 1\,,$$
we must have
$$\lim_{k\to\infty} \sup_{x\in[0,1]} \big|f_k(x) - f(x)\big| = 1 \neq 0\,.$$
Therefore, $\{f_k\}_{k=1}^{\infty}$ does not converge uniformly to f on $[0, 1]$.

On the other hand, if $0 < a < 1$, then
$$\sup_{x\in[0,a]} \big|f_k(x) - f(x)\big| \leqslant a^k\,;$$
thus by the Sandwich lemma,
$$\lim_{k\to\infty} \sup_{x\in[0,a]} \big|f_k(x) - f(x)\big| = 0\,.$$
Therefore, $\{f_k\}_{k=1}^{\infty}$ converges to uniformly f on $[0, a]$ if $0 < a < 1$.

Example 5.4. Let $f_k : \mathbb{R} \to \mathbb{R}$ be given by $f_k(x) = \dfrac{\sin x}{k}$. Then for each $x \in \mathbb{R}$, $|f_k(x)| \leqslant \dfrac{1}{k}$ which converges to 0 as $k \to \infty$. By the Sandwich lemma,
$$\lim_{k\to\infty} \big|f_k(x)\big| = 0 \qquad \forall\, x \in \mathbb{R}\,.$$
Therefore, $f_k \to 0$ p.w.. Moreover, since $\sup\limits_{x\in\mathbb{R}} \big|f_k(x)\big| \leqslant \dfrac{1}{k}$, $\lim\limits_{k\to\infty} \sup\limits_{x\in\mathbb{R}} \big|f_k(x)\big| = 0$. Therefore, $\{f_k\}_{k=1}^{\infty}$ converges uniformly to 0 on $\mathbb{R}$.

Proposition 5.5. *Let (M, d) and (N, ρ) be two metric spaces, $A \subseteq M$ be a set, and $f_k, f : A \to N$ be functions for $k = 1, 2, \cdots$. If $\{f_k\}_{k=1}^\infty$ converges uniformly to f on A, then $\{f_k\}_{k=1}^\infty$ converges pointwise to f.*

Proof. For each $a \in A$,

$$\rho\big(f_k(a), f(a)\big) \leqslant \sup_{x \in A} \rho\big(f_k(x), f(x)\big)\,;$$

thus the Sandwich lemma suggests that

$$\lim_{k \to \infty} \rho\big(f_k(a), f(a)\big) = 0$$

since $\{f_k\}_{k=1}^\infty$ converges uniformly to f on A. □

Proposition 5.6 (Cauchy criterion for uniform convergence)**.** *Let (M, d) and (N, ρ) be two metric spaces, $A \subseteq M$ be a set, and $f_k : A \to N$ be a sequence of functions. Suppose that (N, ρ) is complete. Then $\{f_k\}_{k=1}^\infty$ converges uniformly on $B \subseteq A$ if and only if for every $\varepsilon > 0$, $\exists\, N > 0$ such that*

$$\rho\big(f_k(x), f_\ell(x)\big) < \varepsilon \qquad \forall\, k, \ell \geqslant N \text{ and } x \in B\,.$$

Proof. "$\Rightarrow$" Let $\varepsilon > 0$ be given, and $\{f_k\}_{k=1}^\infty$ converges uniformly to f on B. Then $\exists\, N > 0$ such that

$$\rho\big(f_k(x), f(x)\big) < \frac{\varepsilon}{2} \qquad \forall\, k \geqslant N \text{ and } x \in B\,.$$

Then if $k, \ell \geqslant N$ and $x \in B$,

$$\rho\big(f_k(x), f_\ell(x)\big) \leqslant \rho\big(f_k(x), f(x)\big) + \rho\big(f(x), f_\ell(x)\big) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.$$

"$\Leftarrow$" Let $b \in B$. By assumption, $\big\{f_k(b)\big\}_{k=1}^\infty$ is a Cauchy sequence in (N, ρ); thus is convergent. Let $f(b)$ denote the limit of $\big\{f_k(b)\big\}_{k=1}^\infty$. Then $\{f_k\}_{k=1}^\infty$ converges pointwise to f on B. We claim that the convergence is indeed uniform on B.

Let $\varepsilon > 0$ be given. Then $\exists\, N > 0$ such that

$$\rho\big(f_k(x), f_\ell(x)\big) < \frac{\varepsilon}{2} \qquad \forall\, k, \ell \geqslant N \text{ and } x \in B\,.$$

Moreover, for each $x \in B$ there exists $N_x > 0$ such that

$$\rho\big(f_\ell(x), f(x)\big) < \frac{\varepsilon}{2} \qquad \forall\, \ell \geqslant N_x\,.$$

Then for all $k \geqslant N$ and $x \in B$,

$$\rho\big(f_k(x), f(x)\big) \leqslant \rho\big(f_k(x), f_\ell(x)\big) + \rho\big(f_\ell(x), f(x)\big) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

in which we choose $\ell \geqslant \max\{N, N_x\}$ to conclude the inequality. □

Theorem 5.7. *Let (M, d) and (N, ρ) be two metric spaces, $A \subseteq M$ be a set, and $f_k : A \to N$ be a sequence of continuous functions converging to $f : A \to N$ uniformly on A. Then f is continuous on A; that is,*

$$\lim_{x\to a} f(x) = \lim_{x\to a} \lim_{k\to\infty} f_k(x) = \lim_{k\to\infty} \lim_{x\to a} f_k(x) = f(a)\,.$$

Proof. Let $a \in A$ and $\varepsilon > 0$ be given. Since $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f on A, $\exists\, N > 0$ such that

$$\rho\big(f_k(x), f(x)\big) < \frac{\varepsilon}{3} \qquad \forall\, k \geqslant N \text{ and } x \in A\,.$$

By the continuity of f_N, $\exists\, \delta > 0$ such that

$$\rho\big(f_N(x), f_N(a)\big) < \frac{\varepsilon}{3} \text{ whenever } x \in D(a, \delta) \cap A\,.$$

Therefore, if $x \in D(a, \delta) \cap A$, by the triangle inequality

$$\begin{aligned}\rho\big(f(x), f(a)\big) &\leqslant \rho\big(f(x), f_N(x)\big) + \rho\big(f_N(x), f_N(a)\big) + \rho\big(f_N(a), f(a)\big) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\,;\end{aligned}$$

thus f is continuous at a. □

Example 5.8. Let $f_k : [0, 2] \to \mathbb{R}$ be given by $f_k(x) = \dfrac{x^k}{1 + x^k}$. Then

1. For each $a \in [0, 1)$, $f_k(a) \to 0$ as $k \to \infty$;
2. For each $a \in (1, 2]$, $f_k(a) \to 1$ as $k \to \infty$;
3. If $a = 1$, then $f_k(a) = \dfrac{1}{2}$ for all k.

Let $f(x) = \begin{cases} 0 & \text{if } x \in [0, 1)\,, \\ \frac{1}{2} & \text{if } x = 1\,, \\ 1 & \text{if } x \in (1, 2]\,. \end{cases}$ Then $\{f_k\}_{k=1}^{\infty}$ converges pointwise to f. However, $\{f_k\}_{k=1}^{\infty}$ does not converge uniformly to f on $[0, 2]$ since f_k are continuous functions for all $k \in \mathbb{N}$ but f is not.

Remark 5.9. The uniform limit of sequence of continuous function might not be uniformly continuous. For example, let $A = (0,1)$ and $f_k(x) = \dfrac{1}{x}$ for all $k \in \mathbb{N}$. Then $\{f_k\}_{k=1}^{\infty}$ converges uniformly to $f(x) = \dfrac{1}{x}$, but the limit function is not uniformly continuous on A.

Theorem 5.10 (Dini's Theorem)**.** *Let K be a compact set, and $f_k : K \to \mathbb{R}$ be continuous for all $k \in \mathbb{N}$ such that $f_k \leqslant f_{k+1}$ for all $k \in \mathbb{N}$. If $\{f_k\}_{k=1}^{\infty}$ converges pointwise to a continuous function $f : K \to \mathbb{R}$, then $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f on K.*

Proof. Suppose the contrary that there exists $\varepsilon > 0$ such that for all $k \in \mathbb{N}$, the set

$$F_k \equiv \big\{x \in K \,\big|\, f(x) - f_k(x) \geqslant \varepsilon\big\}$$

is non-empty. Note that since $f_k \leqslant f_{k+1}$ for all $n \in \mathbb{N}$, $F_k \supseteq F_{k+1}$ for all $k \in \mathbb{N}$. Moreover, by the continuity of f_k and f, F_k is a closed subset of K; thus F_k is compact. Therefore, the nested set property for compact sets (Theorem 3.32) implies that $\bigcap_{n=1}^{\infty} F_k$ is non-empty. In other words, there exists $x \in K$ such that $f(x) - f_k(x) \geqslant \varepsilon$ for all $n \in \mathbb{N}$ which contradicts to the fact that $f_k \to f$ p.w. on K. □

Theorem 5.11. *Let $I \subseteq \mathbb{R}$ be a finite interval, $f_k : I \to \mathbb{R}$ be a sequence of differentiable functions, and $g : I \to \mathbb{R}$ be a function. Suppose that $\big\{f_k(a)\big\}_{k=1}^{\infty}$ converges for some $a \in I$, and $\{f_k'\}_{k=1}^{\infty}$ converges uniformly to g on I. Then*

1. *$\{f_k\}_{k=1}^{\infty}$ converges uniformly to some function f on I.*

2. *The limit function f is differentiable on I, and $f'(x) = g(x)$ for all $x \in I$; that is,*

$$\lim_{k\to\infty} f_k'(x) = \lim_{k\to\infty} \frac{d}{dx} f_k(x) = \frac{d}{dx} \lim_{k\to\infty} f_k(x) = f'(x)\,.$$

Proof. 1. Let $\varepsilon > 0$ be given. Since $\big\{f_k(a)\big\}_{k=1}^{\infty}$ converges to $f(a)$, $\big\{f_k(a)\big\}_{k=1}^{\infty}$ is a Cauchy sequence. Therefore, $\exists\, N_1 > 0$ such that

$$\big|f_k(a) - f_\ell(a)\big| < \frac{\varepsilon}{2} \qquad \forall\, k, \ell \geqslant N_1\,.$$

By the uniform convergence of $\{f_k'\}_{k=1}^{\infty}$ on I and Proposition 5.6, $\exists\, N_2 > 0$ such that

$$\big|f_k'(x) - f_\ell'(x)\big| < \frac{\varepsilon}{2|I|} \qquad \forall\, k, \ell \geqslant N_2 \text{ and } x \in I\,,$$

where $|I|$ is the length of the interval.

Let $N = \max\{N_1, N_2\}$. By the mean value theorem, for all $k, \ell \geqslant N$ and $x \in I$, there exists ξ in between x and a such that

$$\big|f_k(x) - f_\ell(x) - f_k(a) + f_\ell(a)\big| = |f'_k(\xi) - f'_\ell(\xi)||x-a| < \frac{\varepsilon|x-a|}{2|I|} \leqslant \frac{\varepsilon}{2};$$

thus for all $k, \ell \geqslant N$ and $x \in I$,

$$\big|f_k(x) - f_\ell(x)\big| \leqslant \big|f_k(a) - f_\ell(a)\big| + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, Proposition 5.6 suggests that $\{f_k\}_{k=1}^\infty$ converges uniformly on I.

2. Suppose that the uniform limit of $\{f_k\}_{k=1}^\infty$ is f. Let $x \in I$ be a fixed point, and define

$$\phi_k(t) = \begin{cases} \dfrac{f_k(t) - f_k(x)}{t-x} & \text{if } t \in I,\ t \neq x\,, \\ f'_k(x) & \text{if } t = x\,, \end{cases} \quad \text{and} \quad \phi(t) = \begin{cases} \dfrac{f(t) - f(x)}{t-x} & \text{if } t \in I,\ t \neq x\,, \\ g(x) & \text{if } t = x\,. \end{cases}$$

Then ϕ_k is continuous on I for all $k \in \mathbb{N}$, and $\{\phi_k\}_{k=1}^\infty$ converges pointwise to ϕ.

Claim: $\{\phi_k\}_{k=1}^\infty$ converges uniformly to ϕ on I.

Proof of claim: Let $\varepsilon > 0$ be given. Since $\{f'_k\}_{k=1}^\infty$ converges uniformly on I, $\exists\, N > 0$ such that

$$\sup_{t \in I} \big|f'_k(t) - f'_\ell(t)\big| < \varepsilon \qquad \forall\, k, \ell \geqslant N\,.$$

Since

$$\big|\phi_k(t) - \phi_\ell(t)\big| = \begin{cases} \dfrac{\big|f_k(t) - f_k(x) - f_\ell(t) + f_\ell(x)\big|}{|t-x|} & \text{if } t \neq x,\ t \in I\,, \\ \big|f'_k(x) - f'_\ell(x)\big| & \text{if } t = x\,, \end{cases}$$

by the mean value theorem we obtain that

$$\big|\phi_k(t) - \phi_\ell(t)\big| \leqslant \sup_{s \in I} \big|f'_k(s) - f'_\ell(s)\big| < \varepsilon \qquad \forall\, k, \ell \geqslant N \text{ and } t \in I\,.$$

Finally, by Theorem 5.7, ϕ is continuous on I; thus

$$f'(x) = \lim_{t \to x} \phi(t) = \phi(x) = g(x)\,. \qquad \square$$

Example 5.12. Assume that $f_k : I \to \mathbb{R}$ is differentiable for all $k \in \mathbb{N}$, and $\{f'_k\}_{k=1}^\infty$ converges uniformly to g on I. Then $\{f_k\}_{k=1}^\infty$ might **NOT** converge. For example, consider $f_k(x) = k$. Then $f'_k \equiv 0$ but $\{f_k\}_{k=1}^\infty$ does not converge.

Example 5.13. Assume that $f_k : I \to \mathbb{R}$ is differentiable for all $k \in \mathbb{N}$, and $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f on I. Then f might **NOT** be differentiable. In fact, there are differentiable functions $f_k : [a,b] \to \mathbb{R}$ such that f_k converges uniformly to f on $[a,b]$ but f is not differentiable. For example, consider

$$f_k(x) = \begin{cases} \dfrac{k}{2}x^2 & \text{if } |x| \leqslant \dfrac{1}{k}, \\ |x| - \dfrac{1}{2k} & \text{if } \dfrac{1}{k} \leqslant |x| \leqslant 1. \end{cases}$$

Observe that $f_k(-x) = f_k(x)$, so it suffices to consider $x \geqslant 0$.

1. Let $f(x) = |x|$. Then $f_k \to f$ uniformly:

$$\begin{aligned} \sup_{x\in[-1,1]} \big|f_k(x) - f(x)\big| &= \sup_{x\in[0,1]} \big|f_k(x) - x\big| = \max\Big\{ \sup_{x\in[0,\frac{1}{k}]} \big|f_k(x) - x\big|, \sup_{x\in[\frac{1}{k},1]} \big|f_k(x) - x\big|\Big\} \\ &= \max\Big\{ \sup_{x\in[0,\frac{1}{k}]} \Big|\frac{kx^2}{2} - x\Big|, \sup_{x\in[\frac{1}{k},1]} \Big|x - \frac{1}{2k} - x\Big|\Big\} \\ &\leqslant \sup_{x\in[0,\frac{1}{k}]} \Big|\frac{kx^2}{2}\Big| + |x| \leqslant \frac{k}{2}(\frac{1}{k})^2 + \frac{1}{k} = \frac{3}{2k} \to 0 \text{ as } k \to \infty\,. \end{aligned}$$

2. To see if f_k are differentiable, it suffices to show $f_k'(\frac{1}{k})$ exists.

$$\begin{aligned} f_k'(\frac{1}{k}) = \lim_{h\to 0} \frac{f_k(\frac{1}{k}+h) - f_k(\frac{1}{k})}{h} &= \lim_{h\to 0} \frac{1}{h} \begin{cases} \big(\dfrac{1}{k} + h\big) - \dfrac{1}{2k} - \dfrac{1}{2k} & \text{if } h > 0 \\ \dfrac{k}{2}(\dfrac{1}{k} + h)^2 - \dfrac{1}{2k} & \text{if } h < 0 \end{cases} \\ = \lim_{h\to 0} \frac{1}{h} \begin{cases} h & \text{if } h > 0 \\ h + \dfrac{k}{2}h^2 & \text{if } h < 0 \end{cases} &= 1\,. \end{aligned}$$

Example 5.14. Assume that $f_k : [-1,1] \to \mathbb{R}$ be given by

$$f_k(x) = \begin{cases} 0 & \text{if } x \in [-1,0]\,, \\ \dfrac{k^2}{2}x^2 & \text{if } x \in \big(0, \dfrac{1}{k}\big]\,, \\ 1 - \dfrac{k^2}{2}\big(x - \dfrac{2}{k}\big)^2 & \text{if } x \in \big(\dfrac{1}{k}, \dfrac{2}{k}\big]\,, \\ 1 & \text{if } x \in \big(\dfrac{2}{k}, 1\big]\,. \end{cases}$$

Then $f_k'(x) = \begin{cases} 0 & \text{if } x \in [-1,0]\,, \\ k^2 x & \text{if } x \in \big(0, \dfrac{1}{k}\big]\,, \\ -k^2\big(x - \dfrac{2}{k}\big) & \text{if } x \in \big(\dfrac{1}{k}, \dfrac{2}{k}\big]\,, \\ 0 & \text{if } x \in \big(\dfrac{2}{k}, 1\big]\,, \end{cases}$ and $\{f_k'\}_{k=1}^\infty$ converges pointwise to 0 but not uniformly on $[-1,1]$. We note that $\{f_k\}_{k=1}^\infty$ converges to a discontinuous function

$$f(x) = \begin{cases} 0 & \text{if } x \in [-1,0]\,, \\ 1 & \text{if } x \in (0,1]\,, \end{cases}$$

so the convergence of $\{f_k\}_{k=1}^\infty$ cannot be uniform on $[-1,1]$.

Example 5.15. Suppose $f_k : [0,1] \to \mathbb{R}$ are differentiable on $(0,1)$ and f_k converges uniformly to f on $[0,1]$ for some $f : [0,1] \to \mathbb{R}$. Does f_k' converge uniformly?

Answer: No! Take $f_k = \dfrac{\sin(k^2 x)}{k}$, $k = 1, 2, \cdots$, then $f_k \to 0$ uniformly on $[0,1]$ since

$$\sup_{x\in[0,1]} \big|f_k(x) - 0\big| = \sup_{x\in[0,1]} \Big|\frac{\sin(k^2x)}{k}\Big| \leqslant \frac{1}{k} \Rightarrow \lim_{k\to\infty} \sup_{x\in[0,1]} \big|f_k(x) - 0\big| = 0\,.$$

Now $f_k'(x) = k\cos(k^2x)$ and $f_k'(0) = k \to \infty$ as $k \to \infty$.

Example 5.16. There are differentiable functions $f_k : [a,b] \to \mathbb{R}$ such that f_k converges uniformly to f on $[a,b]$ but $\lim\limits_{k\to\infty} f_k' \neq (\lim\limits_{k\to\infty} f_k)'$. For example, take $f_k(x) = \dfrac{x}{1+k^2x^2}$ on $[-1,1]$. Then $f_k'(x) = \dfrac{1-k^2x^2}{(1+k^2x^2)^2}$.

1. Since $\lim\limits_{k\to\infty} \sup\limits_{x\in[-1,1]} \Big|\dfrac{x}{1+k^2x^2} - 0\Big| = \lim\limits_{k\to\infty} \dfrac{1}{2k} = 0$, f_k converges uniformly to 0 on $[-1,1]$.

2. $(\lim\limits_{k\to\infty} f_k(x))' = 0' = 0$.

3. $\lim\limits_{k\to\infty} f_k'(x) = \lim\limits_{k\to\infty} \dfrac{1-k^2x^2}{(1+k^2x^2)^2} = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0, |x| < 1. \end{cases}$ Note that f_k' does not converge uniformly.

Theorem 5.17. *Let $f_k : [a,b] \to \mathbb{R}$ be a sequence of Riemann integrable functions which converges uniformly to f on $[a,b]$. Then f is Riemann integrable, and*

$$\lim_{k\to\infty} \int_a^b f_k(x)dx = \int_a^b \lim_{k\to\infty} f_k(x)dx = \int_a^b f(x)dx\,. \tag{5.1.1}$$

Proof. Let $\varepsilon > 0$ be given. Since $\{f_k\}_{k=1}^\infty$ converges uniformly to f on $[a,b]$, $\exists\, N > 0$ such that

$$\big|f_k(x) - f(x)\big| < \frac{\varepsilon}{4(b-a)} \qquad \forall\, k \geqslant N \text{ and } x \in [a,b]\,. \tag{5.1.2}$$

Since f_N is Riemann integrable on $[a,b]$, by Riemann's condition there exists a partition $\mathcal{P}$ of $[a,b]$ such that

$$U(f_N,\mathcal{P}) - L(f_N,\mathcal{P}) < \frac{\varepsilon}{2}\,.$$

Using (4.7.3), we find that

$$\begin{aligned} U(f,\mathcal{P}) - L(f,\mathcal{P}) &= U(f - f_N + f_N,\mathcal{P}) - L(f - f_N + f_N,\mathcal{P}) \\ &\leqslant U(f - f_N,\mathcal{P}) + U(f_N,\mathcal{P}) - L(f - f_N,\mathcal{P}) - L(f_N,\mathcal{P}) \\ &\leqslant \frac{\varepsilon}{4(b-a)}(b-a) + \frac{\varepsilon}{4(b-a)}(b-a) + U(f_N,\mathcal{P}) - L(f_N,\mathcal{P}) \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon\,; \end{aligned}$$

thus by Riemann's condition f is Riemann integrable on $[a,b]$.

Now, if $k \geqslant N$, (5.1.2) implies that

$$\begin{aligned} \Big|\int_a^b f_k(x)dx - \int_a^b f(x)dx\Big| = \Big|\int_a^b \big(f_k(x) - f(x)\big)dx\Big| &\leqslant \int_a^b \big|f_k(x) - f(x)\big|dx \\ &\leqslant \frac{\varepsilon}{4(b-a)}(b-a) = \frac{\varepsilon}{4} < \varepsilon \end{aligned}$$

which suggests (5.1.1). □

Example 5.18. Let $\{q_k\}_{k=1}^\infty$ be the rational numbers in $[0,1]$, and

$$f_k(x) = \begin{cases} 0 & \text{if } x \in \{q_1, q_2, \cdots, q_k\}\,, \\ 1 & \text{otherwise}\,. \end{cases}$$

Then f_k converges pointwise to the Dirichlet function

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \cap [0,1]\,, \\ 1 & \text{if } x \in [0,1]\backslash\mathbb{Q}\,. \end{cases}$$

However, $\{f_k\}_{k=1}^\infty$ does not converge uniformly to f since f_k are Riemann integrable on $[0,1]$ for all $k \in \mathbb{N}$ but f is not.

Example 5.19. Let $f_k : [0,1] \to \mathbb{R}$ be functions given in Example 5.14, and let $g_k = f_k'$. Then $\{g_k\}_{k=1}^\infty$ converges pointwise to 0, but $\int_0^1 g_k(x)dx = 1$ for all $k \in \mathbb{N}$.

5.2 Series of Functions and The Weierstrass M-Test

Definition 5.20. Let (M,d) be a metric space, $(\mathcal{V},\|\cdot\|)$ be a norm space, $A \subseteq M$ be a subset, and $g_k, g : A \to \mathcal{V}$ be functions. We say that the series $\sum\limits_{k=1}^{\infty} g_k$ converges pointwise to g, and write $\sum\limits_{k=1}^{\infty} g_k = g$ p.w. if the sequence of partitial sum $\{s_n\}_{n=1}^{\infty}$ given by

$$s_n = \sum_{k=1}^{n} g_k$$

converges pointwise to g. We say that $\sum\limits_{k=1}^{\infty} g_k$ converges to g uniformly on $B \subseteq A$ if $\{s_n\}_{n=1}^{\infty}$ converges uniformly to g on B.

Example 5.21. Consider the geometric series $\sum\limits_{k=0}^{\infty} x^k$. The partial sum s_n is given by

$$s_n(x) = \begin{cases} \dfrac{1-x^{n+1}}{1-x} & \text{if } x \neq 1\,, \\ n+1 & \text{if } x = 1\,. \end{cases}$$

Then

1. $\sum\limits_{k=0}^{\infty} x^k$ converges pointwise to $g(x) = \dfrac{1}{1-x}$ in $(-1,1)$.

2. $\sum\limits_{k=0}^{\infty} x^k$ does not converge pointwise in $(-\infty,-1] \cup [1,\infty)$.

3. $\sum\limits_{k=0}^{\infty} x^k$ converges uniformly on $(-a,a)$ if $0 < a < 1$ since

$$\sup_{x\in(-a,a)} |s_n(x) - g(x)| = \sup_{x\in(-a,a)} \frac{|x|^{n+1}}{1-x} \leqslant \frac{|a|^{n+1}}{1-a} \to 0 \text{ as } n \to \infty.$$

4. $\sum\limits_{k=0}^{\infty} x^k$ does not converge uniformly on $(-1,1)$ since $\sup\limits_{x\in(-1,1)} |s_n(x) - g(x)| = \infty$.

The following two corollaries are direct consequences of Proposition 5.6 and Theorem 5.7.

Corollary 5.22. *Let (M, d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a complete normed vector space, $A \subseteq M$ be a subset, and $g_k : A \to \mathcal{V}$ be functions. Then $\sum\limits_{k=1}^{\infty} g_k$ converges uniformly on A if and only if*

$$\forall\, \varepsilon > 0, \exists\, N > 0 \ni \Big\| \sum_{k=m+1}^{n} g_k(x) \Big\| < \varepsilon \qquad \forall\, n > m \geqslant N \text{ and } x \in A\,.$$

Corollary 5.23. *Let (M, d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a normed vector space, $A \subseteq M$ be a subset, and $g_k, g : A \to \mathcal{V}$ be functions. If $g_k : A \to \mathcal{V}$ are continuous and $\sum\limits_{k=1}^{\infty} g_k(x)$ converges to g uniformly on A, then g is continuous.*

Theorem 5.24. *Let $f : (a, b) \to \mathbb{R}$ be an infinitely differentiable functions; that is, $f^{(k)}(x)$ exists for all $k \in \mathbb{N}$ and $x \in (a, b)$. Let $c \in (a, b)$ and suppose that for some $0 < h < \infty$, $\big|f^{(k)}(x)\big| \leqslant M$ for all $x \in (c-h, c+h) \subseteq (a, b)$. Then*

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \qquad \forall\, x \in (c-h, c+h)\,.$$

Proof. First, we claim that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + (-1)^n \int_c^x \frac{(y-x)^n}{n!} f^{(n+1)}(y)dy \qquad \forall\, x \in (a, b)\,. \tag{5.2.1}$$

By the fundamental theorem or Calculus (Theorem 4.89) It is clear that (5.2.1) holds for $n = 0$. Suppose that (5.2.1) holds for $n = m$. Then

$$\begin{aligned} f(x) &= \sum_{k=0}^{m} \frac{f^{(k)}(c)}{k!}(x-c)^k + (-1)^m \Big[\frac{(y-x)^{m+1}}{(m+1)!} f^{(m+1)}(y)\Big|_{y=c}^{y=x} - \int_c^x \frac{(y-x)^{m+1}}{(m+1)!} f^{(m+2)}(y)dy\Big] \\ &= \sum_{k=0}^{m+1} \frac{f^{(k)}(c)}{k!}(x-c)^k + (-1)^{m+1} \int_c^x \frac{(y-x)^{m+1}}{(m+1)!} f^{(m+2)}(y)dy \end{aligned}$$

which implies that (5.2.1) also holds for $n = m+1$. By induction (5.2.1) holds for all $n \in \mathbb{N}$.

Letting $s_n(x) = \sum\limits_{k=0}^{n} \dfrac{f^{(k)}(c)}{k!}(x-c)^k$, then if $x \in (c-h, c+h)$,

$$\big|s_n(x) - f(x)\big| \leqslant \Big|\int_c^x \frac{h^n}{n!} M dy\Big| \leqslant \frac{h^{n+1}}{n!} M\,.$$

Let $\varepsilon > 0$ be given. Since $\lim\limits_{n\to\infty} \dfrac{h^{n+1}}{n!} M = 0$, $\exists\, N > 0$ such that $\Big|\dfrac{h^{n+1}}{n!}\Big| M < \varepsilon$ if $n \geqslant N$. As a consequence, if $n \geqslant N$,

$$\big|s_n(x) - f(x)\big| < \varepsilon \quad \text{whenever } n \geqslant N\,. \qquad \square$$

Example 5.25. The series $\sum\limits_{k=0}^{\infty}(-1)^k\dfrac{x^{2k+1}}{(2k+1)!}$ converges to $\sin x$ uniformly on any bounded subset of $\mathbb{R}$.

Theorem 5.26 (Weierstrass M-test)**.** *Let (M,d) be a metric space, $(\mathcal{V},\|\cdot\|)$ be a complete normed vector space, $A\subseteq M$ be a subset, and $g_k : A\to\mathcal{V}$ be a sequence of functions. Suppose that $\exists\, M_k>0$ such that $\sup\limits_{x\in A}\|g_k(x)\|\leqslant M_k$ for all $k\in\mathbb{N}$ and $\sum\limits_{k=1}^{\infty}M_k$ converges. Then $\sum\limits_{k=1}^{\infty}g_k$ converges uniformly and absolutely (that is, $\sum\limits_{k=1}^{\infty}\|g_k\|$ converges uniformly) on A.*

Proof. We show that the partial sum $s_n=\sum\limits_{k=1}^{n}g_k$ satisfies the Cauchy criterion. Let $\varepsilon>0$ be given. Since $\sum\limits_{k=1}^{\infty}M_k$ converges (which means $\sum\limits_{k=1}^{n}M_k$ converges as $n\to\infty$), there exists $N>0$ such that

$$\sum_{k=m+1}^{n}M_k=\Big|\sum_{k=m+1}^{n}M_k\Big|<\varepsilon\quad\forall\, n>m\geqslant N\,.$$

Therefore,

$$\Big\|\sum_{k=m+1}^{n}g_k(x)\Big\|\leqslant\sum_{k=m+1}^{n}\|g_k(x)\|\leqslant\sum_{k=m+1}^{n}M_k<\varepsilon\quad\forall\, n>m\geqslant N \text{ and } x\in A\,.$$

Apply Proposition 5.6 to the sequence $\{s_n\}_{n=1}^{\infty}$, we conclude the theorem. □

Theorem 5.7 and 5.26 together imply the following

Corollary 5.27. *Let (M,d) be a metric space, $(\mathcal{V},\|\cdot\|)$ be a complete normed vector space, $A\subseteq M$ be a subset, and $g_k : A\to\mathcal{V}$ be a sequence of continuous functions. Suppose that $\exists\, M_k>0$ such that $\sup\limits_{x\in A}\|g_k(x)\|\leqslant M_k$ for all $k\in\mathbb{N}$ and $\sum\limits_{k=1}^{\infty}M_k$ converges. Then $\sum\limits_{k=1}^{\infty}g_k$ is continuous on A.*

Example 5.28. Consider the series $f(x)=\sum\limits_{k=0}^{\infty}\Big(\dfrac{x^k}{k!}\Big)^2$. For all $x\in[-R,R]$, $\Big(\dfrac{x^k}{k!}\Big)^2\leqslant\dfrac{R^{2k}}{(k!)^2}$. Moreover,

$$\limsup_{k\to\infty}\frac{R^{2(k+1)}}{((k+1)!)^2}\Big/\frac{R^{2k}}{(k!)^2}=\limsup_{k\to\infty}\frac{R^2}{(k+1)^2}=0\,;$$

thus the ratio test and the Weierstrass M-test suggest that the series $\sum\limits_{k=0}^{\infty}\Big(\dfrac{x^k}{k!}\Big)^2$ converges uniformly on $[-R,R]$. Corollary 5.27 suggests that f is continuous on $[-R,R]$. Since R is arbitrary, we find that f is continuous on $\mathbb{R}$.

Example 5.29. Let $\{a_k\}_{k=0}^{\infty}$ be a bounded sequence. Then $\sum\limits_{k=0}^{\infty} \frac{a_k}{k!} x^k$ converges to a continuous function.

Example 5.30. Consider the function $f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum\limits_{k=0}^{\infty} \frac{\cos(2k+1)x}{(2k+1)^2}$. We can in fact show (much later) that $f(x) = |x|$ for all $x \in [-\pi, \pi]$, and by the Weierstrass M-test it is easy to see that the convergence is uniform on $\mathbb{R}$.

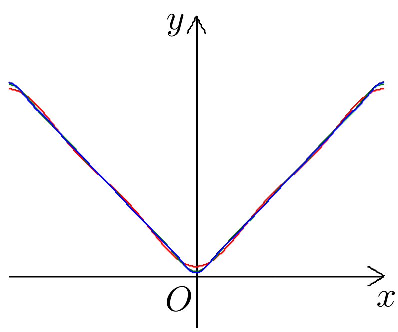


Figure 5.1: The graph of some partial sums

5.3 Integration and Differentiation of Series

The following two theorems are direct consequences of Theorem 5.11 and 5.17.

Theorem 5.31. *Let $g_k : [a, b] \to \mathbb{R}$ be a sequence of Riemann integrable functions. If $\sum\limits_{k=1}^{\infty} g_k$ converges uniformly on $[a, b]$, then*

$$\int_a^b \sum_{k=1}^{\infty} g_k(x)dx = \sum_{k=1}^{\infty} \int_a^b g_k(x)dx\,.$$

Theorem 5.32. *Let $g_k : (a, b) \to \mathbb{R}$ be a sequence of differentiable functions. Suppose that $\sum\limits_{k=1}^{\infty} g_k$ converges for some $c \in (a, b)$, and $\sum\limits_{k=1}^{\infty} g_k'$ converges uniformly on (a, b). Then*

$$\sum_{k=1}^{\infty} g_k'(x) = \frac{d}{dx} \sum_{k=1}^{\infty} g_k(x)\,.$$

Definition 5.33. A series is called a ***power series about*** c or ***centered at*** c if it is of the form $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ for some sequence $\{a_k\}_{k=0}^{\infty} \subseteq \mathbb{R}$ and $c \in \mathbb{R}$.

Proposition 5.34. *If a power series centered at c is convergent at some point $b \neq c$, then the power series converges pointwise on $(c-|b-c|, c+|b-c|)$, and converges uniformly on $[\alpha, \beta]$ if $[\alpha, \beta] \subseteq (c-|b-c|, c+|b-c|)$.*

Proof. Since the series $\sum\limits_{k=0}^{\infty} a_k(b-c)^k$ converges, $|a_k||b-c|^k \to 0$ as $k \to \infty$; thus $\exists\, M > 0$ such that $|a_k||b-c|^k \leqslant M$ for all k.

1. $x \in (c-|b-c|, c+|b-c|)$, the series $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ converges absolutely since

$$\sum_{k=0}^{\infty} |a_k(x-c)^k| \leqslant \sum_{k=0}^{\infty} |a_k||x-c|^k = \sum_{k=0}^{\infty} |a_k||b-c|^k \frac{|x-c|^k}{|b-c|^k} \leqslant M \sum_{k=0}^{\infty} \Big(\frac{|x-c|}{|b-c|}\Big)^k$$

   which converges (because of the geometric series test or ratio test).

2. Let $r = \max\Big\{\dfrac{|\beta-c|}{|b-c|}, \dfrac{|\alpha-c|}{|b-c|}\Big\}$. Then $0 < r < 1$, and $|a_k(x-c)^k| \leqslant Mr^k$ if $x \in [\alpha, \beta]$. Therefore, the Weierstrass M-test implies that the series $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ converges uniformly on $[\alpha, \beta]$. □

By the proposition above, we immediately conclude that the collection of all x at which the power series converges must be connected; thus is an interval or a point. Moreover, if the interval contains point other than c, the interior of this interval must be symmetric about c. These observations induce the following

Definition 5.35. A number R is called the ***radius of convergence*** of the power series $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ if the series converges for all $x \in (c-R, c+R)$ but diverges if $x > c+R$ or $x < c-R$. In other words,

$$R = \sup\Big\{r \geqslant 0 \,\Big|\, \sum_{k=0}^{\infty} a_k(x-c)^k \text{ converges in } [c-r, c+r]\Big\}\,.$$

The ***interval of convergence*** or ***convergence interval*** of a power series is the collection of all x at which the power series converges.

Remark 5.36. A power series converges pointwise on its interval of convergence.

Theorem 5.37. *Let $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ be a power series with convergence interval I, and $[\alpha, \beta] \subseteq int(I) \equiv (c-R, c+R)$ be a closed interval (in which R is the radius of convergence). Then*

1. *The power series* $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ *converges uniformly on* $[\alpha,\beta]$.

2. *The power series* $\sum\limits_{k=0}^{\infty}(k+1)a_{k+1}(x-c)^k$ *converges pointwise on* $(c-R,c+R)$*, and converges uniformly on* $[\alpha,\beta]$.

Proof. 1. It is simply a restatement of Proposition 5.34.

2. By 1, it suffices to show that the power series $\sum\limits_{k=0}^{\infty}(k+1)a_{k+1}(x-c)^k$ converges pointwise on $(c-R,c+R)$. Clearly the series converges at $x=c$. Let $x\in(c-R,c+R)$ and $x\neq c$. Define
$$b=\begin{cases}\dfrac{x+C+R}{2} & \text{if } x>c\,,\\ \dfrac{x+C-R}{2} & \text{if } x<c\,.\end{cases}$$
Then if $r=\dfrac{|x-c|}{|b-c|}$, $0<r<1$ and
$$\sum_{k=0}^{\infty}(k+1)|a_{k+1}||x-c|^k\leqslant\sum_{k=0}^{\infty}(k+1)|a_{k+1}||b-c|^k\Big(\frac{|x-c|}{|b-c|}\Big)^k\leqslant M\sum_{k=0}^{\infty}(k+1)r^k$$
for some $M>0$. Note that the ratio test implies that the series $\sum\limits_{k=0}^{\infty}(k+1)r^k$ converges if $0<r<1$; thus the series $\sum\limits_{k=0}^{\infty}(k+1)|a_{k+1}||x-c|^k$ converges by the comparison test. □

Corollary 5.38. *A power series* $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ *is differentiable in the interior of its interval of convergence* I*. Moreover,*
$$\frac{d}{dx}\sum_{k=0}^{\infty}a_k(x-c)^k=\sum_{k=1}^{\infty}ka_k(x-c)^{k-1}\qquad\forall\,x\in int(I)\,.$$

Corollary 5.39. *A power series* $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ *is Riemann integrable on any closed intervals contained in the interior of its interval of convergence* I*. Moreover, if* $[\alpha,\beta]\subseteq int(I)$*, then*
$$\int_\alpha^\beta\sum_{k=0}^{\infty}a_k(x-c)^k dx=\sum_{k=0}^{\infty}a_k\int_\alpha^\beta(x-c)^k dx$$

Example 5.40. Let $\{a_k\}_{k=0}^{\infty}$ be a bounded sequence. Then
$$\frac{d}{dx}\Big(\sum_{k=0}^{\infty}\frac{a_k}{k!}x^k\Big)=\sum_{k=1}^{\infty}\frac{a_k}{(k-1)!}x^{k-1}=\sum_{k=0}^{\infty}\frac{a_{k+1}}{k!}x^k\,.$$

Example 5.41. We show $\displaystyle\int_0^t e^x dx = e^t - 1$ as follows. By Theorem 5.24, $e^x = \sum\limits_{k=0}^{\infty} \dfrac{x^k}{k!}$ and the convergence is uniform on any bounded sets of $\mathbb{R}$; thus Corollary 5.39 suggests that

$$\int_0^t e^x dx = \int_0^t \sum_{k=0}^{\infty} \frac{x^k}{k!} dx = \sum_{k=0}^{\infty} \int_0^t \frac{x^k}{k!} dx = \sum_{k=0}^{\infty} \frac{t^{k+1}}{(k+1)!} = \sum_{k=1}^{\infty} \frac{t^k}{k!} = e^t - 1\,.$$

Example 5.42. $\dfrac{d}{dx}\Big(\sum\limits_{k=1}^{\infty} \dfrac{x^k}{k}\Big) = \sum\limits_{k=1}^{\infty} x^{k-1} = \sum\limits_{k=0}^{\infty} x^k$ for all $x \in (-1,1)$; thus

$$\frac{d}{dx}\Big(\sum_{k=1}^{\infty} \frac{x^k}{k}\Big) = \frac{1}{1-x} \qquad \forall\, x \in (-1,1)\,.$$

As a consequence,

$$\sum_{k=1}^{\infty} \frac{t^k}{k} = \int_0^t \frac{d}{dx}\Big(\sum_{k=1}^{\infty} \frac{x^k}{k}\Big) dx = -\log(1-t) \qquad \forall\, t \in (-1,1)\,. \tag{5.3.1}$$

Using the alternating series test, it is clear that the left-hand side of (5.3.1) converges at $t = -1$. What is the value of

$$-\sum_{k=1}^{\infty} \frac{(-1)^k}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots ?$$

Consider the partial sum $\dfrac{d}{dx}\Big(\sum\limits_{k=1}^{n} \dfrac{x^k}{k}\Big) = \sum\limits_{k=0}^{n-1} x^k = \dfrac{1-x^n}{1-x} = \dfrac{1}{1-x} - \dfrac{x^n}{1-x}$. Integrating both sides over $[-1,0]$,

$$\Big|\sum_{k=1}^{n} \frac{(-1)^k}{k} + \log 2\Big| \leqslant \int_{-1}^{0} \frac{|x|^n}{1-x} dx \leqslant \int_{-1}^{0} (-x)^n dx = \frac{1}{n+1} \to 0 \text{ as } n \to \infty;$$

thus

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \log 2\,.$$

In other words,

$$\sum_{k=1}^{\infty} \frac{t^k}{k} = -\log(1-t) \qquad \forall\, t \in [-1,1)\,.$$

Example 5.43. It is clear that $\dfrac{1}{1+x^2} = \sum\limits_{k=0}^{\infty} (-x^2)^k = \sum\limits_{k=0}^{\infty} (-1)^k x^{2k}$ for all $x \in (-1,1)$. So if $x \in (-1,1)$,

$$\begin{aligned}\tan^{-1} x &= \int_0^x \frac{dt}{1+t^2} = \int_0^x \sum_{k=0}^{\infty} (-1)^k t^{2k} dt = \sum_{k=0}^{\infty} \int_0^x (-1)^k t^{2k} dt \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} t^{2k+1}\Big|_{t=0}^{t=x} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .\end{aligned}$$

The right-hand side of the identity above converges at $x = 1$. What is the value of

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots ?$$

Mimic the previous example, we consider

$$\begin{aligned}
\tan^{-1} x &= \int_0^x \frac{dt}{1+t^2} = \int_0^x \frac{1-(-t^2)^{n+1}}{1+t^2}\,dt + \int_0^x \frac{(-t^2)^{n+1}}{1+t^2}\,dt \\
&= \int_0^x \sum_{k=0}^{n} (-1)^k t^{2k}\,dt + \int_0^x \frac{(-t^2)^{n+1}}{1+t^2}\,dt \\
&= \sum_{k=0}^{n} \int_0^x (-1)^k t^{2k}\,dt + \int_0^x \frac{(-t^2)^{n+1}}{1+t^2}\,dt = \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} x^{2k+1} + \int_0^x \frac{(-t^2)^{n+1}}{1+t^2}\,dt\,;
\end{aligned}$$

thus plugging $x = 1$,

$$\Big| \tan^{-1} 1 - \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} \Big| \leqslant \int_0^1 \frac{t^{2(n+1)}}{1+t^2}\,dt \leqslant \int_0^1 t^{2(n+1)}\,dt = \frac{1}{2n+3} \to 0 \text{ as } n \to \infty.$$

Therefore,

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \tan^{-1} 1 = \frac{\pi}{4}\,.$$

5.4 The Space of Continuous Functions

Definition 5.44. Let (M, d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a normed vector space, and $A \subseteq M$ be a subset. We define $\mathscr{C}(A;\mathcal{V})$ as the collection of all continuous functions on A with value in $\mathcal{V}$; that is,

$$\mathscr{C}(A;\mathcal{V}) = \big\{ f : A \to \mathcal{V} \,\big|\, f \text{ is continuous on } A \big\}\,.$$

Let $\mathscr{C}_b(A;\mathcal{V})$ be the subspace of $\mathscr{C}(A;\mathcal{V})$ which consists of all bounded continuous functions on A; that is,

$$\mathscr{C}_b(A;\mathcal{V}) = \big\{ f \in \mathscr{C}(A;\mathcal{V}) \,\big|\, f \text{ is bounded} \big\}\,.$$

Every $f \in \mathscr{C}_b(A;\mathcal{V})$ is associated with a non-negative real number $\|f\|_\infty$ given by

$$\|f\|_\infty = \sup\big\{ \|f(x)\| \,\big|\, x \in A \big\} = \sup_{x\in A} \|f(x)\|$$

The number $\|f\|_\infty$ is called the ***sup-norm*** of f.

Proposition 5.45. *Let (M,d) be a metric space, $(\mathcal{V},\|\cdot\|)$ be a normed vector space, $A \subseteq M$ be a subset.*

1. *$\mathscr{C}(A;\mathcal{V})$ and $\mathscr{C}_b(A;\mathcal{V})$ are vector spaces.*
2. *$\big(\mathscr{C}_b(A;\mathcal{V}),\|\cdot\|_\infty\big)$ is a normed vector space.*
3. *If $K \subseteq M$ is compact, then $\mathscr{C}(K;\mathcal{V}) = \mathscr{C}_b(K;\mathcal{V})$.*

Proof. 1 and 2 are trivial, and 3 is concluded by Theorem 4.21. ▫

Remark 5.46. In general $\|\cdot\|_\infty$ is not a "norm" on $\mathscr{C}(A;\mathcal{V})$. For example, the function $f(x) = \dfrac{1}{x}$ belongs to $\mathscr{C}((0,1);\mathbb{R})$ and $\|f\|_\infty = \infty$. Note that to be a norm $\|f\|_\infty$ has to take values in $\mathbb{R}$, and $\infty \notin \mathbb{R}$.

Proposition 5.47. *Let (M,d) be a metric space, $(\mathcal{V},\|\cdot\|)$ be a normed vector space, $A \subseteq M$ be a subset, and $f_k, f \in \mathscr{C}_b(A;\mathcal{V})$ for all $k \in \mathbb{N}$. Then $\{f_k\}_{k=1}^\infty$ converges uniformly to f on A if and only if $\{f_k\}_{k=1}^\infty$ converges to f in $\big(\mathscr{C}(A;\mathcal{V}),\|\cdot\|_\infty\big)$.*

Proof. The equivalency is obvious since $\lim\limits_{k\to\infty}\sup\limits_{x\in A}\|f_k(x)-f(x)\| = \lim\limits_{k\to\infty}\|f_k - f\|_\infty$. ▫

Theorem 5.48. *Let (M,d) be a metric space, $(\mathcal{V},\|\cdot\|)$ be a normed vector space, and $A \subseteq M$ be a subset. If $(\mathcal{V},\|\cdot\|)$ is complete, so is $\big(\mathscr{C}_b(A;\mathcal{V}),\|\cdot\|_\infty\big)$.*

Proof. Let $\{f_k\}_{k=1}^\infty$ be a Cauchy sequence in $\big(\mathscr{C}_b(A;\mathcal{V}),\|\cdot\|_\infty\big)$. Then

$$\forall\,\varepsilon > 0, \exists\,N > 0 \ni \|f_k - f_\ell\|_\infty < \varepsilon \text{ if } k,\ell \geqslant N\,.$$

By the definition of the sup-norm, the statement above suggests that

$$\forall\,\varepsilon > 0, \exists\,N > 0 \ni \|f_k(x) - f_\ell(x)\| < \varepsilon \quad \text{if } k,\ell \geqslant N \text{ and } x \in A$$

which implies that $\{f_k\}_{k=1}^\infty$ satisfies the Cauchy criterion. By Proposition 5.6, $\{f_k\}_{k=1}^\infty$ converges uniformly to some function f on A. By Theorem 5.7, $f \in \mathscr{C}(A;\mathcal{V})$.
Claim: f is bounded on A (which further suggests that $f \in \mathscr{C}_b(A;\mathcal{V})$).
Proof of claim: Since $\{f_k\}_{k=1}^\infty$ converges uniformly to f on A, $\exists\,N > 0$ such that

$$\|f_k(x) - f(x)\| < 1 \quad \forall\,k \geqslant N \text{ and } x \in A\,.$$

Suppose that $\|f_N(x)\| \leqslant M$ for all $x \in A$. Then if $x \in A$,

$$\|f(x)\| \leqslant \|f_N(x) - f(x)\| + \|f_N(x)\| \leqslant 1 + M$$

which implies that f is bounded. Finally, we conclude the theorem by Proposition 5.47. ▫

Definition 5.49. A ***Banach space*** is a complete normed vector space.

Example 5.50. The set $B = \big\{f \in \mathscr{C}([0,1];\mathbb{R}) \,\big|\, f(x) > 0 \text{ for all } x \in [0,1]\big\}$ is open in $\big(\mathscr{C}([0,1];\mathbb{R}), \|\cdot\|_\infty\big)$.

Reason: Let $f \in B$ be given. Since $[0,1]$ is compact and f is continuous, by the extreme value theorem $\exists\, x_0 \in [0,1]$ so that $\inf\limits_{x\in[0,1]} f(x) = f(x_0) > 0$. Take $\varepsilon = \dfrac{f(x_0)}{2}$. Now if g is such that $\|g - f\|_\infty = \sup\limits_{x\in[0,1]} \big|g(x) - f(x)\big| < \varepsilon = \dfrac{f(x_0)}{2}$, we have for any $y \in [0,1]$,

$$
\begin{aligned}
&\big|g(y) - f(y)\big| \leqslant \sup_{x\in[0,1]} \big|g(x) - f(x)\big| < \frac{f(x_0)}{2} \\
\Rightarrow\ & f(y) - \frac{f(x_0)}{2} \leqslant g(y) \leqslant f(y) + \frac{f(x_0)}{2} \\
\Rightarrow\ & g(y) \geqslant f(y) - \frac{f(x_0)}{2} \geqslant f(x_0) - \frac{f(x_0)}{2} = \frac{f(x_0)}{2} > 0\,.
\end{aligned}
$$

Therefore, $g \in B$; thus $D(f,\varepsilon) \subseteq B$.

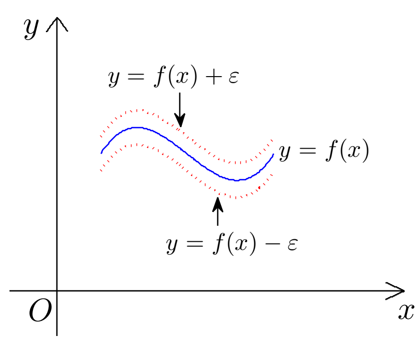


Figure 5.2: $g \in D(f,\varepsilon)$ if the graph of g lies in between the two red dash lines

Example 5.51. Find the closure of B given in the previous example.

Proof. Claim: $\mathrm{cl}(B) = \big\{f \in \mathscr{C}([0,1],\mathbb{R}) \,\big|\, f(x) \geqslant 0\big\}$.

Proof of claim: We show $\forall\, f \in \big\{f \in \mathscr{C}([0,1],\mathbb{R}) \,\big|\, f(x) \geqslant 0\big\}$, $\exists\, f_k \in B \ni \|f_k - f\|_\infty \to 0$ as $k \to \infty$. Take $f_k(x) = f(x) + \dfrac{1}{k}$, then $f_k \in B$ ($\because f_k(x) > 0$), and

$$
\|f_k - f\|_\infty = \sup_{x\in[0,1]} \big|f_k(x) - f(x)\big| \leqslant \sup_{x\in[0,1]} \frac{1}{k} = \frac{1}{k} \to 0 \text{ as } k \to \infty\,. \qquad \square
$$

5.5 The Arzelà-Ascoli Theorem

5.5.1 Equi-continuous family of functions

The first part of this section is devoted to the investigation of the difference between the pointwise convergence and the uniform convergence of sequence of functions.

Definition 5.52. Let (M, d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a normed vector space, and $A \subseteq M$ be a subset. A subset $B \subseteq \mathscr{C}_b(A; \mathcal{V})$ is said to be ***equi-continuous*** if

$$\forall\, \varepsilon > 0, \exists\, \delta > 0 \ni \|f(x_1) - f(x_2)\| < \varepsilon \quad \text{whenever } d(x_1, x_2) < \delta,\ x_1, x_2 \in A, \text{ and } f \in B\,.$$

Remark 5.53. 1. If $B \subseteq \mathscr{C}_b(A; \mathcal{V})$ is equi-continuous, and B' is a subset of B, then B' is also equi-continuous.

2. In an equi-continuous set of functions B, every $f \in B$ is uniformly continuous.

Remark 5.54. For a uniformly continuous function f, let $\delta_f(\varepsilon)$ (we have defined this number in Remark 4.51) denote the largest δ that can be used in the definition of the uniform continuity; that is, $\delta_f(\varepsilon)$ has the property that

$$\|f(x) - f(y)\| < \varepsilon \text{ whenever } d(x, y) < \delta, x, y \in A \quad \Leftrightarrow \quad 0 < \delta \leqslant \delta_f(\varepsilon)\,.$$

Suppose that every element in $B \subseteq \mathscr{C}_b(A; \mathcal{V})$ is uniformly continuous on A. Then B is equi-continuous if and only if $\inf\limits_{f \in B} \delta_f(\varepsilon) > 0$.

Example 5.55. Let $B = \big\{f \in \mathscr{C}_b((0,1); \mathcal{V}) \,\big|\, |f'(x)| \leqslant 1 \text{ for all } x \in (0,1)\big\}$. Then B is equi-continuous (by choosing $\delta = \epsilon$ for any given ϵ, and applying the mean value theorem).

Example 5.56. Let $f_k : [0,1] \to \mathbb{R}$ be a sequence of functions given by

$$f_k(x) = \begin{cases} kx & \text{if } 0 \leqslant x \leqslant \dfrac{1}{k}\,, \\ 2 - kx & \text{if } \dfrac{1}{k} \leqslant x \leqslant \dfrac{2}{k}\,, \\ 0 & \text{if } x \geqslant \dfrac{2}{k}\,, \end{cases}$$

and $B = \{f_k\}_{k=1}^{\infty}$. Then B is not equi-continuous since the largest δ for each k is $\dfrac{\varepsilon}{k}$ which converges to 0.

Lemma 5.57. *Let (M, d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a normed vector space, and $K \subseteq M$ be a compact subset. If $B \subseteq \mathscr{C}(K;\mathcal{V})$ is pre-compact, then B is equi-continuous.*

Proof. Suppose the contrary that B is not equi-continuous. Then $\exists\, \varepsilon > 0$ such that

$$\forall\, k \in \mathbb{N}, \exists\, x_k, y_k \in K \text{ and } f_k \in B \ni d(x_k, y_k) < \frac{1}{k} \text{ but } \|f_k(x_k) - f_k(y_k)\| \geqslant \varepsilon\,.$$

Since B is pre-compact in $\big(\mathscr{C}(K;\mathcal{V}), \|\cdot\|_\infty\big)$ and K is compact in (M, d), there exists a subsequence $\big\{f_{k_j}\big\}_{j=1}^\infty$ and $\{x_{k_j}\}_{j=1}^\infty$ such that $\big\{f_{k_j}\big\}_{j=1}^\infty$ converges uniformly to some function $f \in \big(\mathscr{C}(K;\mathcal{V}), \|\cdot\|_\infty\big)$ and $\{x_{k_j}\}_{j=1}^\infty$ converges to some $a \in K$. We must also have $\{y_{k_j}\}_{j=1}^\infty$ converges to a since $d(x_{k_j}, y_{k_j}) < \dfrac{1}{k_j}$.

Since f is continuous at a,

$$\exists\, \delta > 0 \ni \|f(x) - f(a)\| < \frac{\varepsilon}{5} \qquad \text{if } x \in D(a, \delta) \cap K.$$

Moreover, since $\big\{f_{k_j}\big\}_{j=1}^\infty$ converges to f uniformly on K and $x_{k_j}, y_{k_j} \to a$ as $j \to \infty$, $\exists\, N > 0$ such that

$$\|f_{k_j}(x) - f(x)\| < \frac{\varepsilon}{5} \qquad \text{if } j \geqslant N \text{ and } x \in K$$

and

$$\|x_{k_j} - a\| < \delta \quad \text{and} \quad \|y_{k_j} - a\| < \delta \qquad \text{if } j \geqslant N\,.$$

As a consequence, for all $j \geqslant N$,

$$\begin{aligned}\varepsilon \leqslant \|f_{k_j}(x_{k_j}) - f_{k_j}(y_{k_j})\| &\leqslant \|f_{k_j}(x_{k_j}) - f(x_{k_j})\| + \|f(x_{k_j}) - f(a)\| \\ &\quad + \|f(y_{k_j}) - f(a)\| + \|f(y_{k_j}) - f_{k_j}(y_{k_j})\| < \frac{4\varepsilon}{5}\end{aligned}$$

which is a contradiction. □

Alternative proof of Lemma 5.57. Suppose the contrary that B is not equi-continuous. Then $\exists\, \varepsilon > 0$ such that

$$\forall\, k \in \mathbb{N}, \exists\, x_k, y_k \in K \text{ and } f_k \in B \ni d(x_k, y_k) < \frac{1}{k} \text{ but } \|f_k(x_k) - f_k(y_k)\| \geqslant \varepsilon\,.$$

Since B is pre-compact in $\big(\mathscr{C}(K;\mathcal{V}), \|\cdot\|_\infty\big)$, there exists a subsequence $\big\{f_{k_j}\big\}_{j=1}^\infty$ converges to some function f in $\big(\mathscr{C}(K;\mathcal{V}), \|\cdot\|_\infty\big)$. By Proposition 5.47, $\big\{f_{k_j}\big\}_{j=1}^\infty$ converges uniformly to f on K; thus there exists $N_1 > 0$ such that

$$\big\|f_{k_j}(x) - f(x)\big\| < \frac{\varepsilon}{4} \quad \forall\, j \geqslant N_1 \text{ and } x \in K\,.$$

Since $f \in \mathscr{C}(K;\mathcal{V})$, by Theorem 4.52, f is uniformly continuous on K; thus

$$\exists\, \delta > 0 \ni \|f(x) - f(y)\| < \frac{\varepsilon}{4} \quad \text{if } d(x,y) < \delta \text{ and } x, y \in K\,.$$

For one of such a δ, there exists $N_2 > 0$ such that

$$d(x_k, y_k) < \delta \quad \forall\, k \geqslant N_2\,.$$

Therefore, $d(x_{k_j}, y_{k_j}) < \delta$ if $j \geqslant N_2$ (this is because $k_j \geqslant j$ for all $j \in \mathbb{N}$); thus for all $j \geqslant \max\{N_1, N_2\}$,

$$\varepsilon \leqslant \|f_{k_j}(x_{k_j}) - f_{k_j}(y_{k_j})\| \leqslant \|f_{k_j}(x_{k_j}) - f(x_{k_j})\| + \|f(x_{k_j}) - f(y_{k_j})\| + \|f(y_{k_j}) - f_{k_j}(y_{k_j})\| < \frac{3\varepsilon}{4}$$

which is a contradiction. □

Corollary 5.58. *Let (M,d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a normed vector space, and $K \subseteq M$ be a compact subset. If $\{f_k\}_{k=1}^\infty$ converges uniformly on K, then $\{f_k\}_{k=1}^\infty$ is equi-continuous.*

Example 5.59. Corollary 5.58 fails to hold if the compactness of K is removed. For example, let $\{f_k\}_{k=1}^\infty$ be a sequence of identical functions $f_k(x) = \dfrac{1}{x}$ on $(0,1)$. Then $\{f_k\}_{k=1}^\infty$ converges uniformly on $(0,1)$ but $\{f_k\}_{k=1}^\infty$ is not equi-continuous since none of f_k is uniformly continuous on $(0,1)$ which violates Remark 5.53.

We have just shown that if $\{f_k\}_{k=1}^\infty$ converges uniformly on a compact set K, then $\{f_k\}_{k=1}^\infty$ must be equi-continuous. The inverse statement, on the other hand, cannot be true. For example, taking $\{f_k\}_{k=1}^\infty$ to be a sequence of constant functions $f_k(x) = k$. Then $\{f_k\}_{k=1}^\infty$ obviously does not converge, not even any subsequence. Therefore, we would like to study under what additional conditions, equi-continuity of a sequence of functions (defined on a compact set K) indeed converges uniformly. The following lemma is an answer to the question.

Lemma 5.60. *Let (M,d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a Banach space, $K \subseteq M$ be a compact set, and $\{f_k\}_{k=1}^\infty \subseteq \mathscr{C}(K;\mathcal{V})$ be a sequence of equi-continuous functions. If $\{f_k\}_{k=1}^\infty$ converges pointwise on a dense subset E of K (that is, $E \subseteq K \subseteq \mathrm{cl}(E)$), then $\{f_k\}_{k=1}^\infty$ converges uniformly on K.*

Proof. Let $\varepsilon > 0$ be given. By the equi-continuity of $\{f_k\}_{k=1}^\infty$,

$$\exists\, \delta > 0 \ni \|f_k(x) - f_k(y)\| < \frac{\varepsilon}{3} \quad \text{if } d(x,y) < \delta,\ x, y \in K \text{ and } k \in \mathbb{N}\,.$$

Since K is compact, K is totally bounded; thus

$$\exists \{y_1, \cdots, y_m\} \subseteq K \ni K \subseteq \bigcup_{j=1}^{m} D\big(y_j, \frac{\delta}{2}\big).$$

By the denseness of E in K, for each $j = 1, \cdots, m$, $\exists\, z_j \in E$ such that $d(z_j, y_j) < \dfrac{\delta}{2}$. Moreover, $D\big(y_j, \dfrac{\delta}{2}\big) \subseteq D(z_j, \delta)$; thus $K \subseteq \bigcup\limits_{j=1}^{m} D(z_j, \delta)$. Since $\{f_k\}_{k=1}^\infty$ converges pointwise on E, $\{f_k(z_j)\}_{k=1}^\infty$ converges as $k \to \infty$ for all $j = 1, \cdots, m$. Therefore,

$$\exists\, N_j > 0 \ni \|f_k(z_j) - f_\ell(z_j)\| < \frac{\varepsilon}{3} \qquad \forall\, k, \ell \geqslant N_j.$$

Let $N = \max\{N_1, \cdots, N_m\}$, then

$$\|f_k(z_j) - f_\ell(z_j)\| < \frac{\varepsilon}{3} \qquad \forall\, k, \ell \geqslant N \text{ and } j = 1, \cdots, m.$$

Now we are in the position of concluding the lemma. If $x \in K$, there exists $z_j \in E$ such that $d(x, z_j) < \delta$; thus if we further assume that $k, \ell \geqslant N$,

$$\|f_k(x) - f_\ell(x)\| \leqslant \|f_k(x) - f_k(z_j)\| + \|f_k(z_j) - f_\ell(z_j)\| + \|f_\ell(z_j) - f_\ell(x)\| < \varepsilon.$$

By Proposition 5.6, $\{f_k\}_{k=1}^\infty$ converges uniformly on K. □

Remark 5.61. Corollary 5.58 and Lemma 5.60 suggest that "a sequence $\{f_k\}_{k=1}^\infty \subseteq \mathscr{C}(K; \mathcal{V})$ converges uniformly on K if and only if $\{f_k\}_{k=1}^\infty$ is equi-continuous and pointwise convergent (on a dense subset of K)".

5.5.2 Compact sets in $\mathscr{C}(K; \mathcal{V})$

The next subject in this section is to obtain a (useful) criterion of determining the compactness (or pre-compactness) of a subset $B \subseteq \mathscr{C}(K; \mathcal{V})$ which guarantees the existence of a convergent subsequence $\{f_{k_j}\}_{j=1}^\infty$ of a given sequence $\{f_k\}_{k=1}^\infty \subseteq B$ in $\big(\mathscr{C}(K; \mathcal{V}), \|\cdot\|_\infty\big)$.

Lemma 5.62 (Diagonal Process)**.** *Let E be a countable set, $(\mathcal{V}, \|\cdot\|)$ be a Banach space, and $f_k : E \to \mathcal{V}$ be a sequence of functions. Suppose that for each $x \in E$, $\{f_k(x)\}_{k=1}^\infty$ is pre-compact in $\mathcal{V}$. Then there exists a subsequence of $\{f_k\}_{k=1}^\infty$ that converges pointwise on E.*

Proof. Since E is countable, $E = \{x_\ell\}_{\ell=1}^\infty$.

1. Since $\{f_k(x_1)\}_{k=1}^{\infty}$ is pre-compact in $(\mathcal{V}, \|\cdot\|)$, there exists a subsequence $\{f_{k_j}\}_{j=1}^{\infty}$ such that $\{f_{k_j}(x_1)\}_{j=1}^{\infty}$ converges in $(\mathcal{V}, \|\cdot\|)$.

2. Since $\{f_k(x_2)\}_{k=1}^{\infty}$ is pre-compact in $(\mathcal{V}, \|\cdot\|)$, the sequence $\{f_{k_j}(x_2)\}_{j=1}^{\infty} \subseteq \{f_k(x_2)\}_{k=1}^{\infty}$ has a convergent subsequence $\{f_{k_{j_\ell}}(x_2)\}_{\ell=1}^{\infty}$.

Continuing this process, we obtain a sequence of sequences S_1, S_2, $\cdots$ such that

1. S_k consists of a subsequence of $\{f_k\}_{k=1}^{\infty}$ which converges at x_k, and

2. $S_k \supseteq S_{k+1}$ for all $k \in \mathbb{N}$.

Let g_k be the k-th element of S_k. Then the sequence $\{g_k\}_{k=1}^{\infty}$ is a subsequence of $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ converges at each point of E. □

The condition that "$\{f_k(x)\}_{k=1}^{\infty}$ is pre-compact in $\mathcal{V}$ for each $x \in E$" in Lemma 5.62 motivates the following

Definition 5.63. Let (M, d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a normed vector space, and $A \subseteq M$ be a subset. A subset $B \subseteq \mathscr{C}_b(A;\mathcal{V})$ is said to be ***pointwise*** $\begin{array}{c}\textbf{\textit{compact}}\\ \textbf{\textit{pre-compact}}\\ \textbf{\textit{bounded}}\end{array}$ if the set $B_x \equiv \{f(x) \,|\, f \in B\}$ is $\begin{array}{c}\text{compact}\\ \text{pre-compact}\\ \text{bounded}\end{array}$ in $(\mathcal{V}, \|\cdot\|)$ for all $x \in A$.

Example 5.64. Let $f_k : [0,1] \to \mathbb{R}$ be given in Example 5.56, and $B = \{f_k\}_{k=1}^{\infty}$. Then B is pointwise compact: for each $x \in [0,1]$, B_x is a finite set since if $f_k(0) = 0$ for all $k \in \mathbb{N}$, and if $x > 0$, $f_k(x) = 0$ for all k large enough which implies that $\#B_x < \infty$.

是時候可以來看 $\mathscr{C}(K;\mathcal{V})$ 裡面的 compact sets 有什麼等價條件了。首先我們先看何時 $B \subseteq \mathscr{C}(K;\mathcal{V})$ 是 compact set。給定一個函數列 $\{f_k\}_{k=1}^{\infty} \subseteq B$，我們想知道能不能找到一個在 sup-norm 下收斂的 subsequence $\{f_{k_j}\}_{j=1}^{\infty}$（即 sequentially compact）。由 Diagonal Process (Lemma 5.62) 知，我們得在 K 中找一個稠密的子集合 E 使得 $\{f_k\}_{k=1}^{\infty}$ 在 E 上是 pointwise pre-compact（這個部份只保證了可以找到 subsequence 逐點收斂），然後加上 Lemma 5.60 的幫助，馬上知道加上 equi-continuity 的條件之後，逐點收斂會變均勻收斂。因此，很自然地我們會要求 B 滿足 pointwise pre-compact 還有 equi-continuous 這兩個條件來證出 B 是 $\mathscr{C}(K;\mathcal{V})$ 中的 compact set。而在一個 compact set K 中能不能找到一個稠密子集合則是由下面這個 Lemma 所提供。

Lemma 5.65. *A compact set K in a metric space (M, d) is separable; that is, there exists a countable subset E of K such that* $\mathrm{cl}(E) = K$.

Proof. Since K is compact, K is totally bounded; thus $\forall\, n \in \mathbb{N}$, $\exists\, E_n \subseteq K$ such that

$$\#E_n < \infty \quad \text{and} \quad K \subseteq \bigcup_{y \in E_n} D\big(y, \frac{1}{n}\big)\,.$$

Let $E = \bigcup_{n=1}^{\infty} E_n$. Then E is countable by Theorem 1.42. We claim that $\mathrm{cl}(E) = K$.

To see this, first by the definition of the closure of a set, $\mathrm{cl}(E) \subseteq K$ (since K is closed). Let $x \in K$. Since $K \subseteq \bigcup_{y \in E_n} D\big(y, \frac{1}{n}\big)$, $x \in D\big(y, \frac{1}{n}\big)$ for some $y \in E_n$. Therefore, $D\big(x, \frac{1}{n}\big) \cap E \neq \varnothing$ for all $n \in \mathbb{N}$. This implies that $x \in \bar{E} = \mathrm{cl}(E)$. ▫

Theorem 5.66. *Let (M, d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a Banach space, $K \subseteq M$ be a compact set, and $B \subseteq \mathscr{C}(K; \mathcal{V})$ be equi-continuous and pointwise pre-compact. Then B is pre-compact in $\big(\mathscr{C}(K; \mathcal{V}), \|\cdot\|_\infty\big)$.*

Proof. We show that every sequence $\{f_k\}_{k=1}^{\infty}$ in B has a convergent subsequence. Since K is compact, there is a countable dense subset E of K (Lemma 5.65), and the diagonal process (Lemma 5.62) suggests that there exists $\big\{f_{k_j}\big\}_{j=1}^{\infty}$ that converges pointwise on E. Since E is dense in K, by Lemma 5.60 $\big\{f_{k_j}\big\}_{j=1}^{\infty}$ converges uniformly on K; thus $\big\{f_{k_j}\big\}_{j=1}^{\infty}$ converges in $\big(\mathscr{C}(K; \mathcal{V}), \|\cdot\|_\infty\big)$ by Proposition 5.47. ▫

Remark 5.67. Lemma 5.57 and Theorem 5.66 suggest that "a set $B \subseteq \mathscr{C}(K; \mathcal{V})$ is pre-compact if and only if B is equi-continuous and pointwise pre-compact". (That B is pre-compact implies that B is pointwise pre-compact is left as an exercise).

Corollary 5.68. *Let (M, d) be a metric space, and $K \subseteq M$ be a compact set. Assume that $B \subseteq \mathscr{C}(K; \mathbb{R})$ is equi-continuous and pointwise bounded on K. Then every sequence in B has a uniformly convergent subsequence.*

Proof. By the Bolzano-Weierstrass theorem the boundedness of $\big\{f_k(x)\big\}_{k=1}^{\infty}$ suggests that $\big\{f_k(x)\big\}_{k=1}^{\infty}$ is pre-compact for all $x \in E$. Therefore, we can apply Theorem 5.66 under the setting $(\mathcal{V}, \|\cdot\|) = (\mathbb{R}, |\cdot|)$ to conclude the corollary. ▫

The following theorem provides how compact sets look like in $\mathscr{C}(K; \mathcal{V})$.

Theorem 5.69 (The Arzelà-Ascoli Theorem)**.** *Let (M, d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a Banach space, $K \subseteq M$ be a compact set, and $B \subseteq \mathscr{C}(K;\mathcal{V})$. Then B is compact in $\big(\mathscr{C}(K;\mathcal{V}), \|\cdot\|_\infty\big)$ if and only if B is closed, equi-continuous, and pointwise compact.*

Proof. "$\Leftarrow$" This direction is conclude by Theorem 5.66 and the fact that B is closed.

"$\Rightarrow$" By Lemma 3.10 and Lemma 5.57, it suffices to shows that B is pointwise compact. Let $x \in K$ and $\big\{f_k(x)\big\}_{k=1}^\infty$ be a sequence in B_x. Since B is compact, there exists a subsequence $\big\{f_{k_j}\big\}_{j=1}^\infty$ that converges uniformly to some function $f \in B$. In particular, $\big\{f_{k_j}(x)\big\}_{j=1}^\infty$ converges to $f(x) \in B_x$. In other words, we find a subsequence $\big\{f_{k_j}(x)\big\}_{j=1}^\infty$ of $\big\{f_k(x)\big\}_{k=1}^\infty$ that converges to a point in B_x. This implies that B_x is sequentially compact; thus B_x is compact. □

Example 5.70. Let $f_k : [0,1] \to \mathbb{R}$ be a sequence of functions such that

(1) $|f_k(x)| \leqslant M_1$ for all $k \in \mathbb{N}$ and $x \in [0,1]$; (2) $|f_k'(x)| \leqslant M_2$ for all $k \in \mathbb{N}$ and $x \in [0,1]$.

Then $\{f_k\}_{k=1}^\infty$ is clearly pointwise bounded. Moreover, by the mean value theorem

$$\big|f_k(x) - f_k(y)\big| \leqslant M_2|x-y| \qquad \forall\, x, y \in [0,1], k \in \mathbb{N}$$

which suggests that $\{f_k\}_{k=1}^\infty$ is equi-continuous. Therefore, by Corollary 5.68 there exists a subsequence $\big\{f_{k_j}\big\}_{j=1}^\infty$ that converges uniformly on $[0,1]$.

Question: If assumption (1) of Example 5.70 is omitted, can $\{f_k\}_{k=1}^\infty$ still have a convergent subsequence?

Answer: No! Take $f_k(x) = k$, then $\{f_k\}_{k=1}^\infty$ does not have a convergent subsequence (note that f_k is continuous and $f_k'(x) = 0$; that is, Assumptions (1) (2) are fulfilled).

Example 5.71. We show that Assumption (1) of Example 5.70 can be replaced by $f_k(0) = 0$ for all $k \in \mathbb{N}$.

Proof. (a) If $f_n(0) = 0$, then by the mean value theorem we have for all $x \in (0,1]$ and $k \in \mathbb{N}$, $f_k(x) - f_k(0) = f_k'(c_k)(x-0)$. Then Assumption (2) of Example 5.70 suggests that

$$\big|f_k(x) - f_k(0)\big| = \big|f_k'(c_k)\big|\big|x\big| \leqslant M_2|x| \leqslant M_2$$

which suggests that $\{f_k\}_{k=1}^\infty$ is uniformly bounded by M_2.

(b) $\{f_k\}_{k=1}^\infty$ are equi-continuous (same proof as in Example 5.70). □

5.6 The Contraction Mapping Principle（收縮映射原理）and its Applications

Definition 5.72. Let (M,d) be a metric space, and $\Phi : M \to M$ be a mapping. Φ is said to be a ***contraction mapping*** if there exists a constant $k \in [0,1)$ such that

$$d\big(\Phi(x),\Phi(y)\big) \leqslant k d(x,y) \qquad \forall\, x,y \in M\,.$$

Remark 5.73. A contraction mapping must be (uniformly) continuous.
Reason: Given $\varepsilon > 0$, take $\delta = \dfrac{\varepsilon}{k}$, where k is set as in the definition of contraction. Now if $d(x,y) < \delta$, then

$$d\big(\Phi(x),\Phi(y)\big) \leqslant k d(x,y) < k\cdot\frac{\varepsilon}{k} = \varepsilon\,.$$

Example 5.74. For what $r < 1$ do we have $f : [0,r] \to [0,r]$ where $f(x) = x^2$ a contraction?
Answer: By the mean value theorem, $f(x) - f(y) = f'(c)(x-y)$, c between x,y; thus

$$\big|f(x)-f(y)\big| = \big|f'(c)\big||x-y| = 2c|x-y| < 2r|x-y| < 2R|x-y|\,.$$

Hence $\forall\, r < R < \dfrac{1}{2}$, the map $f : [0,r] \to [0,r]$ is a contraction where $f(x) = x^2$.

On the other hand, suppose $\exists\, k \in [0,1) \ni \forall\, x,y \in \big[0,\frac{1}{2}\big]$, $\big|x^2-y^2\big| \leqslant k\big|x-y\big|$, then

$$\sup_{x\neq y, x,y\in[0,\frac{1}{2}]} \frac{\big|x^2-y^2\big|}{|x-y|} \leqslant k < 1\,.$$

But we can take $x = \dfrac{1}{2}$, $y_n = \dfrac{1}{2} - \dfrac{1}{n}$, $n = 1,2,\cdots, x, y_n \in \big[0,\frac{1}{2}\big]$. So

$$\lim_{n\to\infty} \frac{\big|x^2-{y_n}^2\big|}{\big|x-y_n\big|} = \lim_{n\to\infty}\big|x+y_n\big| = \lim_{n\to\infty}\Big(\frac{1}{2}+\frac{1}{2}-\frac{1}{n}\Big) = 1\,.$$

This means $\displaystyle\sup_{x\neq y, x,y\in[0,\frac{1}{2}]} \frac{\big|x^2-y^2\big|}{|x-y|} < 1$ is not possible.

Definition 5.75. Let (M,d) be a metric space, and $\Phi : M \to M$ be a mapping. A point $x_0 \in M$ is called a ***fixed-point*** for Φ if $\Phi(x_0) = x_0$.

Example 5.76. Let $\Phi : \mathbb{R} \to \mathbb{R}$ be given by $\Phi(x) = \dfrac{x^2+2}{3}$. Then 1 is a fixed-point, and 2 is also a fixed-point.

Theorem 5.77 (Contraction Mapping Principle)**.** *Let (M,d) be a complete metric space, and $\Phi : M \to M$ be a contraction mapping. Then Φ has a unique fixed-point.*

Proof. Let $x_0 \in M$, and define $x_{n+1} = \Phi(x_n)$ for all $n \in \mathbb{N} \cup \{0\}$. Then

$$d(x_{n+1}, x_n) = d\big(\Phi(x_n), \Phi(x_{n-1})\big) \leqslant kd(x_n, x_{n-1}) \leqslant k^n d(x_1, x_0)\,;$$

thus if $n > m$,

$$\begin{aligned} d(x_n, x_m) &\leqslant d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ &\leqslant (k^m + k^{m+1} + \cdots + k^{n-1}) d(x_1, x_0) \\ &\leqslant k^m(1 + k + k^2 + \cdots) d(x_1, x_0) = \frac{k^m}{1-k} d(x_1, x_0)\,. \end{aligned} \tag{5.6.1}$$

Since $k \in [0,1)$, $\lim\limits_{m\to\infty} \dfrac{k^m}{1-k} d(x_1, x_0) = 0$; thus

$$\forall\, \varepsilon > 0, \exists\, N > 0 \ni d(x_n, x_m) < \varepsilon \quad \forall\, n, m \geqslant N\,.$$

In other words, $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence. Since (M,d) is complete, $x_n \to x$ as $n \to \infty$ for some $x \in M$. Finally, since $\Phi(x_n) = x_{n+1}$ for all $n \in \mathbb{N}$, by the continuity of Φ we obtain that

$$\Phi(x) = \lim_{n\to\infty} \Phi(x_n) = \lim_{n\to\infty} x_{n+1} = x$$

which guarantees the existence of a fixed-point.

Suppose that for some $x, y \in M$, $\Phi(x) = x$ and $\Phi(y) = y$. Then

$$d(x,y) = d\big(\Phi(x), \Phi(y)\big) \leqslant kd(x,y)$$

which suggests that $d(x,y) = 0$ or $x = y$. Therefore, the fixed-point of Φ is unique. □

Remark 5.78. The proof of the contraction mapping principle also suggests an iterative way, $x_{k+1} = \Phi(x_k)$, of finding the fixed-point of a contraction mapping Φ. Using (5.6.1), the convergence rate of $\{x_m\}_{m=1}^\infty$ to the fixed-point x is measured by

$$d(x_m, x) = \lim_{n\to\infty} d(x_m, x_n) \leqslant \frac{k^m}{1-k} d(x_1, x_0)\,.$$

Therefore, the smaller the contraction constant k, the faster the convergence.

Remark 5.79. Theorem 5.77 sometimes is also called the ***Banach fixed-point theorem***.

Example 5.80. The condition $k < 1$ in Theorem 5.77 is necessary. For example, let $M = \mathbb{R}$, $d(x,y) = |x-y|$, and $\Phi : \mathbb{R} \to \mathbb{R}$ be given by $\Phi(x) = x+1$. Then $\big|\Phi(x) - \Phi(y)\big| = \big|x-y\big|$. Suppose $x_\star$ is a fixed-point of Φ. Then $x_\star = \Phi(x_\star) = x_\star + 1$ which leads to a contradiction that $0 = 1$.

Example 5.81. Let $\Phi : [1,\infty) \to [1,\infty)$ be given by $\Phi(x) = x + \dfrac{1}{x}$. Then if $x \neq y$,

$$\big|\Phi(x) - \Phi(y)\big| = \big|x - y + \frac{1}{x} - \frac{1}{y}\big| = \big|(x-y)\big(1 - \frac{1}{xy}\big)\big| < |x-y|\,.$$

However, there is no fixed-point of Φ.

Example 5.82 (The secant method)**.** Suppose that f is continuously differentiable, $f'(x) > 0$ for all $x \in [a,b]$ and $f(a)f(b) < 0$. By the intermediate value theorem there must be a (unique) zero of f. How do we find this zero?

Assume that $\sup\limits_{x\in[a,b]} f'(x) < \infty$. Let

$$M = \max\Big\{\sup_{x\in[a,b]} f'(x),\ -\frac{f(a)}{b-a}, \frac{f(b)}{b-a}\Big\} + 1$$

be a positive constant, and consider $\Phi(x) = x - \dfrac{f(x)}{M}$. Then by the mean value theorem,

$$\big|\Phi(x) - \Phi(y)\big| = \big|(x-y)(1 - \frac{f'(\xi)}{M})\big| \leqslant \big(1 - \frac{\min_{\xi\in[a,b]} f'(\xi)}{M}\big)|x-y| \leqslant k|x-y|\,,$$

where $k \in [0,1)$ is a fixed constant. Moreover, $\Phi'(x) = 1 - \dfrac{f'(x)}{M} > 0$; thus Φ is strictly increasing. Since the choice of M implies that $a < \Phi(a) < \Phi(b) < b$; thus $\Phi : [a,b] \to [a,b]$. Therefore, the contraction mapping principle suggests that one can find the fixed-point of Φ (which is the zero of f) using the iterative scheme $x_{k+1} = \Phi(x_k)$ (by picking any arbitrary initial guess $x_0 \in [a,b]$).

5.6.1 The existence and uniqueness of the solution to ODEs

In this sub-section we are concerned with if there is a solution to the initial value problem of ordinary differential equation:

$$x'(t) = f\big(x(t),t\big) \qquad \forall\, t \in [t_0, t_0 + \Delta t], \tag{5.6.2a}$$

$$x(t_0) = x_0\,, \tag{5.6.2b}$$

where $x : [t_0, t_0 + \Delta t] \to \mathbb{R}^n$ and $f : \mathbb{R}^n \times [t_0, t_0 + \Delta t] \to \mathbb{R}^n$ are vector-valued functions, and $x_0 \in \mathbb{R}^n$ is a vector. Another question we would like to answer is "if (5.6.2) indeed has a solution, is the solution unique?"

Theorem 5.83 (Fundamental Theorem of ODE)**.** *Suppose that for some $r > 0$, $f : D(x_0, r) \times [t_0, T] \to \mathbb{R}^n$ is continuous and is Lipschitz in the spatial variable; that is,*

$$\exists K > 0 \ni \big\|f(x,t) - f(y,t)\big\|_2 \leqslant K\|x-y\|_2 \qquad \forall\, x, y \in D(x_0, r) \textit{ and } t \in [t_0, T]\,.$$

Then there exists $0 < \Delta t \leqslant T - t_0$ such that there exists a unique solution to (5.6.2).

Proof. For any $x \in \mathscr{C}([t_0, T]; \mathbb{R}^n)$, define

$$\Phi(x)(t) = x_0 + \int_{t_0}^{t} f\big(x(s), s\big) ds\,.$$

We note that if $x(t)$ is a solution to (5.6.2), then x is a fixed point of Φ (for $t \in [t_0, t_0 + \Delta t]$). Therefore, the problem of finding a solution to (5.6.2) transforms to a problem of finding a fixed-point of Φ.

To guarantee the existence of a unique fixed-point, we appeal to the contraction mapping principle. To be able to apply the contraction mapping principle, we need to specify the metric space (M, d). Let

$$\Delta t = \min\Big\{T - t_0, \frac{r}{Kr + 2\|f(x_0, \cdot)\|_\infty}, \frac{1}{2K}\Big\}\,, \tag{5.6.3}$$

and define

$$M = \Big\{x \in \mathscr{C}\big([t_0, t_0 + \Delta t]; \mathbb{R}^n\big) \,\Big|\, \|x - x_0\|_\infty \leqslant \frac{r}{2}\Big\}$$

with the metric induced by the sup-norm $\|\cdot\|_\infty$ of $\mathscr{C}\big([t_0, t_0 + \Delta t]; \mathbb{R}^n\big)$. Then

1. We first show that $\Phi : M \to M$. To see this, we observe that

$$\begin{aligned}
\big\|\Phi(x) - x_0\big\|_\infty &= \Big\|\int_{t_0}^{t} f\big(x(s), s\big) ds\Big\|_\infty = \Big\|\int_{t_0}^{t} \big[f\big(x(s), s\big) - f(x_0, s)\big] ds + \int_{t_0}^{t} f(x_0, s) ds\Big\|_\infty \\
&\leqslant \int_{t_0}^{t_0+\Delta t} \big\|f\big(x(s), s\big) - f(x_0, s)\big\|_2 ds + \int_{t_0}^{t_0+\Delta t} \big\|f(x_0, s)\big\|_2 ds \\
&\leqslant K \int_{t_0}^{t_0+\Delta t} \|x(s) - x_0\|_2 ds + \Delta t \big\|f(x_0, \cdot)\big\|_\infty \\
&\leqslant \Delta t \Big[K\|x - x_0\|_\infty + \big\|f(x_0, \cdot)\big\|_\infty\Big]\,;
\end{aligned}$$

thus if $x \in M$, (5.6.3) implies that $\|\Phi(x) - x_0\|_\infty \leqslant \dfrac{r}{2}$.

2. Next we show that Φ is a contraction mapping. To see this, we compute $\big\|\Phi(x)-\Phi(y)\big\|_\infty$ for $x, y \in M$ and find that
$$\begin{aligned}\big\|\Phi(x)-\Phi(y)\big\|_\infty &\leqslant \Big\|\int_{t_0}^{t}\big[f\big(x(s),s\big)-f\big(y(s),s\big)\big]ds\Big\|_\infty \\ &\leqslant \int_{t_0}^{t_0+\Delta t} K\|x(s)-y(s)\|_2 ds \leqslant K\Delta t\|x-y\|_\infty \leqslant \frac{1}{2}\|x-y\|_\infty\,;\end{aligned}$$
thus $\Phi : M \to M$ is a contraction mapping.

3. Finally we show that (M, d) is complete. It suffices to show that M is a closed subset of $\mathscr{C}([t_0, t_0 + \Delta t]; \mathbb{R}^n)$. Let $\{x_k\}_{k=1}^\infty$ be a uniformly convergent sequence with limit x. Since $\|x_k(t) - x_0\|_2 \leqslant \dfrac{r}{2}$ for all $t \in [t_0, t_0 + \Delta t]$, passing k to the limit we find that $\|x(t) - x_0\|_2 \leqslant \dfrac{r}{2}$ for all $t \in [t_0, t_0 + \Delta t]$ which implies that $\|x - x_0\|_\infty \leqslant \dfrac{r}{2}$; thus $x \in M$.

Therefore, by the contraction mapping principle, there exists a unique fixed point $x \in M$ which suggests that there exists a unique solution to (5.6.2). □

Example 5.84. Let
$$x_c(t) = \begin{cases} 0 & \text{if } 0 \leqslant t < c\,, \\ \dfrac{1}{4}(t-c)^2 & \text{if } t \geqslant c\,. \end{cases}$$
Then for all $c > 0$, $x_c(t)$ is a solution to $x'(t) = x(t)^{\frac{1}{2}}$ for all $t > 0$ with initial value $x(0) = 0$. The reason for not having unique solution is that if $f(x,t) = \sqrt{x}$, $f : D(0,r) \times \mathbb{R} \to \mathbb{R}$ is not Lipschitz in the spatial variabvle for all $r > 0$. In other words, for all $r, K > 0$, there exists $x, y \in D(0,r)$ satisfying $\big|f(x) - f(y)\big| \geqslant K|x-y|$.

Example 5.85. Find a function $x(t)$ satisfying $x'(t) = x(t)$ with initial value $x(0) = 1$. Define $\Phi(x)(t) = 1 + \displaystyle\int_0^t x(s)ds$, $x_0(t) = 1$ and $x_{k+1}(t) = \Phi(x_k)(t)$. Then
$$\begin{aligned}& x_1(t) = 1 + \int_0^t x_0(s)ds = 1 + t \Rightarrow x_2(t) = 1 + \int_0^t x_1(s)ds = 1 + t + \frac{t^2}{2} \\ &\Rightarrow x_3(t) = 1 + \int_0^t x_2(s)ds = 1 + t + \frac{t^2}{2} + \frac{t^3}{3\cdot 2} \\ &\Rightarrow \cdots\cdots \\ &\Rightarrow \text{By induction, we have } x_k(t) = 1 + t + \frac{t^2}{2} + \cdots + \frac{t^k}{k!}\end{aligned}$$
which converges to $x(t) = \sum\limits_{k=0}^\infty \dfrac{t^k}{k!} = e^t$.

Example 5.86. Find a function $x(t)$ satisfying $x'(t) = tx(t)$ with initial value $x(0) = 3$. Define $\Phi(x)(t) = 3 + \int_0^t sx(s)ds$, $x_0(t) = 3$ and $x_{k+1}(t) = \Phi(x_k)(t)$. Then

$$
\begin{aligned}
x_1(t) = 3 + \int_0^t 3sds = 3 + \frac{3t^2}{2} &\Rightarrow x_2(t) = 3 + \int_0^t sx_1(s)ds = 3 + \frac{3t^2}{2} + \frac{3t^4}{2\cdot 4} \\
\Rightarrow x_3(t) = 3 + \int_0^t sx_2(s)ds = 3 + \frac{3t^2}{2} + \frac{3t^4}{2\cdot 4} + \frac{3t^6}{2\cdot 4\cdot 6}\,.
\end{aligned}
$$

We can conjecture and prove that

$$x_k(t) = 3 + \frac{3t^2}{2} + \frac{3t^4}{2\cdot 4} + \cdots + \frac{3t^{2k}}{2\cdot 4\cdots(2k)}\,;$$

thus $x_k(t) \to x(t) = 3 + 3\sum_{k=1}^{\infty} \frac{t^{2k}}{2\cdot 4\cdots(2k)}$. To see what $x(t)$ is, we observe that

$$1 + \sum_{k=1}^{\infty} \frac{t^{2k}}{2\cdot 4\cdots(2k)} = \sum_{k=0}^{\infty} \frac{t^{2k}}{2^k k!} = \sum_{k=0}^{\infty} \frac{(t^2/2)^k}{k!} = \exp\Big(\frac{t^2}{2}\Big)\,;$$

thus the solution is $x(t) = 3\exp\Big(\dfrac{t^2}{2}\Big)$.

Remark 5.87. In the iterative process above of solving ODE, the iterative relation

$$x_{k+1}(t) = x_0 + \int_{t_0}^t f\big(x_k(s), s\big)ds$$

is called the ***Picard iteration***.

Example 5.88. Is there a solution to the Fredholm equation

$$x(t) = \lambda \int_a^b K(t,s)x(s)ds + \varphi(t)\,? \tag{5.6.4}$$

Define $\Phi : \mathscr{C}([a,b];\mathbb{R}) \to \mathscr{C}([a,b];\mathbb{R})$ by

$$\Phi(x)(t) = \lambda \int_a^b K(t,s)x(s)ds + \varphi(t)\,.$$

Then if $K : [a,b]\times[a,b] \to \mathbb{R}$ is continuous, and $\varphi : [a,b] \to \mathbb{R}$ is continuous, $\Phi(x) \in \mathscr{C}([a,b];\mathbb{R})$ as long as $x \in \mathscr{C}([a,b];\mathbb{R})$. Moreover,

$$\big|\Phi(x)(t) - \Phi(y)(t)\big| \leqslant \Big|\lambda \int_a^b K(t,s)\big(x(s) - y(s)\big)ds\Big| \leqslant |\lambda|\|K\|_\infty |b-a|\|x-y\|_\infty\,;$$

thus if $|\lambda|\|K\|_\infty|b-a| < 1$, Φ is a contraction mapping. As a consequence, if

1. $K : [a,b] \times [a,b] \to \mathbb{R}$ is continuous;

2. $\varphi : [a,b] \to \mathbb{R}$ is continuous;

3. $|\lambda| \|K\|_\infty |b-a| < 1$,

there exists a unique function $x(t)$ satisfying (5.6.4).

5.7 The Stone-Weierstrass Theorem

Theorem 5.89 (Weierstrass)**.** *Let $f : [0,1] \to \mathbb{R}$ be continuous and let $\varepsilon > 0$ be given. Then there is a polynomial $p : [0,1] \to \mathbb{R}$ such that $\|f - p\|_\infty < \varepsilon$. In other words, the collection of all polynomials is dense in the space $\big(\mathscr{C}([0,1];\mathbb{R}), \|\cdot\|_\infty\big)$.*

Proof. Let $r_k(x) = C^n_k x^k (1-x)^{n-k}$. By looking at the partial derivatives with respect to x of the identity $(x+y)^n = \sum\limits_{k=0}^{n} C^n_k x^k y^{n-k}$, we find that

1. $\sum\limits_{k=0}^{n} r_k(x) = 1$; 2. $\sum\limits_{k=0}^{n} k r_k(x) = nx$; 3. $\sum\limits_{k=0}^{n} k(k-1) r_k(x) = n(n-1)x^2$.

As a consequence,

$$\sum_{k=0}^{n} (k-nx)^2 r_k(x) = \sum_{k=0}^{n} \big[k(k-1) + (1-2nx)k + n^2x^2\big] r_k(x) = nx(1-x)\,.$$

Since $f : [0,1] \to \mathbb{R}$ is continuous on a compact $[0,1]$, f is uniformly continuous on $[0,1]$ (by Theorem 4.52); thus

$$\exists\, \delta > 0 \ni \big|f(x) - f(y)\big| < \frac{\varepsilon}{2} \qquad \text{if } |x-y| < \delta,\ x, y \in [0,1]\,.$$

Consider the ***Bernstein polynomial*** $p_n(x) = \sum\limits_{k=0}^{\infty} f\big(\frac{k}{n}\big) r_k(x)$. Note that

$$\begin{aligned}
\big|f(x) - p_n(x)\big| &= \Big|\sum_{k=0}^{n} \Big(f(x) - f\big(\frac{k}{n}\big)\Big) r_k(x)\Big| \leqslant \sum_{k=0}^{n} \Big|f(x) - f\big(\frac{k}{n}\big)\Big| r_k(x) \\
&\leqslant \Big(\sum_{|k-nx|<\delta n} + \sum_{|k-nx|\geqslant \delta n}\Big) \Big|f(x) - f\big(\frac{k}{n}\big)\Big| r_k(x) \\
&< \frac{\varepsilon}{2} + 2\|f\|_\infty \sum_{|k-nx|\geqslant \delta n} \frac{(k-nx)^2}{(k-nx)^2} r_k(x) \\
&\leqslant \frac{\varepsilon}{2} + \frac{2\|f\|_\infty}{n^2\delta^2} \sum_{k=0}^{n} (k-nx)^2 r_k(x) \leqslant \frac{\varepsilon}{2} + \frac{2\|f\|_\infty}{n\delta^2} x(1-x) \leqslant \frac{\varepsilon}{2} + \frac{\|f\|_\infty}{2n\delta^2}\,.
\end{aligned}$$

Choose N large enough such that $\dfrac{\|f\|_\infty}{2N\delta^2} < \dfrac{\varepsilon}{2}$. Then for all $n \geqslant N$,

$$\|f - p_n\|_\infty = \sup_{x\in[0,1]} \big|f(x) - p_n(x)\big| < \varepsilon\,. \qquad \square$$

Remark 5.90. A polynomial of the form $p_n(x) = \sum\limits_{k=0}^{n} \beta_k r_k(x)$ is called a ***Bernstein polynomial of degree*** n, and the coefficients β_k are called Bernstein coefficients.

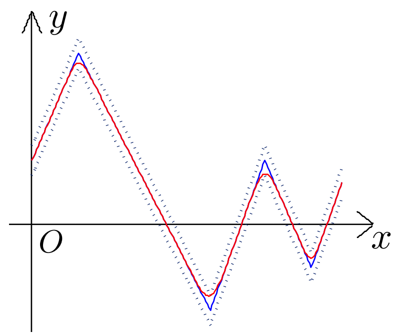


Figure 5.3: Using a Bernstein polynomial of degree 350 (the red curve) to approximate a "saw-tooth" function (the blue curve)

Corollary 5.91. *The collection of polynomials on* $[a, b]$ *is dense in* $\big(\mathscr{C}([a, b]; \mathbb{R}), \|\cdot\|_\infty\big)$.

Proof. We note that $g \in \mathscr{C}([a, b]; \mathbb{R})$ if and only if $f(x) = g\big(x(b-a)+a\big) \in \mathscr{C}([0, 1]; \mathbb{R})$; thus

$$\big|f(x) - p(x)\big| < \varepsilon \;\forall\, x \in [0, 1] \Leftrightarrow \Big|g(x) - p(\frac{x-a}{b-a})\Big| < \varepsilon \;\forall\, x \in [a, b]\,. \qquad \square$$

Example 5.92.
Question: Let $f \in \mathscr{C}([0, 1], \mathbb{R})$ be such that $\|p_n - f\|_\infty \to 0$ as $n \to \infty$; that is, $\{p_n\}_{n=1}^\infty$ converges uniformly to f on $[0, 1]$, where $p_n \in \mathscr{P}([0, 1])$. Is f differentiable?
Answer: No! Take any continuous but not differentiable function f (for example, let $f(x) = \big|x - \frac{1}{2}\big|$). By Theorem 5.89, $\exists\, p_n$: polynomial $\ni \|p_n - f\|_\infty \to 0$ as $n \to \infty$.

Definition 5.93. Let (M, d) be a metric space, and $E \subseteq M$ be a subset. A family $\mathcal{A}$ of functions defined on E is called an ***algebra*** if

1. $f + g \in \mathcal{A}$ for all $f, g \in \mathcal{A}$;
2. $f \cdot g \in \mathcal{A}$ for all $f, g \in \mathcal{A}$;
3. $\alpha f \in \mathcal{A}$ for all $f \in \mathcal{A}$ and $\alpha \in \mathbb{R}$.

In other words, $\mathcal{A}$ is an algebra if $\mathcal{A}$ is closed under addition, multiplication, and scalar multiplication.

Example 5.94. A function $g : [a,b] \to \mathbb{R}$ is called ***simple*** if we can divide up $[a,b]$ into sub-intervals on which g is constant except perhaps at the end-points. In other words, g is called simple if there is a partition $\mathcal{P} = \{x_0, x_1, \cdots, x_N\}$ of $[a,b]$ such that

$$g(x) = g\Big(\frac{x_{i-1}+x_i}{2}\Big) \qquad \text{if} \ \ x \in (x_{i-1}, x_i)\,.$$

Then the collection of all simple functions is an algebra.

Proposition 5.95. *Let (M,d) be a metric space, and $A \subseteq M$ be a subset. If $\mathcal{A} \subseteq \mathscr{C}_b(A;\mathbb{R})$ is an algebra, then* $\mathrm{cl}(\mathcal{A})$ *is also an algebra.*

Proof. Let $f, g \in \mathrm{cl}(\mathcal{A})$. Then $\exists\, \{f_k\}_{k=1}^\infty, \{g_k\}_{k=1}^\infty \subseteq \mathcal{A}$ such that $\{f_k\}_{k=1}^\infty$ converges uniformly to f on A, and $\{g_k\}_{k=1}^\infty$ converges uniformly to g on A. Since $\mathcal{A}$ is an algebra, $f_k + g_k$, $f_k \cdot g_k$ and αf_k belong to $\mathcal{A}$ for all $k \in \mathbb{N}$. As a consequence, the uniform limit of $f_k + g_k$, $f_k \cdot g_k$ and αf_k belong to $\mathrm{cl}(\mathcal{A})$ which suggests that $f+g$, $f \cdot g$ and αf belong to $\mathrm{cl}(\mathcal{A})$. As a consequence, $\mathrm{cl}(\mathcal{A})$ is an algebra. □

Definition 5.96. Let (M,d) be a metric space, and $A \subseteq M$ be a subset. A family $\mathscr{F}$ of functions defined on A is said to

1. ***separate points*** on A if for all $x, y \in A$ and $x \neq y$, there exists $f \in \mathscr{F}$ such that $f(x) \neq f(y)$.
2. ***vanish at no point*** of A if for each $x \in A$ there is $f \in \mathscr{F}$ such that $f(x) \neq 0$.

Example 5.97. Let $\mathscr{P}([a,b])$ denote the collection of polynomials defined on $[a,b]$ is an algebra. Moreover, $\mathscr{P}([a,b])$ separates points on $[a,b]$ since $p(x) = x$ does the separation, and $\mathscr{P}([a,b])$ vanishes at no point of $[a,b]$.

Example 5.98. Let $\mathscr{P}_{\text{even}}([a,b])$ denote the collection of all polynomials $p(x)$ of the form

$$p(x) = \sum_{k=0}^n a_k x^{2k} = a_n x^{2n} + a_{n-1}x^{2n-2} + \cdots + a_0\,.$$

Then $\mathscr{P}_{\text{even}}([a,b])$ is an algebra. Moreover, $\mathscr{P}_{\text{even}}([a,b])$ vanishes at no point of $[a,b]$ since the constant functions are polynomials (since constant functions belongs to $\mathscr{P}([a,b])$. However, if $ab < 0$, $\mathscr{P}_{\text{even}}([a,b])$ does not separate points on $[a,b]$. On the other hand, if $ab \geqslant 0$, then $\mathscr{P}_{\text{even}}([a,b])$ separates points on $[a,b]$ since $p(x) = x^2$ does the job.

Lemma 5.99. *Let (M,d) be a metric space, and $A \subseteq M$ be a subset. Suppose that $\mathcal{A}$ is an algebra of functions defined on A, $\mathcal{A}$ separates points on A, and $\mathcal{A}$ vanishes at no point of A. Then for all $x_1, x_2 \in A$, $x_1 \neq x_2$, and $c_1, c_2 \in \mathbb{R}$ (c_1, c_2 could be the same), there exists $f \in \mathcal{A}$ such that $f(x_1) = c_1$ and $f(x_2) = c_2$.*

Proof. Since $\mathcal{A}$ separates points on A, $\exists\, g \in \mathcal{A}$ such that $g(x_1) \neq g(x_2)$, and since $\mathcal{A}$ vanishes at no point of A, $\exists\, h, k \in \mathcal{A}$ such that $h(x_1) \neq 0$ and $k(x_2) \neq 0$. Then

$$f(x) = c_1 \frac{\big[g(x) - g(x_2)\big]h(x)}{\big[g(x_1) - g(x_2)\big]h(x_1)} + c_2 \frac{\big[g(x) - g(x_1)\big]k(x)}{\big[g(x_2) - g(x_1)\big]k(x_2)}$$

has the desired property. □

Theorem 5.100 (Stone). *Let (M,d) be a metric space, $K \subseteq M$ be a compact set, and $\mathcal{A} \subseteq \mathscr{C}(K;\mathbb{R})$ satisfying*

1. *$\mathcal{A}$ is an algebra.* 2. *$\mathcal{A}$ vanishes at no point of K.* 3. *$\mathcal{A}$ separates points on K.*

Then $\mathcal{A}$ is dense in $\mathscr{C}(K;\mathbb{R})$.

Example 5.101. Let $K = [-1,1] \times [-1,1] \subseteq \mathbb{R}^2$. Consider the set $\mathscr{P}(K)$ of all polynomials $p(x,y)$ in two variables $(x,y) \in K$. Then $\mathscr{P}(K)$ is dense in $\mathscr{C}(K;\mathbb{R})$.
Reason: Since K is compact, and $\mathscr{P}(K)$ is definitely an algebra and the constant function $p(x,y) = 1 \in \mathscr{P}(K)$ vanishes at no point of K, it suffices to show that $\mathscr{P}(K)$ separates points. Let (a_1, b_1) and (a_2, b_2) be two different points in K. Then the polynomial

$$p(x,y) = (x - a_1)^2 + (y - b_1)^2$$

has the property that $p(a_1, b_1) \neq p(a_2, b_2)$. Therefore, $\mathscr{P}(K)$ separates points in K,

Proof of Theorem 5.100. We divide the proof into the following four steps:

Step 1: We claim that if $f \in \bar{\mathcal{A}}$, then $|f| \in \bar{\mathcal{A}}$.

Proof of claim: Let $M = \sup\limits_{x \in K} |f(x)|$, and $\varepsilon > 0$ be given. By Corollary 5.91, for every $\varepsilon > 0$ there is a polynomial $p(y)$ such that $\big|p(y) - |y|\big| < \varepsilon$ for all $y \in [-M, M]$. Since $\mathcal{A}$ is an algebra, by Proposition 5.95 $\mathrm{cl}(\mathcal{A})$ is also an algebra; thus $g \equiv p(f) \in \mathrm{cl}(\mathcal{A})$ if $f \in \mathrm{cl}(\mathcal{A})$. Nevertheless,

$$\big|g(x) - |f(x)|\big| < \varepsilon \qquad \forall\, x \in K$$

which suggests that $|f| \in \bar{\mathcal{A}}$.

Step 2: Let the functions $\max\{f,g\}$ and $\min\{f,g\}$ be defined by

$$\max\{f,g\}(x) = \max\big\{f(x), g(x)\big\}, \quad \min\{f,g\}(x) = \min\big\{f(x), g(x)\big\}.$$

Since $\max\{f,g\} = \dfrac{f+g}{2} + \dfrac{|f-g|}{2}$ and $\min\{f,g\} = \dfrac{f+g}{2} - \dfrac{|f-g|}{2}$, we find that if $f, g \in \bar{\mathcal{A}}$, then $\max\{f,g\} \in \bar{\mathcal{A}}$ and $\min\{f,g\} \in \bar{\mathcal{A}}$. As a consequence, if $f_1, \cdots, f_n \in \bar{\mathcal{A}}$,

$$\max\{f_1, \cdots, f_n\} \in \bar{\mathcal{A}} \quad \text{and} \quad \min\{f_1, \cdots, f_n\} \in \bar{\mathcal{A}}.$$

Step 3: We claim that for any given $f \in \mathscr{C}(K;\mathbb{R})$, $a \in K$ and $\varepsilon > 0$, there exists a function $g_a \in \bar{\mathcal{A}}$ such that

$$g_a(a) = f(a) \quad \text{and} \quad g_a(x) > f(x) - \varepsilon \qquad \forall\, x \in K. \tag{5.7.1}$$

Proof of claim: Since $\mathcal{A}$ separates points on K and $\mathcal{A}$ vanishes at no point of K, so does $\bar{\mathcal{A}}$. Therefore, Lemma 5.99 suggests that for every $b \in K$ with $b \neq a$, there exists $h_b \in \bar{\mathcal{A}}$ such that $h_b(a) = f(a)$ and $h_b(b) = f(b)$. Note that every function in $\bar{\mathcal{A}}$ is continuous (by Theorem 5.7); thus the continuity of h_b implies that there exists $\delta = \delta_b > 0$ such that

$$h_b(x) > f(x) - \varepsilon \qquad \forall\, x \in \big[D\big(b, \delta_b\big) \cup D\big(a, \delta_b\big)\big] \cap K.$$

Let $\mathcal{U}_b = D\big(b, \delta_b\big) \cup D\big(a, \delta_b\big)$. Then $\mathcal{U}_b$ is open. Since $K \subseteq \bigcup\limits_{\substack{b \in K \\ b \neq a}} \mathcal{U}_b$ and K is compact, there exists a finite set $\{b_1, \cdots, b_n\} \subseteq K$ such that $K \subseteq \bigcup\limits_{j=1}^{n} \mathcal{U}_{b_j}$. Define $g_a = \max\big\{h_{b_1}, \cdots h_{b_n}\big\}$. Then $g_a \in \bar{\mathcal{A}}$, and $g_a(a) = f(a)$. Moreover, if $x \in K$, $x \in \mathcal{U}_{b_j}$ for some j; thus

$$g_a(x) \geqslant h_{b_j}(x) > f(x) - \varepsilon$$

which implies that g satisfies (5.7.1).

Step 4: Let $f \in \mathscr{C}(K;\mathbb{R})$ and $\varepsilon > 0$ be given. For any $a \in K$, let $g_a \in \bar{\mathcal{A}}$ be a function provided in Step 3 satisfying

$$g_a(a) = f(a) \quad \text{and} \quad g_a(x) > f(x) - \frac{\varepsilon}{2} \qquad \forall\, x \in K. \tag{5.7.2}$$

By the continuity of g_a, there exists $\delta = \delta_a > 0$ such that

$$g_a(x) < f(x) + \frac{\varepsilon}{2} \qquad \forall\, x \in D(a, \delta_a) \cap K. \tag{5.7.3}$$

Similar to Step 3, $\exists \{a_1, \cdots, a_m\} \subseteq K$ such that

$$K \subseteq \bigcup_{j=1}^{m} D\big(a_j, \delta_{a_j}\big). \tag{5.7.4}$$

Define $h = \min\big\{g_{a_1}, \cdots, g_{a_m}\big\}$. Then $h \in \bar{\mathcal{A}}$, and (5.7.2) suggests that

$$h(x) > f(x) - \frac{\varepsilon}{2} \qquad \forall\, x \in K.$$

Moreover, similar to Step 3 (5.7.3) and (5.7.4) imply that

$$h(x) < f(x) + \frac{\varepsilon}{2} \qquad \forall\, x \in K.$$

On the other hand, since $h \in \bar{\mathcal{A}}$, there exists $p \in \mathcal{A}$ such that

$$\big|p(x) - h(x)\big| < \frac{\varepsilon}{2} \qquad \forall\, x \in K\,;$$

thus

$$\big|p(x) - f(x)\big| \leqslant \big|p(x) - h(x)\big| + \big|h(x) - f(x)\big| < \varepsilon \qquad \forall\, x \in K$$

which concludes the theorem. □

Example 5.102. Consider $\mathscr{P}_{\rm even}([0,1]) = \Big\{p(x) = \sum\limits_{k=0}^{n} a_k x^{2k} \,\Big|\, a_k \in \mathbb{R}\Big\}$ (see Example 5.98). Then $\mathcal{A} = \mathscr{P}_{\rm even}([0,1])$ satisfies all the conditions in the Stone theorem, so $\mathscr{P}_{\rm even}([0,1])$ is dense in $\mathscr{C}([0,1];\mathbb{R})$.

On the other hand, if $K = [-1,1]$, then $\mathscr{P}_{\rm even}([-1,1])$ does not separate points on K since if $p \in \mathscr{P}_{\rm even}([-1,1])$, $p(x) = p(-x)$; thus the Stone theorem cannot be applied to conclude the denseness of $\mathscr{P}_{\rm even}([-1,1])$ in $\mathscr{C}([-,1];\mathbb{R})$. In fact, $\mathscr{P}_{\rm even}([-1,1])$ is not dense in $\mathscr{C}([-1,1];\mathbb{R})$ since polynomials in $\mathscr{P}_{\rm even}([-1,1])$ are all even functions and $f(x) = x$ cannot be approximated by even functions.

Corollary 5.103. *Let $\mathscr{C}(\mathbb{T})$ be the collection of all 2π-periodic continuous functions, and $\mathscr{P}_n(\mathbb{T})$ be the collection of all trigonometric polynomials of degree n; that is,*

$$\mathscr{P}_n(\mathbb{T}) = \Big\{\frac{c_0}{2} + \sum_{k=1}^{n} c_k \cos kx + s_k \sin kx \,\Big|\, \{c_k\}_{k=0}^{n}, \{s_k\}_{k=1}^{n} \subseteq \mathbb{R}\Big\}.$$

Let $\mathscr{P}(\mathbb{T}) = \bigcup\limits_{n=0}^{\infty} \mathscr{P}_n(\mathbb{T})$. Then $\mathscr{P}(\mathbb{T})$ is dense in $\mathscr{C}(\mathbb{T})$. In other words, if $f \in \mathscr{C}(\mathbb{T})$ and $\varepsilon > 0$ is given, there exists $p \in \mathscr{P}(\mathbb{T})$ such that

$$\big|f(x) - p(x)\big| < \varepsilon \qquad \forall\, x \in \mathbb{R}\,.$$

Proof. We note that $\mathscr{C}(\mathbb{T})$ can be viewed as the collection of all continuous functions defined on the unit circle $\mathbb{S}^1$ in the sense that every $f \in \mathscr{C}(\mathbb{T})$ corresponds to a unique $F \in \mathscr{C}(\mathbb{S}^1;\mathbb{R})$ such that $f(x) = F(\cos x, \sin x)$, and vice versa. Since $\mathbb{S}^1 \subseteq [-1,1] \times [-1,1]$ is compact, Example 5.101 suggests that $\mathscr{P}(\mathbb{S}^1)$, the collection of all polynomials defined on $\mathbb{S}^1$, is an algebra that separates points of $\mathbb{S}^1$ and vanishes at no point on $\mathbb{S}^1$. The Stone-Weierstrass Theorem then suggests that there exists $P \in \mathscr{P}(\mathbb{S}^1)$ such that

$$\big|F(x,y) - P(x,y)\big| < \varepsilon \qquad \forall\,(x,y) \in \mathbb{S}^1 \text{ (that is, } x^2+y^2=1).$$

Let $p(x) = P(\cos x, \sin x)$. Note that

$$\begin{aligned}\cos^n x &= \Big(\frac{e^{ix}+e^{-ix}}{2}\Big)^n = \sum_{k=0}^{n} \frac{1}{2^n} C^n_k e^{ikx} e^{-i(n-k)x} = \sum_{k=0}^{n} \frac{1}{2^n} C^n_k e^{i(2k-n)x} \\ &= \sum_{k=0}^{n} \frac{1}{2^n} C^n_k \big(\cos(2k-n)x + i\sin(2k-n)x\big) = \sum_{k=0}^{n} \frac{1}{2^n} C^n_k \cos(2k-n)x \in \mathscr{P}_n(\mathbb{T})\,,\end{aligned}$$

and similarly, $\sin^m x \in \mathscr{P}_m(\mathbb{T})$. Therefore, if $P(x,y) = \sum\limits_{k,\ell=0}^{n} a_{k,\ell} x^k y^\ell$, then $P(\cos x, \sin x) \in \mathscr{P}_{2n}(\mathbb{T})$ because of the identities

$$\begin{aligned}\cos\theta\cos\varphi &= \frac{1}{2}\Big[\cos(\theta-\varphi) + \cos(\theta+\varphi)\Big]\,,\\ \sin\theta\cos\varphi &= \frac{1}{2}\Big[\sin(\theta+\varphi) + \sin(\theta-\varphi)\Big]\,,\\ \sin\theta\sin\varphi &= \frac{1}{2}\Big[\cos(\theta-\varphi) - \cos(\theta+\varphi)\Big]\,.\end{aligned}$$

As a consequence, we conclude that

$$\big|f(x) - p(x)\big| = \big|F(\cos x, \sin x) - P(\cos x, \sin x)\big| < \varepsilon \qquad \forall\, x \in \mathbb{R}\,.$$

□

Chapter 6

Differentiable Maps

6.1 Bounded Linear Maps

Definition 6.1. A map L from a vector space X into a vector space Y is said to be ***linear*** if $L(cx_1 + x_2) = cL(x_1) + L(x_2)$ for all $x_1, x_2 \in X$ and $c \in \mathbb{R}$. We often write Lx instead of $L(x)$, and the collection of all linear maps from X to Y is denoted by $\mathscr{L}(X, Y)$.

Suppose further that X and Y are normed spaces equipped with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. A linear map $L : X \to Y$ is said to be bounded if

$$\sup_{\|x\|_X=1} \|Lx\|_Y < \infty\,.$$

The collection of all bounded linear maps from X to Y is denoted by $\mathscr{B}(X, Y)$, and the number $\sup\limits_{\|x\|_X=1} \|Lx\|_Y$ is often denoted by $\|L\|_{\mathscr{B}(X,Y)}$.

Example 6.2. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be given by $Lx = Ax$, where A is an $m \times n$ matrix. Then Example 1.133 suggests that $\|L\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)}$ is the square root of the largest eigenvalue of $A^{\mathrm{T}}A$ which is certainly a finite number. Therefore, any linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is bounded.

Proposition 6.3. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces, and $L \in \mathscr{B}(X, Y)$. Then*

$$\|L\|_{\mathscr{B}(X,Y)} = \sup_{x\neq 0} \frac{\|Lx\|_Y}{\|x\|_X} = \inf\big\{M > 0 \,\big|\, \|Lx\|_Y \leqslant M\|x\|_X\big\}\,.$$

In particular, the first equality implies that

$$\|Lx\|_Y \leqslant \|L\|_{\mathscr{B}(X,Y)}\|x\|_X \qquad \forall\, x \in X\,.$$

Proposition 6.4. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces, and $L \in \mathscr{L}(X, Y)$. Then L is continuous on X if and only if $L \in \mathscr{B}(X, Y)$.*

Proof. "$\Rightarrow$" Since L is continuous at $0 \in X$, there exists $\delta > 0$ such that

$$\|Lx\|_Y = \|Lx - L0\|_Y < 1 \quad \text{if} \quad \|x\|_X < \delta\,.$$

Then $\big\|L\big(\frac{\delta}{2}x\big)\big\|_Y \leqslant 1$ if $\big\|\frac{\delta}{2}x\big\|_X < \delta$; thus by the properties of norm,

$$\|Lx\|_Y \leqslant \frac{2}{\delta} \qquad \text{if} \quad \|x\|_X < 2\,.$$

Therefore, $\sup\limits_{\|x\|_X=1} \|Lx\|_Y \leqslant \frac{2}{\delta}$ which suggests that $L \in \mathscr{B}(X, Y)$.

"$\Leftarrow$" If $L \in \mathscr{B}(X, Y)$, then $M = \|L\|_{\mathscr{B}(X,Y)} < \infty$, and

$$\|Lx_1 - Lx_2\|_Y = \|L(x_1 - x_2)\|_Y \leqslant M\|x_1 - x_2\|_X$$

which suggests that L is uniformly continuous on X. □

Proposition 6.5. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces. Then $\big(\mathscr{B}(X, Y), \|\cdot\|_{\mathscr{B}(X,Y)}\big)$ is a normed space. Moreover, if $(Y, \|\cdot\|_Y)$ is a Banach space, so is $\big(\mathscr{B}(X, Y), \|\cdot\|_{\mathscr{B}(X,Y)}\big)$.*

Proof. That $\big(\mathscr{B}(X, Y), \|\cdot\|_{\mathscr{B}(X,Y)}\big)$ is a normed space is left as an exercise. Now suppose that $\big(Y, \|\cdot\|_Y\big)$ is a Banach space. Let $\{L_k\}_{k=1}^\infty \subseteq \mathscr{B}(X, Y)$ be a Cauchy sequence. Then by Proposition 6.3, for each $x \in X$ we have

$$\|L_k x - L_\ell x\|_Y = \|(L_k - L_\ell)x\|_Y \leqslant \|L_k - L_\ell\|_{\mathscr{B}(X,Y)}\|x\|_X \to 0 \quad \text{as} \quad k, \ell \to \infty\,.$$

Therefore, $\{L_k x\}_{k=1}^\infty$ is a Cauchy sequence in Y; thus convergent. Suppose that $\lim\limits_{k\to\infty} L_k x = y$. We then establish a map $x \mapsto y$ which we denoted by L; that is, $Lx = y$. Then L is linear since if $x_1, x_2 \in X$ and $c \in \mathbb{R}$,

$$L(cx_1 + x_2) = \lim_{k\to\infty} L_k(cx_1 + x_2) = \lim_{k\to\infty} \big(cL_k x_1 + L_k x_2\big) = cLx_1 + Lx_2\,.$$

Moreover, since $\{L_k\}_{k=1}^\infty$ is a Cauchy sequence, $\exists\, M > 0$ such that $\|L_k\|_{\mathscr{B}(X,Y)} \leqslant M$ for all $k \in N$. If $\varepsilon > 0$ is given, for each $x \in X$ there exists $N = N_x > 0$ such that

$$\|L_k x - Lx\|_Y < \varepsilon \qquad \forall\, k \geqslant N_x\,.$$

Therefore,

$$\|Lx\|_Y < \|L_k x\|_Y + \varepsilon \leqslant \|L_k\|_{\mathscr{B}(X,Y)}\|x\|_X + \varepsilon \leqslant M\|x\|_X + \varepsilon$$

which suggests that $\sup\limits_{\|x\|_X=1} \|Lx\|_Y \leqslant M + \varepsilon$; thus $L \in \mathscr{B}(X,Y)$.

Finally, we show that $\lim\limits_{k\to\infty} \|L_k - L\|_{\mathscr{B}(X,Y)} = 0$. Let $x \in X$ and $\varepsilon > 0$ be given. Since $\{L_k\}_{k=1}^{\infty}$ is a Cauchy sequence, there exists $N > 0$ such that $\|L_k - L_\ell\|_{\mathscr{B}(X,Y)} < \dfrac{\varepsilon}{2}$ if $k, \ell \geqslant N$. Then if $k \geqslant N$,

$$\|L_k x - Lx\|_Y = \lim_{\ell\to\infty} \|L_k x - L_\ell x\|_Y \leqslant \limsup_{\ell\to\infty} \|L_k - L_\ell\|_{\mathscr{B}(X,Y)}\|x\|_X \leqslant \frac{\varepsilon}{2}\|x\|_X$$

which suggests that $\|L_k - L\|_{\mathscr{B}(X,Y)} < \varepsilon$ if $k \geqslant N$. □

Proposition 6.6. *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, $(Z, \|\cdot\|_Z)$ be normed spaces, and $L \in \mathscr{B}(X,Y)$, $K \in \mathscr{B}(Y,Z)$. Then $K \circ L \in \mathscr{B}(X,Z)$, and*

$$\|K \circ L\|_{\mathscr{B}(X,Z)} \leqslant \|K\|_{\mathscr{B}(Y,Z)}\|L\|_{\mathscr{B}(X,Y)}\,.$$

We often write $K \circ L$ as KL if K and L are linear.

Proof. By the properties of the norm of a bounded linear map,

$$\|K \circ L(x)\|_Z = \|K(Lx)\|_Z \leqslant \|K\|_{\mathscr{B}(Y,Z)}\|Lx\|_Y \leqslant \|K\|_{\mathscr{B}(Y,Z)}\|L\|_{\mathscr{B}(X,Y)}\|x\|_X\,.$$ □

From now on, when the domain X and the target Y of a linear map L is clear, we use $\|L\|$ instead of $\|L\|_{\mathscr{B}(X,Y)}$ to simplify the notation.

Theorem 6.7. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces, and X be finite dimensional. Then every linear map from X to Y is bounded; that is, $\mathscr{L}(X,Y) = \mathscr{B}(X,Y)$.*

Proof. Suppose that $\dim(X) = n$. Let $\{\mathrm{e}_k\}_{k=1}^n \subseteq X$ be a linearly independent set of vectors. From Example 4.24, every two norms on X are equivalent; thus we only focus on the norm $\|\cdot\|_2$ on X induced by the inner product

$$(\mathrm{e}_i, \mathrm{e}_j)_X = \delta_{ij} \qquad \forall\, i = 1, \cdots n.$$

Since $\{\mathrm{e}_k\}_{k=1}^n$ is a linear independent set of vectors, every $x \in X$ can be expressed as a unique linear combination of e_k's; that is, for all $x \in X$, $\exists\, c_1 = c_1(x), \cdots, c_n = c_n(x) \in \mathbb{R}$ such that

$$x = c_1\mathrm{e}_1 + \cdots c_n\mathrm{e}_n\,.$$

These coefficients c_k's in fact are determined by $c_k = (x, \mathrm{e}_k)_X$, and, by Example 4.24 and the Schwartz inequality, satisfy

$$\big|c_k(x)\big| \leqslant \|x\|_2 \|\mathrm{e}_k\|_2 \leqslant C\|x\|_X .$$

As a consequence, if L is a linear map from X to Y, then

$$\begin{aligned}\|Lx\|_Y &= \big\|L(c_1(x)\mathrm{e}_1 + \cdots c_n(x)\mathrm{e}_n)\big\|_Y \leqslant |c_1(x)|\|L\mathrm{e}_1\|_Y + \cdots |c_n(x)|\|L\mathrm{e}_n\|_Y \\ &\leqslant nC\|x\|_X \max\big\{\|L\mathrm{e}_1\|_Y, \cdots \|L\mathrm{e}_n\|_Y\big\} \leqslant M\|x\|_X\end{aligned}$$

for some constant $M > 0$; thus $\|L\|_{\mathscr{B}(X,Y)} \leqslant M < \infty$ which suggests that $L \in \mathscr{B}(X, Y)$. □

Theorem 6.8. *Let* $\mathrm{GL}(n)$ *be the set of all invertible linear maps on* $\mathbb{R}^n$*; that is,*

$$\mathrm{GL}(n) = \big\{L \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n) \,\big|\, L \textit{ is one-to-one (and onto)}\big\} .$$

1. *If* $L \in \mathrm{GL}(n)$ *and* $K \in \mathscr{B}(\mathbb{R}^n, \mathbb{R}^n)$ *satisfying* $\|K - L\|\|L^{-1}\| < 1$, *then* $K \in \mathrm{GL}(n)$.

2. $\mathrm{GL}(n)$ *is an open set of* $\mathscr{B}(\mathbb{R}^n, \mathbb{R}^n)$.

3. *The mapping* $L \mapsto L^{-1}$ *is continuous on* $\mathrm{GL}(n)$.

Proof. 1. Let $\|L^{-1}\| = \dfrac{1}{\alpha}$ and $\|K - L\| = \beta$. Then $\beta < \alpha$; thus for every $x \in \mathbb{R}^n$,

$$\begin{aligned}\alpha\|x\|_{\mathbb{R}^n} &= \alpha\|L^{-1}Lx\|_{\mathbb{R}^n} \leqslant \alpha\|L^{-1}\|\|Lx\|_{\mathbb{R}^n} = \|Lx\|_{\mathbb{R}^n} \leqslant \|(L-K)x\|_{\mathbb{R}^n} + \|Kx\|_{\mathbb{R}^n} \\ &\leqslant \beta\|x\|_{\mathbb{R}^n} + \|Kx\|_{\mathbb{R}^n} .\end{aligned}$$

As a consequence, $(\alpha - \beta)\|x\|_{\mathbb{R}^n} \leqslant \|Kx\|_{\mathbb{R}^n}$ and this implies that $K : \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one hence invertible.

2. By 1, we find that if $\|K - L\| < \dfrac{1}{\|L^{-1}\|}$, then $K \in \mathrm{GL}(n)$. Then $D\big(L, \dfrac{1}{\|L^{-1}\|}\big) \subseteq \mathrm{GL}(n)$ if $L \in \mathrm{GL}(n)$. Therefore, $\mathrm{GL}(n)$ is open.

3. Let $L \in \mathrm{GL}(n)$ and $\varepsilon > 0$ be given. Choose $\delta = \min\Big\{\dfrac{1}{2\|L^{-1}\|}, \dfrac{\varepsilon}{2\|L^{-1}\|^2}\Big\}$. If $\|K-L\| < \delta$, then $K \in \mathrm{GL}(n)$. Since $L^{-1} - K^{-1} = K^{-1}(K - L)L^{-1}$, we find that if $\|K - L\| < \delta$,

$$\|K^{-1}\| - \|L^{-1}\| \leqslant \|K^{-1} - L^{-1}\| \leqslant \|K^{-1}\|\|K - L\|\|L^{-1}\| < \frac{1}{2}\|K^{-1}\|$$

which implies that $\|K^{-1}\| < 2\|L^{-1}\|$. Therefore, if $\|K - L\| < \delta$,

$$\|L^{-1} - K^{-1}\| \leqslant \|K^{-1}\|\|K - L\|\|L^{-1}\| < 2\|L^{-1}\|^2\delta < \varepsilon .$$ □

6.1.1 The matrix representation of linear maps between finite dimensional normed spaces

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two finite dimensional normed spaces. Suppose that $\mathcal{B} = \{\mathrm{e}_k\}_{k=1}^n$ and $\widetilde{\mathcal{B}} = \{\widetilde{\mathrm{e}}_k\}_{k=1}^m$ are basis of X and Y, respectively. Then every $x \in X$, and $y \in Y$, there exists unique vectors $(c_1, \cdots, c_n) \in \mathbb{R}^n$ and $(d_1, \cdots, d_m) \in \mathbb{R}^m$ such that

$$x = c_1\mathrm{e}_1 + \cdots + c_n\mathrm{e}_n \quad \text{and} \quad y = d_1\widetilde{\mathrm{e}}_1 + \cdots + d_m\widetilde{\mathrm{e}}_m\,.$$

We write $[x]_{\mathcal{B}}$ for the column vector $[c_1, \cdots, c_n]^{\mathrm{T}}$ and $[y]_{\widetilde{\mathcal{B}}}$ for the column vector $[d_1, \cdots, d_m]^{\mathrm{T}}$. Then for each $L \in \mathscr{L}(X, Y)$, the matrix representation of L with respect to basis $\mathcal{B}$ and $\widetilde{\mathcal{B}}$, denoted by $[L]_{\mathcal{B},\widetilde{\mathcal{B}}}$, is the matrix $\Big[[L\mathrm{e}_1]_{\widetilde{\mathcal{B}}} \,\vdots\, [L\mathrm{e}_2]_{\widetilde{\mathcal{B}}} \,\vdots\, \cdots \,\vdots\, [L\mathrm{e}_n]_{\widetilde{\mathcal{B}}}\Big]$. The matrix $[L]_{\mathcal{B},\widetilde{\mathcal{B}}}$ has the property that

$$[Lx]_{\widetilde{\mathcal{B}}} = [L]_{\mathcal{B},\widetilde{\mathcal{B}}}[x]_{\mathcal{B}}\,.$$

6.2 Definition of Derivatives and the Matrix Representation of Derivatives

Definition 6.9. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces. A map $f : A \subseteq X \to Y$ is said to be ***differentiable*** at $x_0 \in A$ if there is a bounded linear map, denoted by $(Df)(x_0) : X \to Y$ and called the ***derivative*** of f at x_0, such that

$$\lim_{\substack{x\to x_0\\ x\in A}} \frac{\big\|f(x) - f(x_0) - (Df)(x_0)(x - x_0)\big\|_Y}{\|x - x_0\|_X} = 0\,,$$

where $(Df)(x_0)(x - x_0)$ denotes the value of the linear map $(Df)(x_0)$ applied to the vector $x - x_0 \in X$ (so $(Df)(x_0)(x - x_0) \in Y$). In other words, f is differentiable at $x_0 \in A$ if there exists $L \in \mathscr{B}(X, Y)$ such that

$$\forall\,\varepsilon > 0, \exists\,\delta > 0 \ni \|f(x) - f(x_0) - L(x - x_0)\|_Y \leqslant \varepsilon\|x - x_0\|_X \text{ whenever } x \in D(x_0, \delta) \cap A\,.$$

If f is differentiable at each point of A, we say that f is differentiable on A.

Example 6.10. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces. Then every bounded linear map $L : X \to Y$ is differentiable. In fact, $(DL)(x_0) = L$ for all $x_0 \in X$ since

$$\lim_{x\to x_0} \frac{\|Lx - Lx_0 - L(x - x_0)\|_Y}{\|x - x_0\|_X} = 0\,.$$

Remark 6.11. Let $f:(a,b)\to\mathbb{R}$ be "differentiable" at $x_0\in(a,b)$ in the sense of Definition 4.55. The "derivative" $f'(x_0)$ and the derivative $(Df)(x_0)$ is related by $(Df)(x_0)(h)=f'(x_0)h$ since

$$\lim_{x\to x_0}\frac{\big|f(x)-f(x_0)-f'(x_0)(x-x_0)\big|}{|x-x_0|}=0\,.$$

Example 6.12. Let $f:(0,\infty)\to\mathbb{R}$ be given by $f(x)=\dfrac{1}{x}$. Then f is differentiable at any $x_0\in(0,\infty)$ and $f'(x_0)=-\dfrac{1}{{x_0}^2}$ and $(Df)(x_0):\mathbb{R}\to\mathbb{R}$ is the linear map given by

$$(Df)(x_0)(x)=-\frac{1}{{x_0}^2}\cdot x\,.$$

Then

$$\begin{aligned}\lim_{x\to x_0}\frac{\Big|\dfrac{1}{x}-\dfrac{1}{x_0}-\dfrac{-1}{{x_0}^2}(x-x_0)\Big|}{|x-x_0|}&=\lim_{x\to x_0}\frac{\Big|\dfrac{{x_0}^2-xx_0+x^2-xx_0}{x{x_0}^2}\Big|}{|x-x_0|}=\lim_{x\to x_0}\frac{{x_0}^2-2xx_0+x^2}{x{x_0}^2|x-x_0|}\\&=\lim_{x\to x_0}\frac{|x-x_0|}{x{x_0}^2}=0\,.\end{aligned}$$

Example 6.13. Let $f:\mathrm{GL}(n)\to\mathrm{GL}(n)$ be given by $f(L)=L^{-1}$, where $\mathrm{GL}(n)$ is defined in Theorem 6.8. Then f is differentiable at any "point" $L\in\mathrm{GL}(n)$ with derivative $(Df)(K)\in\mathscr{B}(\mathrm{GL}(n),\mathrm{GL}(n))$ given by $(Df)(L)(K)=-L^{-1}KL^{-1}$ for all $K\in\mathrm{GL}(n)$. The proof is left as an exercise.

Theorem 6.14. *Let $(X,\|\cdot\|_X)$, $(Y,\|\cdot\|_Y)$ be normed vector spaces, $\mathcal{U}\subseteq X$ be an open set, and $f:\mathcal{U}\to Y$ be differentiable at $x_0\in\mathcal{U}$. Then $(Df)(x_0)$ is uniquely determined by f.*

Proof. Suppose $L_1,L_2\in\mathscr{B}(X,Y)$ are derivatives of f at x_0. Let $\varepsilon>0$ be given and $\mathrm{e}\in X$ be a unit vector; that is, $\|x\|_X=1$. Since $\mathcal{U}$ is open, there exists $r>0$ such that $D(x_0,r)\subseteq\mathcal{U}$. By Definition 6.9, there exists $0<\delta<r$ such that

$$\frac{\|f(x)-f(x_0)-L_1(x-x_0)\|_Y}{\|x-x_0\|_X}<\frac{\varepsilon}{2}\quad\text{and}\quad\frac{\|f(x)-f(x_0)-L_2(x-x_0)\|_Y}{\|x-x_0\|_X}<\frac{\varepsilon}{2}$$

if $0<\|x-x_0\|_X<\delta$. Letting $x=x_0+\lambda\mathrm{e}$ with $0<|\lambda|<\delta$, we have

$$\begin{aligned}\|L_1\mathrm{e}-L_2\mathrm{e}\|_Y&=\frac{1}{|\lambda|}\|L_1(x-x_0)-L_2(x-x_0)\|_Y\\&\leqslant\frac{1}{|\lambda|}\big(\big\|f(x)-f(x_0)-L_1(x-x_0)\big\|_Y+\big\|f(x)-f(x_0)-L_2(x-x_2)\big\|_Y\big)\\&=\frac{\big\|f(x)-f(x_0)-L_1(x-x_0)\big\|_Y}{\|x-x_0\|_X}+\frac{\big\|f(x)-f(x_0)-L_2(x-x_0)\big\|_Y}{\|x-x_0\|_X}\\&<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon\,.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $L_1\mathrm{e} = L_2\mathrm{e}$ for all unit vectors $\mathrm{e} \in X$ which guarantees that $L_1 = L_2$ (since if $x \neq 0$, $L_1 x = \|x\|_X L_1\big(\frac{x}{\|x\|_X}\big) = \|x\|_X L_2\big(\frac{x}{\|x\|_X}\big) = L_2 x$). $\square$

Example 6.15. $(Df)(x_0)$ may not be unique if the domain of f is not open. For example, let $A = \big\{(x,y) \,\big|\, 0 \leqslant x \leqslant 1, y = 0\big\}$ be a subset of $\mathbb{R}^2$, and $f : A \to \mathbb{R}$ be given by $f(x,y) = 0$. Fix $x_0 = (h,0) \in A$, then both of the linear map

$$L_1(x,y) = 0 \quad \text{and} \quad L_2(x,y) = hy \qquad \forall\,(x,y) \in \mathbb{R}^2$$

satisfy Definition 6.9 since

$$\lim_{(x,0)\to(h,0)} \frac{\big|f(x,0) - f(h,0) - L_1(x-h,0)\big|}{\big\|(x,0) - (h,0)\big\|_{\mathbb{R}^2}} = \lim_{(x,0)\to(h,0)} \frac{\big|f(x,0) - f(h,0) - L_2(x-h,0)\big|}{\big\|(x,0) - (h,0)\big\|_{\mathbb{R}^2}} = 0\,.$$

Remark 6.16. Let $\mathcal{U} \subseteq \mathbb{R}^n$ be an open set and suppose that $f : \mathcal{U} \to \mathbb{R}^m$ is differentiable on $\mathcal{U}$. Then $Df : \mathcal{U} \to \mathscr{B}(\mathbb{R}^n, \mathbb{R}^m)$. Treating Df as a map from $\mathcal{U}$ to the normed space $\big(\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m), \|\cdot\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)}\big)$, and suppose that Df is also differentiable on $\mathcal{U}$. Then the derivative of Df, denoted by $D^2 f$, is a map from $\mathcal{U}$ to $\mathscr{B}(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n,\mathbb{R}^m))$. In other words, for each $a \in \mathcal{U}$, $(D^2 f)(a) \in \mathscr{B}(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n,\mathbb{R}^m))$ satisfying

$$\lim_{x\to a} \frac{\big\|(Df)(x) - (Df)(a) - (D^2 f)(a)(x-a)\big\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)}}{\|x-a\|_{\mathbb{R}^n}} = 0\,,$$

here $(D^2 f)(a)$ is bounded linear map from $\mathbb{R}^n$ to $\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)$; thus $(D^2 f)(a)(x-a) \in \mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)$.

Definition 6.17. Let $\{\mathrm{e}_k\}_{k=1}^n$ be the standard basis of $\mathbb{R}^n$, $\mathcal{U} \subseteq \mathbb{R}^n$ be an open set, $a \in \mathcal{U}$ and $f : \mathcal{U} \to \mathbb{R}$ be a function. The partial derivative of f at a in the direction e_j, denoted by $\dfrac{\partial f}{\partial x_j}(a)$, is the limit

$$\lim_{h\to 0} \frac{f(a + h\mathrm{e}_j) - f(a)}{h}$$

if it exists. In other words, if $a = (a_1, \cdots, a_n)$, then

$$\frac{\partial f}{\partial x_j}(a) = \lim_{h\to 0} \frac{f(a_1,\cdots,a_{j-1}, a_j + h, a_{j+1},\cdots,a_n) - f(a_1,\cdots,a_n)}{h}\,.$$

Theorem 6.18. *Suppose $\mathcal{U} \subseteq \mathbb{R}^n$ is an open set and $f : \mathcal{U} \to \mathbb{R}^m$ is differentiable at $a \in \mathcal{U}$. Then the partial derivatives $\dfrac{\partial f_i}{\partial x_j}(a)$ exists for all $i = 1, \cdots m$ and $j = 1, \cdots n$, and the matrix* *representation of the linear map $Df(a)$ with respect to the standard basis of $\mathbb{R}^n$ and $\mathbb{R}^m$ is given by*

$$\big[Df(a)\big] = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1}(a) & \cdots & \dfrac{\partial f_1}{\partial x_n}(a) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1}(a) & \cdots & \dfrac{\partial f_m}{\partial x_n}(a) \end{bmatrix} \quad \textit{or} \quad \big(Df(a)\big)_{ij} = \frac{\partial f_i}{\partial x_j}(a)\,.$$

Proof. Since $\mathcal{U}$ is open and $a \in \mathcal{U}$, there exists $r > 0$ such that $D(a,r) \subseteq \mathcal{U}$. By the differentiability of f at a, there is $L \in \mathscr{B}(\mathbb{R}^n, \mathbb{R}^m)$ such that for any given $\varepsilon > 0$, there exists $0 < \delta < r$ such that

$$\|f(x) - f(a) - L(x-a)\|_{\mathbb{R}^m} \leqslant \varepsilon\|x-a\|_{\mathbb{R}^n} \text{ whenever } x \in D(a,\delta)\,.$$

In particular, for each $i = 1, \cdots, m$,

$$\Big|\frac{f_i(a+h\mathrm{e}_j) - f_i(a)}{h} - (L\mathrm{e}_j)_i\Big| \leqslant \Big\|\frac{f(a+h\mathrm{e}_j) - f(a)}{h} - L\mathrm{e}_j\Big\|_{\mathbb{R}^m} \leqslant \varepsilon \quad \forall\, 0 < |h| < \delta, h \in \mathbb{R}\,,$$

where $(L\mathrm{e}_j)_i$ denotes the i-th component of $L\mathrm{e}_j$ in the standard basis. As a consequence, for each $i = 1, \cdots, m$,

$$\lim_{h \to 0} \frac{f_i(a+h\mathrm{e}_j) - f_i(a)}{h} = (L\mathrm{e}_j)_i \text{ exists}$$

and by definition, we must have $(L\mathrm{e}_j)_i = \dfrac{\partial f_i}{\partial x_j}(a)$. Therefore, $L_{ij} = \dfrac{\partial f_i}{\partial x_j}(a)$. □

Definition 6.19. Let $\mathcal{U} \subseteq \mathbb{R}^n$ be an open set, and $f : \mathcal{U} \to \mathbb{R}^m$. The matrix

$$(Jf)(x) \equiv \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{bmatrix}(x) \equiv \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1}(x) & \cdots & \dfrac{\partial f_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1}(x) & \cdots & \dfrac{\partial f_m}{\partial x_n}(x) \end{bmatrix}$$

is called the ***Jacobian matrix*** of f at x (if each entry exists).

Remark 6.20. A function f might not be differential even if the Jacobian matrix Jf exists; however, if f is differentiable at x_0, then $(Df)(x)$ can be represented by $(Jf)(x)$; that is, $[(Df)(x)] = (Jf)(x)$.

Example 6.21. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $f(x_1, x_2) = (x_1^2, x_1^3 x_2, x_1^4 x_2^2)$. Suppose that f is differentiable at $x = (x_1, x_2)$, then

$$\big[(Df)(x)\big] = \left[\begin{array}{cc} 2x_1 & 0 \\ 3x_1^2 x_2 & x_1^3 \\ 4x_1^3 x_2^2 & 2x_1^4 x_2 \end{array}\right].$$

Remark 6.22. For each $x \in A$, $Df(x)$ is a linear map, but Df in general is not linear in x.

Example 6.23. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

Then $\dfrac{\partial f}{\partial x}(0,0) = \dfrac{\partial f}{\partial y}(0,0) = 0$; thus if f is differentiable at $(0,0)$, then $(Df)(0,0) = \begin{bmatrix} 0 & 0 \end{bmatrix}$. However,

$$\Big| f(x,y) - f(0,0) - \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Big| = \frac{|xy|}{x^2+y^2} = \frac{|xy|}{(x^2+y^2)^{\frac{3}{2}}} \sqrt{x^2+y^2}\,;$$

thus f is not differentiable at $(0,0)$ since $\dfrac{|xy|}{(x^2+y^2)^{\frac{3}{2}}}$ cannot be arbitrarily small even if x^2+y^2 is small.

Example 6.24. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} x & \text{if } y = 0\,, \\ y & \text{if } x = 0\,, \\ 1 & \text{otherwise}\,. \end{cases}$$

Then $\dfrac{\partial f}{\partial x}(0,0) = \lim\limits_{h \to 0} \dfrac{f(h,0) - f(0,0)}{h} = \lim\limits_{h \to 0} \dfrac{h}{h} = 1$. Similarly, $\dfrac{\partial f}{\partial y}(0,0) = 1$; thus if f is differentiable at $(0,0)$, then $(Df)(0,0) = \begin{bmatrix} 1 & 1 \end{bmatrix}$. However,

$$\Big| f(x,y) - f(0,0) - \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Big| = \big| f(x,y) - (x+y) \big|\,;$$

thus if $xy \neq 0$,

$$\big| f(x,y) - (x+y) \big| = |1 - x - y| \nrightarrow 0 \text{ as } (x,y) \to (0,0), xy \neq 0.$$

Therefore, f is not differentiable at $(0,0)$.

Definition 6.25. Let $\mathcal{U} \subseteq \mathbb{R}^n$ be an open set. The derivative of a scalar function $f : \mathcal{U} \to \mathbb{R}$ is called the ***gradient*** of f and is denoted by grad$f$ or ∇f.

6.3 Continuity of Differentiable Maps

Theorem 6.26. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces, $\mathcal{U} \subseteq X$ be open, and $f : \mathcal{U} \to Y$ be differentiable at $x_0 \in \mathcal{U}$. Then f is continuous at x_0.*

Proof. Since f is differentiable at x_0, there exists $L \in \mathscr{B}(X, Y)$ such that

$$\exists\, \delta_1 > 0 \ni \big\|f(x) - f(x_0) - L(x - x_0)\big\|_Y \leqslant \|x - x_0\|_X \quad \forall\, x \in D(x_0, \delta_1)\,.$$

As a consequence,

$$\big\|f(x) - f(x_0)\big\|_Y \leqslant \big(\|L\| + 1\big)\|x - x_0\|_X \quad \forall\, x \in D(x_0, \delta_1)\,. \tag{6.3.1}$$

For a given $\varepsilon > 0$, let $\delta = \min\Big\{\delta_1, \dfrac{\varepsilon}{2(\|L\| + 1)}\Big\}$. Then $\delta > 0$, and if $x \in D(x_0, \delta)$,

$$\big\|f(x) - f(x_0)\big\|_Y \leqslant \frac{\varepsilon}{2} < \varepsilon\,.$$ □

Remark 6.27. In fact, if f is differentiable at x_0, then f satisfies the "local Lipschitz property"; that is,

$$\exists\, M = M(x_0) > 0 \text{ and } \delta = \delta(x_0) > 0 \ni \text{if } \|x - x_0\|_X < \delta, \text{ then } \|f(x) - f(x_0)\|_Y \leqslant M\|x - x_0\|_X$$

since we can choose $M = \|L\| + 1$ and $\delta = \delta_1$ (see (6.3.1)).

Example 6.28. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given in Example 6.23. We have shown that f is not differentiable at $(0, 0)$. In fact, f is not even continuous at $(0, 0)$ since when approaching the origin along the straight line $x_2 = mx_1$,

$$\lim_{(x_1, mx_1) \to (0,0)} f(x_1, mx_1) = \lim_{x_1 \to 0} \frac{mx_1^2}{(m^2 + 1)x_1^2} = \frac{m^2}{m^2 + 1} \neq f(0, 0) \text{ if } m \neq 0\,.$$

Example 6.29. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given in Example 6.24. Then f is not continuous at $(0, 0)$; thus not differentiable at $(0, 0)$.

Example 6.30. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = \begin{cases} \dfrac{x^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0)\,, \\ 0 & \text{if } (x, y) = (0, 0)\,. \end{cases}$$

Then $f_x(0,0) = 1$ and $f_y(0,0) = 0$. However,

$$\frac{\left\| f(x,y) - f(0,0) - \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right\|_{\mathbb{R}^2}}{\sqrt{x^2+y^2}} = \frac{|x|y^2}{(x^2+y^2)^{\frac{3}{2}}} \nrightarrow 0 \text{ as } (x,y) \to (0,0).$$

Therefore, f is not differentiable at $(0,0)$. On the other hand, f is continuous at $(0,0)$ since

$$\big|f(x,y) - f(0,0)\big| = \big|f(x,y)\big| \leqslant |x| \to 0 \text{ as } (x,y) \to (0,0).$$

6.4 Conditions for Differentiability

Proposition 6.31. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, $a \in \mathcal{U}$, and $f = (f_1, \cdots, f_m) : \mathcal{U} \to \mathbb{R}^m$. Then f is differentiable at a if and only if f_i is differentiable at a for all $i = 1, \cdots, m$. In other words, for vector-valued functions defined on an open subset of $\mathbb{R}^n$,*

Componentwise differentiable $\Leftrightarrow$ Differentiable.

Proof. "$\Rightarrow$" Let $(Df)(a)$ be the Jacobian matrix of f at a. Then

$$\forall\, \varepsilon > 0, \exists\, \delta > 0 \ni \big\|f(x) - f(a) - (Df)(a)(x-a)\big\|_{\mathbb{R}^m} \leqslant \varepsilon \|x-a\|_{\mathbb{R}^n} \text{ if } \|x-a\|_{\mathbb{R}^n} < \delta\,.$$

Let $\{\mathrm{e}_j\}_{j=1}^m$ be the standard basis of $\mathbb{R}^m$, and $L_i \in \mathscr{L}(\mathbb{R}^n, \mathbb{R})$ be given by $L_i(h) = \mathrm{e}_i^{\mathrm{T}}[(Df)(a)]h$. Then $L_i \in \mathscr{B}(\mathbb{R}^n, \mathbb{R})$ by Theorem 6.7, and if $\|x-a\|_{\mathbb{R}^n} < \delta$,

$$\begin{aligned} \big|f_i(x) - f_i(a) - L_i(x-a)\big| &= \big|\mathrm{e}_i \cdot \big(f(x) - f(a) - (Df)(a)(x-a)\big)\big| \\ &\leqslant \big\|f(x) - f(a) - (Df)(a)(x-a)\big\|_{\mathbb{R}^m} \leqslant \varepsilon \|x-a\|_{\mathbb{R}^n}\,; \end{aligned}$$

thus f_i is differentiable at a with derivatives L_i.

"$\Leftarrow$" Suppose that $f_i : \mathcal{U} \to \mathbb{R}$ is differentiable at a for each $i = 1, \cdots, m$. Then there exists $L_i \in \mathscr{B}(\mathbb{R}^n, \mathbb{R})$ such that

$$\forall\, \varepsilon > 0, \exists\, \delta_i > 0 \ni \big|f_i(x) - f_i(a) - L_i(x-a)\big| \leqslant \frac{\varepsilon}{m} \|x-a\|_{\mathbb{R}^n} \text{ if } \|x-a\|_{\mathbb{R}^n} < \delta_i\,.$$

Let $L \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^m)$ be given by $Lx = (L_1x, L_2x, \cdots, L_mx) \in \mathbb{R}^m$ if $x \in \mathbb{R}^n$. Then $L \in \mathscr{B}(\mathbb{R}^n, \mathbb{R}^m)$ by Theorem 6.7, and

$$\big\|f(x) - f(a) - L(x-a)\big\|_{\mathbb{R}^m} \leqslant \sum_{i=1}^m \big|f_i(x) - f_i(a) - L_i(x-a)\big| \leqslant \varepsilon \|x-a\|_{\mathbb{R}^n}$$

if $\|x-a\|_{\mathbb{R}^n} < \delta = \min\big\{\delta_1, \cdots, \delta_m\big\}$. $\square$

Theorem 6.32. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, $a \in \mathcal{U}$, and $f : \mathcal{U} \to \mathbb{R}$. If each entry of the Jacobian matrix* $\left[\dfrac{\partial f}{\partial x_1} \ \cdots \ \dfrac{\partial f}{\partial x_n}\right]$ *of f*

1. *exists in a neighborhood of a, and*

2. *is continuous at a except perhaps one entry.*

Then f is differentiable at a.

Proof. W.L.O.G. we may assume that the entry $\dfrac{\partial f}{\partial x_n}$ is not continuous at a. Let $\{\mathrm{e}_j\}_{j=1}^n$ be the standard basis of $\mathbb{R}^n$, and $\varepsilon > 0$ be given. Since $\dfrac{\partial f}{\partial x_i}$ is continuous at a for $i = 1, \cdots, n-1$,

$$\exists\, \delta_i > 0 \ni \left|\frac{\partial f}{\partial x_i}(x) - \frac{\partial f}{\partial x_i}(a)\right| < \frac{\varepsilon}{\sqrt{n}} \text{ whenever } \|x - a\|_{\mathbb{R}^n} < \delta_i\,.$$

On the other hand, by the definition of the partial derivatives,

$$\exists\, \delta_n > 0 \ni \left|\frac{f(a + h\mathrm{e}_n) - f(a)}{h} - \frac{\partial f}{\partial x_n}(a)\right| < \frac{\varepsilon}{\sqrt{n}} \text{ whenever } 0 < |h| < \delta_n\,.$$

Let $k = x - a$ and $\delta = \min\big\{\delta_1, \cdots, \delta_n\big\}$. Then

$$\begin{aligned}
&\left|f(x) - f(a) - \left[\frac{\partial f}{\partial x_1}(a)(x_1 - a_1) + \cdots + \frac{\partial f}{\partial x_n}(a)(x_n - a_n)\right]\right| \\
&\quad = \left|f(a + k) - f(a) - \frac{\partial f}{\partial x_1}(a)k_1 - \cdots - \frac{\partial f}{\partial x_n}(a)k_n\right| \\
&\quad = \left|f(a_1 + k_1, \cdots, a_n + k_n) - f(a_1, \cdots, a_n) - \frac{\partial f}{\partial x_1}(a)k_1 - \cdots - \frac{\partial f}{\partial x_n}(a)k_n\right| \\
&\quad \leqslant \left|f(a_1 + k_1, \cdots, a_n + k_n) - f(a_1, a_2 + k_2, \cdots, a_n + k_n) - \frac{\partial f}{\partial x_1}(a)k_1\right| \\
&\qquad + \left|f(a_1, a_2 + k_2, \cdots, a_n + k_n) - f(a_1, a_2, a_3 + k_3, \cdots, a_n + k_n) - \frac{\partial f}{\partial x_2}(a)k_2\right| \\
&\qquad + \cdots + \left|f(a_1, \cdots, a_{n-1}, a_n + k_n) - f(a_1, \cdots, a_n) - \frac{\partial f}{\partial x_n}(a)k_n\right|.
\end{aligned}$$

By the mean value theorem,

$$\begin{aligned}
&f(a_1, \cdots, a_{j-1}, a_j + k_j, \cdots, a_n + k_n) - f(a_1, \cdots, a_j, a_{j+1} + k_{j+1}, \cdots, a_n + k_n) \\
&\quad = k_j \frac{\partial f}{\partial x_j}(a_1, \cdots, a_{j-1}, a_j + \theta_j k_j, a_{j+1} + k_{j+1}, \cdots, a_n + k_n)
\end{aligned}$$

for some $0 < \theta_j < 1$; thus for $j = 1, \cdots, n-1$, if $\|x-a\|_{\mathbb{R}^n} = \|k\|_{\mathbb{R}^n} < \delta$,

$$\begin{aligned}&\Big|f(a_1,\cdots,a_{j-1},a_j+k_j,\cdots,a_n+k_n) - f(a_1,\cdots,a_j,a_{j+1}+k_{j+1},\cdots,a_n+k_n) - \frac{\partial f}{\partial x_j}(a)k_j\Big| \\ &\quad = \Big|\frac{\partial f}{\partial x_j}(a_1,\cdots,a_{j-1},a_j+\theta_j k_j,a_{j+1}+k_{j+1},\cdots,a_n+k_n) - \frac{\partial f}{\partial x_j}(a)\Big||k_j| \leqslant \frac{\varepsilon}{\sqrt{n}}|k_j|\,.\end{aligned}$$

Moreover, if $\|x-a\|_{\mathbb{R}^n} < \delta$, then $|k_n| \leqslant \|k\|_{\mathbb{R}^n} = \|x-a\|_{\mathbb{R}^n} < \delta \leqslant \delta_n$; thus

$$\Big|f(a_1,\cdots,a_{n-1},a_n+k_n) - f(a_1,\cdots,a_n) - \frac{\partial f}{\partial x_n}(a)k_n\Big| \leqslant \frac{\varepsilon}{\sqrt{n}}|k_n|\,.$$

As a consequence, if $\|x-a\|_{\mathbb{R}^n} < \delta$, by Cauchy's inequality,

$$\begin{aligned}&\Big|f(x) - f(a) - \Big[\frac{\partial f}{\partial x_1}(a)(x_1-a_1) + \cdots + \frac{\partial f}{\partial x_n}(a)(x_n-a_n)\Big]\Big| \\ &\qquad \leqslant \frac{\varepsilon}{\sqrt{n}}\sum_{j=1}^n |k_j| \leqslant \varepsilon\|k\|_{\mathbb{R}^n} = \varepsilon\|x-a\|_{\mathbb{R}^n}\end{aligned}$$

which implies that f is differentiable at a. □

Remark 6.33. When two or more components of the Jacobian matrix $\Big[\frac{\partial f}{\partial x_1} \ \cdots \ \frac{\partial f}{\partial x_n}\Big]$ of a scalar function f are discontinuous at a point $x_0 \in \mathcal{U}$, in general f is not differentiable at x_0. For example, both components of the Jacobian matrix of the functions given in Example 6.23, 6.24, 6.30 are discontinuous at $(0,0)$, and these functions are not differentiable at $(0,0)$.

Example 6.34. Let $\mathcal{U} = \mathbb{R}^2 \backslash \big\{(x,0) \in \mathbb{R}^2 \,\big|\, x \geqslant 0\big\}$, and $f : \mathcal{U} \to \mathbb{R}$ be given by

$$f(x,y) = \arg(x+iy) = \left\{\begin{array}{cl} \cos^{-1}\dfrac{x}{\sqrt{x^2+y^2}} & \text{if } y > 0\,, \\ \pi & \text{if } y = 0\,, \\ 2\pi - \cos^{-1}\dfrac{x}{\sqrt{x^2+y^2}} & \text{if } y < 0\,. \end{array}\right.$$

Then

$$\frac{\partial f}{\partial x}(x,y) = \left\{\begin{array}{cl} -\dfrac{y}{x^2+y^2} & \text{if } y \neq 0\,, \\ 0 & \text{if } y = 0\,, \end{array}\right. \quad \text{and} \quad \frac{\partial f}{\partial y}(x,y) = \left\{\begin{array}{cl} \dfrac{x}{x^2+y^2} & \text{if } y \neq 0\,, \\ \dfrac{1}{x} & \text{if } y = 0\,. \end{array}\right.$$

Since $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are both continuous on $\mathcal{U}$, f is differentiable on $\mathcal{U}$.

Definition 6.35. Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, and $f : \mathcal{U} \to \mathbb{R}^m$ be differentiable on $\mathcal{U}$. f is said to be ***continuously differentiable*** on $\mathcal{U}$ if $Df : \mathcal{U} \to \mathscr{B}(\mathbb{R}^n, \mathbb{R}^m)$ is continuous on $\mathcal{U}$. The collection of all continuously differentiable mappings from $\mathcal{U}$ to $\mathbb{R}^m$ is denoted by $\mathscr{C}'(\mathcal{U}; \mathbb{R}^m)$. The collection of all bounded differentiable functions from $\mathcal{U}$ to $\mathbb{R}^m$ whose derivative is continuous and bounded is denoted by $\mathscr{C}^1(\mathcal{U}; \mathbb{R}^m)$. In other words,

$$\mathscr{C}'(\mathcal{U}; \mathbb{R}^m) = \big\{ f : \mathcal{U} \to \mathbb{R}^m \text{ is differentiable on } \mathcal{U} \,\big|\, Df : \mathcal{U} \to \mathscr{B}(\mathbb{R}^n, \mathbb{R}^m) \text{ is continuous} \big\}$$

and

$$\mathscr{C}^1(\mathcal{U}; \mathbb{R}^m) = \Big\{ f \in \mathscr{C}'(\mathcal{U}; \mathbb{R}^m) \,\Big|\, \sup_{x \in \mathcal{U}} |f(x)| + \sup_{x \in \mathcal{U}} \|Df(x)\|_{\mathscr{B}(\mathbb{R}^n, \mathbb{R}^m)} < \infty \Big\}.$$

Corollary 6.36. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, and $f : \mathcal{U} \to \mathbb{R}^m$. Then $f \in \mathscr{C}'(\mathcal{U})$ if and only if the partial derivatives $\dfrac{\partial f_i}{\partial x_j}$ exist and are continuous on $\mathcal{U}$ for $i = 1, \cdots, m$ and $j = 1, \cdots, n$.*

Proof. Note that for any matrix $A = [a_{ij}]_{m \times n}$, $\|A\|_{\mathscr{B}(\mathbb{R}^n, \mathbb{R}^m)} \leqslant \sum\limits_{i,j} |a_{ij}| \leqslant nm\|A\|$; thus

$$\begin{aligned} \big\|(Df)(x) - (Df)(x_0)\big\|_{\mathscr{B}(\mathbb{R}^n, \mathbb{R}^m)} &\leqslant \sum_{i=1}^m \sum_{j=1}^n \Big| \frac{\partial f_i}{\partial x_j}(x) - \frac{\partial f_i}{\partial x_j}(x_0) \Big| \\ &\leqslant nm \big\|(Df)(x) - (Df)(x_0)\big\|_{\mathscr{B}(\mathbb{R}^n, \mathbb{R}^m)}. \end{aligned}$$

Therefore, the continuity of Df is equivalent to the continuity of the partial derivatives $\dfrac{\partial f_i}{\partial x_j}$ for all i, j. The corollary is then concluded by Proposition 6.31 and Theorem 6.32. $\square$

Example 6.37. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at x_0, must f' be continuous at x_0? In other words, is it always true that $\lim\limits_{x \to x_0} f'(x) = f'(x_0)$?

Answer: No! For example, take

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

1° Show $f(x)$ is differentiable at $x = 0$:

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin \frac{1}{h}}{h} = \lim_{h \to 0} h \sin \frac{1}{h} = 0.$$

2° We compute the derivative of f and find that

$$f'(x) = \begin{cases} 2x \sin \dfrac{1}{x} - \cos \dfrac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

However, $\lim\limits_{x \to 0} f'(x)$ does not exist.

Proposition 6.38. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open. Given $f \in \mathscr{C}^1(\mathcal{U};\mathbb{R}^m)$, define*

$$\|f\|_{\mathscr{C}^1(\mathcal{U};\mathbb{R}^m)} = \sup_{x\in\mathcal{U}} \Big[|f(x)| + \sum_{i=1}^{m}\sum_{j=1}^{n}\Big|\frac{\partial f_i}{\partial x_j}(x)\Big|\Big].$$

Then $\big(\mathscr{C}^1(\mathcal{U};\mathbb{R}^m), \|\cdot\|_{\mathscr{C}^1(\mathcal{U};\mathbb{R}^m)}\big)$ is a Banach space.

Proof. Left as an exercise. □

Definition 6.39. Let f be real-valued and defined on a neighborhood of $x_0 \in \mathbb{R}^n$, and let $\mathrm{v} \in \mathbb{R}^n$ be a unit vector. Then

$$(D_{\mathrm{v}}f)(x_0) \equiv \frac{d}{dt}\Big|_{t=0} f(x_0 + t\mathrm{v}) = \lim_{t\to 0}\frac{f(x_0+t\mathrm{v}) - f(x_0)}{t}$$

is called the （方向導數）of f at x_0 in the direction v.

Remark 6.40. Let $\{\mathrm{e}_j\}_{j=1}^n$ be the standard basis of $\mathbb{R}^n$. Then the partial derivative $\dfrac{\partial f}{\partial x_j}(x_0)$ (if it exists) is the directional derivative of f at x_0 in the direction e_j.

Theorem 6.41. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, and $f : \mathcal{U} \to \mathbb{R}$ be differentiable at x_0. Then the directional derivative of f at x_0 in the direction* v *is $(Df)(x_0)(\mathrm{v})$.*

Proof. Since f is differentiable at x_0, $\forall\,\varepsilon > 0$, $\ni \delta > 0$ such that

$$\big|f(x) - f(x_0) - (Df)(x_0)(x - x_0)\big| \leqslant \frac{\varepsilon}{2}\|x - x_0\|_{\mathbb{R}^n} \text{ whenever } \|x - x_0\|_{\mathbb{R}^n} < \delta\,.$$

In particular, if $x = x_0 + t\mathrm{v}$ with v being a unit vector in $\mathbb{R}^n$ and $0 < |t| < \delta$, then

$$\begin{aligned}\Big|\frac{f(x_0+t\mathrm{v}) - f(x_0)}{t} - (Df)(x_0)(\mathrm{v})\Big| &= \frac{\big|f(x_0+t\mathrm{v}) - f(x_0) - (Df)(x_0)(t\mathrm{v})\big|}{|t|}\\ &= \frac{\big|f(x) - f(x_0) - (Df)(x_0)(x - x_0)\big|}{|t|} \leqslant \frac{\varepsilon}{2} < \varepsilon\,;\end{aligned}$$

thus $(D_{\mathrm{v}}f)(x_0) = (Df)(x_0)(\mathrm{v})$. □

Remark 6.42. When $v \in \mathbb{R}^n$ but $0 < \|v\|_{\mathbb{R}^n} \neq 1$, we let $\mathrm{v} = \dfrac{v}{\|v\|_{\mathbb{R}^n}}$. Then the direction derivatives of a function $f : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}$ at $a \in \mathcal{U}$ in the direction v is

$$(D_{\mathrm{v}}f)(a) = \lim_{t\to 0}\frac{f(a+t\mathrm{v}) - f(a)}{t}\,.$$

Making a change of variable $s = \dfrac{t}{\|v\|_{\mathbb{R}^n}}$. Then

$$(Df)(x_0)(v) = \|v\|_{\mathbb{R}^n}(Df)(x_0)(\mathrm{v}) = \|v\|_{\mathbb{R}^n} \lim_{t\to 0} \frac{f(a+t\mathrm{v}) - f(a)}{t} = \lim_{s\to 0} \frac{f(a+sv) - f(a)}{s}\,.$$

We sometimes also call the value $(Df)(x_0)(v)$ the "directional derivative" of f in the "direction" v.

Example 6.43. The existence of directional derivatives of a function f at x_0 in all directions does not guarantee the differentiability of f at x_0. For example, let $f : \mathbb{R}^2 \to \mathbb{R}$ be given as in Example 6.30, and $\mathrm{v} = (\mathrm{v}_1, \mathrm{v}_2) \in \mathbb{R}^2$ be a unit vector. Then

$$(D_{\mathrm{v}} f)(0) = \lim_{t\to 0} \frac{f(t\mathrm{v}_1, t\mathrm{v}_2) - f(0,0)}{t} = \mathrm{v}_1^3\,.$$

However, f is not differentiable at $(0,0)$. We also note that in this example, $(D_{\mathrm{v}} f)(0) \neq (Jf)(0)\mathrm{v}$, where $(Jf)(0) = \begin{bmatrix} \dfrac{\partial f}{\partial x}(0,0) & \dfrac{\partial f}{\partial y}(0,0) \end{bmatrix}$ is the Jacobian matrix of f at $(0,0)$.

Example 6.44. The existence of directional derivatives of a function f at x_0 in all directions does not even guarantee the continuity of f at x_0. For example, let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2 + y^4} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,, \end{cases}$$

and $\mathrm{v} = (\mathrm{v}_1, \mathrm{v}_2) \in \mathbb{R}^2$ be a unit vector. Then if $\mathrm{v}_1 \neq 0$,

$$(D_{\mathrm{v}} f)(0) = \lim_{t\to 0} \frac{f(t\mathrm{v}_1, t\mathrm{v}_2) - f(0,0)}{t} = \lim_{t\to 0} \frac{t^3 \mathrm{v}_1 \mathrm{v}_2^2}{t(t^2\mathrm{v}_1^2 + t^4 \mathrm{v}_2^4)} = \frac{\mathrm{v}_2^2}{\mathrm{v}_1}$$

while if $\mathrm{v}_1 = 0$,

$$(D_{\mathrm{v}} f)(0) = \lim_{t\to 0} \frac{f(t\mathrm{v}_1, t\mathrm{v}_2) - f(0,0)}{t} = 0\,.$$

However, f is not continuous at $(0,0)$ since if (x,y) approaches $(0,0)$ along the curve $x = my^2$ with $m \neq 0$, we have

$$\lim_{y\to 0} f(my^2, y) = \lim_{y\to 0} \frac{my^4}{m^2y^4 + y^4} = \frac{m}{m^2+1}$$

which depends on m. Therefore, f is not continuous at $(0,0)$.

Example 6.45. Here comes another example showing that a function having directional derivative in all directions might not be continuous. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} \dfrac{xy}{x+y^2} & \text{if } x+y^2 \neq 0\,, \\ 0 & \text{if } x+y^2 = 0\,, \end{cases}$$

and $\mathrm{v} = (\mathrm{v}_1, \mathrm{v}_2) \in \mathbb{R}^2$ be a unit vector. Then if $\mathrm{v}_1 \neq 0$,

$$(D_{\mathrm{v}} f)(0) = \lim_{t\to 0} \frac{f(t\mathrm{v}_1, t\mathrm{v}_2) - f(0,0)}{t} = \lim_{t\to 0} \frac{t^2 \mathrm{v}_1 \mathrm{v}_2}{t(t\mathrm{v}_1 + t^2 \mathrm{v}_2^2)} = \mathrm{v}_2$$

while if $\mathrm{v}_1 = 0$,

$$(D_{\mathrm{v}} f)(0) = \lim_{t\to 0} \frac{f(t\mathrm{v}_1, t\mathrm{v}_2) - f(0,0)}{t} = 0\,.$$

However, f is not continuous at $(0,0)$ since if (x,y) approaches $(0,0)$ along the polar curve

$$\theta(r) = \frac{\pi}{2} + \sin^{-1}(r - mr^2) \qquad 0 < r \ll 1\,,$$

we have

$$\begin{aligned} \lim_{\substack{(x,y)\to(0,0) \\ x = r\cos\theta(r), y = r\sin\theta(r)}} f(x,y) &= \lim_{r\to 0^+} \frac{r^2 \cos\theta(r)\sin\theta(r)}{r^2\sin^2\theta(r) + r\cos\theta(r)} = \lim_{r\to 0^+} \frac{r(-r+mr^2)\sin\theta(r)}{r\sin^2\theta(r) - r + mr^2} \\ &= \lim_{r\to 0^+} \frac{(-r+mr^2)\sin\theta(r)}{\sin^2\theta(r) - 1 + mr} = \frac{-1}{m} \end{aligned}$$

which depends on m. Therefore, f is not continuous at $(0,0)$.

6.5 The Product Rules and Gradients

Proposition 6.46. *Let $A \subseteq \mathbb{R}^n$, and $f : A \to \mathbb{R}^m$ and $g : A \to \mathbb{R}$ be differentiable at $x_0 \in A$. Then $gf : A \to \mathbb{R}^m$ is differentiable at x_0, and*

$$D(gf)(x_0)(v) = g(x_0)(Df)(x_0)(v) + (Dg)(x_0)(v)f(x_0)\,. \tag{6.5.1}$$

Moreover, if $g(x_0) \neq 0$, then $\dfrac{f}{g} : A \to \mathbb{R}^m$ is also differentiable at x_0, and $D(\dfrac{f}{g})(x_0) : \mathbb{R}^n \to \mathbb{R}^m$ is given by

$$D\Big(\frac{f}{g}\Big)(x_0)(v) = \frac{g(x_0)\big((Df)(x_0)(v)\big) - (Dg)(x_0)(v)f(x_0)}{g^2(x_0)}\,. \tag{6.5.2}$$

Proof. We only prove (6.5.1), and (6.5.2) is left as an exercise.

Let A be the Jacobian matrix of gf at x_0; that is, the (i,j)-th entry of A is

$$\frac{\partial(gf_i)}{\partial x_j}(x_0) = g(x_0)\frac{\partial f_i}{\partial x_j}(x_0) + \frac{\partial g}{\partial x_j}(x_0)f_i(x_0)\,.$$

Then $Av = g(x_0)(Df)(x_0)(v) + (Dg)(x_0)(v)f(x_0)$; thus

$$\begin{aligned}(gf)(x) - (gf)(x_0) - A(x-x_0) &= g(x_0)\big(f(x) - f(x_0) - (Df)(x_0)(x-x_0)\big)\\ &\quad + \big(g(x) - g(x_0) - (Dg)(x_0)(x-x_0)\big)f(x)\\ &\quad + \big((Dg)(x_0)(x-x_0)\big)\big(f(x) - f(x_0)\big)\,.\end{aligned}$$

Since $(Dg)(x_0) \in \mathscr{B}(\mathbb{R}^n,\mathbb{R})$, $\|(Dg)(x_0)\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R})} < \infty$; thus using the inequality

$$\big|(Dg)(x_0)(x-x_0)\big| \leqslant \big\|(Dg)(x_0)\big\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R})}\|x-x_0\|_{\mathbb{R}^n}$$

and the continuity of f at x_0 (due to Theorem 6.26), we find that

$$\lim_{x\to x_0}\bigg|\frac{\big|(Dg)(x_0)(x-x_0)\big|}{\|x-x_0\|_{\mathbb{R}^n}}\big\|f(x)-f(x_0)\big\|_{\mathbb{R}^m}\bigg| \leqslant \lim_{x\to x_0}\big\|(Dg)(x_0)\big\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R})}\big\|f(x)-f(x_0)\big\|_{\mathbb{R}^m} = 0\,.$$

As a consequence,

$$\begin{aligned}&\lim_{x\to x_0}\frac{\big\|(gf)(x) - (gf)(x_0) - A(x-x_0)\big\|_{\mathbb{R}^m}}{\|x-x_0\|_{\mathbb{R}^n}}\\ &\qquad\leqslant \big|g(x_0)\big|\lim_{x\to x_0}\frac{\big\|f(x)-f(x_0)-(Df)(x_0)(x-x_0)\big\|_{\mathbb{R}^m}}{\|x-x_0\|_{\mathbb{R}^n}}\\ &\qquad\quad + \lim_{x\to x_0}\bigg[\frac{\big|g(x)-g(x_0)-(Dg)(x_0)(x-x_0)\big|}{\|x-x_0\|_{\mathbb{R}^n}}\|f(x)\|_{\mathbb{R}^m}\bigg]\\ &\qquad\quad + \lim_{x\to x_0}\bigg[\frac{\big|(Dg)(x_0)(x-x_0)\big|}{\|x-x_0\|_{\mathbb{R}^n}}\big\|f(x)-f(x_0)\big\|_{\mathbb{R}^m}\bigg] = 0\end{aligned}$$

which implies that gf is differentiable at x_0 with derivative $D(gf)(x_0)$ given by (6.5.1). □

Proposition 6.47. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open and $f \in \mathscr{C}'(\mathcal{U};\mathbb{R})$; that is, $f : \mathcal{U} \to \mathbb{R}$ is continuously differentiable. Then if $(\nabla f)(x_0) \neq 0$, the vector $\dfrac{(\nabla f)(x_0)}{\|(\nabla f)(x_0)\|_{\mathbb{R}^n}}$ is the unit normal to the level set $\big\{x \in \mathcal{U}\,\big|\, f(x) = f(x_0)\big\}$ at x_0.*

Proof. Let $\gamma : (-\delta,\delta) \to \mathbb{R}^n$ be a curve such that

1. $\gamma(t) \in \big\{x \in \mathcal{U}\,\big|\, f(x) = f(x_0)\big\}$; that is, $f(\gamma(t)) = f(x_0)$ for all $t \in (-\delta,\delta)$;

2. $\gamma(0) = x_0$; 3. $\gamma'(t) \neq 0$.

Then by the chain rule (or Example 6.55),

$$(\nabla f)\big(\gamma(t)\big) \cdot \gamma'(t) = (Df)(\gamma(t))\big(\gamma'(t)\big) = 0\,.$$

In particular, $(\nabla f)(x_0) \cdot \gamma'(0) = 0$; thus $(\nabla f)(x_0)$ is normal to the level set $\big\{x \in \mathcal{U} \,\big|\, f(x) = f(x_0)\big\}$ at x_0. □

Example 6.48. Find the normal to $\mathcal{S} = \big\{(x, y, z) \,\big|\, x^2 + y^2 + z^2 = 3\big\}$ at $(1, 1, 1) \in \mathcal{S}$.
Solution: Take $f(x, y, z) = x^2 + y^2 + z^2 - 3$. Then $(\nabla f)(x, y, z) = (2x, 2y, 2z)$; thus $(\nabla f)(1, 1, 1) = (2, 2, 2)$ is normal to S at $(1, 1, 1)$.

Example 6.49. Consider the surface

$$\mathcal{S} = \big\{(x, y, z) \in \mathbb{R}^3 \,\big|\, x^2 - y^2 + xyz = 1\big\}.$$

Find the tangent plane of $\mathcal{S}$ at $(1, 0, 1)$.

Solution: Let $f(x, y, z) = x^2 - y^2 + xyz$. Then

$$\mathcal{S} = \big\{(x, y, z) \in \mathbb{R}^3 \,|\, f(x, y, z) = f(1, 0, 1)\big\}\,;$$

that is, $\mathcal{S}$ is a level set of f. Since $(\nabla f)(1, 0, 1) = (2, 1, 0) \neq (0, 0, 0)$, $(2, 1, 0)$ is normal to $\mathcal{S}$ at $(1, 0, 1)$; thus the tangent plane of S at $(1, 0, 1)$ is $2(x - 1) + y = 0$. □

Proposition 6.50. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable. Then $\pm\dfrac{\nabla f}{\|\nabla f\|_{\mathbb{R}^n}}$ is the direction in which the function f increases/decreases most rapidly*（最速上升／下降方向）.

Proof. Let $x_0 \in \mathbb{R}^n$ be given. Suppose that f increases most rapidly in the direction v, then $(D_{\rm v} f)(x_0) = \sup\limits_{\|w\|_{\mathbb{R}^n}=1} (D_w f)(x_0)$. Since f is differentiable, $(D_w f)(x_0) = (Df)(x_0)(w) = (\nabla f)(x_0) \cdot w$ which is maximized in the direction $\dfrac{(\nabla f)(x_0)}{\|(\nabla f)(x_0)\|_{\mathbb{R}^n}}$. □

Example 6.51. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x, y, z) = x^2 y \sin z$. Find the direction of the greatest rate of change at $(3, 2, 0)$.
Solution: We compute the gradient of f at $(3, 2, 0)$ as follows:

$$\begin{aligned}(\nabla f)(3, 2, 0) &= \Big(\frac{\partial f}{\partial x}(3, 2, 0), \frac{\partial f}{\partial y}(3, 2, 0), \frac{\partial f}{\partial z}(3, 2, 0)\Big)\\ &= (2xy \sin z, x^2 \sin z, x^2 y \cos z)\big|_{(x,y,z)=(3,,2,0)} = (0, 0, 18).\end{aligned}$$

Therefore, the greatest rate of change of f at $(3, 2, 0)$ is $(0, 0, 1)$.

6.6 The Chain Rule

Theorem 6.52. *Suppose that* $\mathcal{U} \subseteq \mathbb{R}^n$ *is open,* $f : \mathcal{U} \to \mathbb{R}^m$ *and* f *is differentiable at* $x_0 \in \mathcal{U}$*,* $g : f(\mathcal{U}) \to \mathbb{R}^\ell$ *and* g *is differentiable at* $f(x_0)$*. Then the map* $F = g \circ f$ *defined by*

$$F(x) = g\big(f(x)\big) \qquad \forall\, x \in \mathcal{U}$$

is differentiable at x_0*, and*

$$(DF)(x_0)(h) = (Dg)\big(f(x_0)\big)\big((Df)(x_0)(h)\big) \quad \textit{or} \quad \big((DF)(x_0)\big)_{ij} = \sum_{k=1}^{m} \frac{\partial g_i}{\partial y_k}\big(f(x_0)\big)\frac{\partial f_k}{\partial x_j}(x_0)\,.$$

Proof. To simplify the notation, let $y_0 = f(x_0)$, $A = (Df)(x_0) \in \mathscr{B}(\mathbb{R}^n, \mathbb{R}^m)$, and $B = (Dg)(y_0) \in \mathscr{B}(\mathbb{R}^m, \mathbb{R}^\ell)$. Let $\varepsilon > 0$ be given. By the differentiability of f and g at x_0 and y_0, there exists $\delta_1, \delta_2 > 0$ such that if $\|x - x_0\|_{\mathbb{R}^n} < \delta_1$ and $\|y - y_0\|_{\mathbb{R}^m} < \delta_2$, we have

$$\begin{aligned}\|f(x) - f(x_0) - A(x - x_0)\|_{\mathbb{R}^m} &\leqslant \min\big\{1, \frac{\varepsilon}{2(\|B\|+1)}\big\}\|x - x_0\|_{\mathbb{R}^n}\,,\\ \|g(y) - g(y_0) - B(y - y_0)\|_{\mathbb{R}^\ell} &\leqslant \frac{\varepsilon}{2(\|A\|+1)}\|y - y_0\|_{\mathbb{R}^m}\,.\end{aligned}$$

Define

$$\begin{aligned} u(h) &= f(x_0 + h) - f(x_0) - Ah \qquad \forall\, \|h\|_{\mathbb{R}^n} < \delta_1\,,\\ v(k) &= g(y_0 + k) - g(y_0) - Bk \qquad \forall\, \|k\|_{\mathbb{R}^m} < \delta_2\,.\end{aligned}$$

Then if $\|h\|_{\mathbb{R}^n} < \delta_1$ and $\|k\|_{\mathbb{R}^m} < \delta_2$,

$$\|u(h)\|_{\mathbb{R}^m} \leqslant \frac{\varepsilon}{2(\|B\|+1)}\|h\|_{\mathbb{R}^n} \quad \text{and} \quad \|v(k)\|_{\mathbb{R}^\ell} \leqslant \|k\|_{\mathbb{R}^m}\,.$$

Let $k = f(x_0 + h) - f(x_0) = Ah + u(h)$. Then $\lim\limits_{h \to 0} k = 0$; thus there exists $\delta_3 > 0$ such that

$$\|k\|_{\mathbb{R}^m} < \delta_2 \quad \text{whenever} \quad \|h\|_{\mathbb{R}^n} < \delta_3\,.$$

Since

$$\begin{aligned} F(x_0 + h) - F(x_0) = g(y_0 + k) - g(y_0) &= Bk + v(k) = B\big(Ah + u(h)\big) + v(k)\\ &= BAh + Bu(h) + v(k)\,,\end{aligned}$$

we conclude that if $\|h\|_{\mathbb{R}^n} < \delta = \min\{\delta_1, \delta_3\}$,

$$\begin{aligned}\|F(x_0 + h) - F(x_0) - BAh\|_{\mathbb{R}^\ell} &\leqslant \|Bu(h)\|_{\mathbb{R}^\ell} + \|v(k)\|_{\mathbb{R}^\ell} \leqslant \|B\|\|u(h)\|_{\mathbb{R}^m} + \frac{\varepsilon}{2(\|A\|+1)}\|k\|_{\mathbb{R}^m}\\ &\leqslant \frac{\varepsilon}{2}\|h\|_{\mathbb{R}^n} + \frac{\varepsilon}{2(\|A\|+1)}\big(\|A\|\|h\|_{\mathbb{R}^n} + \|u(h)\|_{\mathbb{R}^m}\big) \leqslant \frac{\varepsilon}{2}\|h\|_{\mathbb{R}^n} + \frac{\varepsilon}{2}\|h\|_{\mathbb{R}^n} = \varepsilon\|h\|_{\mathbb{R}^n}\end{aligned}$$

which implies that F is differentiable at x_0 and $(DF)(x_0) = BA$. □

Example 6.53. Consider the polar coordinate $x = r\cos\theta$, $y = r\sin\theta$. Then every function $f : \mathbb{R}^2 \to \mathbb{R}$ is associated with a function $F : [0,\infty) \times [0, 2\pi) \to \mathbb{R}$ satisfying

$$F(r,\theta) = f(r\cos\theta, r\sin\theta)\,.$$

Suppose that f is differentiable. Then F is differentiable, and the chain rule implies that

$$\begin{bmatrix} \dfrac{\partial F}{\partial r} & \dfrac{\partial F}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \end{bmatrix} \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \end{bmatrix} \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}.$$

Example 6.54. Let $f : \mathbb{R} \to \mathbb{R}$ and $F : \mathbb{R}^2 \to \mathbb{R}$ be differentiable, and $F\big(x, f(x)\big) = 0$ and $\dfrac{\partial F}{\partial y} \neq 0$. Then $f'(x) = -\dfrac{F_x\big(x, f(x)\big)}{F_y\big(x, f(x)\big)}$, where $F_x = \dfrac{\partial F}{\partial x}$ and $F_y = \dfrac{\partial F}{\partial y}$.

Example 6.55. Let $\gamma : (0,1) \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable. Let $F(t) = f\big(\gamma(t)\big)$. Then $F'(t) = (Df)\big(\gamma(t)\big)\gamma'(t)$.

Example 6.56. Let $f(u,v,w) = u^2v + wv^2$ and $g(x,y) = (xy, \sin x, e^x)$. Let $h = f \circ g : \mathbb{R}^2 \to \mathbb{R}$. Find $\dfrac{\partial h}{\partial x}$.

Way I: Compute $\dfrac{\partial h}{\partial x}$ directly: Since

$$h(x,y) = f(g(x,y)) = f(xy, \sin x, e^x) = x^2y^2\sin x + e^x\sin^2 x\,,$$

we have

$$\frac{\partial h}{\partial x} = 2xy^2\sin x + x^2y^2\cos x + e^x\sin^2 x + 2e^x\cos x\,.$$

Way II: Use the chain rule:

$$\begin{aligned} \frac{\partial h}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial g_1}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial g_2}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial g_3}{\partial x} = 2uv \cdot y + (u^2 + 2wv)\cdot\cos x + v^2 \cdot e^x \\ &= 2xy^2\sin x + (x^2y^2 + 2e^x\sin x)\cos x + e^x\sin^2 x. \end{aligned}$$

Example 6.57. Let $F(x,y) = f(x^2+y^2)$, $f : \mathbb{R} \to \mathbb{R}$, $F : \mathbb{R}^2 \to \mathbb{R}$. Show that $x\dfrac{\partial F}{\partial y} = y\dfrac{\partial F}{\partial x}$.

Proof: Let $g(x,y) = x^2 + y^2$, $g : \mathbb{R}^2 \to \mathbb{R}$, then $F(x,y) = (f \circ g)(x,y)$. By the chain rule,

$$\begin{bmatrix} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial y} \end{bmatrix} = f'(g(x,y)) \cdot \begin{bmatrix} \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{bmatrix} = f'(g(x,y)) \begin{bmatrix} 2x & 2y \end{bmatrix}$$

which implies that

$$\frac{\partial F}{\partial x} = 2xf'(g(x,y)), \qquad \frac{\partial F}{\partial y} = 2yf'(g(x,y))\,.$$

So $y\dfrac{\partial F}{\partial x} = f'(g(x,y))2xy = x\dfrac{\partial F}{\partial y}$.

6.7 The Mean Value Theorem

Theorem 6.58. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, and $f : \mathcal{U} \to \mathbb{R}^m$ with $f = (f_1, \cdots, f_m)$. Suppose that f is differentiable on $\mathcal{U}$ and the line segment joining x and y lies in $\mathcal{U}$. Then there exist points $c_1, \cdots, c_m$ on that segment such that*

$$f_i(y) - f_i(x) = (Df_i)(c_i)(y - x) \qquad \forall\, i = 1, \cdots, m.$$

Proof. Let $\gamma : [0, 1] \to \mathbb{R}^n$ be given by $\gamma(t) = (1 - t)x + ty$. Then by Theorem 6.52, for each $i = 1, \cdots, m$, $(f_i \circ \gamma) : [0, 1] \to \mathbb{R}$ is differentiable on $(0, 1)$. By the mean value theorem (Corollary 4.65), there exists $t_i \in (0, 1)$ such that

$$f_i(y) - f_i(x) = (f_i \circ \gamma)(1) - (f_i \circ \gamma)(0) = (f_i \circ \gamma)'(t_i) = (Df_i)(c_i)\big(\gamma'(t_i)\big)\,,$$

where $c_i = \gamma(t_i)$. On the other hand, $\gamma'(t_i) = y - x$. □

Corollary 6.59. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open and convex, and $f : \mathcal{U} \to \mathbb{R}^m$ be differentiable. Then for all $x, y \in \mathcal{U}$, there exists $c_1, \cdots, c_m$ on $\overline{xy}$ such that*

$$f_i(y) - f_i(x) = (Df_i)(c_i)(y - x).$$

Example 6.60. Let $f : [0, 1] \to \mathbb{R}^2$ be given by $f(t) = (t^2, t^3)$. Then there is no $s \in (0, 1)$ such that

$$(1, 1) = f(1) - f(0) = f'(s)(1 - 0) = f'(s)$$

since $f'(s) = (2s, 3s^2) \neq (1, 1)$ for all $s \in (0, 1)$.

Example 6.61. Let $f : \mathbb{R} \to \mathbb{R}^2$ be given by $f(x) = (\cos x, \sin x)$. Then $f(2\pi) - f(0) = (0, 0)$; however, $f'(x) = (-\sin x, \cos x)$ which cannot be a zero vector.

Example 6.62. Let f be given in Example 6.34, and $\mathcal{U}$ be a small neighborhood of the curve

$$\mathcal{C} = \big\{(x, y) \,\big|\, x^2 + y^2 = 1, x \leqslant 0\big\} \cup \big\{(x, \pm 1) \,|\, 0 \leqslant x \leqslant 1\big\}\,.$$

Then

$$f(1, -1) - f(1, 1) = \frac{3\pi}{2}\,.$$

On the other hand,

$$(Df)(x, y)(0, -2) = \begin{bmatrix} \dfrac{-y}{x^2 + y^2} & \dfrac{x}{x^2 + y^2} \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix} = -\frac{2x}{x^2 + y^2}$$

which can never be $\frac{3\pi}{2}$ since $\left|\frac{2x}{x^2+y^2}\right| \leqslant 3$ if $(x,y) \in \mathcal{U}$ while $\frac{3\pi}{2} > 3$. Therefore, no point (x,y) in $\mathcal{U}$ validates

$$(Df)(x,y)\big((1,-1)-(1,1)\big) = f(1,-1) - f(1,1)\,.$$

Example 6.63. Suppose that $A \subseteq \mathbb{R}^n$ is an open convex set, and $f : A \to \mathbb{R}^m$ is differentiable and $Df(x) = 0$ for all $x \in A$. Then f is a constant; that is, $\exists\, \alpha \in \mathbb{R}^m \ni f(x) = \alpha$ for all $x \in A$.

Reason: Since A is convex, then the Mean Value Theorem can be applied to any $x, y \in A$ such that $f_i(x) - f_i(y) = Df_i(c_i)(x-y) = 0$ ($\because Df_i = 0$) for $i = 1, 2, \cdots, m$; thus $f(x) = f(y)$ for any $x, y \in A$. Let $\alpha = f(x) \in \mathbb{R}^m$, then we reach the conclusion.

Example 6.64. Let $f : [0,\infty) \to \mathbb{R}$ be continuous and be differentiable on $(0,\infty)$. Suppose that $f(0) = 0$ and $f'(x)$ is non-decreasing (that is, if $x < y$, then $f'(x) \leqslant f'(y)$). Show that $g : (0,\infty) \to \mathbb{R}$, $g(x) = \frac{f(x)}{x}$ is also non-decreasing.

Proof: It suffices to prove $g'(x) \geqslant 0$. Since f is differentiable on $(0,\infty)$, then g is differentiable on $(0,\infty)$, and $g'(x) = \frac{xf'(x) - f(x)}{x^2}$. Hence

$$g'(x) \geqslant 0 \Leftrightarrow xf'(x) \geqslant f(x)\,.$$

Let $x > 0$ be fixed. Applying the Mean Value Theorem to f we find that

$$\exists\, c \in (0,x) \ni f(x) = f(x) - f(0) = f'(c)(x-0) \leqslant xf'(x)\,.$$

Theorem 6.65. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, $K \subseteq \mathcal{U}$ be compact, and $f : \mathcal{U} \to \mathbb{R}$ be of class $\mathscr{C}^1$. Then for each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\big|f(y) - f(x) - (Df)(x)(y-x)\big| \leqslant \varepsilon \|y-x\|_{\mathbb{R}^n} \quad \textit{if} \quad \|y-x\|_{\mathbb{R}^n} < \delta \textit{ and } x, y \in K\,.$$

Proof. Define $g : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$ by

$$g(x,y) = \begin{cases} \dfrac{\big|f(y) - f(x) - (Df)(x)(y-x)\big|}{\|y-x\|_{\mathbb{R}^n}} & \text{if } y \neq x\,, \\ 0 & \text{if } y = x\,. \end{cases}$$

Since f is of class $\mathscr{C}^1$, g is continuous on $\mathcal{U} \times \mathcal{U}$. In fact, it is clear that g is continuous at (x,y) if $x \neq y$, while the mean value theorem implies that $f(w) - f(z) = (Df)(\xi)(w-z)$

for some ξ on the line segment joining w and z; thus

$$\begin{aligned}
&\limsup_{\substack{(z,w)\to(x,x)\\ z\neq w}} \frac{\big|f(w)-f(z)-(Df)(z)(w-z)\big|}{\|w-z\|_{\mathbb{R}^n}} \\
&\quad = \limsup_{\substack{(z,w)\to(x,x)\\ z\neq w}} \frac{\big|\big((Df)(\xi)-(Df)(z)\big)(w-z)\big|}{\|w-z\|_{\mathbb{R}^n}} \leqslant \limsup_{\substack{(z,w)\to(x,x)\\ z\neq w}} \big\|(Df)(\xi)-(Df)(z)\big\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R})} = 0\,.
\end{aligned}$$

Now by the compactness of $K \times K$, for each given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|g(z,w)-g(x,y)| < \varepsilon \qquad \text{if } \ \|(z,w)-(x,y)\|_{\mathbb{R}^{2n}} < \delta \text{ and } x,y,z,w \in K\,.$$

In particular, with $(z,w) = (x,x)$ we find that $|g(x,y)| < \varepsilon$ if $\|x-y\|_{\mathbb{R}^n} < \delta$; thus

$$\frac{\big|f(y)-f(x)-(Df)(x)(y-x)\big|}{\|y-x\|_{\mathbb{R}^n}} < \varepsilon \qquad \text{if } \ 0 < \|x-y\|_{\mathbb{R}^n} < \delta\,,\ x,y \in K\,. \qquad \square$$

Corollary 6.66. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, $K \subseteq \mathcal{U}$ be compact, and $f : \mathcal{U} \to \mathbb{R}^m$ be of class $\mathscr{C}^1$. Then for each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\big\|f(y)-f(x)-(Df)(x)(y-x)\big\|_{\mathbb{R}^m} \leqslant \varepsilon\|y-x\|_{\mathbb{R}^n} \quad \textit{if} \ \ \|y-x\|_{\mathbb{R}^n} < \delta \textit{ and } x,y \in K\,.$$

6.8 Higher Derivatives and Taylor's Theorem

6.8.1 Higher derivatives of functions

Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, and $f : \mathcal{U} \to \mathbb{R}^m$ is differentiable. By Proposition 6.5, the space $\big(\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m), \|\cdot\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)}\big)$ is a normed space (in fact, it is a Banach space), so it is legitimate to ask if $Df : \mathcal{U} \to \mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)$ is differentiable or not. If Df is differentiable at x_0, we call f twice differentiable at x_0, and denote the twice derivative of f at x_0 as $(D^2f)(x_0)$. If Df is differentiable on $\mathcal{U}$, then $D^2f : \mathcal{U} \to \mathscr{B}\big(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)\big)$. Similar, we can talk about three times differentiability of a function if it is twice differentiable. In general, we have the following

Definition 6.67. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces, and $\mathcal{U} \subseteq X$ be open. A function $f : \mathcal{U} \to Y$ is said to be ***twice differentiable*** at $a \in \mathcal{U}$ if

1. f is (once) differentiable in a neighborhood of a;

2. there exists $L_2 \in \mathscr{B}\big(X, \mathscr{B}(X,Y)\big)$, usually denoted by $(D^2 f)(a)$ and called the ***second derivative*** of f at a, such that

$$\lim_{x \to a} \frac{\big\|(Df)(x) - (Df)(a) - L_2(x-a)\big\|_{\mathscr{B}(X,Y)}}{\|x-a\|_X} = 0\,.$$

For any two vectors $u, v \in X$, $(D^2 f)(a)(v) \in \mathscr{B}(X,Y)$ and $(D^2 f)(a)(v)(u) \in Y$. The vector $(D^2 f)(a)(v)(u)$ is usually denoted by $(D^2 f)(a)(u,v)$.

In general, a function f is said to be k***-times differentiable*** at $a \in \mathcal{U}$ if

1. f is $(k-1)$-times differentiable in a neighborhood of a;

2. there exists $L_k \in \mathscr{B}(\underbrace{X, \mathscr{B}(X, \cdots, \mathscr{B}(X}_{k \text{ copies of "}X\text{"}}, Y\underbrace{)\cdots))}_{k \text{ copies of ")"}}$, usually denoted by $(D^k f)(a)$ and called the k***-th derivative*** of f at a, such that

$$\lim_{x \to a} \frac{\big\|(D^{k-1} f)(x) - (D^{k-1} f)(a) - L_k(x-a)\big\|_{\mathscr{B}(X,\mathscr{B}(X,\cdots,\mathscr{B}(X,Y)\cdots))}}{\|x-a\|_X} = 0\,.$$

For any k vectors $u^{(1)}, \cdots u^{(k)} \in X$, the vector $(D^k f)(a)(u^{(1)}, \cdots, u^{(k)})$ is defined as the vector

$$(D^k f)(a)(u^{(k)})(u^{(k-1)}) \cdots (u^{(1)})\,.$$

Example 6.68. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces, and $f(x) = Lx$ for some $L \in \mathscr{B}(X,Y)$. From Example 6.10, $(Df)(x_0) = L$ for all $x_0 \in X$; thus $(D^2 f)(x_0) = 0$ since $Df : \mathcal{U} \in \mathscr{B}(X,Y)$ is a "constant" map. In fact, one can also conclude from

$$\lim_{x \to x_0} \frac{\big\|(Df)(x) - (Df)(x_0) - 0(x-x_0)\big\|_{\mathscr{B}(X,Y)}}{\|x-x_0\|_X} = 0$$

that $(D^2 f)(x_0) = 0$ for all $x_0 \in X$.

Remark 6.69. We focus on what $(D^k f)(a)(u_k)(\cdots)(u_1)$ means in this remark. We first look at the case that f is twice differentiable at a. With $x = a + tv$ for $v \in X$ with $\|v\|_X = 1$ in the definition, we find that

$$\lim_{t \to 0} \frac{\big\|(Df)(a+tv) - (Df)(a) - t(D^2 f)(a)(v)\big\|_{\mathscr{B}(X,Y)}}{|t|} = 0\,.$$

Since $(Df)(a+tv)-(Df)(a)-tD^2f)(a)(v) \in \mathscr{B}(X,Y)$, for all $u \in X$ with $\|u\|_X = 1$ we have

$$\begin{aligned}
&\lim_{t\to 0} \frac{\|(Df)(a+tv)(u)-(Df)(a)(u)-t(D^2f)(a)(v)(u)\|_Y}{|t|} \\
&\qquad = \lim_{t\to 0} \frac{\big\|\big[(Df)(a+tv)-(Df)(a)-t(D^2f)(a)(v)\big](u)\big\|_Y}{|t|} \\
&\qquad \leqslant \lim_{t\to 0} \frac{\|(Df)(a+tv)-(Df)(a)-t(D^2f)(a)(v)\|_{\mathscr{B}(X,Y)}}{|t|} = 0\,.
\end{aligned}$$

On the other hand, by the definition of the direction derivative,

$$(Df)(a+tv)(u)-(Df)(a)(u) = \lim_{s\to 0}\Big[\frac{f(a+tv+su)-f(a+tv)}{s} - \frac{f(a+su)-f(a)}{s}\Big]\,;$$

thus the limit above suggests that

$$\begin{aligned}
(D^2f)(a)(v)(u) &= \lim_{t\to 0}\lim_{s\to 0} \frac{f(a+tv+su)-f(a+tv)-f(a+su)+f(a)}{st} \\
&= \lim_{t\to 0} \frac{\displaystyle\lim_{s\to 0}\frac{f(a+tv+su)-f(a+tv)}{s} - \lim_{s\to 0}\frac{f(a+su)-f(a)}{s}}{t} \\
&= D_v(D_u f)(a)\,.
\end{aligned}$$

Therefore, $(D^2f)(a)(v)(u)$ is obtained by first differentiating f near a in the u-direction, then differentiating (Df) at a in the v-direction.

In general, $(D^kf)(a)(u_k)\cdots(u_1)$ is obtained by first differentiating f near a in the u_1-direction, then differentiating (Df) near a in the u_2-direction, and so on, and finally differentiating $(D^{k-1}f)$ at a in the u_k-direction.

Remark 6.70. Since $(D^2f)(a) \in \mathscr{B}(X,\mathscr{B}(X,Y))$, if $v_1, v_2 \in X$ and $c \in \mathbb{R}$, we have $(D^2f)(a)(cv_1+v_2) = c(D^2f)(a)(v_1)+(D^2f)(a)(v_2)$ (treated as "vectors" in $\mathscr{B}(X,Y)$); thus

$$(D^2f)(a)(cv_1+v_2)(u) = c(D^2f)(a)(v_1)(u)+(D^2f)(a)(v_2)(u) \qquad \forall\, u, v_1, v_2 \in X\,.$$

On the other hand, since $(D^2f)(a)(v) \in \mathscr{B}(X,Y)$,

$$(D^2f)(a)(v)(cu_1+u_2) = c(D^2f)(a)(v)(u_1)+(D^2f)(a)(v)(u_2) \qquad \forall\, u_1, u_2, v \in X\,.$$

Therefore, $(D^2f)(a)(v)(u)$ is linear in both u and v variables. A map with such kind of property is called a ***bilinear*** map (meaning 2-linear). In particular, $(D^2f)(a) : X \times X \to Y$ is a bilinear map.

In general, the vector $(D^k f)(a)(u^{(1)}, \cdots, u^{(k)})$ is linear in $u^{(1)}, \cdots, u^{(k)}$; that is,

$$\begin{aligned}
&(D^k f)(a)(u^{(1)}, \cdots, u^{(i-1)}, \alpha v + \beta w, u^{(i+1)}, \cdots, u^{(k)}) \\
&\quad = \alpha (D^k f)(a)(u^{(1)}, \cdots, u^{(i-1)}, v, u^{(i+1)}, \cdots, u^{(k)}) \\
&\qquad + \beta (D^k f)(a)(u^{(1)}, \cdots, u^{(i-1)}, w, u^{(i+1)}, \cdots, u^{(k)})
\end{aligned}$$

for all $v, w \in X$, $\alpha, \beta \in \mathbb{R}$, and $i = 1, \cdots, n$. Such kind of map which is linear in each component when the other $k-1$ components are fixed is called ***k-linear***.

Consider the case that X is finite dimensional with $\dim(X) = n$, $\{e_1, e_2, \dots, e_n\}$ is a basis of X, and $Y = \mathbb{R}$. Then $(D^2 f)(a) : X \times X \to Y$ is a bilinear form (here the term "form" means that $Y = \mathbb{R}$). A bilinear form $B : X \times X \to \mathbb{R}$ can be represented as follows: Let $a_{ij} = B(e_i, e_j) \in \mathbb{R}$ for $i, j = 1, 2, \cdots, n$. Given $x, y \in \mathbb{R}^n$, write $u = \sum\limits_{i=1}^{n} u_i e_i$ and $v = \sum\limits_{j=1}^{n} v_j e_j$. Then by the bilinearity of B,

$$B(u, v) = B\Big(\sum_{i=1}^{n} u_i e_i, \sum_{j=1}^{n} v_j e_j\Big) = \sum_{i,j=1}^{n} u_i v_j a_{ij} = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}.$$

Therefore, if $f : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}$ is twice differentiable at a, then the bilinear form $(D^2 f)(a)$ can be represented as

$$(D^2 f)(a)(u, v) = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \begin{bmatrix} (D^2 f)(e_1, e_1) & \cdots & (D^2 f)(a)(e_1, e_n) \\ \vdots & \ddots & \vdots \\ (D^2 f)(e_n, e_1) & \cdots & (D^2 f)(a)(e_n, e_n) \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}.$$

The following proposition is an analogy of Proposition 6.31. The proof is similar to the one of Proposition 6.31, and is left as an exercise.

Proposition 6.71. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, $x_0 \in \mathcal{U}$, and $f = (f_1, \cdots, f_m) : \mathcal{U} \to \mathbb{R}^m$. Then f is k-times differentiable at x_0 if and only if f_i is k-times differentiable at x_0 for all $i = 1, \cdots, m$.*

Due to the proposition above, when talking about the higher-order differentiability of $f : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and a point $x_0 \in \mathcal{U}$, from now on we only focus on the case $m = 1$.

Example 6.72. In this example, we focus on what the second derivative $(D^2 f)(a)$ of a function f is, or in particular, what $(D^2 f)(a)(e_i, e_j)$ (which appears in the Remark 6.70) is, if $X = \mathbb{R}^2$.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable, then

$$\big[(Df)(x,y)\big] = \big[f_x(x,y) \quad f_y(x,y)\big] = \left[\frac{\partial f}{\partial x}(x,y) \quad \frac{\partial f}{\partial y}(x,y)\right].$$

Suppose that f is twice differentiable at (a,b), and let $L_2 = (D^2 f)(a,b)$. Then

$$\lim_{(x,y)\to(a,b)} \frac{\big\|(Df)(x,y) - (Df)(a,b) - L_2\big((x-a,y-b)\big)\big\|_{\mathscr{B}(\mathbb{R}^2,\mathbb{R})}}{\sqrt{(x-a)^2+(y-b)^2}} = 0$$

or equivalently,

$$\lim_{(x,y)\to(a,b)} \frac{\big\|\,\big[f_x(x,y) \quad f_y(x,y)\big] - \big[f_x(a,b) \quad f_y(a,b)\big] - \big[L_2\big((x-a,y-b)\big)\big]\big\|_{\mathscr{B}(\mathbb{R}^2,\mathbb{R})}}{\sqrt{(x-a)^2+(y-b)^2}} = 0\,,$$

where $\big[L_2\big((x-a,y-b)\big)\big]$ denotes the matrix representation of the linear map $L_2\big((x-a,y-b)\big) \in \mathscr{B}(\mathbb{R}^2,\mathbb{R})$. In particular, we must have

$$\lim_{x\to a}\left\|\left[\frac{f_x(x,b)-f_x(a,b)}{x-a} \quad \frac{f_y(x,b)-f_y(a,b)}{x-a}\right] - \big[L_2\mathrm{e}_1\big]\right\|_{\mathscr{B}(\mathbb{R}^2,\mathbb{R})} = 0$$

and

$$\lim_{y\to b}\left\|\left[\frac{f_x(a,y)-f_x(a,b)}{y-b} \quad \frac{f_y(a,y)-f_y(a,b)}{y-b}\right] - \big[L_2\mathrm{e}_2\big]\right\|_{\mathscr{B}(\mathbb{R}^2,\mathbb{R})} = 0\,.$$

Using the notation of second partial derivatives, we find that

$$\big[L_2\mathrm{e}_1\big] = \big[f_{xx}(a,b) \quad f_{yx}(a,b)\big] \quad \text{and} \quad \big[L_2\mathrm{e}_2\big] = \big[f_{xy}(a,b) \quad f_{yy}(a,b)\big]\,,$$

where $f_{xy} = (f_x)_y = \dfrac{\partial}{\partial y}\Big(\dfrac{\partial f}{\partial x}\Big)$ and $f_{yx} = (f_y)_x = \dfrac{\partial}{\partial x}\Big(\dfrac{\partial f}{\partial y}\Big)$. Therefore, if $v = v_1\mathrm{e}_1 + v_2\mathrm{e}_2$,

$$[L_2 v] = \big[L_2(v_1\mathrm{e}_1+v_2\mathrm{e}_2)\big] = \big[v_1 f_{xx}(a,b) + v_2 f_{xy}(a,b) \quad v_1 f_{yx}(a,b) + v_2 f_{yy}(a,b)\big]\,. \tag{6.8.1}$$

Symbolically, we can write

$$\big[L_2\big] = \Big[\,\big[f_{xx}(a,b) \quad f_{yx}(a,b)\big] \quad \big[f_{xy}(a,b) \quad f_{yy}(a,b)\big]\,\Big]$$

so that

$$\begin{aligned}\big[L_2(v_1\mathrm{e}_1+v_2\mathrm{e}_2)\big] = \big[L_2\big]\begin{bmatrix}v_1\\ v_2\end{bmatrix} &= \Big[\,\big[f_{xx}(a,b) \quad f_{yx}(a,b)\big] \quad \big[f_{xy}(a,b) \quad f_{yy}(a,b)\big]\,\Big]\begin{bmatrix}v_1\\ v_2\end{bmatrix}\\ &= v_1\,\big[f_{xx}(a,b) \quad f_{yx}(a,b)\big] + v_2\,\big[f_{xy}(a,b) \quad f_{yy}(a,b)\big]\,.\end{aligned}$$

For two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, what does $(D^2f)(a,b)(\boldsymbol{v})(\boldsymbol{u})$ or $(D^2f)(a,b)(\boldsymbol{u},\boldsymbol{v})$ mean? To see this, let $\boldsymbol{u} = u_1\mathrm{e}_1 + u_2\mathrm{e}_2$ and $\boldsymbol{v} = v_1\mathrm{e}_1 + v_2\mathrm{e}_2$. Then

$$\begin{aligned}\big[(D^2f)(a,b)(\boldsymbol{v})(\boldsymbol{u})\big] &= \big[(D^2f)(a,b)(\boldsymbol{v})\big]\big[\boldsymbol{u}\big] = \big[L_2(v_1\mathrm{e}_1 + v_2\mathrm{e}_2)\big]\begin{bmatrix}u_1\\u_2\end{bmatrix}\\ &= v_1\begin{bmatrix}f_{xx}(a,b) & f_{yx}(a,b)\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix} + v_2\begin{bmatrix}f_{xy}(a,b) & f_{yy}(a,b)\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}\\ &= \begin{bmatrix}v_1 & v_2\end{bmatrix}\begin{bmatrix}f_{xx}(a,b) & f_{yx}(a,b)\\ f_{xy}(a,b) & f_{yy}(a,b)\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}.\end{aligned}$$

Therefore, $(D^2f)(a,b)(\mathrm{e}_1,\mathrm{e}_1) = f_{xx}(a,b)$, $(D^2f)(a,b)(\mathrm{e}_1,\mathrm{e}_2) = f_{xy}(a,b)$, $(D^2f)(a,b)(\mathrm{e}_2,\mathrm{e}_1) = f_{yx}(a,b)$ and $(D^2f)(a,b)(\mathrm{e}_2,\mathrm{e}_2) = f_{yy}(a,b)$.

On the other hand, we can identify $\mathscr{B}(\mathbb{R}^2;\mathbb{R})$ as $\mathbb{R}^2$ (every 1×2 matrix is a "row" vector), and treat $g \equiv \big[Df\big]^{\mathrm{T}} : \mathbb{R}^2 \to \mathbb{R}^2$ as a vector-valued function. By Theorem 6.18 $(Dg)(a,b)$ can be represented as a 2×2 matrix given by

$$\big[(Dg)(a,b)\big] = \begin{bmatrix}f_{xx}(a,b) & f_{xy}(a,b)\\ f_{yx}(a,b) & f_{yy}(a,b)\end{bmatrix}.$$

We note that the representation above means

$$\lim_{(x,y)\to(a,b)} \frac{\Big\|\begin{bmatrix}f_x(x,y)\\f_y(x,y)\end{bmatrix} - \begin{bmatrix}f_x(a,b)\\f_y(a,b)\end{bmatrix} - \begin{bmatrix}f_{xx}(a,b) & f_{xy}(a,b)\\ f_{yx}(a,b) & f_{yy}(a,b)\end{bmatrix}\begin{bmatrix}x-a\\y-b\end{bmatrix}\Big\|_{\mathbb{R}^2}}{\sqrt{(x-a)^2+(y-b)^2}} = 0\,.$$

The equality above is equivalent to that

$$\lim_{(x,y)\to(a,b)} \frac{\Big\|\big[(Df)(x,y)\big] - \big[(Df)(a,b)\big] - \begin{bmatrix}x-a & y-b\end{bmatrix}\begin{bmatrix}f_{xx}(a,b) & f_{yx}(a,b)\\ f_{xy}(a,b) & f_{yy}(a,b)\end{bmatrix}\Big\|_{\mathbb{R}^2}}{\sqrt{(x-a)^2+(y-b)^2}} = 0$$

According to the equality above, $L_2 = (D^2f)(a,b)$ should be defined by

$$\big[L_2(v_1\mathrm{e}_1 + v_2\mathrm{e}_2)\big] = \begin{bmatrix}v_1 & v_2\end{bmatrix}\begin{bmatrix}f_{xx}(a,b) & f_{yx}(a,b)\\ f_{xy}(a,b) & f_{yy}(a,b)\end{bmatrix} = \left(\begin{bmatrix}f_{xx}(a,b) & f_{xy}(a,b)\\ f_{yx}(a,b) & f_{yy}(a,b)\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}\right)^{\mathrm{T}}$$

which agrees with what (6.8.1) provides.

Proposition 6.73. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, and $f : \mathcal{U} \to \mathbb{R}$. Suppose that f is k-times differentiable at a. Then for k vectors $u^{(1)},\cdots,u^{(k)} \in \mathbb{R}^n$,*

$$\begin{aligned}(D^kf)(a)(u^{(1)},\cdots,u^{(k)}) &= \sum_{j_1,\cdots,j_k=1}^{n} \frac{\partial^k f}{\partial x_{j_k}\partial x_{j_{k-1}}\cdots\partial x_{j_1}}(a)u^{(1)}_{j_1}u^{(2)}_{j_2}\cdots u^{(k)}_{j_k}\\ &= \sum_{j_1,\cdots,j_k=1}^{n} \frac{\partial}{\partial x_{j_k}}\Big(\frac{\partial}{\partial x_{j_{k-1}}}\Big(\cdots\frac{\partial}{\partial x_{j_2}}\Big(\frac{\partial f}{\partial x_{j_1}}\Big)\cdots\Big)\Big)(a)u^{(1)}_{j_1}u^{(2)}_{j_2}\cdots u^{(k)}_{j_k}\,,\end{aligned}$$

where $u^{(i)} = (u^{(i)}_1, u^{(i)}_2, \cdots, u^{(i)}_n)$ *for all* $i = 1, \cdots, k$.（上標括號中的數字指所給定的 k 個向量中的第幾個向量，下標指每一個固定向量的第幾個分量）

Proof. We prove the proposition by induction. Let $\{\mathrm{e}_j\}_{j=1}^n$ be the standard basis of $\mathbb{R}^n$. By Remark 6.70 (on multi-linearity), it suffices to show that

$$(D^k f)(a)(\mathrm{e}_{j_k})(\mathrm{e}_{j_{k-1}})\cdots(\mathrm{e}_{j_2})(\mathrm{e}_{j_1}) = (D^k f)(a)(\mathrm{e}_{j_1}, \cdots, \mathrm{e}_{j_k}) = \frac{\partial^k f}{\partial x_{j_k}\partial x_{j_{k-1}}\cdots\partial x_{j_1}}(a) \tag{6.8.2}$$

since if so, we must have

$$\begin{aligned}(D^k f)(a)(u^{(1)}, \cdots, u^{(k)}) &= (D^k f)(a)\Big(\sum_{j_1=1}^n u^{(1)}_{j_1}\mathrm{e}_{j_1}, \cdots, \sum_{j_k=1}^n u^{(k)}_{j_k}\mathrm{e}_{j_k}\Big)\\ &= \sum_{j_1=1}^n\sum_{j_2=1}^n\cdots\sum_{j_k=1}^n (D^k f)(a)(\mathrm{e}_{j_1}, \cdots, \mathrm{e}_{j_k})u^{(1)}_{j_1}u^{(2)}_{j_2}\cdots u^{(k)}_{j_k}\\ &= \sum_{j_1,\cdots,j_k=1}^n \frac{\partial^k f}{\partial x_{j_k}\partial x_{j_{k-1}}\cdots\partial x_{j_1}}(a)u^{(1)}_{j_1}u^{(2)}_{j_2}\cdots u^{(k)}_{j_k}\,.\end{aligned}$$

Note that the case $k = 1$ is true because of Theorem 6.18. Next we assume that (6.8.2) holds true for $k = \ell$ if f is $(\ell-1)$-times differentiable in a neighborhood of a and f is ℓ-times differentiable at a. Now we show that (6.8.2) also holds true for $k = \ell+1$ if f is ℓ-times differentiable in a neighborhood of a, and f is $(\ell+1)$-times differentiable at a. By the definition of $(\ell+1)$-times differentiability at a,

$$\lim_{x\to a}\frac{\big\|(D^\ell f)(x) - (D^\ell f)(a) - (D^{\ell+1} f)(a)(x-a)\big\|_{\mathscr{B}(\mathbb{R}^n,\mathscr{B}(\mathbb{R}^n,\cdots,\mathscr{B}(\mathbb{R}^n,\mathbb{R})\cdots))}}{\|x-a\|_{\mathbb{R}^n}} = 0\,.$$

Since

$$\begin{aligned}&\Big|\big[(D^\ell f)(x) - (D^\ell f)(a) - (D^{\ell+1} f)(a)(x-a)\big](\mathrm{e}_{j_\ell})\cdots(\mathrm{e}_{j_2})(\mathrm{e}_{j_1})\Big|\\ &\quad\leqslant \Big\|\big[(D^\ell f)(x) - (D^\ell f)(a) - (D^{\ell+1} f)(a)(x-a)\big](\mathrm{e}_{j_\ell})\cdots(\mathrm{e}_{j_2})\Big\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R})}\|\mathrm{e}_{j_1}\|_{\mathbb{R}^n}\\ &\quad\leqslant \big\|(D^\ell f)(x) - (D^\ell f)(a) - (D^{\ell+1} f)(a)(x-a)\big\|_{\mathscr{B}(\mathbb{R}^n,\mathscr{B}(\mathbb{R}^n,\cdots,\mathscr{B}(\mathbb{R}^n,\mathbb{R})\cdots))}\|\mathrm{e}_{j_1}\|_{\mathbb{R}^n}\cdots\|\mathrm{e}_{j_\ell}\|_{\mathbb{R}^n}\\ &\quad= \big\|(D^\ell f)(x) - (D^\ell f)(a) - (D^{\ell+1} f)(a)(x-a)\big\|_{\mathscr{B}(\mathbb{R}^n,\mathscr{B}(\mathbb{R}^n,\cdots,\mathscr{B}(\mathbb{R}^n,\mathbb{R})\cdots))}\,,\end{aligned}$$

using (6.8.2) (for the case $k = \ell$) we conclude that

$$\begin{aligned}
&\lim_{x\to a} \frac{\Big|\dfrac{\partial^\ell f}{\partial x_{j_\ell}\partial x_{j_{k-1}}\cdots\partial x_{j_1}}(x) - \dfrac{\partial^\ell f}{\partial x_{j_\ell}\partial x_{j_{k-1}}\cdots\partial x_{j_1}}(a) - (D^{\ell+1}f)(a)(\mathrm{e}_{j_1},\cdots,\mathrm{e}_{j_\ell}, x-a)\Big|}{\|x-a\|_{\mathbb{R}^n}} \\
&\quad = \lim_{x\to a} \frac{\big|(D^\ell f)(x)(\mathrm{e}_{j_1},\cdots,\mathrm{e}_{j_\ell}) - (D^\ell f)(a)(\mathrm{e}_{j_1},\cdots,\mathrm{e}_{j_\ell}) - (D^{\ell+1}f)(a)(x-a)(\mathrm{e}_{j_1},\cdots,\mathrm{e}_{j_\ell})\big|}{\|x-a\|_{\mathbb{R}^n}} \\
&\quad \leqslant \lim_{x\to a} \frac{\big\|(D^\ell f)(x) - (D^\ell f)(a) - (D^{\ell+1}f)(a)(x-a)\big\|_{\mathscr{B}(\mathbb{R}^n,\mathscr{B}(\mathbb{R}^n,\cdots,\mathscr{B}(\mathbb{R}^n,\mathbb{R})\cdots))}}{\|x-a\|_{\mathbb{R}^n}} = 0\,.
\end{aligned}$$

In particular, if $x = a + t\mathrm{e}_{j_{\ell+1}}$ for some $j_{\ell+1} = 1,\cdots,n$, by the definition of partial derivatives we conclude that

$$\begin{aligned}
(D^{\ell+1}f)(a)(\mathrm{e}_{j_1},\cdots,\mathrm{e}_{j_\ell},\mathrm{e}_{j_{\ell+1}}) &= \lim_{t\to 0} \frac{\dfrac{\partial^\ell f}{\partial x_{j_\ell}\partial x_{j_{k-1}}\cdots\partial x_{j_1}}(x) - \dfrac{\partial^\ell f}{\partial x_{j_\ell}\partial x_{j_{k-1}}\cdots\partial x_{j_1}}(a)}{t} \\
&= \frac{\partial^{\ell+1} f}{\partial x_{j_{\ell+1}}\partial x_{j_\ell}\partial x_{j_{k-1}}\cdots\partial x_{j_1}}(a)
\end{aligned}$$

which is (6.8.2) for the case $k = \ell + 1$. □

Example 6.74. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x_1, x_2) = x_1^2\cos x_2$, and $u^{(1)} = (2, 0)$, $u^{(2)} = (1, 1)$, $u^{(3)} = (0, -1)$. Suppose that f is three-times differentiable at $a = (0, 0)$ (in fact it is, but we have not talked about this yet). Then

$$\begin{aligned}
(D^3 f)(a)(u^{(1)}, u^{(2)}, u^{(3)}) &= \sum_{i,j,k=1}^{2} \frac{\partial^3 f}{\partial x_k\partial x_j\partial x_i}(a)u_i^{(1)}u_j^{(2)}u_k^{(3)} = \sum_{j=1}^{2} \frac{\partial^3 f}{\partial x_2\partial x_j\partial x_1}(a)\cdot 2\cdot u_j^{(2)}\cdot(-1) \\
&= \frac{\partial^3 f}{\partial x_2\partial x_1^2}(0,0)\cdot 2\cdot 1\cdot(-1) + \frac{\partial^3 f}{\partial x_2^2\partial x_1}(0,0)\cdot 2\cdot 1\cdot(-1) = 0\,.
\end{aligned}$$

Corollary 6.75. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, and $f : \mathcal{U} \to \mathbb{R}$ be $(k+1)$-times differentiable at a. Then for $u^{(1)},\cdots,u^{(k)},u^{(k+1)} \in \mathbb{R}^n$,*

$$(D^{k+1}f)(a)(u^{(1)},\cdots,u^{(k)},u^{(k+1)}) = \sum_{j=1}^{n} u_j^{(k+1)} \frac{\partial}{\partial x_j}\Big|_{x=a}(D^k f)(x)(u^{(1)},\cdots,u^{(k)})\,.$$

In other words, (using the terminology in Remark 6.42) $(D^{k+1}f)(a)(u^{(1)},\cdots,u^{(k)},u^{(k+1)})$ is the "directional derivative" of the function $(D^k f)(\cdot)(u^{(1)},\cdots,u^{(k)})$ at a in the "direction" $u^{(k+1)}$.

Proof. By Proposition 6.73,

$$\begin{aligned}
(D^{k+1}f)(a)(u^{(1)},\cdots,u^{(k)},u^{(k+1)}) &= \sum_{j_1,\cdots,j_k,j_{k+1}=1}^{n} \frac{\partial^{k+1} f}{\partial x_{j_{k+1}}\partial x_{j_k}\cdots\partial x_{j_1}}(a)u^{(1)}_{j_1}\cdots u^{(k)}_{j_k}u^{(k+1)}_{j_{k+1}} \\
&= \sum_{j_{k+1}=1}^{n} u^{(k+1)}_{j_{k+1}} \sum_{j_1,\cdots,j_k=1}^{n} \frac{\partial^{k+1} f}{\partial x_{j_{k+1}}\partial x_{j_k}\cdots\partial x_{j_1}}(a)u^{(1)}_{j_1}\cdots u^{(k)}_{j_k} \\
&= \sum_{j_{k+1}=1}^{n} u^{(k+1)}_{j_{k+1}} \frac{\partial}{\partial x_{j_{k+1}}}\Big|_{x=a} \sum_{j_1,\cdots,j_k=1}^{n} \frac{\partial^{k} f}{\partial x_{j_k}\cdots\partial x_{j_1}}(x)u^{(1)}_{j_1}\cdots u^{(k)}_{j_k} \\
&= \sum_{j_{k+1}=1}^{n} u^{(k+1)}_{j_{k+1}} \frac{\partial}{\partial x_{j_{k+1}}}\Big|_{x=a} (D^k f)(x)(u^{(1)},\cdots,u^{(k)})\,.
\end{aligned}$$

□

Example 6.76. Let $f:\mathbb{R}^2\to\mathbb{R}$ be twice differentiable at $a=(a_1,a_2)\in\mathbb{R}^2$. Then the proposition above suggests that for $u=(u_1,u_2), v=(v_1,v_2)\in\mathbb{R}^2$,

$$\begin{aligned}
(D^2f)(a)(v)(u) = (D^2f)(a)(u,v) &= \sum_{i,j=1}^{2} \frac{\partial^2 f}{\partial x_j\partial x_i}(a)u_iv_j \\
&= \frac{\partial^2 f}{\partial x_1^2}(a)u_1v_1 + \frac{\partial^2 f}{\partial x_2\partial x_1}(a)u_1v_2 + \frac{\partial^2 f}{\partial x_1\partial x_2}(a)u_2v_1 + \frac{\partial^2 f}{\partial x_2^2}(a)u_2v_2 \\
&= \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(a) & \frac{\partial^2 f}{\partial x_2\partial x_1}(a) \\ \frac{\partial^2 f}{\partial x_1\partial x_2}(a) & \frac{\partial^2 f}{\partial x_2^2}(a) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.
\end{aligned}$$

In general, if $f:\mathbb{R}^n\to\mathbb{R}$ be twice differentiable at $a=(a_1,\cdots,a_n)\in\mathbb{R}^n$. Then for $u=(u_1,\cdots,u_n), v=(v_1,\cdots,v_n)\in\mathbb{R}^2$

$$(D^2f)(a)(v)(u) = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(a) & \cdots & \frac{\partial^2 f}{\partial x_n\partial x_1}(a) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1\partial x_n}(a) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(a) \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}.$$

The bilinear form $B:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ given by

$$B(u,v) = (D^2f)(a)(v)(u) \qquad \forall\, u,v\in\mathbb{R}^n$$

is called the ***Hessian*** of f, and is represented (in the matrix form) as an $n \times n$ matrix by

$$\begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2}(a) & \cdots & \dfrac{\partial^2 f}{\partial x_n \partial x_1}(a) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_1 \partial x_n}(a) & \cdots & \dfrac{\partial^2 f}{\partial x_n^2}(a) \end{bmatrix}.$$

If the second partial derivatives $\dfrac{\partial^2 f}{\partial x_j \partial x_i}(a)$ of f at a exists for all $i, j = 1, \cdots, n$ (here the twice differentiability of f at a is ignored), the matrix (on the right-hand side of equality) above is also called the ***Hessian matrix*** of f at a.

Even though there is no reason to believe that $(D^2 f)(a)(u, v) = (D^2 f)(a)(v, u)$ (since the left-hand side means first differentiating f in u-direction and then differentiating Df in v-direction, while the right-hand side means first differentiating f in v-direction then differentiating Df in u-direction), it is still reasonable to ask whether $(D^2 f)(a)$ is symmetric or not; that is, could it be true that $(D^2 f)(a)(u, v) = (D^2 f)(a)(v, u)$ for all $u, v \in \mathbb{R}^n$? When f is twice differentiable at a, this is equivalent of asking (by plugging in $u = \mathrm{e}_i$ and $v = \mathrm{e}_j$) that whether or not

$$\frac{\partial^2 f}{\partial x_j \partial x_i}(a) = \frac{\partial^2 f}{\partial x_i \partial x_j}(a). \tag{6.8.3}$$

The following example provides a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that (6.8.3) does not hold at $a = (0, 0)$. We remark that the function in the following example is not twice differentiable at a even though the Hessian matrix of f at a can still be computed.

Example 6.77. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Then

$$f_x(x, y) = \begin{cases} \dfrac{x^4 y + 4x^2 y^3 - y^5}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0), \end{cases}$$

and

$$f_y(x, y) = \begin{cases} \dfrac{x^5 - 4x^3 y^2 - xy^4}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0), \end{cases}$$

It is clear that f_x and f_y are continuous on $\mathbb{R}^2$; thus f is differentiable on $\mathbb{R}^2$. However,

$$f_{xy}(0,0) = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k} = -1\,,$$

while

$$f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h} = 1\,;$$

thus the Hessian matrix of f at the origin is not symmetric.

Definition 6.78. A function is said to be ***of class*** $\mathscr{C}^r$ if the first r derivatives exist and are continuous. A function is said to be ***smooth*** or ***of class*** $\mathscr{C}^\infty$ if it is of class $\mathscr{C}^r$ for all positive integer r.

The following theorem is an analogy of Corollary 6.36.

Theorem 6.79. *Let* $\mathcal{U} \to \mathbb{R}^n$ *and* $f : \mathcal{U} \to \mathbb{R}$. *Suppose that the partial derivative* $\dfrac{\partial^k f}{\partial x_{j_k}\partial x_{j_{k-1}}\cdots\partial x_{j_1}}$ *exists in a neighborhood of* $a \in \mathcal{U}$ *and is continuous at* a *for all* $j_1, \cdots, j_k = 1, \cdots, n$. *Then* f *is* k*-times differentiable at* a. *Moreover, if* $\dfrac{\partial^k f}{\partial x_{j_k}\partial x_{j_{k-1}}\cdots\partial x_{j_1}}$ *is continuous on* $\mathcal{U}$, *then* f *is of class* $\mathscr{C}^k$.

Theorem 6.80. *Let* $\mathcal{U} \subseteq \mathbb{R}^n$ *be open, and* $f : \mathcal{U} \to \mathbb{R}$. *Suppose that the mixed partial derivatives* $\dfrac{\partial f}{\partial x_i}$, $\dfrac{\partial f}{\partial x_j}$, $\dfrac{\partial^2 f}{\partial x_j \partial x_i}$, $\dfrac{\partial^2 f}{\partial x_j \partial x_i}$ *exist in a neighborhood of* a, *and are continuous at* a. *Then*

$$\frac{\partial^2 f}{\partial x_j \partial x_i}(a) = \frac{\partial^2 f}{\partial x_i \partial x_j}(a)\,. \tag{6.8.4}$$

Proof. Let $S(a,h,k) = f(a + h\mathrm{e}_i + k\mathrm{e}_j) - f(a + h\mathrm{e}_i) - f(a + k\mathrm{e}_j) + f(a)$, and define $\varphi(x) = f(x + h\mathrm{e}_i) - f(x)$ as well as $\psi(x) = f(x + k\mathrm{e}_j) - f(x)$ for x in a neighborhood of a. Then $S(a,h,k) = \varphi(a + k\mathrm{e}_j) - \varphi(a) = \psi(a + h\mathrm{e}_i) - \psi(a)$; thus the mean value theorem implies that there exists c on the line segment joining a and $a + k\mathrm{e}_j$ and d on the line segment joining a and $a + h\mathrm{e}_i$ such that

$$S(a,h,k) = \varphi(a + k\mathrm{e}_j) - \varphi(a) = k\frac{\partial \varphi}{\partial x_j}(c) = k\Big(\frac{\partial f}{\partial x_j}(c + h\mathrm{e}_i) - \frac{\partial f}{\partial x_j}(c)\Big)\,,$$

$$S(a,h,k) = \psi(a + h\mathrm{e}_i) - \psi(a) = h\frac{\partial \psi}{\partial x_i}(d) = h\Big(\frac{\partial f}{\partial x_i}(d + k\mathrm{e}_j) - \frac{\partial f}{\partial x_i}(d)\Big)\,.$$

As a consequence, if $h \neq 0 \neq k$,

$$\frac{1}{k}\Big(\frac{\partial f}{\partial x_i}(d + k\mathrm{e}_j) - \frac{\partial f}{\partial x_i}(d)\Big) = \frac{S(a,h,k)}{hk} = \frac{1}{h}\Big(\frac{\partial f}{\partial x_j}(c + h\mathrm{e}_i) - \frac{\partial f}{\partial x_j}(c)\Big)$$

By the mean value theorem again, there exists c_1 and d_1 on the line segment joining c, $c + h\mathrm{e}_i$ and d, $d + k\mathrm{e}_j$, respectively, such that

$$\frac{\partial^2 f}{\partial x_j \partial x_i}(d_1) = \frac{\partial^2 f}{\partial x_i \partial x_j}(c_1)\,.$$

The theorem is then concluded by the continuity of $\dfrac{\partial^2 f}{\partial x_i \partial x_j}$ and $\dfrac{\partial^2 f}{\partial x_j \partial x_i}$ at a, and $c_1 \to a$ and $d_1 \to a$ as $(h, k) \to (0, 0)$. □

Corollary 6.81. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, and f is of class $\mathscr{C}^2$. Then*

$$(D^2 f)(a)(u, v) = (D^2 f)(a)(v, u) \qquad \forall\, a \in \mathcal{U} \text{ and } u, v \in \mathbb{R}^n\,.$$

Remark 6.82. In view of Remark 6.69, (6.8.4) is the same as the following identity

$$\begin{aligned}&\lim_{h\to 0}\lim_{k\to 0} \frac{f(a + h\mathrm{e}_i + k\mathrm{e}_j) - f(a + h\mathrm{e}_i) - f(a + k\mathrm{e}_j) + f(a)}{hk}\\ &\quad = \lim_{k\to 0}\lim_{h\to 0} \frac{f(a + h\mathrm{e}_i + k\mathrm{e}_j) - f(a + h\mathrm{e}_i) - f(a + k\mathrm{e}_j) + f(a)}{hk}\end{aligned}$$

which implies that the order of the two limits $\lim\limits_{h\to 0}$ and $\lim\limits_{k\to 0}$ can be interchanged without changing the value of the limit (under certain conditions).

Example 6.83. Let $f(x, y) = yx^2 \cos y^2$. Then

$$\begin{aligned} f_{xy}(x, y) &= (2xy \cos y^2)_y = 2x \cos y^2 - 2xy(2y) \sin y^2 = 2x \cos y^2 - 4xy^2 \sin y^2\,,\\ f_{yx}(x, y) &= (x^2 \cos y^2 - yx^2(2y) \sin y^2)_x = (x^2 \cos y^2 - 2x^2y^2 \sin y^2)_x\\ &= 2x \cos y^2 - 4xy^2 \sin y^2 = f_{xy}(x, y)\,. \end{aligned}$$

6.8.2 Taylor's Theorem

Theorem 6.84. *Let $f : (a, b) \to \mathbb{R}$ be of class $\mathscr{C}^{k+1}$ for some $k \in \mathbb{N}$, and $c \in (a, b)$. Then for all $x \in (a, b)$, there exists d in between c and x such that*

$$f(x) = \sum_{j=0}^{k} \frac{f^{(j)}(c)}{j!}(x - c)^j + \frac{f^{(k+1)}(d)}{(k + 1)!}(x - c)^{(k+1)},$$

where $f^{(j)}(c)$ denotes the j-th derivative of f at c.

Proof. Let $g(x) = f(x) - \sum\limits_{j=0}^{k} \dfrac{f^{(j)}(c)}{j!}(x-c)^j$, and $h(x) = (x-c)^{k+1}$. Then for $1 \leqslant j \leqslant k$,

$$g^{(j)}(c) = h^{(j)}(c) = 0\,;$$

thus by the Cauchy mean value theorem (Theorem 4.64), there exists ξ_1 in between x and c, ξ_2 in between ξ_1 and c, $\cdots$, ξ_{k+1} in between ξ_k and c such that

$$\begin{aligned}\frac{g(x)}{h(x)} &= \frac{g(x)-g(c)}{h(x)-h(c)} = \frac{g'(\xi_1)}{h'(\xi_1)} = \frac{g'(\xi_1)-g'(c)}{h'(\xi_1)-h'(c)} = \frac{g''(\xi_2)}{h''(\xi_2)} = \cdots \\ &= \frac{g^{(k)}(\xi_k)}{h^{(k)}(\xi_k)} = \frac{g^{(k)}(\xi_k)-g^{(k)}(c)}{h^{(k)}(\xi_k)-h^{(k)}(c)} = \frac{g^{(k+1)}(\xi_{k+1})}{h^{(k+1)}(\xi_k)} = \frac{f^{(k+1)}(\xi_{k+1})}{(k+1)!}\,.\end{aligned}$$

Letting $d = \xi_{k+1}$ we conclude the theorem. □

Theorem 6.85 (Taylor)**.** *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, and $f : \mathcal{U} \to \mathbb{R}$ be of class $\mathscr{C}^{k+1}$. Let $x, a \in \mathcal{U}$ and suppose that the line segment joining x and a lies in $\mathcal{U}$. Then there exists a point c on the line segment joining x and a such that*

$$f(x) - f(a) = \sum_{j=1}^{k} \frac{1}{j!}(D^j f)(a)(\underbrace{x-a,\cdots,x-a}_{j\text{ copies of }x-a}) + \frac{1}{(k+1)!}(D^{k+1}f)(c)(\underbrace{x-a,\cdots,x-a}_{(k+1)\text{ copies of }x-a})\,.$$

Proof. Let $g(t) = f\big((1-t)a+tx\big)$. Then $g : (-\delta, 1+\delta) \to \mathbb{R}$ is of class $\mathscr{C}^{k+1}$; thus Theorem 6.84 implies that for some $t_0 \in (0,1)$,

$$g(1) - g(0) = \sum_{j=1}^{k} \frac{g^{(j)}(0)}{j!} + \frac{g^{(k+1)}(t_0)}{(k+1)!}\,. \tag{6.8.5}$$

By the chain rule,

$$g'(t) = (Df)\big((1-t)a+tx\big)(x-a) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\big((1-t)a+tx\big)(x_i-a_i)\,;$$

thus

$$g''(t) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i}\big((1-t)a+tx\big)(x_i-a_i)(x_j-a_j) = (D^2 f)\big((1-t)a+tx\big)(x-a, x-a).$$

By induction, we conclude that

$$g^{(j)}(t) = (D^j f)\big((1-t)a+tx\big)(\underbrace{x-a,\cdots,x-a}_{j\text{ copies of }x-a})\,;$$

thus with $c = (1-t_0)a + t_0 x$, (6.8.5) implies that

$$f(x) - f(a) = \sum_{j=1}^{k} \frac{1}{j!}(D^j f)(a)(x-a, \cdots, x-a) + \frac{1}{(k+1)!}(D^{k+1} f)(c)(x-a, \cdots, x-a). \quad \square$$

Definition 6.86. Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, and $f : \mathcal{U} \to \mathbb{R}$ be of class $\mathscr{C}^k$. The k-th degree Taylor polynomial for f centered at a is the polynomial

$$\sum_{j=0}^{k} \frac{1}{j!}(D^j f)(a)(\underbrace{x-a, \cdots, x-a}_{j \text{ copies } x-a}).$$

Corollary 6.87. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, $f : \mathcal{U} \to \mathbb{R}$ be of class $\mathscr{C}^{k+1}$, and define the remainder*

$$R_k(a,h) = f(a+h) - \sum_{j=0}^{k} \frac{1}{j!}(D^j f)(a)(h, \cdots, h).$$

Then $\lim\limits_{h\to 0} \dfrac{R_k(a,h)}{\|h\|_{\mathbb{R}^n}^k} = 0$, or in notation, $R_k(a,h) = o(\|h\|_{\mathbb{R}^n}^k)$ as $h \to 0$.

Example 6.88. Let $f(x,y) = e^x \cos y$. Compute the fourth degree Taylor polynomial for f centered at $(0,0)$.

Solution: We compute the zeroth, the first, the second, the third and the fourth mixed derivatives of f at $(0,0)$ as follows:

$$\begin{aligned}
f(0,0) &= 1\,, \qquad f_x(0,0) = 1\,, \qquad f_y(0,0) = 0\,,\\
f_{xx}(0,0) &= 1\,, \qquad f_{xy}(0,0) = f_{yx}(0,0) = 0\,, \qquad f_{yy}(0,0) = -1\,,\\
f_{xxx}(0,0) &= 1\,, \qquad f_{xxy}(0,0) = f_{xyx}(0,0) = f_{yxx}(0,0) = 0\,,\\
f_{yyy}(0,0) &= 0\,, \qquad f_{yyx}(0,0) = f_{yxy}(0,0) = f_{xyy}(0,0) = -1\,,
\end{aligned}$$

and

$$\begin{aligned}
f_{xxxx}(0,0) &= 1\,, \qquad f_{yyyy}(0,0) = 1\,,\\
f_{xxxy}(0,0) &= f_{xxyx}(0,0) = f_{xyxx}(0,0) = f_{yxxx}(0,0) = 0\,,\\
f_{xyyy}(0,0) &= f_{yxyy}(0,0) = f_{yyxy}(0,0) = f_{yyyx}(0,0) = 0\,,\\
f_{xxyy}(0,0) &= f_{xyxy}(0,0) = f_{xyyx}(0,0) = f_{yxxy}(0,0)\\
&= f_{yxyx}(0,0) = f_{yyxx}(0,0) = -1\,.
\end{aligned}$$

Then the fourth degree Taylor polynomial for f centered at $(0,0)$ is

$$\begin{aligned} & f(0,0)+f_x(0,0)x+f_y(0,0)y+\frac{1}{2}\Big[f_{xx}(0,0)x^2+2f_{xy}(0,0)xy+f_{yy}(0,0)y^2\Big] \\ &\qquad +\frac{1}{6}\Big[f_{xxx}(0,0)x^3+3f_{xxy}(0,0)x^2y+3f_{xyy}(0,0)xy^2+f_{yyy}(0,0)y^3\Big] \\ &\qquad +\frac{1}{24}\Big[f_{xxxx}(0,0)x^4+4f_{xxxy}(0,0)x^3+6f_{xxyy}(0,0)x^2y^2 \\ &\qquad\qquad +4f_{xyyy}(0,0)xy^3+f_{yyyy}(0,0)y^4\Big] \\ &\quad = 1+x+\frac{1}{2}\big(x^2-y^2\big)+\frac{1}{6}\big(x^3-3xy^2\big)+\frac{1}{24}\big(x^4-6x^2y^2+y^4\big)\,. \end{aligned}$$

Observing that using the Taylor expansions

$$e^x = 1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\cdots \quad \text{and} \quad \cos y = 1-\frac{1}{2}y^2+\frac{1}{24}y^4+\cdots\,,$$

we can "formally" compute $e^x\cos y$ by multiplying the two "polynomials" above and obtain that

$$e^x\cos y \text{ “=” } 1+x+\frac{1}{2}\big(x^2-y^2\big)+\big(\frac{1}{6}x^3-\frac{1}{2}xy^2\big)+\big(\frac{1}{24}x^4-\frac{1}{4}x^2y^2+\frac{1}{24}y^2\big)+\text{h.o.t.}\,;$$

where h.o.t. stands for the higher order terms which are terms with fifth or higher degree.

Definition 6.89. Let $\mathcal{U}\subseteq\mathbb{R}^n$ be open. A function $f:\mathcal{U}\to\mathbb{R}$ is said to be ***real analytic*** at $a\in\mathcal{U}$ if $f(x)=\sum\limits_{k=0}^{\infty}\frac{1}{k!}(D^kf)(a)(x-a,\cdots,x-a)$ in a neighborhood of a.

Example 6.90. Let $f:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(x)=\begin{cases} \exp\big(-\dfrac{1}{|x|^2}\big) & \text{if } x>0\,, \\ 0 & \text{if } x<0\,. \end{cases}$$

Then f is of class $\mathscr{C}^\infty$, and $f^{(k)}(0)=0$ for all $k\in\mathbb{N}$. Therefore, f is not real analytic at 0.

6.9 Maxima and Minima

Definition 6.91. Let $\mathcal{U}\subseteq\mathbb{R}^n$ be open, and $f:\mathcal{U}\to\mathbb{R}$.

1. If there is a neighborhood of $x_0\in\mathcal{U}$ such that $f(x_0)$ is a maximum in this neighborhood, then x_0 is called a ***local maximum point*** of f.

2. If there is a neighborhood of $x_0 \in \mathcal{U}$ such that $f(x_0)$ is a minimum in this neighborhood, then x_0 is called a ***local minimum point*** of f.

3. A point is called an ***extreme point*** of f if it is either a local maximum point or a local minimum point of f.

4. A point x_0 is a ***critical point*** of f if f is differentiable at x_0 and $(Df)(x_0) = 0$; that is, $(Df)(x_0) \in \mathscr{B}(\mathbb{R}^n, \mathbb{R})$ is the trivial map.

5. A point x_0 is a ***saddle point*** of f if x_0 is a critical point of f but not an extreme point of f.

Theorem 6.92. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, $f : \mathcal{U} \to \mathbb{R}$ be differentiable, and $x_0 \in \mathcal{U}$ is an extreme point of f. Then x_0 is a critical point of f.*

Proof. Suppose the contrary that the linear map $(Df)(x_0) : \mathbb{R}^n \to \mathbb{R}$ is not the zero map; that is, $\exists\, u \in \mathbb{R}^n$, $u \neq 0 \ni Df(x_0)(u) = c \neq 0$ for some constant $c \in \mathbb{R}$. W.L.O.G, we can assume that $c > 0$ (for otherwise change u to $-u$). By the differentiability of f,

$$\exists\, \delta > 0 \ni \big|f(x_0 + h) - f(x_0) - Df(x_0)(h)\big| \leqslant \frac{c}{2\|u\|_{\mathbb{R}^n}} \|h\|_{\mathbb{R}^n} \quad \text{whenever} \quad \|h\|_{\mathbb{R}^n} < \delta\,.$$

Take $\lambda_0 > 0$ such that $\lambda_0 \|u\|_{\mathbb{R}^n} < \delta$. Then for any $0 < \lambda \leqslant \lambda_0$, $\|\lambda u\|_{\mathbb{R}^n} < \delta$; thus

$$\big|f(x_0 \pm \lambda u) - f(x_0) \mp \lambda Df(x_0)(u)\big| \leqslant \frac{c}{2\|u\|_{\mathbb{R}^n}} \cdot \lambda \|u\|_{\mathbb{R}^n} = \frac{\lambda c}{2}\,.$$

Therefore, $-\dfrac{\lambda c}{2} \leqslant f(x_0 \pm \lambda u) - f(x_0) \mp \lambda c \leqslant \dfrac{\lambda c}{2}$ which further implies that

$$f(x_0) \leqslant f(x_0 + \lambda u) - \frac{\lambda c}{2} < f(x_0 + \lambda u) \ \text{ and } \ f(x_0) \geqslant f(x_0 - \lambda u) + \frac{\lambda c}{2} > f(x_0 - \lambda u)$$

for all $\lambda > 0$ small enough. As a consequence, x_0 cannot be a local extreme point of f, a contradiction. □

Definition 6.93. If $f : \mathcal{U} \to \mathbb{R}$ is of class $\mathscr{C}^2$, the ***Hessian of f at*** x_0 is the bilinear function $H_{x_0}(f) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by

$$H_{x_0}(f)(u, v) = (D^2 f)(x_0)(u, v) \qquad \forall\, u, v \in \mathbb{R}^n\,.$$

The matrix representation of $H_{x_0}(g)(\cdot,\cdot)$ is given by

$$\big[H_{x_0}(f)\big] = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2}(x_0) & \cdots & \dfrac{\partial^2 f}{\partial x_n \partial x_1}(x_0) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_1 \partial x_n}(x_0) & \cdots & \dfrac{\partial^2 f}{\partial x_n^2}(x_0) \end{bmatrix}$$

in the sense that $H_{x_0}(f)(u,v) = [u]^{\mathrm{T}}\big[H_{x_0}(f)\big][v] = [v]^{\mathrm{T}}\big[H_{x_0}(f)\big][u]$.

Definition 6.94. A bilinear form $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is called ***positive definite*** / ***negative definite*** if $B(u,u) \begin{smallmatrix} > \\ < \end{smallmatrix} 0$ for all $u \neq 0$, and is called ***positive semi-definite*** / ***negative semi-definite*** if $B(u,u) \begin{smallmatrix} \geqslant \\ \leqslant \end{smallmatrix} 0$ for all $u \in \mathbb{R}^n$.

Theorem 6.95. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, and $f : \mathcal{U} \to \mathbb{R}$ be a function of class $\mathscr{C}^2$.*

1. *If x_0 is a critical point of f such that the Hessian $H_{x_0}(f)$ is negative / positive definite, then f has a local maximum / minimum point at x_0.*

2. *If f has a local maximum / minimum point at x_0, then $H_{x_0}(f)$ is negative / positive semi-definite.*

Proof. 1. Suppose that $H_{x_0}(f)$ is negative definite.

Claim: There exists $-\infty < \lambda < 0$ such that

$$H_{x_0}(f)(u,u) \leqslant \lambda \|u\|_{\mathbb{R}^n}^2 \qquad \forall\, u \in \mathbb{R}^n. \tag{6.9.1}$$

Proof of claim: Let $\lambda = \max\limits_{u\in\mathbb{R}^n, \|u\|_{\mathbb{R}^n}=1} H_{x_0}(f)(u,u)$. Then $-\infty < \lambda < 0$, and for all $u \in \mathbb{R}^n$ with $u \neq 0$,

$$H_{x_0}(f)\Big(\frac{u}{\|u\|_{\mathbb{R}^n}}, \frac{u}{\|u\|_{\mathbb{R}^n}}\Big) \leqslant \lambda \qquad \forall\, u \in \mathbb{R}^n, u \neq 0\,.$$

The inequality (6.9.1) follows from that the Hessian $H_{x_0}(f)$ is bilinear.

Since $f \in \mathscr{C}^2$, there exists $\delta > 0$ such that $D(x_0,\delta) \subseteq \mathcal{U}$ and

$$\big\|H_x(f) - H_{x_0}(f)\big\|_{\mathscr{B}(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n,\mathbb{R}))} \leqslant -\lambda \qquad \forall\, x \in D(x_0,\delta).$$

Note that the inequality above suggests that

$$\big|H_x(f)(u,u) - H_{x_0}(f)(u,u)\big| = \big|\big(H_x(f) - H_{x_0}(f)\big)(u,u)\big| < -\lambda\|u\|^2_{\mathbb{R}^2}. \tag{6.9.2}$$

Now since x_0 is a critical point of f, $(Df)(x_0) = 0$. As a consequence, by Taylor's theorem (Theorem 6.85), for any $x \in D(x_0,\delta)$, we can find $c = c(x) \in \overline{xx_0}$ such that

$$\begin{aligned}
f(x) &= f(x_0) + (Df)(x_0)(x-x_0) + \frac{1}{2}(D^2f)(c)(x-x_0,x-x_0)\\
&= f(x_0) + \frac{1}{2}(D^2f)(x_0)(x-x_0,x-x_0) + \frac{1}{2}\big[(D^2f)(c) - (D^2f)(x_0)\big](x-x_0,x-x_0)\\
&\leqslant f(x_0) + \frac{1}{2}\lambda\|x-x_0\|^2_{\mathbb{R}^n} + \frac{1}{2}\big[(D^2f)(c) - (D^2f)(x_0)\big](x-x_0,x-x_0)\\
&= f(x_0) + \frac{1}{2}\lambda\|x-x_0\|^2_{\mathbb{R}^n} + \frac{1}{2}(x-x_0)^{\mathrm{T}}\big[H_c(f) - H_{x_0}(f)\big](x-x_0)\,.
\end{aligned}$$

Note that $c = c(x) \in D(x_0,\delta)$ if $x \in D(x_0,\delta)$; thus (6.9.2) implies that

$$(x-x_0)^{\mathrm{T}}\big[H_c(f) - H_{x_0}(f)\big](x-x_0) \leqslant -\lambda\|x-x_0\|^2_{\mathbb{R}^n}.$$

As a consequence, for all $x \in D(x_0,\delta)$, $f(x) \leqslant f(x_0)$ which validates that x_0 is a local maximum point of f.

2. Suppose the contrary that f has a local maximum point at x_0 but for some $u \in \mathbb{R}^n$,

$$H_{x_0}(f)(u,u) > 0\,.$$

By Theorem 6.92, $(Df)(x_0) = 0$; thus Taylor's Theorem suggests that

$$f(x) = f(x_0) + \frac{1}{2}(D^2f)(c)(x-x_0,x-x_0) = f(x_0) + \frac{1}{2}(x-x_0)^{\mathrm{T}}\big[H_c(f)\big](x-x_0)\,.$$

Since x_0 is a local maximum point of f, there exists $\delta > 0$ such that $f(x) \leqslant f(x_0)$ for all $x \in D(x_0,\delta)$. As a consequence,

$$(x-x_0)^{\mathrm{T}}\big[H_c(f)\big](x-x_0) = 2\big[f(x) - f(x_0)\big] \leqslant 0 \qquad \forall\, x \in D(x_0,\delta)\,.$$

Let $0 < t < \delta$ and $x = x_0 + t\dfrac{u}{\|u\|_{\mathbb{R}^n}}$. Then $x \in D(x_0,\delta)$; thus

$$H_c(f)(u,u) \leqslant 0 \qquad \forall\, t \in (0,\delta)\,.$$

We note that c depends on t, and $c \to x_0$ as $t \to 0$. Therefore, by the continuity of $H_\bullet(f)$, passing $t \to 0$ in the inequality above we find that

$$H_{x_0}(f)(u,u) \leqslant 0$$

which is a contradiction. □

Remark 6.96. Inequality (6.9.1) can also be obtained by studying the largest eigenvalue of $H_{x_0}(f)$. Note that since $f \in \mathscr{C}^2$, $H_{x_0}(f)$ is symmetric by Theorem 6.80. As a consequence, there exists an orthonormal matrix $\mathbb{O} \in \mathrm{GL}(n)$ whose columns are (real) eigenvectors of $H_{x_0}(f)$

$$H_{x_0}(f) = \mathbb{O}\Lambda\mathbb{O}^{\mathrm{T}},$$

where Λ is a diagonal matrix whose diagonal entries are eigenvalues of $H_{x_0}(f)$. Note that by the orthonormality of $\mathbb{O}$, every vector $u \in \mathbb{R}^n$ satisfies $\|\mathbb{O}^{\mathrm{T}}u\|_{\mathbb{R}^n} = \|u\|_{\mathbb{R}^n}$. Therefore,

$$H_{x_0}(f)(u,u) = u^{\mathrm{T}}\mathbb{O}\Lambda\mathbb{O}^{\mathrm{T}}u = (\mathbb{O}^{\mathrm{T}}u)^{\mathrm{T}}\Lambda(\mathbb{O}^{\mathrm{T}}u) \leqslant \lambda\|\mathbb{O}^{\mathrm{T}}u\|^2_{\mathbb{R}^n} = \lambda\|u\|^2_{\mathbb{R}^n},$$

where λ is the largest eigenvalue of Λ.

Remark 6.97 (Sylvester's criterion)**.** To justify if a matrix $\big[H_{x_0}(f)\big]$ is positive/negative definite, let

$$\Delta_k = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \cdots & \dfrac{\partial^2 f}{\partial x_k \partial x_1} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_1 \partial x_k} & \cdots & \dfrac{\partial^2 f}{\partial x_k^2} \end{bmatrix} (x_0).$$

Then $H_{x_0}(f)$ is $\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}$ definite if and only if $\begin{matrix}\det(\Delta_k) > 0\\ (-1)^k\det(\Delta_k) > 0\end{matrix}$ for all $k = 1, \cdots, n$.

Chapter 7

The Inverse and Implicit Function Theorems

7.1 The Inverse Function Theorem（反函數定理）

反函數定理是用來探討一個函數的反函數是否存在的問題。只要一個函數不是一對一的，一般來說都不能定義其反函數，例如三角函數中，正弦、餘弦及正切函數都是周期函數，所以全域的反函數不存在。但是我們也知道有所謂的反三角函數 $\sin^{-1}$（或 arcsin），$\cos^{-1}$（或 arctan）及 $\tan^{-1}$（或 arctan），這是因為我們限制了原三角函數的定義域使其在新的定義域上是一對一的（因此反函數存在）。因此，要討論一個定在某一個（大範圍的）定義域的函數的反函數，常常我們最多只能說反函數只在某一小塊區域上存在。

如何知道一個函數在一小塊區域上的反函數存在，我們首先該問的是在定義域是一維（或是指單變數函數）的情況下發生什麼事？由一維的反函數定理 (Theorem 4.71) 我們知道首先應該要保留的條件是類似於微分不為零的這個條件。但是在多變數函數之下，微分不為零的條件該怎麼呈現，這是第一個問題。而當我們觀察 (4.6.1)，應該可以猜出在多變數版本裡面所該對應到的條件，即是 $(Df)(x)$ 這個 bounded linear map 的可逆性。

另外，假設 $f \in \mathscr{C}'$，那麼由 Theorem 6.8 我們知道在一個點 x_0 如果 $(Df)(x_0)$ 可逆的話，那麼在一個鄰域裡 $(Df)(x)$ 都可逆。所以下面這個反函數定理的條件中只有 (Df) 在一個點可逆這個條件，因為我們目前想先知道小區域的反函數存不存在。

Theorem 7.1 (Inverse Function Theorem). *Let $\mathcal{D} \subseteq \mathbb{R}^n$ be open, $x_0 \in \mathcal{D}$, $f : \mathcal{D} \to \mathbb{R}^n$ be of class $\mathscr{C}^1$, and $(Df)(x_0)$ be invertible. Then there exist an open neighborhood $\mathcal{U}$ of x_0 and an open neighborhood $\mathcal{V}$ of $f(x_0)$ such that*

1. *$f : \mathcal{U} \to \mathcal{V}$ is one-to-one and onto;*

2. *The inverse function* $f^{-1} : \mathcal{V} \to \mathcal{U}$ *is of class* $\mathscr{C}^1$*;*

3. *If* $x = f^{-1}(y)$*, then* $(Df^{-1})(y) = \big((Df)(x)\big)^{-1}$*;*

4. *If* f *is of class* $\mathscr{C}^r$ *for some* $r > 1$*, so is* f^{-1}*.*

Proof. Assume that $A = (Df)(x_0)$. Then $\|A^{-1}\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)} \neq 0$. Choose $\lambda > 0$ such that $2\lambda\|A^{-1}\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)} = 1$. Since $f \in \mathscr{C}^1$, there exists $\delta > 0$ such that

$$\big\|(Df)(x) - A\big\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)} = \big\|(Df)(x) - (Df)(x_0)\big\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)} < \lambda \quad \text{whenever } x \in D(x_0,\delta) \cap \mathcal{D}\,.$$

By choosing δ even smaller if necessary, we can assume that $D(x_0,\delta) \subseteq \mathcal{D}$. Let $\mathcal{U} = D(x_0,\delta)$.
Claim: $f : \mathcal{U} \to \mathbb{R}^n$ is one-to-one (hence $f : \mathcal{U} \to f(\mathcal{U})$ is one-to-one and onto).
Proof of claim: For each $y \in \mathbb{R}^n$, define $\varphi_y(x) = x + A^{-1}\big(y - f(x)\big)$ (and we note that every fixed-point of φ_y corresponds to a solution to $f(x) = y$). Then

$$(D\varphi_y)(x) = \mathrm{Id} - A^{-1}(Df)(x) = A^{-1}\big(A - (Df)(x)\big)\,,$$

where Id is the identity map on $\mathbb{R}^n$. Therefore,

$$\big\|(D\varphi_y)(x)\big\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)} \leqslant \|A^{-1}\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)}\big\|A - (Df)(x)\big\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)} < \frac{1}{2} \quad \forall\, x \in D(x_0,\delta)\,.$$

By the mean value theorem (Theorem 6.58), (see Remark 7.2)

$$\big\|\varphi_y(x_1) - \varphi_y(x_2)\big\|_{\mathbb{R}^n} < \frac{1}{2}\|x_1 - x_2\|_{\mathbb{R}^n} \qquad \forall\, x_1, x_2 \in D(x_0,\delta),\ x_1 \neq x_2\,; \tag{7.1.1}$$

thus at most one x satisfies $\varphi_y(x) = x$; that is, φ_y has at most one fixed-point. As a consequence, $f : D(x_0,\delta) \to \mathbb{R}^n$ is one-to-one.
Claim: The set $\mathcal{V} = f(\mathcal{U})$ is open.
Proof of claim: Let $b \in \mathcal{V}$. Then there is $a \in \mathcal{V}$ with $f(a) = b$. Choose $r > 0$ such that $\overline{D(a,r)} \subseteq \mathcal{U}$. We observe that if $y \in D(b,\lambda r)$, then

$$\|\varphi_y(a) - a\|_{\mathbb{R}^n} \leqslant \|A^{-1}\big(y - f(a)\big)\|_{\mathbb{R}^n} \leqslant \|A^{-1}\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)}\|y - b\|_{\mathbb{R}^n} < \lambda\|A^{-1}\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)}r = \frac{r}{2}\,;$$

thus if $y \in D(b,\lambda r)$ and $x \in D(a,r)$,

$$\|\varphi_y(x) - a\|_{\mathbb{R}^n} \leqslant \big\|\varphi_y(x) - \varphi_y(a)\big\|_{\mathbb{R}^n} + \|\varphi_y(a) - a\|_{\mathbb{R}^n} < \frac{1}{2}\|x - a\|_{\mathbb{R}^n} + \frac{r}{2} < r\,.$$

Therefore, if $y \in D(b,\lambda r)$, then $\varphi_y : D(a,r) \to D(a,r)$. By the continuity of φ_y,

$$\varphi_y : \overline{D(a,r)} \to \overline{D(a,r)}\,.$$

On the other hand, (7.1.1) implies that φ_y is a contraction mapping if $y \in D(b, \lambda r)$; thus by the contraction mapping principle 5.77 φ_y has a unique fixed-point $x \in D(a, r)$. As a result, every $y \in D(b, \lambda r)$ corresponds to a unique $x \in D(a, r)$ such that $\varphi_y(x) = x$ or equivalently, $f(x) = y$. Therefore,

$$D(b, \lambda r) \subseteq f\big(D(a, r)\big) \subseteq f(\mathcal{U}) = \mathcal{V}\,.$$

Next we show that $f^{-1} : \mathcal{V} \to \mathcal{U}$ is differentiable. We note that if $x \in D(x_0, \delta)$,

$$\|(Df)(x_0) - (Df)(x)\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)} \|A^{-1}\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)} < \lambda \|A^{-1}\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)} = \frac{1}{2}\,;$$

thus Theorem 6.8 implies that $(Df)(x)$ is invertible if $x \in D(x_0, \delta)$.

Let $b \in \mathcal{V}$ and $k \in \mathbb{R}^n$ such that $b + k \in \mathcal{V}$. Then there exists a unique $a \in \mathcal{U}$ and $h = h(k) \in \mathbb{R}^n$ such that $a + h \in \mathcal{U}$, $b = f(a)$ and $b + k = f(a + h)$. By the mean value theorem and (7.1.1),

$$\big\|\varphi_y(a + h) - \varphi_y(a)\big\|_{\mathbb{R}^n} < \frac{1}{2}\|h\|_{\mathbb{R}^n}\,;$$

thus the fact that $f(a + h) - f(a) = k$ implies that

$$\|h - A^{-1}k\|_{\mathbb{R}^n} < \frac{1}{2}\|h\|_{\mathbb{R}^n}$$

which further suggests that

$$\frac{1}{2}\|h\|_{\mathbb{R}^n} \leqslant \|A^{-1}k\|_{\mathbb{R}^n} \leqslant \|A^{-1}\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)}\|k\|_{\mathbb{R}^n} \leqslant \frac{1}{2\lambda}\|k\|_{\mathbb{R}^n}\,. \tag{7.1.2}$$

As a consequence, if k is such that $b + k \in \mathcal{V}$,

$$\begin{aligned}
\frac{\big\|f^{-1}(b + k) - f^{-1}(b) - \big((Df)(a)\big)^{-1}k\big\|_{\mathbb{R}^n}}{\|k\|_{\mathbb{R}^n}} &= \frac{\big\|a + h - a - \big((Df)(a)\big)^{-1}k\big\|_{\mathbb{R}^n}}{\|k\|_{\mathbb{R}^n}} \\
&\leqslant \big\|\big((Df)(a)\big)^{-1}\big\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)} \frac{\big\|k - (Df)(a)(h)\big\|_{\mathbb{R}^n}}{\|k\|_{\mathbb{R}^n}} \\
&\leqslant \big\|\big((Df)(a)\big)^{-1}\big\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)} \frac{\big\|f(a + h) - f(a) - (Df)(a)(h)\big\|_{\mathbb{R}^n}}{\|h\|_{\mathbb{R}^n}} \frac{\|h\|_{\mathbb{R}^n}}{\|k\|_{\mathbb{R}^n}} \\
&\leqslant \frac{\big\|\big((Df)(a)\big)^{-1}\big\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)}}{\lambda} \frac{\big\|f(a + h) - f(a) - (Df)(a)(h)\big\|_{\mathbb{R}^n}}{\|h\|_{\mathbb{R}^n}}\,.
\end{aligned}$$

Using (7.1.2), $h \to 0$ as $k \to 0$; thus passing $k \to 0$ on the left-hand side of the inequality above, by the differentiability of f we conclude that

$$\lim_{k \to 0} \frac{\big\|f^{-1}(b + k) - f^{-1}(b) - \big((Df)(a)\big)^{-1}k\big\|_{\mathbb{R}^n}}{\|k\|_{\mathbb{R}^n}} = 0\,.$$

This proves 3.

To see 4, we note that the map $g : \mathrm{GL}(n) \to \mathrm{GL}(n)$ given by $g(L) = L^{-1}$ is infinitely many time differentiable; thus using the identity

$$(Df^{-1})(y) = \big((Df)(x)\big)^{-1} = \big(g \circ (Df) \circ f^{-1}\big)(y)\,,$$

by the chain rule we find that if $f \in \mathscr{C}^r$, then $Df^{-1} \in \mathscr{C}^{r-1}$ which is the same as saying that $f^{-1} \in \mathscr{C}^r$. □

Remark 7.2. The norm of $\mathbb{R}^n$ used in the proof of the inverse function theorem is given by $\|\cdot\|_{\mathbb{R}^n} \equiv \|\cdot\|_\infty$; that is, if $x = (x_1, \cdots, x_n)$, then

$$\|x\|_\infty = \max\big\{|x_1|, \cdots, |x_n|\big\}\,.$$

Note that the concept of differentiability of a function $f : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ remains unchanged since the infinity-norm $\|\cdot\|_\infty$ is equivalent to the two-norm $\|\cdot\|_2$. Write $\varphi_y = (\varphi_1, \cdots, \varphi_n)$ (ignore the subscript y for a moment). Then Example 1.133 implies that for each $i \in \{1, \cdots, n\}$,

$$\|(D\varphi_i)(x)\|_{\mathbb{R}^n} \leqslant \max_{1\leqslant i\leqslant n} \|(D\varphi_i)(x)\|_{\mathbb{R}^n} \leqslant \max_{1\leqslant i\leqslant n} \sum_{j=1}^n \Big|\frac{\partial \varphi_i}{\partial x_j}(x)\Big| = \|D\varphi_y(x)\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)}\,.$$

Therefore, the mean value theorem implies that if $x_1, x_2 \in D(x_0, \delta)$ and $x_1 \neq x_2$,

$$\begin{aligned}\|\varphi_y(x_1) - \varphi_y(x_2)\|_{\mathbb{R}^n} &= \max_{1\leqslant i\leqslant n} \big\|\varphi_i(x_1) - \varphi_i(x_2)\big\| = \max_{1\leqslant i\leqslant n} \big|D\varphi_i(c_i)(x_1 - x_2)\big| \\ &\leqslant \max_{1\leqslant i\leqslant n} \|D\varphi_y(x)\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)} \|x_1 - x_2\|_{\mathbb{R}^n}\,.\end{aligned}$$

Remark 7.3. Since $f^{-1} : \mathcal{V} \to \mathcal{U}$ is continuous, for any open subset $\mathcal{W}$ of $\mathcal{U}$ $f(\mathcal{W}) = (f^{-1})^{-1}(\mathcal{W})$ is open relative to $\mathcal{V}$, or $f(\mathcal{W}) = \mathcal{O} \cap \mathcal{V}$ for some open set $\mathcal{O} \subseteq \mathbb{R}^n$. In other words, if $\mathcal{U}$ is an open neighborhood of x_0 given by the inverse function theorem, then $f(\mathcal{W})$ is also open for all open subsets $\mathcal{W}$ of $\mathcal{U}$. We call this property as f is a ***local open mapping*** at x_0.

Remark 7.4. Since $(Df)(x_0) \in \mathscr{B}(\mathbb{R}^n, \mathbb{R}^n)$, the condition that $(Df)(x_0)$ is invertible can be replaced by that the determinant of the Jacobian matrix of f at x_0 is not zero; that is,

$$\det\big(\big[(Df)(x_0)\big]\big) \neq 0\,.$$

The determinant of the Jacobian matrix of f at x_0 is called the ***Jacobian*** of f at x_0. The Jacobian of f at x sometimes is denoted by $\mathbf{J}_f(x)$ or $\dfrac{\partial(f_1, \cdots, f_n)}{\partial(x_1, \cdots, x_n)}$.

Example 7.5. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x + 2x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0\,, \\ 0 & \text{if } x = 0\,. \end{cases}$$

Let $0 \in (a, b)$ for some (small) open interval (a, b). Since $f'(x) = 1 - 2\cos\dfrac{1}{x} + 4x \sin\dfrac{1}{x}$ for $x \neq 0$, f has infinitely many critical points in (a, b), and (for whatever reasons) these critical points are local maximum points or local minimum points of f which implies that f is **not** locally invertible even though we have $f'(0) = 1 \neq 0$. One cannot apply the inverse function theorem in this case since f is not $\mathscr{C}^1$.

Corollary 7.6. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, $f : \mathcal{U} \to \mathbb{R}^n$ be of class $\mathscr{C}^1$, and $(Df)(x)$ be invertible for all $x \in \mathcal{U}$. Then $f(\mathcal{W})$ is open for every open set $\mathcal{W} \subseteq \mathcal{U}$.*

在證明了小區域的（local）反函數定理 (Theorem 7.1) 之後，我們接下來要問的是全域的（global）反函數在什麼條件之下會存在的情況。如果照一維的反函數定理，我們會猜測是不是只要 $(Df)(x)$ 在整個區域都可逆就能得到在全域的反函數都存在。以下給個反例說單單在這個條件之下，函數不一定會有一對一的性質。

Example 7.7. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(x, y) = (e^x \cos y, e^x \sin y)\,.$$

Then

$$\big[(Df)(x, y)\big] = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}.$$

It is easy to see that the Jacobian of f at any point is not zero (thus $(Df)(x)$ is invertible for all $x \in \mathbb{R}^2$), and f is not globally one-to-one (thus the inverse of f does not exist globally) since for example, $f(x, y) = f(x, y + 2\pi)$.

要再加什麼條件進來才能得到反函數在全域都存在是個不容易的問題。在一維的情況下，導數是 sign definite 就表示函數在全域是嚴格單調的。在高維度的情況，即使是 $(Df)(x)$ 到處都可逆，仍然有很多情況可能發生（如上例）。下面這個定理（全域的反函數存在定理），從某種角度來說並沒有真的加了什麼條件以確保全域的反函數存在，只是多要求了在所考慮的區域邊界上函數是一對一的。這個條件在一維的情況之下是自動成立的：因為如果一單變數函數的導數是 sign definite，那麼函數在邊界上必定是一對一的（因為嚴格單調的關係）。

Theorem 7.8 (Global Existence of Inverse Function)**.** *Let $\mathcal{D} \subseteq \mathbb{R}^n$ be open, $f : \mathcal{D} \to \mathbb{R}^n$ be of class $\mathscr{C}^1$, and $(Df)(x)$ be invertible for all $x \in K$. Suppose that K is a connected compact subset of $\mathcal{D}$, and $f : \partial K \to \mathbb{R}^n$ is one-to-one. Then $f : K \to \mathbb{R}^n$ is one-to-one.*

Proof. Define $E = \{x \in K \mid \exists\, y \in K, y \neq x \ni f(x) = f(y)\}$. Our goal is to show that $E = \varnothing$.
Claim 1: E is closed.
Proof of claim 1: Suppose the contrary that E is not closed. Then there exists $\{x_k\}_{k=1}^{\infty} \subseteq E$, $x_k \to x$ as $k \to \infty$ but $x \in K \backslash E$. Since $x_k \in E$, by the definition of E there exists $y_k \in E$ such that $y_k \neq x_k$ and $f(x_k) = f(y_k)$. By the compactness of K, there exists a convergent subsequence $\{y_{k_j}\}_{j=1}^{\infty}$ of $\{y_k\}_{k=1}^{\infty}$ with limit $y \in K$. Since $x \notin E$ and $f(x_{k_j}) = f(y_{k_j}) \to f(y)$ as $j \to \infty$, we must have $x = y$; thus $y_{k_j} \to x$ as $j \to \infty$.

Since $(Df)(x)$ is invertible, by the inverse function theorem there exists $\delta > 0$ such that $f : D(x,\delta) \to \mathbb{R}^n$ is one-to-one. By the convergence of sequences $\{x_{k_j}\}_{j=1}^{\infty}$ and $\{y_{k_j}\}_{j=1}^{\infty}$, there exists $N > 0$ such that

$$x_{k_j}, y_{k_j} \in D(x,\delta) \qquad \forall\, j \geqslant N\,.$$

This implies that $f : D(x,\delta) \to \mathbb{R}^n$ cannot be one-to-one $\big($since $x_{k_j} \neq y_{k_j}$ but $f(x_{k_j}) = f(y_{k_j})\big)$, a contradiction. Therefore, E is closed.
Claim 2: E is open relative to K; that is, for every $x \in E$, there exists an open set $\mathcal{U}$ such that $x \in \mathcal{U}$ and $\mathcal{U} \cap K \subseteq E$.
Proof of claim 2: Let $x_1 \in E$. Then there is $x_2 \in E$, $x_2 \neq x_1$, such that $f(x_1) = f(x_2)$. Since $(Df)(x_1)$ and $(Df)(x_2)$ are invertible, by the inverse function theorem there exist open neighborhoods $\mathcal{U}_1$ of x_1 and $\mathcal{U}_2$ of x_2, as well as open neighborhoods $\mathcal{V}_1$, $\mathcal{V}_2$ of $f(x_1)$, such that $f : \mathcal{U}_1 \to \mathcal{V}_1$ and $f : \mathcal{U}_2 \to \mathcal{V}_2$ are both one-to-one and onto. Since $x_1 \neq x_2$, W.L.O.G. we can assume that $\mathcal{U}_1 \cap \mathcal{U}_2 = \varnothing$. Since $\mathcal{V}_1 \cap \mathcal{V}_2$ is open, the continuity of f implies that $f^{-1}(\mathcal{V}_1 \cap \mathcal{V}_2) = \mathcal{O} \cap \mathcal{D}$ for some open set $\mathcal{O}$; thus

$$f : \mathcal{U}_1 \cap \mathcal{O} \cap K \to \mathcal{V}_1 \cap \mathcal{V}_2 \cap f(K) \text{ is one-to-one and onto}\,,$$
$$f : \mathcal{U}_2 \cap \mathcal{O} \cap K \to \mathcal{V}_1 \cap \mathcal{V}_2 \cap f(K) \text{ is one-to-one and onto}\,.$$

Let $\mathcal{U} = \mathcal{U}_1 \cap \mathcal{O}$. Then every $x \in \mathcal{U} \cap K$ corresponds to a unique $\widetilde{x} \in \mathcal{U}_2 \cap \mathcal{O} \cap K$ such that $f(x) = f(\widetilde{x})$. Since $\mathcal{U}_1 \cap \mathcal{U}_2 = \varnothing$, we must have $x \neq \widetilde{x}$. Therefore, $x \in E$, or equivalently, $\mathcal{U} \cap K \subseteq E$.

Now we show that $E = \varnothing$. Since K is connected, E is open relative to K and E is closed, Remark 3.46 implies that $E = K$ or $E = \varnothing$. Suppose the case that $E = K$. Let

$x \in \partial K \subseteq E$. Then there exists $y \in E$ such that $y \neq x$ and $f(x) = f(y)$. Since $f : \partial K \to \mathbb{R}^n$ is one-to-one, $y \notin \partial K$. Therefore, we have shown that if $E = K$, then $f(\partial K) \subseteq f\big(\text{int}(K)\big)$.

By Theorem 4.21, the compactness of K implies that $f(K)$ is compact; thus there is $b \in \mathbb{R}^n$ such that $b \notin f(K)$. Consider the function $\varphi : K \to \mathbb{R}$ defined by

$$\varphi(x) = \frac{1}{2}\|f(x) - b\|^2_{\mathbb{R}^n} = \frac{1}{2}\sum_{j=1}^{n} |f_i(x) - b_i|^2 .$$

Then φ is a continuous function on K; thus φ attains its maximum at $x_0 \in K$. Since $f(\partial K) \subseteq f\big(\text{int}(K)\big)$, we can assume that $x_0 \in \text{int}(K)$; thus Theorem 6.92 implies that $(D\varphi)(x_0) = 0$. As a consequence,

$$\big[(Df)(x_0)\big]^{\mathrm{T}}\big[f(x_0) - b\big] = 0 .$$

By the choice of b, $f(x_0) - b \neq 0$; thus we must have that $(Df)(x_0)$ is not invertible, a contradiction. □

Example 7.9. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given as in Example 7.7, and $\mathcal{D} = \big\{(x, y) \,\big|\, x \in \mathbb{R}, 0 < y < 2\pi\big\}$. Then $f : \mathcal{D} \to \mathbb{R}^2$ is one-to-one. If K is a compact subset of $\mathcal{D}$, then $f : K \to \mathbb{R}^2$ is also one-to-one (thus $f : \partial K \to \mathbb{R}^2$ must be one-to-one as well).

Corollary 7.10. *Let $\mathcal{D} \subseteq \mathbb{R}^n$ be a bounded open convex set, and $f : \mathcal{D} \to \mathbb{R}^n$ be of class $\mathscr{C}^1$ such that*

1. *f and Df are continuous on $\bar{\mathcal{D}}$;*
2. *the Jacobian* $\det\big(\big[(Df)(x)\big]\big) \neq 0$ *for all $x \in \bar{\mathcal{D}}$;*
3. *$f : \partial\mathcal{D} \to \mathbb{R}^n$ is one-to-one.*

Then $f : \bar{\mathcal{D}} \to \mathbb{R}^n$ is one-to-one. Moreover, $f^{-1} : f(\bar{\mathcal{D}}) \to \mathbb{R}^n$ is continuous, and $f^{-1} : f(\mathcal{D}) \to \mathcal{D}$ is of class $\mathscr{C}^1$.

Proof. We first claim that there exists a small $\varepsilon > 0$ such that $f : \mathcal{D}_\varepsilon \to \mathbb{R}^n$ is one-to-one, where

$$\mathcal{D}_\varepsilon \equiv \big\{x \in \mathcal{D} \,\big|\, d(x, \partial\mathcal{D}) < \varepsilon\big\}.$$

Assume the contrary that for every $k > 0$, there exists $x_k, y_k \in \mathcal{D}$ such that

(a) $x_k \neq y_k$; (b) $d(x_k, \partial\Omega) < \dfrac{1}{k}$ and $d(y_k, \partial\Omega) < \dfrac{1}{k}$; (c) $f(x_k) = f(y_k)$.

Since $\{x_k\}_{k=1}^\infty$ and $\{y_k\}_{k=1}^\infty$ are bounded (due to the boundedness of $\mathcal{D}$), by the Bolzano-Weierstrass Theorem (or Corollary 3.29) there exist $\{x_{k_j}\}_{j=1}^\infty$ and $\{y_{k_j}\}_{j=1}^\infty$ such that $x_{k_j} \to x \in \bar{\mathcal{D}}$ and $y_{k_j} \to y \in \bar{\mathcal{D}}$. By (b), $x, y \in \partial\mathcal{D}$; thus the fact that $f : \partial\mathcal{D} \to \mathbb{R}^n$ is one-to-one implies that $x = y$. Therefore, $x_{k_j} \to x$ and $y_{k_j} \to x$ as $j \to \infty$.

Let $u_j = \dfrac{x_{k_j} - y_{k_j}}{\|x_{k_j} - y_{k_j}\|_{\mathbb{R}^n}}$. Since $\{u_j\}_{j=1}^\infty$ is bounded in $\mathbb{R}^n$, by the Bolzano-Weierstrass Theorem again there is a convergent subsequence $\{u_{j_\ell}\}_{\ell=1}^\infty$ with limit $u \neq 0$. Moreover, by the convexity of $\mathcal{D}$, the mean value theorem implies that for each $i = 1, \cdots, n$, there exists $c_{i\ell}$ on the line segment joining $x_{k_{j_\ell}}$ and $y_{k_{j_\ell}}$ such that

$$0 = f_i(x_{k_{j_\ell}}) - f_i(y_{k_{j_\ell}}) = (Df_i)(c_{i\ell})(x_{k_{j_\ell}} - y_{k_{j_\ell}}) = \|x_{k_{j_\ell}} - y_{k_{j_\ell}}\|_{\mathbb{R}^n}(Df_i)(c_{i\ell})\big(u_{j_\ell}\big)$$

which by (a) further suggests that $(Df_i)(c_{i\ell})\big(u_{j_\ell}\big) = 0$ for all $i = 1, \cdots, n$ and $\ell \in \mathbb{N}$. Since $c_{i\ell} \to x$ as $\ell \to \infty$, passing $\ell \to \infty$ we conclude that $(Df_i)(x)(u) = 0$. This holds for each $i = 1, \cdots, n$; thus $(Df)(x)(u) = 0$. Therefore, $\det\big([(Df)(x)]\big) = 0$, a contradiction.

Now suppose that there exists $x, y \in \mathcal{D}$ such that $f(x) = f(y)$. Choose a compact set $K \subseteq \mathcal{D}$ such that $x, y \in K$ and $\partial K \subseteq \mathcal{D}_\varepsilon$ (this can be done, for example, by choosing that $K = \bar{\mathcal{D}}\backslash\mathcal{D}_\delta$ for some small $\delta > 0$). Since $f : \mathcal{D}_\varepsilon \to \mathbb{R}^n$ is one-to-one, $f : \partial K \to \mathbb{R}^n$ is one-to-one. By Theorem 7.8, $f : K \to \mathbb{R}^n$ is one-to-one. Then $x = y$; thus $f : \mathcal{D} \to \mathbb{R}^n$ is one-to-one.

Next, we show that $f : \bar{\mathcal{D}} \to \mathbb{R}^n$ is one-to-one. Assume the contrary that there exists $x \in \mathcal{D}$ and $y \in \partial\mathcal{D}$ such that $f(x) = f(y)$. By the inverse function theorem there exists open neighborhood $\mathcal{U}$ of x and $\mathcal{V}$ of $f(x)$ such that $f : \mathcal{U} \to \mathcal{V}$ is one-to-one and onto. By choosing $\mathcal{U}$ even smaller if necessary, we can assume that there exists $\{y_k\}_{k=1}^\infty \subseteq \mathcal{D}\backslash\mathcal{U}$ and $y_k \to y$ as $k \to \infty$. By the continuity of f, $f(y_k) \to f(y)$ as $k \to \infty$. However, since $f : \mathcal{D} \to \mathbb{R}^n$ is one-to-one, $\{f(y_k)\}_{k=1}^\infty \notin \mathcal{V}$; thus $\{f(y_k)\}_{k=1}^\infty$ cannot converge to $f(y)$ as $k \to \infty$ (since $f(y) \in \mathcal{V}$), a contradiction.

Finally, the inverse function theorem implies that $f^{-1} : f(\mathcal{D}) \to \mathcal{D}$ is of class $\mathscr{C}^1$, and the continuity of f^{-1} on $f(\bar{\mathcal{D}})$ follows from the fact that $(f^{-1})^{-1}(F) = f(F)$ is closed in $\bar{\mathcal{D}}$ for all closed subset F of $\mathbb{R}^n$. □

Remark 7.11. Suppose that $\mathcal{D} \subseteq \mathbb{R}^n$ in Corollary 7.10 is open, bounded, connected but **not** convex. The Whitney extension theorem (which is not covered in this text) implies that there exists a function $F : \mathbb{R}^n \to \mathbb{R}^n$ so that $F = f$ and $DF = Df$ on $\bar{\mathcal{D}}$. Then Theorem 7.8 can be applied to guarantee that F is one-to-one on $\bar{\mathcal{D}}$.

7.2 The Implicit Function Theorem（隱函數定理）

Theorem 7.12 (Implicit Function Theorem). *Let $\mathcal{D} \subseteq \mathbb{R}^n \times \mathbb{R}^m$ be open, and $F : \mathcal{D} \to \mathbb{R}^m$ be a function of class $\mathscr{C}^r$. Suppose that for some $(x_0, y_0) \in \mathcal{D}$, where $x_0 \in \mathbb{R}^n$ and $y_0 \in \mathbb{R}^m$, $F(x_0, y_0) = 0$ and*

$$\big[(D_yF)(x_0,y_0)\big] = \begin{bmatrix} \dfrac{\partial F_1}{\partial y_1} & \cdots & \dfrac{\partial F_1}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial F_m}{\partial y_1} & \cdots & \dfrac{\partial F_m}{\partial y_m} \end{bmatrix} (x_0,y_0)$$

is invertible. Then there exists an open neighborhood $\mathcal{U} \subseteq \mathbb{R}^n$ of x_0, an open neighborhood $\mathcal{V} \subseteq \mathbb{R}^m$ of y_0, and $f : \mathcal{U} \to \mathcal{V}$ such that

1. $F\big(x, f(x)\big) = 0$ *for all* $x \in \mathcal{U}$*;*
2. $y_0 = f(x_0)$*;*
3. $(Df)(x) = -\big((D_yF)(x, f(x))\big)^{-1}(D_xF)\big(x, f(x)\big)$ *for all $x \in \mathcal{U}$, where the matrix representation of $D_xF(x, f(x)) \in \mathscr{B}(\mathbb{R}^n, \mathbb{R}^n)$ is given by*
$$\big[(D_xF)(x,y)\big] = \begin{bmatrix} \dfrac{\partial F_1}{\partial x_1} & \cdots & \dfrac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial F_m}{\partial x_1} & \cdots & \dfrac{\partial F_m}{\partial x_n} \end{bmatrix} (x,y)\,.$$
4. f *is of class* $\mathscr{C}^r$.

Proof. Let $z = (x, y)$ and $w = (u, v)$, where $x, u \in \mathbb{R}^n$ and $y, v \in \mathbb{R}^m$. Define G by $G(x, y) = \big(x, F(x, y)\big)$, and write $w = G(z)$. Then $G : \mathcal{D} \to \mathbb{R}^{n+m}$, and

$$\big[(DG)(x,y)\big] = \begin{bmatrix} \mathbb{I}_n & 0 \\ (D_xF)(x,y) & (D_yF)(x,y) \end{bmatrix},$$

where $\mathbb{I}_n$ is the $n \times n$ identity matrix. We note that the Jacobian of G at (x_0, y_0) is $\det\big([(D_yF)(x_0, y_0)]\big)$ which does not vanish since $(D_yF)(x_0, y_0)$ is invertible, so the inverse function theorem implies that there exists open neighborhoods $\mathcal{O}$ of (x_0, y_0) and $\mathcal{W}$ of $\big(x_0, F(x_0, y_0)\big) = (x_0, 0)$ such that

(a) $G : \mathcal{O} \to \mathcal{W}$ is one-to-one and onto;

(b) the inverse function $G^{-1} : \mathcal{W} \to \mathcal{O}$ is of class $\mathscr{C}^r$;

(c) $(DG^{-1})\big(x, F(x,y)\big) = \big((DG)(x,y)\big)^{-1}$.

By Remark 7.3, W.L.O.G. we can assume that $\mathcal{O} = \mathcal{U} \times \mathcal{V}$, where $\mathcal{U} \subseteq \mathbb{R}^n$ and $\mathcal{V} \subseteq \mathbb{R}^m$ are open, and $x_0 \in \mathcal{U}$, $y_0 \in \mathcal{V}$.

Write $G^{-1}(u,v) = \big(\varphi(u,v), \psi(u,v)\big)$, where $\varphi : \mathcal{W} \to \mathcal{U}$ and $\psi : \mathcal{W} \to \mathcal{V}$. Then

$$(u,v) = G\big(\varphi(u,v), \psi(u,v)\big) = \big(\varphi(u,v), F(u, \psi(u,v))\big)$$

which implies that $\varphi(u,v) = u$ and $v = F(u, \psi(u,v))$. Let $f(x) = \psi(x,0)$. Then $\big(u, f(u)\big) \in \mathcal{U} \times \mathcal{V}$ is the unique point satisfying $F\big(u, f(u)\big) = 0$ if $u \in \mathcal{U}$. Therefore, $f : \mathcal{U} \to \mathcal{V}$, and

$$F\big(x, f(x)\big) = 0 \qquad \forall\, x \in \mathcal{U}\,.$$

Since $G(x_0, y_0) = (x_0, 0) = G\big(x_0, f(x_0)\big)$, $(x_0, y_0), \big(x_0, f(x_0)\big) \in \mathcal{O}$, and $G : \mathcal{O} \to \mathcal{W}$ is one-to-one, we must have $y_0 = f(x_0)$.

By (b) and (c), we have G^{-1} is of class $\mathscr{C}^1$, and

$$(DG^{-1})(u,v) = \big((DG)(x,y)\big)^{-1}\,.$$

As a consequence, $\psi \in \mathscr{C}^1$, and

$$\begin{aligned}\begin{bmatrix}(D_u\varphi)(u,v) & (D_v\varphi)(u,v)\\ (D_u\psi)(u,v) & (D_v\psi)(u,v)\end{bmatrix} &= \begin{bmatrix}\mathbb{I}_n & 0\\ (D_xF)(x,y) & (D_yF)(x,y)\end{bmatrix}^{-1}\\ &= \begin{bmatrix}\mathbb{I}_n & 0\\ -\big((D_yF)(x,y)\big)^{-1}(D_xF)(x,y) & \big((D_yF)(x,y)\big)^{-1}\end{bmatrix}.\end{aligned}$$

Evaluating the equation above at $v = 0$, we conclude that

$$(Df)(u) = (D_u\psi)(u,0) = -\big((D_yF)(u,f(u))\big)^{-1}(D_xF)\big(u,f(u)\big)$$

which implies 3. We also note that 4 follows from (b). □

Alternative proof of Theorem 7.12 ***without*** *applying the inverse function theorem.* Let $z = (x,y)$, $z_0 = (x_0, y_0)$, $A = (D_xF)(x_0,y_0)$ and $B = (D_yF)(x_0,y_0)$. Define

$$r(x,y) = F(x,y) - A(x - x_0) - B(y - y_0)\,.$$

Our goal is to solve the equation

$$0 = A(x - x_0) + B(y - y_0) + r(x, y)$$

for y. By the invertibility of B, this is equivalent of finding a fixed-point of the map

$$\Phi_x(y) = y_0 - B^{-1}\big[A(x - x_0) + r(x, y)\big]\,.$$

Since r is of class $\mathscr{C}^1$ and $(Dr)(x_0, y_0) = 0$,

$$\exists\, \delta > 0 \ni \|(Dr)(x, y)\|_{\mathscr{B}(\mathbb{R}^m,\mathbb{R}^m)} < \min\Big\{\frac{1}{4m\|B^{-1}\|_{\mathscr{B}(\mathbb{R}^m,\mathbb{R}^m)}}, \frac{1}{2m}\Big\} \quad \forall\, z \in D(z_0, \delta)\,.$$

Therefore, the mean value theorem (Theorem 6.58) implies that

$$\begin{aligned}\big\|r(x, y) - r(x_0, y_0)\big\|_{\mathbb{R}^m} &\leqslant \sum_{i=1}^{m} \big|r_i(x, y) - r_i(x_0, y_0)\big| = \sum_{i=1}^{m} \big|(Dr_i)(c_i)(z - z_0)\big| \\ &\leqslant \frac{\|z - z_0\|_{\mathbb{R}^{n+m}}}{4\|B^{-1}\|_{\mathscr{B}(\mathbb{R}^m,\mathbb{R}^m)}} < \frac{\delta}{4\|B^{-1}\|_{\mathscr{B}(\mathbb{R}^m,\mathbb{R}^m)}}\end{aligned}$$

for all $z = (x, y) \in D\big(x_0, \frac{\delta}{2}\big) \times D\big(y_0, \frac{\delta}{2}\big) \subseteq D(z_0, \delta)$, and

$$\big\|\Phi_x(y_1) - \Phi_x(y_2)\big\|_{\mathbb{R}^m} \leqslant \|B^{-1}\|_{\mathscr{B}(\mathbb{R}^m,\mathbb{R}^m)}\|r(x, y_1) - r(x, y_2)\|_{\mathbb{R}^m} < \frac{1}{4}\|y_1 - y_2\|_{\mathbb{R}^m} \tag{7.2.1}$$

if $x \in D\big(x_0, \frac{\delta}{2}\big)$, $y_1, y_2 \in D\big(y_0, \frac{\delta}{2}\big)$ and $y_1 \neq y_2$. As a consequence, for each (fixed) x satisfying $\|x - x_0\|_{\mathbb{R}^n} < r \equiv \min\Big\{\frac{\delta}{4\big(1 + \|A\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)}\big)\|B^{-1}\|_{\mathscr{B}(\mathbb{R}^m,\mathbb{R}^m)}}, \frac{\delta}{2}\Big\}$, if $\|y - y_0\|_{\mathbb{R}^m} < \frac{\delta}{2}$ we have

$$\begin{aligned}\|\Phi_x(y) - y_0\|_{\mathbb{R}^n} &\leqslant \|B^{-1}\|_{\mathscr{B}(\mathbb{R}^m,\mathbb{R}^m)}\big\|A(x - x_0) + r(x, y)\big\|_{\mathbb{R}^m} \\ &\leqslant \|B^{-1}\|_{\mathscr{B}(\mathbb{R}^m,\mathbb{R}^m)}\big[\|A\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)}\|x - x_0\|_{\mathbb{R}^n} + \|r(x, y)\|_{\mathbb{R}^m}\big] < \frac{\delta}{2}\,.\end{aligned} \tag{7.2.2}$$

Let $M = \big\{y \in \mathbb{R}^m \,\big|\, \|y - y_0\|_{\mathbb{R}^m} \leqslant \frac{\delta}{2}\big\}$. Then for each $x \in \mathcal{U} \equiv D(x_0, r)$, (7.2.1) and (7.2.2) suggest that $\Phi_x : M \to M$ is a contraction mapping; thus there is a unique fixed-point $y \in M$. Denote this unique fixed-point as $f(x)$. Then $f : \mathcal{U} \to \mathcal{V} \equiv D\big(y_0, \frac{\sqrt{3}\delta}{2}\big)$ (the choice of this $\mathcal{V}$ guarantees that $\mathcal{U} \times \mathcal{V} \subseteq D(z_0, \delta)$) and

$$F\big(x, f(x)\big) = A(x - x_0) + B\big(f(x) - y_0\big) + r\big(x, f(x)\big) = 0\,.$$

Moreover, since $F(x,y) = 0$ if and only if y is a fixed-point of Φ_x, and the contraction mapping principle provides the uniqueness of the fixed-point if $(x,y) \in \mathcal{U} \times M$. Since $(x_0,y_0),(x_0,f(x_0)) \in \mathcal{U} \times M$, we must have $y_0 = f(x_0)$.

To see the differentiability of f, we first claim that $f : \mathcal{U} \to \mathcal{V}$ is continuous. Since $f(x)$ is the fixed-point of Φ_x,

$$f(x) = y_0 - B^{-1}\big(A(x-x_0) + r(x,f(x))\big)\,.$$

If $x_1, x_2 \in \mathcal{U}$, then $(x_1,f(x_1)),(x_2,f(x_2)) \in D(z_0,\delta)$; thus (7.2.1) implies that

$$\begin{aligned}
\big\|f(x_1)-f(x_2)\big\|_{\mathbb{R}^m} &= \big\|B^{-1}A(x_1-x_2)\big\|_{\mathbb{R}^m} + \big\|r(x_1,f(x_1)) - r(x_2,f(x_2))\big\|_{\mathbb{R}^m}\\
&\leqslant \|B^{-1}A\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)}\|x_1-x_2\|_{\mathbb{R}^n} + \frac{1}{2}\sqrt{\|x_1-x_2\|^2_{\mathbb{R}^n} + \|f(x_1)-f(x_2)\|^2_{\mathbb{R}^m}}\\
&\leqslant \|B^{-1}A\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)}\|x_1-x_2\|_{\mathbb{R}^n} + \frac{1}{2}\|x_1-x_2\|_{\mathbb{R}^n} + \frac{1}{2}\|f(x_1)-f(x_2)\|_{\mathbb{R}^m}\,.
\end{aligned}$$

Therefore,

$$\big\|f(x_1)-f(x_2)\big\|_{\mathbb{R}^m} \leqslant \big(2\|B^{-1}A\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)} + 1\big)\|x_1-x_2\|_{\mathbb{R}^n} \tag{7.2.3}$$

which implies that $f : \mathcal{U} \to \mathcal{V}$ is (Lipschitz) continuous.

Now let $a \in \mathcal{U}$ and $\varepsilon > 0$ be given. Define $b = (a,f(a))$, and $\widetilde{A} = (D_xF)(b)$, $\widetilde{B} = (D_yF)(b)$. We would like to show that there exists $\delta_1 > 0$ such that

$$\big\|f(x) - f(a) + \widetilde{B}^{-1}\widetilde{A}(x-a)\big\|_{\mathbb{R}^m} \leqslant \varepsilon\|x-a\|_{\mathbb{R}^n} \qquad \forall\, x \in D(a,\delta_1)\,.$$

Since $F \in \mathscr{C}^1$ and the map $L \mapsto L^{-1}$ is continuous, there exists $\delta_2 > 0$ such that

$$\big\|(D_yF)(z)^{-1}(D_xF)(z) - (D_yF)(z_0)^{-1}(D_xF)(z_0)\big\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)} \leqslant \frac{\varepsilon}{4} \qquad \forall\, z \in D(z_0,\delta_2)\,.$$

Moreover, since $r \in \mathscr{C}^1$, there exists $\delta_3 > 0$ such that

$$\big\|r(z) - r(b) - (Dr)(b)(z-b)\big\|_{\mathbb{R}^m} \leqslant \frac{\varepsilon}{2\|B^{-1}\|_{\mathscr{B}(\mathbb{R}^m,\mathbb{R}^m)}}\|z-b\|_{\mathbb{R}^{n+m}} \quad \forall\, z \in D(b,\delta_3)$$

and

$$\big\|(Dr)(z) - (Dr)(b)\big\|_{\mathscr{B}(\mathbb{R}^{n+m},\mathbb{R}^m)} \leqslant \frac{\varepsilon}{2\|B^{-1}\|_{\mathscr{B}(\mathbb{R}^m,\mathbb{R}^m)}}\|z-b\|_{\mathbb{R}^{n+m}} \quad \forall\, z \in D(b,\delta_3)\,.$$

Choose $\delta_1 = \min\Big\{\dfrac{\delta_2}{2}, \dfrac{\delta_3}{2}, \dfrac{\delta_3}{2(2\|B^{-1}A\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)}+1)}\Big\}$. Then if $\|x-a\|_{\mathbb{R}^n} < \delta_1$, using (7.2.3) we find that

$$\big\|(x,f(x)) - (a,f(a))\big\|_{\mathbb{R}^{n+m}} \leqslant \|x-a\|_{\mathbb{R}^n} + \|f(x)-f(a)\|_{\mathbb{R}^m} < \min\{\delta_2,\delta_3\}\,;$$

thus if $\|x-a\|_{\mathbb{R}^n} < \delta_1$,

$$
\begin{aligned}
&\big\|f(x)-f(a)+\widetilde{B}^{-1}\widetilde{A}(x-a)\big\|_{\mathbb{R}^m} \\
&\quad = \big\|\big(\widetilde{B}^{-1}\widetilde{A}-B^{-1}A\big)(x-a)+B^{-1}\big(r(x,f(x))-r(a,f(a))\big)\big\|_{\mathbb{R}^m} \\
&\quad \leqslant \big\|\widetilde{B}^{-1}\widetilde{A}-B^{-1}A\big\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)}\|x-a\|_{\mathbb{R}^n} \\
&\qquad + \|B^{-1}\|_{\mathscr{B}(\mathbb{R}^m,\mathbb{R}^m)}\big\|r(x,f(x))-r(a,f(a))-(Dr)(b)\big(x-a,f(x)-f(a)\big)\big\|_{\mathbb{R}^m} \\
&\qquad + \|B^{-1}\|_{\mathscr{B}(\mathbb{R}^m,\mathbb{R}^m)}\big\|(Dr)(b)\big((x-a,f(x)-f(a)\big)\big\|_{\mathbb{R}^m} \leqslant \varepsilon\|x-a\|_{\mathbb{R}^n} \,.
\end{aligned}
$$

Therefore, f is differentiable on $\mathcal{U}$, and

$$
(Df)(x) = -\big((D_yF)(x,f(x))\big)^{-1}(D_xF)\big(x,f(x)\big) \qquad \forall\, x \in \mathcal{U}\,. \tag{7.2.4}
$$

Since F is of class $\mathscr{C}^1$ and f is continuous on $\mathcal{U}$, we find that Df is continuous; thus f is of class $\mathscr{C}^1$. Moreover, if F is of class $\mathscr{C}^r$, (7.2.4) also implies that f is of class $\mathscr{C}^r$. ▫

Example 7.13. Let $F(x,y) = x^2+y^2-1$.

1. If $(x_0,y_0) = (1,0)$, then $F_x(x_0,y_0) = 2 \neq 0$; thus the implicit function theorem implies that locally x can be expressed as a function of y.
2. If $(x_0,y_0) = (0,-1)$, then $F_y(x_0,y_0) = -2 \neq 0$; thus the implicit function theorem implies that locally y can be expressed as a function of x.
3. If $(x_0,y_0) = \big(-\frac{1}{2}, \frac{\sqrt{3}}{2}\big)$, then $F_x(x_0,y_0) = -1 \neq 0$ and $F_y(x_0,y_0) = \sqrt{3} \neq 0$; thus the implicit function theorem implies that locally x can be expressed as a function of y and locally y can be expressed as a function of x.

Example 7.14. Suppose that (x,y,u,v) satisfies the equation

$$
\left\{
\begin{aligned}
xu + yv^2 &= 0 \\
xv^3 + y^2u^6 &= 0
\end{aligned}
\right.
$$

and $(x_0,y_0,u_0,v_0) = (1,-1,1,-1)$. Let $F(x,y,u,v) = (xu+yv^2, xv^3+y^2u^6)$. Then $F(x_0,y_0,u_0,v_0) = 0$.

1. Since $(D_{x,y}F)(x_0,y_0,u_0,v_0) = \begin{bmatrix} \dfrac{\partial F_1}{\partial x} & \dfrac{\partial F_1}{\partial y} \\ \dfrac{\partial F_2}{\partial x} & \dfrac{\partial F_2}{\partial y} \end{bmatrix}(x_0,y_0,u_0,v_0) = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ is invertible, locally (x,y) can be expressed in terms of u,v; that is, locally $x = x(u,v)$ and $y = y(u,v)$.

2. Since $(D_{y,u}F)(x_0,y_0,u_0,v_0) = \begin{bmatrix} \dfrac{\partial F_1}{\partial y} & \dfrac{\partial F_1}{\partial u} \\ \dfrac{\partial F_2}{\partial y} & \dfrac{\partial F_2}{\partial u} \end{bmatrix}(x_0,y_0,u_0,v_0) = \begin{bmatrix} 1 & 1 \\ -2 & 6 \end{bmatrix}$ is invertible, locally (y,u) can be expressed in terms of x, v.

Example 7.15. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$f(x,y,z) = (xe^y + ye^z, xe^z + ze^y)\,.$$

Then f is of class $\mathscr{C}^1$, $f(-1,1,1) = (0,0)$ and

$$\big[(Df)(x,y,z)\big] = \begin{bmatrix} e^y & xe^y + e^z & ye^z \\ e^z & ze^y & xe^z + e^y \end{bmatrix}.$$

Since $(D_{y,z}f)(-1,1,1) = \begin{bmatrix} 0 & e \\ e & 0 \end{bmatrix}$ is invertible, the implicit function theorem implies that the system

$$\left\{ \begin{array}{l} xe^y + ye^z = 0 \\ xe^z + ze^y = 0 \end{array} \right.$$

can be solved for y and z as continuously differentiable function of x for x near -1 and (y,z) near $(1,1)$. Furthermore, if we write $(y,z) = g(x)$ for x near -1, then

$$g'(x) = \begin{bmatrix} xe^y + e^z & ye^z & ye^z \\ ze^y & xe^z + e^y & \end{bmatrix}^{-1} \begin{bmatrix} e^y \\ e^z \end{bmatrix}.$$

Chapter 8

Integration

In this chapter, we focus on the integration of bounded functions on bounded subsets of $\mathbb{R}^n$.

8.1 Integrable Functions

We start with a simpler case $n = 2$.

Definition 8.1. Let $A \subseteq \mathbb{R}^2$ be a bounded set. Define

$$\begin{aligned}
a_1 &= \inf\left\{x \in \mathbb{R} \,\middle|\, (x, y) \in A \text{ for some } y \in \mathbb{R}\right\}, \\
b_1 &= \sup\left\{x \in \mathbb{R} \,\middle|\, (x, y) \in A \text{ for some } y \in \mathbb{R}\right\}, \\
a_2 &= \inf\left\{y \in \mathbb{R} \,\middle|\, (x, y) \in A \text{ for some } x \in \mathbb{R}\right\}, \\
b_2 &= \sup\left\{y \in \mathbb{R} \,\middle|\, (x, y) \in A \text{ for some } x \in \mathbb{R}\right\}.
\end{aligned}$$

A collection of rectangles $\mathcal{P}$ is called a ***partition*** of A if there exists a partition $\mathcal{P}_x$ of $[a_1, b_1]$ and a partition P_y of $[a_2, b_2]$,

$$\mathcal{P}_x = \left\{a_1 = x_0 < x_1 < \cdots < x_n = b_1\right\} \quad \text{and} \quad \mathcal{P}_y = \left\{a_2 = y_0 < y_1 < \cdots < y_m = b_2\right\},$$

such that

$$\mathcal{P} = \left\{\Delta_{ij} \,\middle|\, \Delta_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}] \text{ for } i = 0, 1, \cdots, n-1 \text{ and } j = 0, 1, \cdots, m-1\right\}.$$

The ***mesh size*** of the partition $\mathcal{P}$, denoted by $\|\mathcal{P}\|$, is defined by

$$\|\mathcal{P}\| = \max\left\{\sqrt{(x_{i+1} - x_i)^2 + (y_{j+1} - y_j)^2} \,\middle|\, i = 0, 1, \cdots, n-1, j = 0, 1, \cdots, m-1\right\}.$$

The number $\sqrt{(x_{i+1} - x_i)^2 + (y_{j+1} - y_j)^2}$ is often denoted by $\operatorname{diam}(\Delta_{ij})$, and is called the ***diameter*** of Δ_{ij}.

Similar to the integrability of f on a bounded subset of $\mathbb{R}$, we have the following

Definition 8.2. Let $A \subseteq \mathbb{R}^2$ be a bounded set, and $f : A \to \mathbb{R}$ be a bounded function. For any partition $\mathcal{P} = \big\{\Delta_{ij} \,\big|\, \Delta_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}], i = 0, \cdots, n-1, j = 0, \cdots, m-1\big\}$, the ***upper sum*** and the ***lower sum*** of f with respect to the partition $\mathcal{P}$, denoted by $U(f, \mathcal{P})$ and $L(f, \mathcal{P})$ respectively, are numbers defined by

$$U(f,\mathcal{P}) = \sum_{\substack{0\leqslant i\leqslant n-1\\ 0\leqslant j\leqslant m-1}} \sup_{(x,y)\in\Delta_{ij}} \overline{f}^A(x,y)\mathbb{A}(\Delta_{ij})\,,$$
$$L(f,\mathcal{P}) = \sum_{\substack{0\leqslant i\leqslant n-1\\ 0\leqslant j\leqslant m-1}} \inf_{(x,y)\in\Delta_{ij}} \overline{f}^A(x,y)\mathbb{A}(\Delta_{ij})\,,$$

where $\mathbb{A}(\Delta_{ij}) = (x_{i+1} - x_i)(y_{j+1} - y_j)$ is the area of the rectangle Δ_{ij}, and $\overline{f}^A$ is an extension of f, called the extension of f by zero outside A, given by

$$\overline{f}^A(x) = \begin{cases} f(x) & x \in A\,, \\ 0 & x \notin A\,. \end{cases}$$

The two numbers

$$\overline{\int_A} f(x,y)d\mathbb{A} \equiv \inf\big\{U(f,\mathcal{P}) \,\big|\, \mathcal{P} \text{ is a partition of } A\big\}$$

and

$$\underline{\int_A} f(x,y)d\mathbb{A} \equiv \sup\big\{L(f,\mathcal{P}) \,\big|\, \mathcal{P} \text{ is a partition of } A\big\}$$

are called the ***upper integral*** and ***lower integral*** of f over A, respectively. The function f is said to be ***Riemann*** (***Darboux***) ***integrable*** (over A) if $\overline{\int_A} f(x,y)d\mathbb{A} = \underline{\int_A} f(x,y)d\mathbb{A}$, and in this case, we express the upper and lower integral as $\int_A f(x,y)d\mathbb{A}$, called the ***integral*** of f over A.

In general, we can consider the integrability of a bounded function f defined on a bounded set $A \subseteq \mathbb{R}^n$ as follows

Definition 8.3. Let $A \subseteq \mathbb{R}^n$ be a bounded set. Define the numbers $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ by

$$a_k = \inf\big\{x_k \in \mathbb{R} \,\big|\, x = (x_1, \cdots, x_n) \in A \text{ for some } x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n \in \mathbb{R}\big\},$$
$$b_k = \sup\big\{x_k \in \mathbb{R} \,\big|\, x = (x_1, \cdots, x_n) \in A \text{ for some } x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n \in \mathbb{R}\big\}.$$

A collection of rectangles $\mathcal{P}$ is called a ***partition*** of A if there exists partitions $\mathcal{P}^{(k)}$ of $[a_k, b_k]$, $k = 1, \cdots, n$, $\mathcal{P}^{(k)} = \big\{a_k = x_0^{(k)} < x_1^{(k)} < \cdots < x_{N_k}^{(k)} = b_k\big\}$, such that

$$\mathcal{P} = \Big\{\Delta_{i_1 i_2 \cdots i_n} \,\Big|\, \Delta_{i_1 i_2 \cdots i_n} = [x_{i_1}^{(1)}, x_{i_1+1}^{(1)}] \times [x_{i_2}^{(2)}, x_{i_2+1}^{(2)}] \times \cdots \times [x_{i_n}^{(n)}, x_{i_{n+1}}^{(n+1)}],$$
$$i_k = 0, 1, \cdots, N_k - 1, k = 1, \cdots, n\Big\}.$$

The ***mesh size*** of the partition $\mathcal{P}$, denoted by $\|\mathcal{P}\|$, is defined by

$$\|\mathcal{P}\| = \max\Big\{\sqrt{\sum_{k=1}^{n} (x_{i_k+1}^{(k)} - x_{i_k}^{(k)})^2} \,\Big|\, i_k = 0, 1, \cdots, N_k - 1, k = 1, \cdots, n\Big\}.$$

The number $\sqrt{\sum\limits_{k=1}^{n} (x_{i_k+1}^{(k)} - x_{i_k}^{(k)})^2}$ is often denoted by $\operatorname{diam}(\Delta_{i_1 i_2 \cdots i_n})$, and is called the ***diameter*** of the rectangle $\Delta_{i_1 i_2 \cdots i_n}$.

Definition 8.4. Let $A \subseteq \mathbb{R}^n$ be a bounded set, and $f : A \to \mathbb{R}$ be a bounded function. For any partition

$$\mathcal{P} = \Big\{\Delta_{i_1 i_2 \cdots i_n} \,\Big|\, \Delta_{i_1 i_2 \cdots i_n} = [x_{i_1}^{(1)}, x_{i_1+1}^{(1)}] \times [x_{i_2}^{(2)}, x_{i_2+1}^{(2)}] \times \cdots \times [x_{i_n}^{(n)}, x_{i_{n+1}}^{(n+1)}],$$
$$i_k = 0, 1, \cdots, N_k - 1, k = 1, \cdots, n\Big\},$$

the ***upper sum*** and the ***lower sum*** of f with respect to the partition $\mathcal{P}$, denoted by $U(f, \mathcal{P})$ and $L(f, \mathcal{P})$ respectively, are numbers defined by

$$U(f, \mathcal{P}) = \sum_{\Delta \in \mathcal{P}} \sup_{(x,y) \in \Delta} \overline{f}^A(x, y) \nu(\Delta),$$
$$L(f, \mathcal{P}) = \sum_{\Delta \in \mathcal{P}} \inf_{(x,y) \in \Delta} \overline{f}^A(x, y) \nu(\Delta),$$

where $\nu(\Delta)$ is the ***volume*** of the rectangle Δ given by

$$\nu(\Delta) = (x_{i_1+1}^{(1)} - x_{i_1}^{(1)})(x_{i_2+1}^{(2)} - x_{i_2}^{(2)}) \cdots (x_{i_n+1}^{(n)} - x_{i_n}^{(n)})$$

if $\Delta = [x_{i_1}^{(1)} - x_{i_1+1}^{(1)}] \times [x_{i_2}^{(2)} - x_{i_2+1}^{(2)}] \times \cdots \times [x_{i_n}^{(n)} - x_{i_n+1}^{(n)}]$, and $\overline{f}^A$ is the extension of f by zero outside A given by

$$\overline{f}^A(x) = \begin{cases} f(x) & x \in A, \\ 0 & x \notin A. \end{cases}$$

The two numbers

$$\overline{\int_A} f(x)dx \equiv \inf\left\{U(f,\mathcal{P})\,\middle|\,\mathcal{P} \text{ is a partition of } A\right\},$$

and

$$\underline{\int_A} f(x)dx \equiv \sup\left\{L(f,\mathcal{P})\,\middle|\,\mathcal{P} \text{ is a partition of } A\right\}$$

are called the ***upper integral*** and ***lower integral*** of f over A, respective. The function f is said to be ***Riemann*** (***Darboux***) ***integrable*** (over A) if $\overline{\int_A} f(x)dx = \underline{\int_A} f(x)dx$, and in this case, we express the upper and lower integral as $\int_A f(x)dx$, called the ***integral*** of f over A.

Definition 8.5. A partition $\mathcal{P}'$ of a bounded set $A \subseteq \mathbb{R}^n$ is said to be a ***refinement*** of another partition $\mathcal{P}$ of A if for any $\Delta' \in \mathcal{P}'$, there is $\Delta \in \mathcal{P}$ such that $\Delta' \subseteq \Delta$. A partition $\mathcal{P}$ of a bounded set $A \subseteq \mathbb{R}^n$ is said to be the ***common refinement*** of another partitions $\mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_k$ of A if

1. $\mathcal{P}$ is a refinement of $\mathcal{P}_j$ for all $1 \leqslant j \leqslant k$.

2. If $\mathcal{P}'$ is a refinement of $\mathcal{P}_j$ for all $1 \leqslant j \leqslant k$, then $\mathcal{P}'$ is also a refinement of $\mathcal{P}$.

In other words, $\mathcal{P}$ is a common refinement of $\mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_k$ if it is the coarsest refinement.

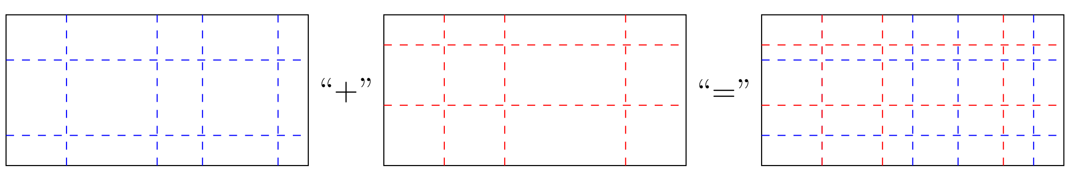


Figure 8.1: The common refinement of two partitions

Quantitatively speaking, $\mathcal{P}$ is a common refinement of $\mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_k$ if for each $j = 1, \cdots n$, the j-th component c_j of the vertex $(c_1, \cdots, c_n)$ of each rectangle $\Delta \in \mathcal{P}$ belongs to $\mathcal{P}_i^{(j)}$ for some $i = 1, \cdots, k$.

Similar to Proposition 4.77, we have

Proposition 8.6. *Let $A \subseteq \mathbb{R}^n$ be a bounded subset, and $f : A \to \mathbb{R}$ be a bounded function. If $\mathcal{P}$ and $\mathcal{P}'$ be partitions of A and $\mathcal{P}'$ is a refinement of $\mathcal{P}$, then*

$$L(f,\mathcal{P}) \leqslant L(f,\mathcal{P}') \leqslant U(f,\mathcal{P}') \leqslant U(f,\mathcal{P}).$$

The proof of the following proposition is identical to the proof of Proposition 4.79.

Proposition 8.7 (Riemann's condition)**.** *Let $A \subseteq \mathbb{R}^n$ be a bounded set, and $f : A \to \mathbb{R}$ be a bounded function. Then f is Riemann integrable over A if and only if*

$$\forall\, \varepsilon > 0, \exists \text{ a partition } \mathcal{P} \text{ of } A \ni U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon\,.$$

Theorem 8.8 (Darboux)**.** *Let $A \subseteq \mathbb{R}^n$ be a bounded set, and $f : A \to \mathbb{R}$ be a bounded function with extension $\overline{f}^A$ given by* (4.7.1)*. Then f is Riemann integrable if and only if $\exists\, \mathrm{I} \in \mathbb{R}$ such that $\forall\, \varepsilon > 0$, $\exists\, \delta > 0 \ni$ if $\mathcal{P} = \{\Delta_1, \cdots, \Delta_N\}$ is a partition of A satisfying $\|\mathcal{P}\| < \delta$ and a set of sample points $\xi_1 \in \Delta_1$, $\xi_2 \in \Delta_2$, $\cdots$, $\xi_N \in \Delta_N$, we have*

$$\Big| \sum_{k=1}^{N} \overline{f}^A(\xi_{k+1})\nu(\Delta_k) - \mathrm{I} \Big| < \varepsilon\,. \tag{8.1.1}$$

*The sum $\sum\limits_{k=1}^{N} \overline{f}^A(\xi_{k+1})\nu(\Delta_k)$ is called a **Reimann sum** of f over A.*

In Section 5.1, we show that if a sequence of Riemann integrable functions $\{f_k\}_{k=1}^{\infty}$ converges to a function f uniformly on $[a, b]$, then f is also Riemann integrable over $[a, b]$ and the integral of the limit function is the same as the limit of the integrals (of the sequences). This theorem can also be established if the domain A under consideration is a bounded subset of $\mathbb{R}^n$. In fact, the same proof used to establish Theorem 5.17 can be applied to conclude the following

Theorem 8.9. *Let $A \subseteq \mathbb{R}^n$ be a bounded set, and $f_k : A \to \mathbb{R}$ be a sequence of Riemann integrable functions over A such that $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f on A. Then f is Riemann integrable over A, and*

$$\lim_{k\to\infty} \int_A f_k(x)dx = \int_A f(x)dx\,. \tag{8.1.2}$$

From now on, we will simply use $\bar{f}$ to denote the zero extension of f when the domain outside which the zero extension is made is clear.

8.2 Volume and Sets of Measure Zero

Definition 8.10. Let $A \subseteq \mathbb{R}^n$ be a bounded set, and 1_A (or χ_A) be the characteristic function of A defined by

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

A is said to ***have volume*** if 1_A is Riemann integrable (over A), and the ***volume*** of A, denoted by $\nu(A)$, is the number $\displaystyle\int_A 1_A(x)dx$. A is said to have ***volume zero*** or ***content zero*** if $\nu(A) = 0$.

Remark 8.11. Not all bounded set has volume.

Proposition 8.12. *Let $A \subseteq \mathbb{R}^n$ be bounded. Then A has volume zero if and only if for every $\varepsilon > 0$, there exists finite (open) rectangles $S_1, \cdots, S_N$ (whose sides are parallel to the coordinate axes) such that*

$$A \subseteq \bigcup_{k=1}^{N} S_k \qquad \textit{and} \qquad \sum_{k=1}^{N} \nu(S_k) < \varepsilon\,.$$

Proof. "$\Rightarrow$" Since A has volume zero, $\displaystyle\overline{\int_A} 1_A(x)dx = 0$; thus for any given $\varepsilon > 0$, there exists a partition $\mathcal{P}$ of A such that

$$U(1_A, \mathcal{P}) < \overline{\int_A} 1_A(x)dx + \frac{\varepsilon}{2} = \frac{\varepsilon}{2}\,.$$

Since $\displaystyle\sup_{x\in\Delta} 1_A(x) = \begin{cases} 1 & \text{if } \Delta \cap A \neq \varnothing\,, \\ 0 & \text{otherwise}\,, \end{cases}$ we must have $\displaystyle\sum_{\substack{\Delta\in\mathcal{P} \\ \Delta\cap A\neq\varnothing}} \nu(\Delta) < \frac{\varepsilon}{2}$. Now if $\Delta \in \mathcal{P}$ and $\Delta \cap A \neq \varnothing$, we can find an open rectangle $\square$ such that $\Delta \subseteq \square$ and $\nu(\square) < 2\nu(\Delta)$. Let $S_1, \cdots, S_N$ be those open rectangles $\square$. Then $A \subseteq \displaystyle\bigcup_{k=1}^{N} S_k$ and $\displaystyle\sum_{k=1}^{N} \nu(S_k) < \varepsilon$.

"$\Leftarrow$" W.L.O.G. we can assume that the ratio of the maximum length and minimum length of sides of S_k is less than 2 for all $k = 1, \cdots, N$ (otherwise we can divide S_k into smaller rectangles so that each smaller rectangle satisfies this requirement). Then each S_k can be covered by a closed rectangle $\square_k$ whose sides are parallel to the coordinate axes with the property that $\nu(\square_k) \leqslant 2^{n-1}\sqrt{n}^{\,n}\nu(S_k)$. Let $\mathcal{P}$ be a partition of A such that for each $\Delta \in \mathcal{P}$ with $\Delta \cap A \neq \varnothing$, $\Delta \subseteq S_k$ for some $k = 1, \cdots, N$. Then

$$U(1_A, \mathcal{P}) = \sum_{\substack{\Delta\in\mathcal{P} \\ \Delta\cap A\neq\varnothing}} \nu(\Delta) \leqslant \sum_{k=1}^{N} \nu(\square_k) \leqslant 2^{n-1}\sqrt{n}^{\,n} \sum_{k=1}^{N} \nu(S_k) < 2^{n-1}\sqrt{n}^{\,n}\varepsilon\,;$$

thus the upper integral $\displaystyle\overline{\int_A} 1_A(x)dx = 0$. Since the lower integral cannot be negative, we must have $\displaystyle\overline{\int_A} 1_A(x)dx = \underline{\int_A} 1_A(x)dx = 0$ which implies that A has volume zero. $\square$

Example 8.13. Each point in $\mathbb{R}^n$ has volume zero.

Example 8.14. The Cantor set (defined in Exercise Problem **??** in Chapter 2) has volume zero.

Definition 8.15. A set $A \subseteq \mathbb{R}^n$ (not necessarily bounded) is said to ***have measure zero***（測度為零）or be ***a set of measure zero***（零測度集）if for every $\varepsilon > 0$, there exist countable many rectangles $S_1, S_2, \cdots$ such that $\{S_k\}_{k=1}^{\infty}$ is a cover of A $\big($that is, $A \subseteq \bigcup\limits_{k=1}^{\infty} S_k\big)$ and $\sum\limits_{k=1}^{\infty} \nu(S_k) < \varepsilon$.

Example 8.16. The real line $\mathbb{R} \times \{0\}$ on $\mathbb{R}^2$ has measure zero: for any given $\varepsilon > 0$, let $S_k = [-k, k] \times \big[\frac{-\varepsilon}{2^{k+3}k}, \frac{\varepsilon}{2^{k+3}k}\big]$. Then

$$\mathbb{R} \times \{0\} \subseteq \bigcup_{k=1}^{\infty} S_k \quad \text{and} \quad \sum_{k=1}^{\infty} \nu(S_k) = \sum_{k=1}^{\infty} 2k \cdot \frac{2\varepsilon}{2^{k+3}k} = \sum_{k=1}^{\infty} \frac{\varepsilon}{2^{k+1}} = \frac{\varepsilon}{2} < \varepsilon\,.$$

Similarly, any hyperplane in $\mathbb{R}^n$ also has measure zero.

Proposition 8.17. *Let $A \subseteq \mathbb{R}^n$ be a set of measure zero. If $B \subseteq A$, then B also has measure zero.*

Modifying the second part (or the "$\Leftarrow$" part) of the proof of Proposition 8.12, we can also conclude the following

Proposition 8.18. *A set $A \subseteq \mathbb{R}^n$ has measure zero if and only if for every $\varepsilon > 0$, there exist countable many open rectangles $S_1, S_2, \cdots$ whose sides are parallel to the coordinate axes such that $A \subseteq \bigcup\limits_{k=1}^{\infty} S_k$ and $\sum\limits_{k=1}^{\infty} \nu(S_k) < \varepsilon$.*

Remark 8.19. If a set A has volume zero, then it has measure zero.

Proposition 8.20. *Let $K \subseteq \mathbb{R}^n$ be a compact set of measure zero. Then K has volume zero.*

Proof. Let $\varepsilon > 0$ be given. Then there are countable open rectangles $S_1, S_2, \cdots$ such that

$$K \subseteq \bigcup_{k=1}^{\infty} S_k \qquad \text{and} \qquad \sum_{k=1}^{\infty} \nu(S_k) < \varepsilon\,.$$

Since $\{S_k\}_{k=1}^{\infty}$ is an open cover of K, by the compactness of K there exists $N > 0$ such that $K \subseteq \bigcup\limits_{k=1}^{N} S_k$, while $\sum\limits_{k=1}^{N} \nu(S_k) \leqslant \sum\limits_{k=1}^{\infty} \nu(S_k) < \varepsilon$. As a consequence, K has volume zero. $\square$

Since the boundary of a rectangle has measure zero, we also have the following

Corollary 8.21. *Let $S \subseteq \mathbb{R}^n$ be a bounded rectangle with positive volume. Then R is not a set of measure zero.*

Theorem 8.22. *If $A_1, A_2, \cdots$ are sets of measure zero in $\mathbb{R}^n$, then $\bigcup\limits_{k=1}^{\infty} A_k$ has measure zero.*

Proof. Let $\varepsilon > 0$ be given. Since A_k's are sets of measure zero, there exist countable rectangles $\{S_j^{(k)}\}_{j=1}^{\infty}$, such that

$$A_k \subseteq \bigcup_{j=1}^{\infty} S_j^{(k)} \quad \text{and} \quad \sum_{j=1}^{\infty} \nu(S_j^{(k)}) < \frac{\varepsilon}{2^{k+1}} \qquad \forall\, k \in \mathbb{N}\,.$$

Consider the collection consisting of all $S_j^{(k)}$'s. Since there are countable many rectangles in this collection, we can label them as $S_1, S_2, \cdots$, and we have

$$\bigcup_{k=1}^{\infty} A_k \subseteq \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{\infty} S_j^{(k)} = \bigcup_{\ell=1}^{\infty} S_\ell$$

and

$$\sum_{k=1}^{\infty} \nu(S_\ell) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \nu(S_j^{(k)}) \leqslant \sum_{k=1}^{\infty} \frac{\varepsilon}{2^{k+1}} = \frac{\varepsilon}{2} < \varepsilon\,.$$

Therefore, $\bigcup\limits_{k=1}^{\infty} A_k$ has measure zero. □

Corollary 8.23. *The set of rational numbers in $\mathbb{R}$ has measure zero.*

8.3 The Lebesgue Theorem

在之前我們提到了函數 Riemann 可積的一個等價條件：Riemann's condition。在這一節中，我們將引進函數是 Riemman 可積的另一個等價條件。這個等價條件說的是一個函數 f 在 A 上是 Riemann 可積的若且唯若 f 的延拓 $\overline{f}^A$（在函數可積分的定義中有定義）的不連續點所構成的集合其測度為零。為了了解這個敘述，我們先對一個函數的連續點做一個量化的刻劃。這個刻劃的方式，可以很容易用來檢驗一個函數在一個點是否連續。

Definition 8.24. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. For any $x \in \mathbb{R}^n$, the ***oscillation*** of f at x is the quantity

$$\operatorname{osc}(f, x) \equiv \inf_{\delta>0} \sup_{x_1, x_2 \in D(x,\delta)} \big|f(x_1) - f(x_2)\big|\,.$$

我們注意到在上述定義中被取 infimum 的這個量 $h(\delta;x) \equiv \sup\limits_{x_1,x_2\in D(x,\delta)} \big|f(x_1)-f(x_2)\big|$ 是個 δ 的遞減函數（x 固定），而 $\operatorname{osc}(f,x)$ 則是 $h(\delta;x)$ 當 $\delta \to 0$ 時的極限。另外，我們也注意到說 $h(\delta;x)$ 也可以寫成 $\sup\limits_{y\in D(x,\delta)} f(y) - \inf\limits_{y\in D(x,\delta)} f(y)$.

以下的 Lemma 是關於如何檢驗一個函數在一個點是連續的。

Lemma 8.25. *Let $f:\mathbb{R}^n \to \mathbb{R}$ be a function, and $x_0 \in \mathbb{R}^n$. Then f is continuous at x_0 if and only if* $\operatorname{osc}(f,x_0)=0$.

Proof. "$\Rightarrow$" Let $\varepsilon>0$ be given. Since f is continuous at x_0,

$$\exists\,\delta>0 \ni \big|f(x)-f(x_0)\big| < \frac{\varepsilon}{3} \qquad \text{whenever } x\in D(x_0,\delta).$$

In particular, for any $x_1,x_2\in D(x_0,\delta)$,

$$\big|f(x_1)-f(x_2)\big| \leqslant \big|f(x_1)-f(x_0)\big| + \big|f(x_0)-f(x_2)\big| < \frac{2\varepsilon}{3}\,;$$

thus $\sup\limits_{x_1,x_2\in D(x_0,\delta)} \big|f(x_1)-f(x_2)\big| \leqslant \dfrac{2\varepsilon}{3}$ which further suggests that

$$0 \leqslant \operatorname{osc}(f,x_0) \leqslant \frac{2\varepsilon}{3} < \varepsilon.$$

Since ε is given arbitrarily, $\operatorname{osc}(f,x_0)=0$.

"$\Leftarrow$" Let $\varepsilon>0$ be given. By the definition of infimum, there exists $\delta>0$ such that

$$\sup_{x_1,x_2\in D(x_0,\delta)} \big|f(x_1)-f(x_2)\big| < \varepsilon\,.$$

In particular, $\big|f(x)-f(x_0)\big| \leqslant \sup\limits_{x_1,x_2\in D(x_0,\delta)} \big|f(x_1)-f(x_2)\big| < \varepsilon$ for all $x\in D(x_0,\delta)$. □

Lemma 8.26. *Let $f:\mathbb{R}^n\to\mathbb{R}$ be a function. Then for all $\varepsilon>0$, the set $D_\varepsilon = \big\{x\in\mathbb{R}^n \,\big|\, \operatorname{osc}(f,x)\geqslant\varepsilon\big\}$ is closed.*

Proof. Suppose that $\{y_k\}_{k=1}^\infty \subseteq D_\varepsilon$ and $y_k\to y$. Then for any $\delta>0$, there exists $N>0$ such that $y_k\in D(y,\delta)$ for all $k\geqslant N$. Since $D(y,\delta)$ is open, for each $k\geqslant N$ there exists $\delta_k>0$ such that $D(y_k,\delta_k)\subseteq D(y,\delta)$; thus we find that

$$\sup_{x_1,x_2\in D(y_k,\delta_k)} \big|f(x_1)-f(x_2)\big| \leqslant \sup_{x_1,x_2\in D(y,\delta)} \big|f(x_1)-f(x_2)\big| \qquad \forall\, k\geqslant N\,.$$

The inequality above implies that $\operatorname{osc}(f,y)\geqslant\varepsilon$; thus $y\in D_\varepsilon$ and D_ε is closed. □

Theorem 8.27 (Lebesgue)**.** *Let $A \subseteq \mathbb{R}^n$ be bounded, $f : A \to \mathbb{R}$ be a bounded function, and $\bar{f}$ be the extension of f by zero outside A; that is,*

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in A\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Then f is Riemann integrable if and only if the collection of discontinuity of $\bar{f}$ is a set of measure zero.

Proof. Let $D = \big\{x \in \mathbb{R}^n \,\big|\, \text{osc}(\bar{f}, x) > 0\big\}$ and $D_\varepsilon = \big\{x \in \mathbb{R}^n \,\big|\, \text{osc}(\bar{f}, x) \geqslant \varepsilon\big\}$. We remark here that $D = \bigcup\limits_{k=1}^{\infty} D_{\frac{1}{k}}$.

"$\Rightarrow$" We show that $D_{\frac{1}{k}}$ has measure zero for all $k \in \mathbb{N}$ (if so, then Theorem 8.22 implies that D has measure zero).

Let $k \in \mathbb{N}$ be fixed, and $\varepsilon > 0$ be given. By Riemann's condition there exists a partition $\mathcal{P}$ of A such that

$$\sum_{\Delta \in \mathcal{P}} \Big[\sup_{x \in \Delta} \bar{f}(x) - \inf_{x \in \Delta} \bar{f}(x)\Big] \nu(\Delta) < \frac{\varepsilon}{k}\,.$$

Define

$$\begin{aligned} D^{(1)}_{\frac{1}{k}} &\equiv \big\{x \in D_{\frac{1}{k}} \,\big|\, x \in \partial\Delta \text{ for some } \Delta \in \mathcal{P}\big\}\,, \\ D^{(2)}_{\frac{1}{k}} &\equiv \big\{x \in D_{\frac{1}{k}} \,\big|\, x \in \text{int}(\Delta) \text{ for some } \Delta \in \mathcal{P}\big\}\,. \end{aligned}$$

Then $D_{\frac{1}{k}} = D^{(1)}_{\frac{1}{k}} \cup D^{(2)}_{\frac{1}{k}}$. We note that $D^{(1)}_{\frac{1}{k}}$ has measure zero since it is contained in $\bigcup_{\Delta \in \mathcal{P}} \partial\Delta$ while each $\partial\Delta$ has measure zero. Now we show that $D^{(2)}_{\frac{1}{k}}$ also has measure zero. Let $C = \big\{\Delta \in \mathcal{P} \,\big|\, \text{int}(\Delta) \cap D_{\frac{1}{k}} \neq \varnothing\big\}$. Then $D^{(2)}_{\frac{1}{k}} \subseteq \bigcup\limits_{\Delta \in C} \Delta$. Moreover, we also note that if $\Delta \in C$, $\sup\limits_{x \in \Delta} \bar{f}(x) - \inf\limits_{x \in \Delta} \bar{f}(x) \geqslant \dfrac{1}{k}$. In fact, if $\Delta \in C$, there exists $y \in \text{int}(\Delta) \cap D_{\frac{1}{k}}$; thus choosing $\delta > 0$ such that $D(y, \delta) \subseteq \text{int}(\Delta)$,

$$\begin{aligned} \sup_{x \in \Delta} \bar{f}(x) - \inf_{x \in \Delta} \bar{f}(x) &= \sup_{x_1, x_2 \in \Delta} \big|\bar{f}(x_1) - \bar{f}(x_2)\big| \geqslant \sup_{x_1, x_2 \in D(y,\delta)} \big|\bar{f}(x_1) - \bar{f}(x_2)\big| \\ &\geqslant \inf_{\delta > 0} \sup_{x_1, x_2 \in D(y,\delta)} \big|\bar{f}(x_1) - \bar{f}(x_2)\big| = \text{osc}(\bar{f}, y) \geqslant \frac{1}{k}\,. \end{aligned}$$

As a consequence,

$$\frac{1}{k} \sum_{\Delta \in C} \nu(\Delta) \leqslant \sum_{\Delta \in \mathcal{P}} \Big[\sup_{x \in \Delta} \bar{f}(x) - \inf_{x \in \Delta} \bar{f}(x)\Big] \nu(\Delta) = U(f, \mathcal{P}) - L(f, \mathcal{P}) < \frac{\varepsilon}{k}$$

which implies that $\sum\limits_{\Delta\in C}\nu(\Delta) < \varepsilon$. In other words, we establish that $D^{(2)}_{\frac{1}{k}}$ has measure zero. Therefore, $D_{\frac{1}{k}}$ has measure zero for all $k\in\mathbb{N}$; thus D has measure zero.

"$\Leftarrow$" Let R be a closed rectangle with sides parallel to the coordinate axes and $\bar{A}\subseteq \operatorname{int}(R)$, and $\varepsilon > 0$ be given. Define $\varepsilon' = \dfrac{\varepsilon}{2\|f\|_\infty + \nu(R)}$, where $\|f\|_\infty = \sup\limits_{x\in A}|f(x)|$.

1. Since $D_{\varepsilon'}$ is a subset of D, Proposition 8.17 implies that $D_{\varepsilon'}$ has measure zero; thus Proposition 8.18 provides open rectangles S_1, S_2, $\cdots$ whose sides are parallel to the coordinate axes such that $D_{\varepsilon'}\subseteq \bigcup\limits_{k=1}^{\infty} S_k$, and $\sum\limits_{k=1}^{\infty}\nu(S_k) < \varepsilon'$. In addition, we can assume that $S_k\subseteq R$ for all $k\in\mathbb{N}$ since $D_{\varepsilon'}\subseteq R$.
2. Since $D_{\varepsilon'}\subseteq R$ is bounded, Lemma 8.26 suggests that $D_{\varepsilon'}$ is compact; thus $D_{\varepsilon'}\subseteq \bigcup\limits_{k=1}^{N} S_k$ for some $N\in\mathbb{N}$.

Let $\square_k = \overline{S_k}$, and $\mathcal{P}$ be a partition of R satisfying

(a) For each $\Delta\in\mathcal{P}$ with $\Delta\cap D_{\varepsilon'}\neq\varnothing$, $\Delta\subseteq\square_k$ for some $k = 1,\cdots,N$.

(b) For each $k = 1,\cdots,N$, $\square_k$ is the union of rectangles in $\mathcal{P}$.

(c) Some collection of $\Delta\in\mathcal{P}$ forms a partition $\widetilde{\mathcal{P}}$ of A.

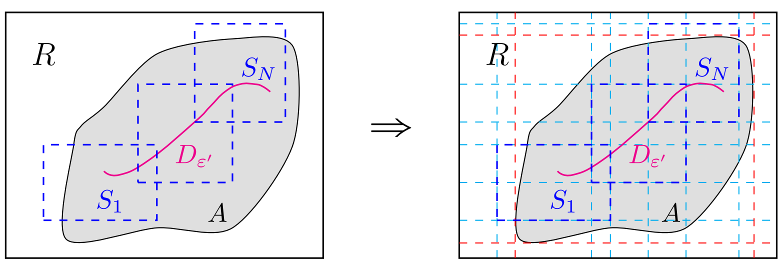


Figure 8.2: Constructing partitions $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ from finite rectangles $S_1,\cdots,S_N$

Rectangles in $\mathcal{P}$ fall into two families: $C_1 = \big\{\Delta\in\mathcal{P}\,\big|\,\Delta\subseteq\square_k \text{ for some } k = 1,\cdots,N\big\}$, and $C_2 = \big\{\Delta\in\mathcal{P}\,\big|\,\Delta\nsubseteq\square_k \text{ for all } k = 1,\cdots,N\big\}$. By the definition of the oscillation function,

$$\forall\, x\notin D_{\varepsilon'}\,, \exists\,\delta_x > 0 \ni \sup_{x_1,x_2\in D(x,\delta_x)}\big|\bar{f}(x_1)-\bar{f}(x_2)\big| < \varepsilon'.$$

Since $K = \bigcup\limits_{\Delta\in C_2}\Delta$ is compact, there exists $r > 0$ (the Lebesgue number associated with K and open cover $\bigcup_{x\in K} D(x,\delta_x)$) such that for each $a\in K$, $D(a,r)\subseteq D(y,\delta_y)$

for some $y \in K$. Let $\mathcal{P}'$ be a refinement of $\mathcal{P}$ such that $\|\mathcal{P}'\| < r$. Then if $\Delta' \in \mathcal{P}'$ such that $\Delta' \subseteq \Delta$ for some $\Delta \in C_2$, for some $y \in K$ we have $\Delta' \subseteq D(y, \delta_y)$; thus

$$\sup_{x\in\Delta'} \bar{f}(x) - \inf_{x\in\Delta'} \bar{f}(x) \leqslant \sup_{x\in D(y,\delta_y)} \bar{f}(y) - \inf_{x\in D(y,\delta_y)} \bar{f}(y) = \sup_{x_1,x_2\in D(y,\delta_y)} \big|\bar{f}(x_1) - \bar{f}(x_2)\big| < \varepsilon'.$$

As a consequence, if $\widetilde{\mathcal{P}}' = \big\{\Delta' \in \mathcal{P}' \,\big|\, \Delta' \subseteq \Delta \text{ for some } \Delta \in \widetilde{\mathcal{P}}\big\}$, then $\widetilde{\mathcal{P}}'$ is a partition of A and

$$\begin{aligned}
U(f, \widetilde{\mathcal{P}}') - L(f, \widetilde{\mathcal{P}}') &= \Big(\sum_{\substack{\Delta'\in\mathcal{P}' \\ \Delta'\subseteq\Delta\in C_1}} + \sum_{\substack{\Delta'\in\mathcal{P}' \\ \Delta'\subseteq\Delta\in C_2}} \Big)\big(\sup_{x\in\Delta'} \bar{f}(x) - \inf_{x\in\Delta'} \bar{f}(x)\big)\nu(\Delta') \\
&\leqslant 2\|f\|_\infty \sum_{\substack{\Delta'\in\mathcal{P}' \\ \Delta'\subseteq\Delta\in C_1}} \nu(\Delta') + \varepsilon' \sum_{\substack{\Delta'\in\mathcal{P}' \\ \Delta'\subseteq\Delta\in C_2}} \nu(\Delta') \\
&\leqslant 2\|f\|_\infty \sum_{\Delta\in\mathcal{P}\cap C_1} \nu(\Delta) + \varepsilon'\nu(R) \\
&\leqslant 2\|f\|_\infty \sum_{k=1}^{N} \nu(S_k) + +\varepsilon'\nu(R) < \big(2\|f\|_\infty + \nu(R)\big)\varepsilon' = \varepsilon\,;
\end{aligned}$$

thus f is Riemann integrable over A by Riemann's condition. □

Example 8.28. Let $A = \mathbb{Q} \cap [0, 1]$, and $f : A \to \mathbb{R}$ be the constant function $f \equiv 1$. Then

$$\bar{f}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0, 1]\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

The collection of points of discontinuity of $\bar{f}$ is $[0, 1]$ which, by Corollary 8.21, cannot be a set of measure zero; thus f is not Riemann integrable.

Another way to see that f is not Riemann integrable is $U(f, \mathcal{P}) = 1$ and $L(f, \mathcal{P}) = 0$ for all partitions $\mathcal{P}$ of A.

Corollary 8.29. *A bounded set $A \subseteq \mathbb{R}^n$ has volume if and only if the boundary of A has measure zero.*

Proof. 1. If $x_0 \notin \partial A$, then there exists $\delta > 0$ such that either $D(x_0, \delta) \subseteq A$ or $D(x_0, \delta) \subseteq A^\complement$; thus $\overline{1_A}$ is continuous at $x_0 \notin \partial A$ since $\overline{1_A}(x)$ is constant for all $x \in D(x_0, \delta)$.

2. On the other hand, if $x_0 \in \partial A$, then there exists $x_k \in A$, $y_k \in A^\complement$ such that $x_k \to x_0$ and $y_k \to x_0$ as $k \to \infty$. This implies that $\overline{1_A}$ cannot be continuous at x_0 since $\overline{1_A}(x_k) = 1$ while $\overline{1_A}(y_k) = 0$ for all $k \in \mathbb{N}$.

As a consequence, the collection of discontinuity of $\overline{1_A}$ is exactly ∂A, and the corollary follows from Lebesgue's theorem. $\square$

Corollary 8.30. *Let $A \subseteq \mathbb{R}^n$ be bounded and have volume. A bounded function $f : A \to \mathbb{R}$ with a finite or countable number of points of discontinuity is Riemann integrable.*

Proof. We note that $\{x \in \mathbb{R}^n \,|\, \text{osc}(\bar{f}, x) > 0\} \subseteq \partial A \cup \{x \in A \,|\, f \text{ is discontinuous at } x\}$. $\square$

Remark 8.31. In addition to the set inclusion listed in the proof of Corollary 8.30, we also have

$$\{x \in A \,|\, f \text{ is discontinuous at } x\} \subseteq \{x \in \mathbb{R}^n \,|\, \text{osc}(\bar{f}, x) > 0\}.$$

Therefore, if $A \subseteq \mathbb{R}^n$ is bounded and has volume, then a bounded function $f : A \to \mathbb{R}$ is Riemann integrable if and only if the collection of points of discontinuity of f has measure zero.

Corollary 8.32. *A bounded function is integrable over a compact set of measure zero.*

Proof. If $f : K \to \mathbb{R}$ is bounded, and K is a compact set of measure zero, then the collection of discontinuities of $\bar{f}$ is a subset of K. $\square$

Corollary 8.33. *Suppose that $A, B \subseteq \mathbb{R}^n$ are bounded sets with volume, and $f : A \to \mathbb{R}$ is Riemann integrable over A. Then f is Riemann integrable over $A \cap B$.*

Proof. By the inclusion

$$\{x \in \text{int}(A \cap B) \,|\, \text{osc}(\overline{f}^{A \cap B}, x) > 0\} \subseteq \{x \in \mathbb{R}^n \,|\, \text{osc}(\overline{f}^{A}, x) > 0\},$$

we find that

$$\begin{aligned}\{x \in \mathbb{R}^n \,|\, \text{osc}(\overline{f}^{A \cap B}, x) > 0\} &\subseteq \partial(A \cap B) \cup \{x \in \text{int}(A \cap B) \,|\, \text{osc}(\overline{f}^{A \cap B}, x) > 0\} \\ &\subseteq \partial A \cup \partial B \cup \{x \in \mathbb{R}^n \,|\, \text{osc}(\overline{f}^{A}, x) > 0\}.\end{aligned}$$

Since ∂A and ∂B both have measure zero, the integrability of f over $A \cap B$ then follows from the integrability of f over A and the Lebesgue Theorem. $\square$

Remark 8.34. Suppose that $A \subseteq \mathbb{R}^n$ is a bounded set of measure zero. Even if $f : A \to \mathbb{R}$ is continuous, f might not be Riemann integrable. For example, the function f given in Example 8.28 is not Riemann integrable even though f is continuous on A.

Remark 8.35. When $f : A \to \mathbb{R}$ is Riemann integrable over A, it is not necessary that A has volume. For example, the zero function is Riemann integrable over $A = \mathbb{Q} \cap [0, 1]$ even though A does not has volume.

8.4 Properties of the Integrals

The proof of the following theorem is essentially the same as the proof of Proposition 4.80, and is left to the readers.

Theorem 8.36. *Let $A \subseteq \mathbb{R}^n$ be bounded, $c \in \mathbb{R}$, and $f, g : A \to \mathbb{R}$ be Riemann integrable. Then*

1. *$f \pm g$ is Riemann integrable, and* $\displaystyle\int_A (f \pm g)(x)dx = \int_A f(x)dx \pm \int_A g(x)dx$.
2. *cf is Riemann integrable, and* $\displaystyle\int_A (cf)(x)dx = c\int_A f(x)dx$.
3. *$|f|$ is Riemann integrable, and* $\displaystyle\Big|\int_A f(x)dx\Big| \leqslant \int_A |f(x)|dx$.
4. *If $f \leqslant g$, then* $\displaystyle\int_A f(x)dx \leqslant \int_A g(x)dx$.
5. *If A has volume and $|f| \leqslant M$, then* $\displaystyle\Big|\int_A f(x)dx\Big| \leqslant M\nu(A)$.

Theorem 8.37. *Let $A \subseteq \mathbb{R}^n$ be bounded, and $f : A \to \mathbb{R}$ be a bounded integrable function.*

1. *If A has measure zero, then* $\displaystyle\int_A f(x)dx = 0$.
2. *If $f(x) \geqslant 0$ for all $x \in A$, and* $\displaystyle\int_A f(x)dx = 0$*, then the set $\big\{x \in A \,\big|\, f(x) \neq 0\big\}$ has measure zero.*

Proof. 1. We show that $L(f, \mathcal{P}) \leqslant 0 \leqslant U(f, \mathcal{P})$ for all partitions $\mathcal{P}$ of A. Let $\mathcal{P} = \big\{\Delta_1, \cdots, \Delta_N\big\}$ be a partition of A. By Corollary 8.21, $\Delta_k \cap A^{\complement} \neq \varnothing$ for $k = 1, \cdots, N$; thus we must have $\inf\limits_{x \in \Delta_k} \bar{f}(x) \leqslant 0$ and $\sup\limits_{x \in \Delta_k} \bar{f}(x) \geqslant 0$. As a consequence, if $\mathcal{P}$ is a partition of A,

$$L(f, \mathcal{P}) = \sum_{k=1}^{N} \inf_{x \in \Delta_k} \bar{f}(x)\nu(\Delta_k) \leqslant 0 \quad \text{and} \quad U(f, \mathcal{P}) = \sum_{k=1}^{N} \sup_{x \in \Delta_k} \bar{f}(x)\nu(\Delta_k) \geqslant 0\,;$$

thus $\displaystyle\underline{\int_A} f(x)dx \leqslant 0 \leqslant \overline{\int_A} f(x)dx$. Since f is integrable over A, $\displaystyle\int_A f(x)dx = 0$.

2. Let $A_k = \left\{x \in A \,\middle|\, f(x) \geqslant \frac{1}{k}\right\}$. We claim that A_k has measure zero for all $k \in \mathbb{N}$.

   Let $\varepsilon > 0$ be given. Since $\displaystyle\int_A f(x)dx = 0$, there exists a partition $\mathcal{P}$ of A such that $U(f, \mathcal{P}) < \frac{\varepsilon}{k}$. Let $C = \left\{\Delta \in \mathcal{P} \,\middle|\, \Delta \cap A_k \neq \varnothing\right\}$. Then $A_k \subseteq \bigcup\limits_{\Delta \in C} \Delta$, and

   $$\frac{1}{k}\sum_{\Delta \in C} \nu(C) \leqslant \sum_{\Delta \in C} \sup_{x \in \Delta} \bar{f}(x)\nu(\Delta) \leqslant \sum_{\Delta \in \mathcal{P}} \sup_{x \in \Delta} \bar{f}(x)\nu(\Delta) = U(f, \mathcal{P}) < \frac{\varepsilon}{k}$$

   which implies that $\sum\limits_{\Delta \in C} \nu(\Delta) < \varepsilon$. Therefore, A_k has measure zero; thus Theorem 8.22 implies that $A = \bigcup\limits_{k=1}^{\infty} A_k$ also has measure zero. □

Remark 8.38. Combining Corollary 8.32 and Theorem 8.37, we conclude that the integral of a bounded function over a compact set of measure zero is zero.

Remark 8.39. Let $A = \mathbb{Q} \cap [0, 1]$ and $f : A \to \mathbb{R}$ be the constant function $f \equiv 1$. We have shown in Example 8.28 that f is not Riemann integrable. We note that A has no volume since $\partial A = [0, 1]$ which is not a set of measure zero. However, A has measure zero since it consists of countable number of points.

1. Since f is continuous on A, the condition that A has volume in Corollary 8.30 cannot be removed.

2. Since A has measure zero, the condition that f is Riemann integrable in Theorem 8.37 cannot be removed.

Theorem 8.40 (Mean Value Theorem for Integrals)**.** *Let A be a subset of $\mathbb{R}^n$ such that A has volume and is compact and connected. Suppose that $f : A \to \mathbb{R}$ is continuous, then there exists $x_0 \in A$ such that*

$$\int_A f(x)dx = f(x_0)\nu(A)\,.$$

*The quantity $\displaystyle\frac{1}{\nu(A)}\int_A f(x)dx$ is called the **average** of f over A.*

Proof. Because of Theorem 8.37, it suffices to show the case that $\nu(A) \neq 0$. Let $m = \min\limits_{x \in A} f(x)$ and $M = \max\limits_{x \in A} f(x)$. Then

$$m1_A(x) \leqslant f(x) \leqslant M1_A(x)\,;$$

thus 2 and 4 of Theorem 8.36 imply that

$$m\nu(A) = \int_A m1_A(x)dx \leqslant \int_A f(x)dx \leqslant \int_A M1_A(x)dx = M\nu(A)\,.$$

By the connectedness of A and continuity of f, Theorem 4.21 and Theorem 3.38 implies that $f(A) = [m, M]$; thus by the fact that the quantity $\dfrac{1}{\nu(A)}\displaystyle\int_A f(x)dx \in [m, M]$, there must be $x_0 \in A$ such that

$$f(x_0) = \frac{1}{\nu(A)}\int_A f(x)dx\,. \qquad \square$$

Definition 8.41. Let $A \subseteq \mathbb{R}^n$ be a set and $f : A \to \mathbb{R}$ be a function. For $B \subseteq A$, the ***restriction of*** f ***to*** B is the function $f\big|_B : A \to \mathbb{R}$ given by $f|_B = f1_B$. In other words,

$$f\big|_B(x) = \begin{cases} f(x) & \text{if } x \in B\,, \\ 0 & \text{if } x \in A\backslash B\,. \end{cases}$$

The proof of the following lemma is not difficult, and is left as an exercise.

Lemma 8.42. *Let $A \subseteq \mathbb{R}^n$ be bounded, and $f : A \to \mathbb{R}$ be a bounded function. Suppose that $B \subseteq A$, and $f\big|_B$ is Riemann integrable over A. Then f is Riemann integrable over B, and*

$$\int_A f\big|_B(x)dx = \int_B f(x)dx\,.$$

Theorem 8.43. *Let A, B be bounded subsets of $\mathbb{R}^n$ be such that $A \cap B$ has measure zero, and $f : A \cup B \to \mathbb{R}$ be such that $f\big|_{A\cap B}$, $f\big|_A$ and $f\big|_B$ are all Riemann integrable over $A \cup B$. Then f is integrable over $A \cup B$, and*

$$\int_{A\cup B} f(x)dx = \int_A f(x)dx + \int_B f(x)dx\,.$$

Proof. Since $1_{A\cup B} = 1_A + 1_B - 1_{A\cap B}$, we have

$$f = f1_{A\cup B} = f\big|_A + f\big|_B - f\big|_{A\cap B}\,;$$

thus Theorem 8.37, Theorem 8.36 and Lemma 8.42 imply that

$$\int_{A\cup B} f(x)dx = \int_{A\cup B} f\big|_A(x)dx + \int_{A\cup B} f\big|_B(x)dx = \int_A f(x)dx + \int_B f(x)dx\,. \qquad \square$$

8.5 The Fubini Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous, the fundamental theorem of Calculus (Theorem 4.89) can be applied to computed the integral of f over $[a, b]$. In the following two sections, we focus on how the integral of f over $A \subseteq \mathbb{R}^n$, where $n \geqslant 2$, can be computed if the integral exists. We start with the special case $n = 2$.

Definition 8.44. Let $S = [a, b] \times [c, d]$ be a rectangle in $\mathbb{R}^2$, and $f : S \to \mathbb{R}$ be bounded. For each fixed $x \in [a, b]$, the lower integral of the function $f(x, \cdot) : [c, d] \to \mathbb{R}$ is denoted by $\underline{\int_c^d} f(x, y)dy$, and the upper integral of $f(x, \cdot) : [c, d] \to \mathbb{R}$ is denoted by $\overline{\int_c^d} f(x, y)dy$. If for each $x \in [a, b]$ the upper integral and the lower integral of $f(x, \cdot) : [c, d] \to \mathbb{R}$ are the same, we simply write $\int_c^d f(x, y)dy$ for the integrals of $f(x, \cdot)$ over $[c, d]$. The integrals $\overline{\int_a^b} f(x, y)dx$, $\underline{\int_a^b} f(x, y)dx$ and $\int_a^b f(x, y)dx$ are defined in a similar way.

Lemma 8.45. *Let $A = [a, b] \times [c, d]$ be a rectangle in $\mathbb{R}^2$, and $f : A \to \mathbb{R}$ be bounded. Then*

$$\underline{\int_A} f(x, y)d\mathbb{A} \leqslant \underline{\int_a^b} \Big(\underline{\int_c^d} f(x, y)dy \Big) dx \leqslant \overline{\int_a^b} \Big(\overline{\int_c^d} f(x, y)dy \Big) dx \leqslant \overline{\int_A} f(x, y)d\mathbb{A} \tag{8.5.1}$$

and

$$\underline{\int_A} f(x, y)d\mathbb{A} \leqslant \underline{\int_c^d} \Big(\underline{\int_a^b} f(x, y)dx \Big) dy \leqslant \overline{\int_c^d} \Big(\overline{\int_a^b} f(x, y)dx \Big) dy \leqslant \overline{\int_A} f(x, y)d\mathbb{A} \,. \tag{8.5.2}$$

Proof. It suffices to prove (8.5.1). Let $\varepsilon > 0$ be given. Choose a partition

$$\mathcal{P} = \big\{ \Delta_{ij} \,\big|\, \Delta_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}] \text{ for } i = 0, 1, \cdots, n-1 \text{ and } j = 0, 1, \cdots, m-1 \big\}$$

of A such that $L(f, \mathcal{P}) > \underline{\int_A} f(x, y)d\mathbb{A} - \varepsilon$. Using (4.7.3) and Remark 4.81, we find that

$$\begin{aligned}
\underline{\int_a^b} \Big(\underline{\int_c^d} f(x, y)dy \Big) dx &= \sum_{i=0}^{n-1} \underline{\int_{x_i}^{x_{i+1}}} \Big(\sum_{j=0}^{m-1} \underline{\int_{y_j}^{y_{j+1}}} f(x, y)dy \Big) dx \\
&\geqslant \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \underline{\int_{x_i}^{x_{i+1}}} \Big(\underline{\int_{y_j}^{y_{j+1}}} f(x, y)dy \Big) dx \\
&\geqslant \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \inf_{(x,y) \in \Delta_{ij}} f(x, y)\nu(\Delta_{ij}) = L(f, \mathcal{P}) > \underline{\int_A} f(x, y)d\mathbb{A} - \varepsilon \,.
\end{aligned}$$

Since $\varepsilon > 0$ is given arbitrary, we must have

$$\underline{\int_a^b}\Big(\underline{\int_c^d} f(x,y)dy\Big)dx \geqslant \underline{\int_A} f(x,y)d\mathbb{A}\,.$$

Similarly, $\overline{\int_a^b}\Big(\overline{\int_c^d} f(x,y)dy\Big)dx \leqslant \overline{\int_A} f(x,y)d\mathbb{A}$, so (8.5.1) is concluded. □

Theorem 8.46 (Fubini's Theorem, the case $n = 2$)**.** *Let $A = [a,b] \times [c,d]$ be a rectangle in $\mathbb{R}^2$, and $f : A \to \mathbb{R}$ be Riemann integrable. Then*

1. *the functions $\underline{\int_c^d} f(\cdot,y)dy$ and $\overline{\int_c^d} f(\cdot,y)dy$ are Riemann integrable over $[a,b]$;*
2. *the functions $\underline{\int_a^b} f(x,\cdot)dx$ and $\overline{\int_a^b} f(x,\cdot)dx$ are Riemann integrable over $[c,d]$, and*
3. *The integral of f over A is the same as the iterated integrals; that is,*

$$\begin{aligned}\int_A f(x,y)d\mathbb{A} &= \int_a^b\Big(\underline{\int_c^d} f(x,y)dy\Big)dx = \int_a^b\Big(\overline{\int_c^d} f(x,y)dy\Big)dx \\ &= \int_c^d\Big(\underline{\int_a^b} f(x,y)dx\Big)dy = \int_c^d\Big(\overline{\int_a^b} f(x,y)dx\Big)dy\,.\end{aligned}$$

Proof. It suffices to prove that $\underline{\int_c^d} f(x,y)dy$ is Riemann integrable over $[a,b]$ and

$$\int_a^b\Big(\underline{\int_c^d} f(x,y)dy\Big)dx = \int_A f(x,y)d\mathbb{A}\,. \tag{8.5.3}$$

Since $\underline{\int_a^b}\Big(\underline{\int_c^d} f(x,y)dy\Big)dx \leqslant \overline{\int_a^b}\Big(\underline{\int_c^d} f(x,y)dy\Big)dx$, Lemma 8.45 implies that

$$\begin{aligned}\underline{\int_A} f(x,y)d\mathbb{A} &\leqslant \underline{\int_a^b}\Big(\underline{\int_c^d} f(x,y)dy\Big)dx \leqslant \overline{\int_a^b}\Big(\underline{\int_c^d} f(x,y)dy\Big)dx \\ &\leqslant \overline{\int_a^b}\Big(\overline{\int_c^d} f(x,y)dy\Big)dx \leqslant \overline{\int_A} f(x,y)d\mathbb{A}\,.\end{aligned}$$

The integrability of $\underline{\int_c^d} f(x,y)dy$ and the validity of (8.5.3) are then concluded by the integrability of f over A. □

Remark 8.47. To simplify the notation, sometimes we use $\int_a^b \int_c^d f(x,y)dydx$ to denote the iterated integral the iterated integral $\int_a^b \Big(\int_c^d f(x,y)dy\Big)dx$. Similar notation applies to the upper and the lower integrals. For example, we also have $\overline{\int_a^b} \underline{\int_c^d} f(x,y)dydx = \overline{\int_a^b} \Big(\underline{\int_c^d} f(x,y)dy\Big)dx$.

Remark 8.48. For each $x \in [a,b]$, define $\varphi(x) = \underline{\int_c^d} f(x,y)dy$ and $\psi(x) = \overline{\int_c^d} f(x,y)dy$. Then $\varphi(x) \leqslant \psi(x)$ for all $x \in [a,b]$, and the Fubini Theorem implies that

$$\int_a^b \big[\psi(x) - \varphi(x)\big]dx = 0\,.$$

By Theorem 8.37, the set $\big\{x \in [a,b] \,\big|\, \psi(x) - \varphi(x) \neq 0\big\}$ has measure zero. In other words, except on a set of measure zero, $f(x,\cdot)$ is Riemann integrable over $[c,d]$ if f is Riemann integrable over $[a,b] \times [c,d]$. This property can be rephrased as that "$f(x,\cdot)$ is Riemann integrable over $[c,d]$ for **almost every** $x \in [a,b]$ if f is Riemann integrable over the rectangle $[a,b] \times [c,d]$". Similarly, $f(\cdot,y)$ is Riemann integrable for almost every $y \in [c,d]$ if f is Riemann integrable over $[a,b] \times [c,d]$.

Remark 8.49. The integrability of f does not guarantee that $f(x,\cdot)$ or $f(\cdot,y)$ is Riemann integrable. In fact, there exists a function $f : [0,1] \times [0,1] \to \mathbb{R}$ such that f is Riemann integrable, $f(\cdot,y)$ is Riemann integrable for each $y \in [0,1]$, but $f(x,\cdot)$ is not Riemann integrable for infinitely many $x \in [0,1]$. For example, let

$$f(x,y) = \begin{cases} 0 & \text{if } x = 0 \text{ or if } x \text{ or } y \text{ is irrational}\,, \\ \dfrac{1}{p} & \text{if } x, y \in \mathbb{Q} \text{ and } x = \dfrac{q}{p} \text{ with } (p,q) = 1\,. \end{cases}$$

Then

1. For each $y \in [0,1]$, $f(\cdot,y)$ is continuous at all irrational numbers. Therefore, $f(\cdot,y)$ is Riemann integrable, and $\int_0^1 f(x,y)dx = \underline{\int_0^1} f(x,y)dx = 0$.

2. For $x = 0$ or $x \notin \mathbb{Q}$, $f(x,\cdot)$ is Riemann integrable, and $\int_0^1 f(x,y)dy = 0$.

3. If $x = \frac{q}{p}$ with $(p, q) = 1$, $f(x, \cdot)$ is nowhere continuous in $[0, 1]$. In fact, for each $y_0 \in [0, 1]$,
$$\lim_{\substack{y \to y_0 \\ y \in \mathbb{Q}}} f(x, y) = \frac{1}{p} \quad \text{while} \quad \lim_{\substack{y \to y_0 \\ y \notin \mathbb{Q}}} f(x, y) = 0\,;$$
thus the limit of $f(x, y)$ as $y \to y_0$ does not exist. Therefore, the Lebesgue theorem implies that $f(x, \cdot)$ is not Riemann integrable if $x \in \mathbb{Q} \cap (0, 1]$. On the other hand, for $x = \frac{q}{p}$ with $(p, q) = 1$ we have
$$\underline{\int_0^1} f(x, y) dy = 0 \quad \text{and} \quad \overline{\int_0^1} f(x, y) dy = \frac{1}{p}\,.$$

4. Define $\varphi(x) = \underline{\int_0^1} f(x, y) dy$ and $\psi(x) = \overline{\int_0^1} f(x, y) dy$. Then 2 and 3 imply that φ and ψ are Riemann integrable over $[0, 1]$, and $\int_0^1 \varphi(x) dx = \int_0^1 \psi(x) dx = 0$.

5. For each $a \notin \mathbb{Q} \cap [0, 1]$ and $b \in [0, 1]$, f is continuous at (a, b). In fact, for any given $\varepsilon > 0$, there exists a prime number p such that $\frac{1}{p} < \varepsilon$. Let
$$\delta = \min\Big\{\big|a - \frac{\ell}{k}\big| \,\Big|\, 0 \leqslant \ell \leqslant k \leqslant p, k \in \mathbb{N}, \ell \in \mathbb{N} \cup \{0\}\Big\}\,.$$
Then $\delta > 0$, and if $(x, y) \in D\big((a, b), \delta\big) \cap ([0, 1] \times [0, 1])$, we have
$$\big|f(x, y) - f(a, b)\big| = \big|f(x, y)\big| < \frac{1}{p} < \varepsilon\,,$$
where we use the fact that if $(x, y) \in D\big((a, b), \delta\big)$ and $x \in \mathbb{Q}$, then $x = \frac{\ell}{k}$ (in reduced form) for some $k > p$.

   As a consequence, $\big\{(a, b) \in \mathbb{R}^2 \,\big|\, \bar{f} \text{ is discontinuous at } (a, b)\big\} \subseteq \mathbb{Q} \times [0, 1]$. Since $\mathbb{Q} \times [0, 1]$ is a countable union of measure zero sets, it has measure zero; thus f is Riemann integrable by the Lebesgue theorem. The Fubini theorem then implies that
$$\int_{[0,1]\times[0,1]} f(x, y) d\mathbb{A} = \int_0^1 \int_0^1 f(x, y) dx dy = 0\,.$$

Remark 8.50. The integrability of $f(x, \cdot)$ and $f(\cdot, y)$ does not guarantee the integrability of f. In fact, there exists a bounded function $f : [0, 1] \times [0, 1] \to \mathbb{R}$ such that $f(x, \cdot)$ and $f(\cdot, y)$

are both Riemann integrable over $[0,1]$, but f is not Riemann integrable over $[0,1]\times[0,1]$. For example, let

$$f(x,y)=\begin{cases}1 & \text{if } (x,y)=\big(\dfrac{k}{2^n},\dfrac{\ell}{2^n}\big),\ 0<k,\ell<2^n \text{ odd numbers, } n\in\mathbb{N}\,,\\ 0 & \text{otherwise}\,.\end{cases}$$

Then for each $x\in[0,1]$, $f(x,\cdot)$ only has finite number of discontinuities; thus $f(x,\cdot)$ is Riemann integrable, and

$$\int_0^1 f(x,y)dy=0\,.$$

Similarly, $f(\cdot,y)$ is Riemann integrable, and $\displaystyle\int_0^1 f(x,y)dx=0$. As a consequence,

$$\int_0^1\int_0^1 f(x,y)dydx=\int_0^1\int_0^1 f(x,y)dxdy=0\,.$$

However, note that f is nowhere continuous on $[0,1]\times[0,1]$; thus the Lebesgue theorem implies that f is not Riemann integrable. One can also see this by the fact that $U(f,\mathcal{P})=1$ and $L(f,\mathcal{P})=0$ for all partition of $[0,1]\times[0,1]$.

Corollary 8.51. 1. *Let $\varphi_1,\varphi_2:[a,b]\to\mathbb{R}$ be continuous maps such that $\varphi_1(x)\leqslant\varphi_2(x)$ for all $x\in[a,b]$, $A=\big\{(x,y)\,\big|\,a\leqslant x\leqslant b,\varphi_1(x)\leqslant y\leqslant\varphi_2(x)\big\}$, and $f:A\to\mathbb{R}$ be continuous. Then f is Riemann integrable over A, and*

$$\int_A f(x,y)d\mathbb{A}=\int_a^b\Big(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy\Big)dx\,.$$

2. *Let $\psi_1,\psi_2:[c,d]\to\mathbb{R}$ be continuous maps such that $\psi_1(y)\leqslant\psi_2(y)$ for all $y\in[c,d]$, $A=\big\{(x,y)\,\big|\,c\leqslant y\leqslant d,\psi_1(y)\leqslant x\leqslant\psi_2(y)\big\}$, and $f:A\to\mathbb{R}$ be continuous. Then f is Riemann integrable over A, and*

$$\int_A f(x,y)d\mathbb{A}=\int_c^d\Big(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx\Big)dy\,.$$

Proof. It suffices to prove 1. First we show that f is Riemann integrable over A. By Lebesgue's theorem, it suffices to show that the set $\big\{(x,y)\in\mathbb{R}^2\,\big|\,\text{osc}\big(\bar{f},(x,y)\big)>0\big\}$ has measure zero, where $\bar{f}$ is the extension of f by zero outside A. Nevertheless, we note that

$$\begin{aligned}\big\{(x,y)\in\mathbb{R}^2\,\big|\,\text{osc}\big(\bar{f},(x,y)\big)>0\big\}\subseteq\{a\}\times\big[\varphi_1(a),\varphi_2(a)\big]\cup\{b\}\times\big[\varphi_1(b),\varphi_2(b)\big]\cup\\ \cup\big\{\big(x,\varphi_1(x)\big)\,\big|\,x\in[a,b]\big\}\cup\big\{\big(x,\varphi_2(x)\big)\,\big|\,x\in[a,b]\big\}\,.\end{aligned}$$

It is clear that $\{a\}\times\big[\varphi_1(a),\varphi_2(a)\big]$ and $\{b\}\times\big[\varphi_1(b),\varphi_2(b)\big]$ have measure zero since they have volume zero. Now we claim that the sets $\big\{\big(x,\varphi_1(x)\big)\,\big|\,x\in[a,b]\big\}$ and $\big\{\big(x,\varphi_2(x)\big)\,\big|\,x\in[a,b]\big\}$ also have measure zero.

Let $\varepsilon>0$ be given. Since φ_1 is continuous on a compact set $[a,b]$, φ_1 is uniformly continuous on $[a,b]$; thus there exists $\delta>0$ such that

$$\big|\varphi_1(x_1)-\varphi_1(x_2)\big|<\frac{\varepsilon}{b-a}\quad\text{whenever}\quad |x_1-x_2|<\delta\,.$$

Let $\mathcal{P}=\{a=x_0<x_1<\cdots<x_{n-1}<x_n=b\}$ be a partition of $[a,b]$ such that $|x_{i+1}-x_i|<\delta$ for all $i=0,\cdots,n-1$, and let $\Delta_i=\big[x_i,x_{i+1}\big]\times\big[\min\limits_{x\in[x_i,x_{i+1}]}\varphi_1(x),\max\limits_{x\in[x_i,x_{i+1}]}\varphi_1(x)\big]$. Then

$$\big\{\big(x,\varphi_1(x)\big)\,\big|\,x\in[a,b]\big\}\subseteq\bigcup_{i=0}^{n-1}\Delta_i$$

and

$$\sum_{i=0}^{n-1}\nu(\Delta_i)<\sum_{i=0}^{n-1}\frac{\varepsilon}{b-a}(x_{i+1}-x_i)=\frac{\varepsilon}{b-a}\sum_{i=0}^{n-1}(x_{i+1}-x_i)=\varepsilon\,.$$

Therefore, $\big\{\big(x,\varphi_1(x)\big)\,\big|\,x\in[a,b]\big\}$ has volume zero; thus $\big\{\big(x,\varphi_1(x)\big)\,\big|\,x\in[a,b]\big\}$ has measure zero. Similarly, $\big\{\big(x,\varphi_2(x)\big)\,\big|\,x\in[a,b]\big\}$ also has measure zero. By Theorem 8.22, $\big\{(x,y)\in\mathbb{R}^2\,\big|\,\text{osc}\big(\bar{f},(x,y)\big)>0\big\}$ has measure zero; thus f is Riemann integrable over A.

Let $m=\min\limits_{x\in[a,b]}\varphi_1(x)$, $M=\max\limits_{x\in[a,b]}\varphi_2(x)$, and $S=[a,b]\times[m,M]$. Then $A\subseteq S$. By Lemma 8.42 and the Fubini Theorem,

$$\int_A f(x,y)d\mathbb{A}=\int_S\bar{f}(x,y)d\mathbb{A}=\int_a^b\Big(\int_m^M\bar{f}(x,y)dy\Big)dx=\int_a^b\Big(\int_{\varphi_1(x)}^{\varphi_2(x)}f(x,y)dy\Big)dx$$

which concludes 1. □

Example 8.52. Let $A=\big\{(x,y)\in\mathbb{R}^2\,\big|\,0\leqslant x\leqslant 1,x\leqslant y\leqslant 1\big\}$, and $f:A\to\mathbb{R}$ be given by $f(x,y)=xy$. Then Corollary 8.51 implies that

$$\int_A f(x,y)d\mathbb{A}=\int_0^1\Big(\int_x^1 xydy\Big)dx=\int_0^1\frac{xy^2}{2}\Big|_{y=x}^{y=1}dx=\int_0^1\Big(\frac{x}{2}-\frac{x^3}{2}\Big)dx=\frac{1}{4}-\frac{1}{8}=\frac{1}{8}\,.$$

On the other hand, since $A=\big\{(x,y)\in\mathbb{R}^2\,\big|\,0\leqslant y\leqslant 1,0\leqslant x\leqslant y\big\}$, we can also evaluate the integral of f over A by

$$\int_A xyd\mathbb{A}=\int_0^1\Big(\int_0^y xydx\Big)dy=\int_0^1\frac{x^2y}{2}\Big|_{x=0}^{x=y}dy=\int_0^1\frac{y^3}{2}dy=\frac{1}{8}\,.$$

Example 8.53. Let $A = \{(x,y) \in \mathbb{R}^2 \,|\, 0 \leqslant x \leqslant 1, \sqrt{x} \leqslant y \leqslant 1\}$, and $f : A \to \mathbb{R}$ be given by $f(x,y) = e^{y^3}$. Then Corollary 8.51 implies that

$$\int_A f(x,y)d\mathbb{A} = \int_0^1 \Big(\int_{\sqrt{x}}^1 e^{y^3} dy\Big) dx\,.$$

Since we do not know how to compute the inner integral, we look for another way of finding the integral. Observing that $A = \{(x,y) \in \mathbb{R}^2 \,|\, 0 \leqslant y \leqslant 1, 0 \leqslant x \leqslant y^2\}$, we have

$$\int_A f(x,y)d\mathbb{A} = \int_0^1 \Big(\int_0^{y^2} e^{y^3} dx\Big) dy = \int_0^1 y^2 e^{y^3} dy = \frac{1}{3} e^{y^3}\Big|_{y=0}^{y=1} = \frac{e-1}{3}\,.$$

Similar to the proof of Lemma 8.45, 8.46 and Corollary 8.51, we can prove the following

Theorem 8.54 (Fubini's Theorem)**.** *Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ be rectangles, and $f : A \times B \to \mathbb{R}$ be bounded. For $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, write $z = (x,y)$. Then*

$$\underline{\int}_{A\times B} f(z)\,dz \leqslant \underline{\int}_A \Big(\underline{\int}_B f(x,y)dy\Big)dx \leqslant \overline{\int}_A \Big(\overline{\int}_B f(x,y)dy\Big)dx \leqslant \overline{\int}_{A\times B} f(z)dz$$

and

$$\underline{\int}_{A\times B} f(z)\,dz \leqslant \underline{\int}_B \Big(\underline{\int}_A f(x,y)dx\Big)dy \leqslant \overline{\int}_B \Big(\overline{\int}_A f(x,y)dx\Big)dy \leqslant \overline{\int}_{A\times B} f(z)\,dz\,.$$

In particular, if $f : A \times B \to \mathbb{R}$ is Riemann integrable, then

$$\begin{aligned}\int_{A\times B} f(z)\,dz &= \int_A \Big(\underline{\int}_B f(x,y)dy\Big)dx = \int_A \Big(\overline{\int}_B f(x,y)dy\Big)dx \\ &= \int_B \Big(\underline{\int}_A f(x,y)dx\Big)dy = \int_B \Big(\overline{\int}_A f(x,y)dx\Big)dy\,.\end{aligned}$$

Corollary 8.55. *Let $S \subseteq \mathbb{R}^n$ be a bounded set with volume, $\varphi_1, \varphi_2 : S \to \mathbb{R}$ be continuous maps such that $\varphi_1(x) \leqslant \varphi_2(x)$ for all $x \in S$, $A = \{(x,y) \in \mathbb{R}^{n+1} \,|\, x \in S, \varphi_1(x) \leqslant y \leqslant \varphi_2(x)\}$, and $f : A \to \mathbb{R}$ be continuous. Then f is Riemann integrable over A, and*

$$\int_A f(x,y)d(x,y) = \int_S \Big(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy\Big)dx\,.$$

The proof of Theorem 8.54 and Corollary 8.55 are left as exercises.

Example 8.56. Let $A \subseteq \mathbb{R}^3$ be the set $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, x_1 \geqslant 0, x_2 \geqslant 0, x_3 \geqslant 0, \text{and } x_1 + x_2 + x_3 \leqslant 1\}$, and $f : A \to \mathbb{R}$ be given by $f(x_1, x_2, x_3) = (x_1 + x_2 + x_3)^2$. Let $S = [0,1] \times [0,1] \times [0,1]$, and $\bar{f} : \mathbb{R}^3 \to \mathbb{R}$ be the extension of f by zero outside A. Then Corollary 8.30 implies that f is Riemann integrable (since ∂A has measure zero). Write $\widehat{x}_1 = (x_2, x_3)$, $\widehat{x}_2 = (x_1, x_3)$ and $\widehat{x}_3 = (x_1, x_2)$. Lemma 8.42 implies that

$$\int_A f(x)dx = \int_S \bar{f}(x)dx\,,$$

and Theorem 8.54 implies that

$$\int_S \bar{f}(x)dx = \int_{[0,1]} \Big(\int_{[0,1]\times[0,1]} \bar{f}(\widehat{x}_3, x_3) d\widehat{x}_3 \Big) dx_3\,.$$

Let $A_{x_3} = \{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_1 \geqslant 0, x_2 \geqslant 0, x_1 + x_2 \leqslant 1 - x_3\}$. Then for each $x_3 \in [0,1]$,

$$\int_{[0,1]\times[0,1]} \bar{f}(\widehat{x}_3, x_3) d\widehat{x}_3 = \int_{A_{x_3}} f(\widehat{x}_3, x_3) d\widehat{x}_3 = \int_0^{1-x_3} \Big(\int_0^{1-x_3-x_2} f(x_1, x_2, x_3) dx_1 \Big) dx_2\,.$$

Computing the iterated integral, we find that

$$\begin{aligned}
\int_A f(x)dx &= \int_0^1 \Big[\int_0^{1-x_3} \Big(\int_0^{1-x_3-x_2} (x_1 + x_2 + x_3)^2 dx_1 \Big) dx_2 \Big] dx_3 \\
&= \int_0^1 \Big[\int_0^{1-x_3} \frac{(x_1 + x_2 + x_3)^3}{3} \Big|_{x_1=0}^{x_1=1-x_3-x_2} dx_2 \Big] dx_3 \\
&= \int_0^1 \Big[\int_0^{1-x_3} \Big(\frac{1}{3} - \frac{(x_2 + x_3)^3}{3} \Big) dx_2 \Big] dx_3 \\
&= \int_0^1 \Big(\frac{1}{4} - \frac{x_3}{3} + \frac{x_3^4}{12} \Big) dx_3 = \frac{1}{4} - \frac{1}{6} + \frac{1}{60} = \frac{15 - 10 + 1}{60} = \frac{1}{10}\,.
\end{aligned}$$

8.6 Change of Variables Formula

Fubini theorem can be used to find the integral of a (Riemann integrable) function over a rectangular domain if the iterated integrals can be evaluated. However, like the integral of a function of one variable, in many cases we need to make use of several change of variables in order to transform the integral to another integral that can be easily evaluated. In this section, we establish the change of variables formula for the integral of functions of several variables.

Theorem 8.57 (Change of Variables Formula)**.** *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be an open bounded set, and $g : \mathcal{U} \to \mathbb{R}^n$ be an one-to-one $\mathscr{C}^1$ mapping with $\mathscr{C}^1$ inverse; that is, $g^{-1} : g(\mathcal{U}) \to \mathcal{U}$ is also continuously differentiable. Assume that the Jacobian of g, $\mathbf{J}_g = \det([Dg])$, does not vanish in $\mathcal{U}$, and $E \subset\subset \mathcal{U}$ has volume. Then $g(E)$ has volume. Moreover, if $f : g(E) \to \mathbb{R}$ is bounded and integrable, then $(f \circ g)\mathbf{J}_g$ is integrable over E, and*

$$\int_{g(E)} f(y)\, dy = \int_E (f \circ g)(x) |\mathbf{J}_g(x)|\, dx = \int_E (f \circ g)(x) \Big| \frac{\partial(g_1, \cdots, g_n)}{\partial(x_1, \cdots, x_n)} \Big|\, dx \,.$$

The proof of the change of variable formula is separated into several steps, and we list each step as a lemma.

First, we show that the map g in Theorem 8.57 has the property that $g^{-1}(Z)$ has measure zero (or volume zero) if Z itself has measure zero (or volume zero). This establishes that if A and B are not overlapping; that is, $\nu(A \cap B) = 0$, then $\nu(g^{-1}(A \cap B)) = 0$.

Lemma 8.58. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be an open set, and $\phi : \mathcal{U} \to \mathbb{R}^n$ be Lipschitz continuous; that is, there exists $L > 0$ such that $\|\phi(x) - \phi(y)\|_{\mathbb{R}^n} \leqslant L\|x - y\|_{\mathbb{R}^n}$ for all $x, y \in \mathcal{U}$. Suppose that $Z \subseteq \mathcal{U}$ is a set of measure zero* (*or a set of volume zero*) *and $\bar{Z} \subseteq \mathcal{U}$. Then $\phi(A)$ has measure zero* (*or volume zero*).

Proof. We prove the case that Z has measure zero, and the proof for the case that Z has volume zero is obtained by changing the countable sum/union to finite sum/union.

First we note that if $S \subseteq \mathcal{U}$ is a rectangle on which the ratio of the maximum length and minimum length of sides is less than 2, then $\phi(S) \subseteq R$ for some n-cube R with side of length $L\sqrt{n}\delta$, where δ is the maximum length of sides of S. Therefore, $\nu(\phi(S)) \leqslant (2\sqrt{n}L)^n \nu(S)$. Let $\varepsilon > 0$ be given. Since Z has measure zero, there exists countable rectangles $S_1, S_2, \cdots$ such that

$$Z \subseteq \bigcup_{k=1}^{\infty} S_k \quad \text{and} \quad \sum_{k=1}^{\infty} \nu(S_k) < \frac{\varepsilon}{(2\sqrt{n}L)^n} \,.$$

Moreover, as in the proof of Proposition 8.12 we can also assume that the ratio of the maximum length and minimum length of sides of S_k is less than 2 for all $k \in \mathbb{N}$; thus

$$\phi(Z) \subseteq \bigcup_{k=1}^{\infty} R_k \quad \text{and} \quad \sum_{k=1}^{\infty} \nu(R_k) < \varepsilon$$

for some rectangles R_k's. ▫

Next, we show that we only have to prove the change of variable formula for the case that f is a constant and E is the pre-image of closed rectangle under g in order to establish the full result.

Lemma 8.59. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be an open bounded set, and $g : \mathcal{U} \to \mathbb{R}^n$ be an one-to-one $\mathscr{C}^1$ mapping that has a $\mathscr{C}^1$ inverse; that is, $g^{-1} : g(\mathcal{U}) \to \mathbb{R}^n$ is also continuously differentiable. Assume that the Jacobian of g, $\mathbf{J}_g = \det([Dg])$, does not vanish in $\mathcal{U}$, and*

$$\nu(R) = \int_{g^{-1}(R)} |\mathbf{J}_g(x)| dx \quad \textit{for all closed rectangle } R \subseteq g(\mathcal{U})\,. \tag{8.6.1}$$

Then if $E \subset\subset \mathcal{U}$ has volume and $f : g(E) \to \mathbb{R}$ is bounded and integrable, then $(f \circ g)|\mathbf{J}_g|$ is Riemann integrable over E, and

$$\int_{g(E)} f(y)dy = \int_E (f \circ g)(x)|\mathbf{J}_g(x)|dx\,.$$

Proof. First we note that since the Jacobian of g does not vanish in $\mathcal{U}$, Remark 7.3 implies that g is an open mapping; thus $g(\mathcal{U})$ is open. Moreover, since $g^{-1} \in \mathscr{C}^1(g(\mathcal{U}))$, for any open set $\mathcal{V} \subset\subset g(\mathcal{U})$, $g \in \mathscr{C}^1(\mathcal{V})$; thus there exists $L_{\mathcal{V}} > 0$ such that $\|g^{-1}(y_1) - g^{-1}(y_2)\|_{\mathbb{R}^n} \leqslant L_{\mathcal{V}} \|y_1 - y_2\|_{\mathbb{R}^n}$ for all $y_1, y_2 \in \mathcal{V}$. Therefore, Lemma 8.58 and Remark 8.38 imply that (8.6.1) holds for all rectangles R that are not necessary closed, as long as $\bar{R} \subseteq g(\mathcal{U})$.

Consider the extensions of f and $(f \circ g)|\mathbf{J}_g|$ given by

$$\overline{f}^{g(E)}(x) = \begin{cases} f(x) & \text{if } x \in g(E)\,, \\ 0 & \text{if } x \in g(E)^{\complement}\,, \end{cases}$$

and

$$\overline{(f \circ g)|\mathbf{J}_g|}^{E}(x) = \begin{cases} (f \circ g)(x)|\mathbf{J}_g|(x) & \text{if } x \in E\,, \\ 0 & \text{if } x \in E^{\complement}\,. \end{cases}$$

By the integrability of f over $g(E)$, the set $\big\{y \in \mathbb{R}^n \,\big|\, \overline{f}^{g(E)} \text{ is discontinuous at } y\big\}$ has measure zero. Since

$$\begin{aligned} \big\{x \in \mathbb{R}^n \,\big|\, \overline{(f \circ g)|\mathbf{J}_g|}^{E} &\text{ is discontinuous at } x\big\} \\ &\subseteq \partial E \cup \big\{x \in \operatorname{int}(E) \,\big|\, f \text{ is discontinuous at } g(x)\big\} \\ &= \partial E \cup \big\{y \in g(\operatorname{int}(E)) \,\big|\, f \text{ is discontinuous at } y\big\} \\ &\subseteq \partial E \cup \big\{y \in \mathbb{R}^n \,\big|\, \overline{f}^{g(E)} \text{ is discontinuous at } y\big\}\,, \end{aligned}$$

we conclude that $\left\{x \in \mathbb{R}^n \,\middle|\, \overline{(f\circ g)|\mathbf{J}_g|}^E \text{ is discontinuous at } x\right\}$ has measure zero. Therefore, $(f\circ g)|\mathbf{J}_g|$ is Riemann integrable over E. On the other hand, by the fact that

$$(\overline{f}^{g(E)}\circ g)|\mathbf{J}_g| = (\overline{f\circ g})^E|\mathbf{J}_g| = \overline{(f\circ g)|\mathbf{J}_g|}^E \quad \text{on} \quad \mathcal{U}\,,$$

we also find that $(\overline{f}^{g(E)}\circ g)|\mathbf{J}_g|$ is Riemann integrable over $\mathcal{U}$, and Corollary 8.33 further suggests that $(\overline{f}^{g(E)}\circ g)|\mathbf{J}_g|$ is Riemann integrable over for every subset of $\mathcal{U}$ that has volume. Moreover,

$$\int_{\mathcal{U}} (\overline{f}^{g(E)}\circ g)(x)|\mathbf{J}_g(x)|dx = \int_E (f\circ g)(x)|\mathbf{J}_g(x)|dx\,. \tag{8.6.2}$$

Fix an open set $\mathcal{V}$ with $g(\bar{E}) \subseteq \mathcal{V} \subset\subset g(\mathcal{U})$. Since $\bar{E}$ is a compact set, by Theorem 4.21 $g(\bar{E})$ is also compact; thus there exists $\delta > 0$ such that $d(x,y) > \delta$ for all $x \in g(E)$ and $y \in \mathcal{V}^{\complement}$. Let $\mathcal{P}$ be a partition of $g(E)$ such that $\operatorname{diam}(\Delta) < \delta$ for all $\Delta \in \mathcal{P}$. Then $\Delta \subseteq \mathcal{V}$ if $\Delta \in \mathcal{P}$ and $\Delta \cap g(E) \neq \varnothing$. Since $\inf\limits_{y\in\Delta} \overline{f}^{g(E)}(y) = \inf\limits_{x\in g^{-1}(\Delta)} (\overline{f}^{g(E)}\circ g)(x)$ if $\Delta \subseteq \mathcal{U}$, we find that

$$L(f,\mathcal{P}) = \sum_{\substack{\Delta\in\mathcal{P}\\ \Delta\cap g(E)\neq\varnothing}} \inf_{y\in\Delta} \overline{f}^{g(E)}(y)\nu(\Delta) = \sum_{\substack{\Delta\in\mathcal{P}\\ \Delta\cap g(E)\neq\varnothing}} \inf_{x\in g^{-1}(\Delta)} (\overline{f}^{g(E)}\circ g)(x)\nu(\Delta)\,;$$

thus (8.6.1) and (8.6.2) imply that

$$\begin{aligned}
L(f,\mathcal{P}) &= \sum_{\substack{\Delta\in\mathcal{P}\\ \Delta\cap g(E)\neq\varnothing}} \inf_{x\in g^{-1}(\Delta)} (\overline{f}^{g(E)}\circ g)(x) \int_{g^{-1}(\Delta)} |\mathbf{J}_g(x)|dx \\
&\leqslant \sum_{\substack{\Delta\in\mathcal{P}\\ \Delta\cap g(E)\neq\varnothing}} \int_{g^{-1}(\Delta)} (\overline{f}^{g(E)}\circ g)(x)|\mathbf{J}_g(x)|dx = \int_{g^{-1}(\bigcup_{\Delta\in\mathcal{P},\Delta\cap g(E)\neq\varnothing}\Delta)} (\overline{f}^{g(E)}\circ g)(x)|\mathbf{J}_g(x)|dx \\
&= \int_E (f\circ g)(x)|\mathbf{J}_g(x)|\,dx\,.
\end{aligned}$$

Similarly, by the fact that $\sup\limits_{y\in\Delta} \overline{f}^{g(E)}(y) = \sup\limits_{x\in g^{-1}(\Delta)} (\overline{f}^{g(E)}\circ g)(x)$ if $\Delta \subseteq \mathcal{U}$, we conclude that

$$\begin{aligned}
U(f,\mathcal{P}) &= \sum_{\substack{\Delta\in\mathcal{P}\\ \Delta\cap g(E)\neq\varnothing}} \sup_{x\in g^{-1}(\Delta)} (\overline{f}^{g(E)}\circ g)(x) \int_{g^{-1}(\Delta)} |\mathbf{J}_g(x)|dx \\
&\geqslant \sum_{\substack{\Delta\in\mathcal{P}\\ \Delta\cap g(E)\neq\varnothing}} \int_{g^{-1}(\Delta)} (\overline{f}^{g(E)}\circ g)(x)|\mathbf{J}_g(x)|dx = \int_{g^{-1}(\bigcup_{\Delta\in\mathcal{P},\Delta\cap g(E)\neq\varnothing}\Delta)} (\overline{f}^{g(E)}\circ g)(x)|\mathbf{J}_g(x)|dx \\
&= \int_E (f\circ g)(x)|\mathbf{J}_g(x)|\,dx\,.
\end{aligned}$$

The integrability of f over $g(E)$ implies that $\displaystyle\int_{g(E)} f(y)dy = \int_E (f\circ g)(x)|\mathbf{J}_g(x)|dx$. $\square$

Since the differentiability of g implies that locally g is very closed to an affine map; that is, $g(x) \approx \mathrm{L}x + c$ for some $\mathrm{L} \in \mathscr{B}(\mathbb{R}^n, \mathbb{R}^n)$ and $c \in \mathbb{R}^n$ (in fact, $g(x) \approx g(x_0) + (Dg)(x_0)(x - x_0)$ in a neighborhood of x_0), our next step is to establish (8.6.1) first for the case that g is an affine map. Since the volume of a set remains unchanged under translation, W.L.O.G. we can assume that g is linear. The following lemma proves a generalized version of (8.6.1) for the case that $g \in \mathscr{B}(\mathbb{R}^n, \mathbb{R}^n)$.

Lemma 8.60. *Let $g \in \mathscr{B}(\mathbb{R}^n, \mathbb{R}^n)$, and $A \subseteq \mathbb{R}^n$ be a set that has volume. Then $g(A)$ has volume, and*

$$\nu\big(g(A)\big) = \int_{g(A)} 1 dy = \int_A |\mathbf{J}_g(x)| dx \,. \tag{8.6.3}$$

Remark 8.61. If $g \in \mathscr{B}(\mathbb{R}^n, \mathbb{R}^n)$, then $g(x) = \mathrm{L}x$ for some $n \times n$ matrix. In this case $\mathbf{J}_g(x) = \det(\mathrm{L})$ for all $x \in A$; thus (8.6.3) is the same as that

$$\nu\big(\mathrm{L}(A)\big) = \int_{\mathrm{L}(A)} 1 dy = \int_A |\det(\mathrm{L})| dx = |\det(\mathrm{L})| \nu(A) \,. \tag{8.6.4}$$

Therefore, in the following we prove (8.6.4) instead of (8.6.3).

Proof of Lemma 8.60. Since any $n \times n$ matrices can be expressed as the product of elementary matrices, it suffices to prove the validity of the lemma for the case that L is an elementary matrix.

Suppose first that $A = [a_1, b_1] \times \cdots \times [a_n, b_n]$ is a rectangle.

1. If L is an elementary matrix of the type

$$\mathrm{L} = \begin{array}{cl} \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & 0 & & & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & & & \vdots \\ \vdots & & 0 & 1 & 0 & & & & \vdots \\ \vdots & & & 0 & c & 0 & & & \vdots \\ \vdots & & & & 0 & 1 & 0 & & \vdots \\ \vdots & & & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & & & 0 & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix} & \begin{array}{l} \\ \\ \\ \\ \leftarrow \text{the } k_0\text{-th row} \\ \\ \\ \\ \\ \end{array} \end{array}$$

then

$$\mathrm{L}(A) = [a_1, b_1] \times \cdots \times [a_{k_0-1}, b_{k_0-1}] \times [ca_{k_0}, cb_{k_0}] \times [a_{k_0+1}, b_{k_0+1}] \times \cdots \times [a_n, b_n]$$

if $c \geqslant 0$ or

$$\mathrm{L}(A) = [a_1, b_1] \times \cdots \times [a_{k_0-1}, b_{k_0-1}] \times [cb_{k_0}, ca_{k_0}] \times [a_{k_0+1}, b_{k_0+1}] \times \cdots \times [a_n, b_n]$$

if $c < 0$. In either case, $\nu\big(\mathrm{L}(A)\big) = |c|\nu(A) = |\det(\mathrm{L})|\nu(A)$.

2. If L is an elementary matrix of the type

$$\mathrm{L} = \begin{array}{c} \left[\begin{array}{ccccccccccc} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & 0 & & & & & & & & \vdots \\ \vdots & \ddots & 1 & \ddots & & & & & & & \vdots \\ \vdots & & 0 & 0 & 0 & & & 1 & & & \vdots \\ \vdots & & & \ddots & 1 & \ddots & & & & & \vdots \\ \vdots & & & & 0 & \ddots & 0 & & & & \vdots \\ \vdots & & & & & \ddots & 1 & \ddots & & & \vdots \\ \vdots & & & 1 & & & 0 & 0 & 0 & & \vdots \\ 0 & & & & & & & \ddots & 1 & \ddots & \vdots \\ 0 & & & & & & & & 0 & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ \\ \leftarrow \text{the } i_0\text{-th row} \\ \\ \\ \\ \leftarrow \text{the } j_0\text{-th row} \\ \\ \\ \\ \end{array} \\ \begin{array}{cc} \uparrow & \uparrow \\ \text{the } i_0\text{-th column} & \text{the } j_0\text{-th column} \end{array} \end{array}$$

then

$$\begin{aligned} \mathrm{L}(A) = [a_1, b_1] \times \cdots \times [a_{i_0-1}, b_{i_0-1}] \times [a_{j_0}, b_{j_0}] \times [a_{i_0+1}, b_{i_0+1}] \times \cdots \times \\ \times [a_{j_0-1}, b_{j_0-1}] \times [a_{i_0}, b_{i_0}] \times [a_{j_0+1}, b_{j_0} + 1] \times \cdots \times [a_n, b_n]\,; \end{aligned}$$

thus $\nu\big(\mathrm{L}(A)\big) = \nu(A) = |\det(\mathrm{L})|\nu(A)$.

3. If L is an elementary matrix of the type

$$\mathrm{L} = \left[\begin{array}{ccccccccc} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & 0 & & & & & & 0 \\ \vdots & \ddots & \ddots & \ddots & & & c & & 0 \\ \vdots & & \ddots & \ddots & \ddots & & & & 0 \\ \vdots & & & 0 & 1 & 0 & & & \vdots \\ \vdots & & & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & & & 0 & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ \leftarrow \text{the } i_0\text{-th row} \\ \\ \\ \\ \\ \\ \\ \end{array}$$

$\uparrow$
the j_0-th column

then $\mathrm{L}(A)$ is a parallelepiped

$$\begin{aligned} L(A) &= \big\{(x_1,\cdots,x_{i_0-1},x_{i_0}+cx_{j_0},x_{i_0+1},\cdots,x_n)\in\mathbb{R}^n\,\big|\,x_i\in[a_i,b_i]\;\forall\,1\leqslant i\leqslant n\big\}\\ &= \big\{(x_1,\cdots,x_{i_0-1},y_{i_0},x_{i_0+1},\cdots,x_n)\in\mathbb{R}^n\,\big|\,a_{i_0}+cx_{j_0}\leqslant y_{i_0}\leqslant b_{i_0}+cx_{j_0},\\ &\qquad\qquad x_i\in[a_i,b_i]\;\forall\,i\neq i_0\big\}\,; \end{aligned}$$

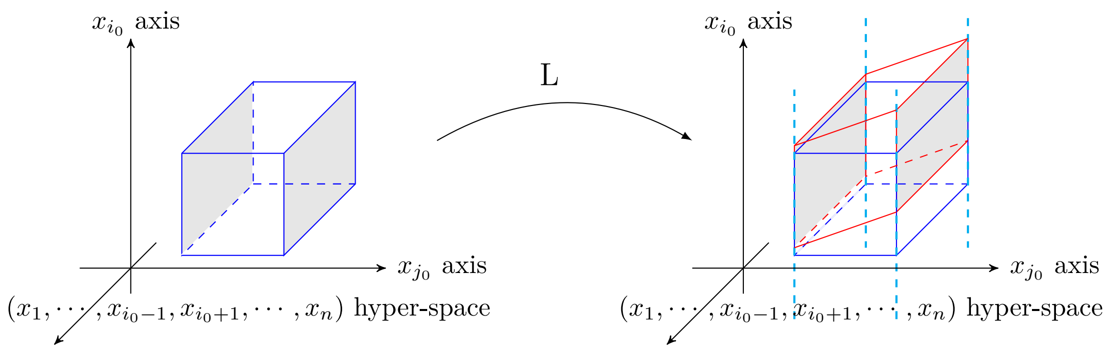


Figure 8.3: The image of a rectangle under a linear map induced by the elementary matrix of the third type

thus the Fubini theorem (or Corollary 8.55) implies that

$$\nu(L(A))=\int_{[a_1,b_1]\times\cdots\times[a_{i_0-1},b_{i_0-1}]\times[a_{i_0+1},b_{i_0+1}]\times\cdots\times[a_n,b_n]}\Big(\int_{a_{i_0}+cx_{j_0}}^{b_{i_0}+cx_{j_0}}1dy_{i_0}\Big)d\widehat{x}_{i_0}=\nu(A)\,.$$

On the other hand, $|\det(\mathrm{L})|=1$, so $\nu\big(\mathrm{L}(A)\big)=|\det(\mathrm{L})|\nu(A)$ is validated.

Therefore, (8.6.4) holds if A is a rectangle and L is an elementary matrix. An immediate consequence of this observation is that if Z is a set of measure zero, so is $\mathrm{L}(Z)$.

Now suppose that A is an arbitrary set with volume, and L is an elementary matrix.

1. If $\det(\mathrm{L})=0$, L must be an elementary matrix of the first type, and in this case,

$$\mathrm{L}(A)\subseteq[-r,r]\times\cdots\times[-r,r]\times\underbrace{[-\varepsilon,\varepsilon]}_{\text{the }k_0\text{-th slot}}\times\cdots\times[-r,r]$$

for some $r>0$ sufficiently large and arbitrary $\varepsilon>0$. Therefore, $\mathrm{L}(A)$ has volume zero; thus $\mathrm{L}(A)$ has volume and $\nu\big(\mathrm{L}(A)\big)=|\det(\mathrm{L})|\nu(A)$.

2. Suppose that $\det(\mathrm{L}) \neq 0$. Let $\varepsilon > 0$ be given. Since A has volume, by Riemann's condition there exists a partition of A such that
$$U(1_A, \mathcal{P}) - L(1_A, \mathcal{P}) < \frac{\varepsilon}{|\det(\mathrm{L})|}.$$
Note that the inequality above also implies that
$$U(1_A, \mathcal{P}) - \nu(A) < \frac{\varepsilon}{|\det(\mathrm{L})|} \quad \text{and} \quad \nu(A) - L(1_A, \mathcal{P}) < \frac{\varepsilon}{|\det(\mathrm{L})|}.$$
Let $C_1 = \big\{\Delta \in \mathcal{P} \,\big|\, \Delta \cap A \neq \varnothing\big\}$ and $C_2 = \big\{\Delta \in \mathcal{P} \,\big|\, \Delta \subseteq A\big\}$, and define $\mathrm{R}_1 = \bigcup\limits_{\Delta \in C_1} \Delta$ and $\mathrm{R}_2 = \bigcup\limits_{\Delta \in C_2} \Delta$. Then $\mathrm{R}_2 \subseteq A \subseteq \mathrm{R}_1$. Moreover,
$$\nu\big(\mathrm{L}(\mathrm{R}_1)\big) = \sum_{\Delta \in C_1} \nu(\mathrm{L}(\Delta)) = \sum_{\Delta \in C_1} |\det(\mathrm{L})|\nu(\Delta) = |\det(\mathrm{L})| U(1_A, \mathcal{P}) < |\det(\mathrm{L})|\nu(A) + \varepsilon$$
and
$$\nu\big(\mathrm{L}(\mathrm{R}_2)\big) = \sum_{\Delta \in C_2} \nu(\mathrm{L}(\Delta)) = \sum_{\Delta \in C_2} |\det(\mathrm{L})|\nu(\Delta) = |\det(\mathrm{L})| L(1_A, \mathcal{P}) > |\det(\mathrm{L})|\nu(A) - \varepsilon.$$
As a consequence, by the fact that $\mathrm{L}(\mathrm{R}_2) \subseteq \mathrm{L}(A) \subseteq \mathrm{L}(\mathrm{R}_1)$ we conclude that
$$\Big| \overline{\int}_{\mathrm{L}(A)} 1dx - \underline{\int}_{\mathrm{L}(A)} 1dx \Big| \leqslant \nu\big(\mathrm{L}(\mathrm{R}_1)\big) - \nu\big(\mathrm{L}(\mathrm{R}_2)\big) = |\det(\mathrm{L})|\big(U(1_A, \mathcal{P}) - L(1_A, \mathcal{P})\big) < \varepsilon.$$
Since $\varepsilon > 0$ is arbitrary, we find that $\overline{\int}_{\mathrm{L}(A)} 1dx = \underline{\int}_{\mathrm{L}(A)} 1dx$ which implies that $1_{\mathrm{L}(A)}$ is Riemann integrable, or equivalently, $\mathrm{L}(A)$ has volume.

On the other hand, observing that
$$|\det(\mathrm{L})|\nu(A) - \varepsilon < \nu\big(\mathrm{L}(\mathrm{R}_2)\big) \leqslant \nu\big(\mathrm{L}(A)\big) \leqslant \nu\big(\mathrm{L}(\mathrm{R}_1)\big) < |\det(\mathrm{L})|\nu(A) + \varepsilon,$$
we conclude that $\nu\big(\mathrm{L}(A)\big) = |\det(\mathrm{L})|\nu(A)$ again because ε is arbitrary. □

Lemma 8.62. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be an open bounded set, and $g : \mathcal{U} \to \mathbb{R}^n$ be an one-to-one $\mathscr{C}^1$ mapping that has a $\mathscr{C}^1$ inverse; that is, $g^{-1} : g(\mathcal{U}) \to \mathbb{R}^n$ is also continuously differentiable. Assume that the Jacobian of g and $\mathbf{J}_g = \det([Dg])$, does not vanish in $\mathcal{U}$. Then*
$$\nu(R) = \int_{g^{-1}(R)} |\mathbf{J}_g(x)| dx \quad \textit{for all closed rectangle } R \subseteq g(\mathcal{U}). \tag{8.6.1}$$

Proof. Since $g : \mathcal{U} \to \mathbb{R}^n$ is of class $\mathscr{C}^1$ and $g^{-1}(R) \subseteq \mathcal{U}$ is compact, there exist $m > 0$ and $\Lambda > 0$ such that

$$|\mathbf{J}_g(x)| \geqslant m \quad \text{and} \quad \big\|(Dg)(x)\big\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)} \leqslant \Lambda \qquad \forall\, x \in g^{-1}(R)\,.$$

Let $\varepsilon > 0$ be given. Since $g^{-1}(R)$ is compact, Theorem 4.52 implies that $\mathbf{J}_g$ is uniformly continuous on $g^{-1}(R)$. Therefore, there exists $\delta_1 > 0$ such that

$$\big|\mathbf{J}_g(x_1) - \mathbf{J}_g(x_2)\big| < m\varepsilon \quad \text{if} \quad \|x_1 - x_2\|_{\mathbb{R}^n} < \delta_1\,.$$

Since g^{-1} is of class $\mathscr{C}^1$, the continuity of g^{-1} and Corollary 6.66 guarantee the existence of $\delta > 0$ such that if $\|y_1 - y_2\|_{\mathbb{R}^n} < \delta$ and $y_1, y_2 \in R$,

$$\big\|g^{-1}(y_1) - g^{-1}(y_2)\big\|_{\mathbb{R}^n} < \delta_1$$

and

$$\big\|g^{-1}(y_2) - g^{-1}(y_1) - (Dg^{-1})(y_1)(y_2 - y_1)\big\|_{\mathbb{R}^n} \leqslant \frac{\varepsilon}{\Lambda}\|y_1 - y_2\|_{\mathbb{R}^n}\,.$$

Let $\mathcal{P}$ be a partition of R with mesh size $\|\mathcal{P}\| < \delta$. Then if $\Delta \in \mathcal{P}$,

$$\big|\mathbf{J}_g(g^{-1}(y_1)) - \mathbf{J}_g(g^{-1}(y_2))\big| < m\varepsilon \quad \forall\, y_1, y_2 \in \Delta$$

and by the fact that $(Dg^{-1})(y_1) = \big((Dg)(g^{-1}(y_1))\big)^{-1}$ (which is due to the inverse function theorem (Theorem 7.1)), for all $y_1, y_2 \in \Delta$ we have

$$\begin{aligned}
\big\|g^{-1}(y_2) - g^{-1}(y_1) - \big((Dg)(g^{-1}(y_1))\big)^{-1}(y_2 - y_1)\big\|_{\mathbb{R}^n} &\leqslant \frac{\varepsilon}{\Lambda}\|y_1 - y_2\|_{\mathbb{R}^n} \\
&\leqslant \frac{\varepsilon}{\Lambda}\big\|(Dg)(g^{-1}(y_1))\big\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)}\big\|\big((Dg)(g^{-1}(y_1))\big)^{-1}(y_1 - y_2)\big\|_{\mathbb{R}^n} \\
&\leqslant \varepsilon\big\|\big((Dg)(g^{-1}(y_1))\big)^{-1}(y_1 - y_2)\big\|_{\mathbb{R}^n}\,.
\end{aligned}$$

The inequality above suggests that g^{-1} is approximately an affine map on Δ, and we must have $S_1 \subseteq g^{-1}(\Delta) \subseteq S_2$ for some parallelepipeds S_1 and S_2 whose sides are parallel to the sides of the parallelepiped $S \equiv \big((Dg)(\widetilde{x})\big)^{-1}\Delta$ in which $\widetilde{x} \in g^{-1}(\Delta)$ is the image of the center of Δ under g^{-1}, and $\nu(S_1) = (1-\varepsilon)^n\nu(S)$, $\nu(S_2) = (1+\varepsilon)^n\nu(S)$. By Lemma 8.60,

$$\frac{(1-\varepsilon)^n}{\big|\mathbf{J}_g(\widetilde{x})\big|}\nu(\Delta) \leqslant \nu\big(g^{-1}(\Delta)\big) \leqslant \frac{(1+\varepsilon)^n}{\big|\mathbf{J}_g(\widetilde{x})\big|}\nu(\Delta)\,;$$

thus

$$\begin{aligned}\int_{g^{-1}(\Delta)} |\mathbf{J}_g(x)|dx &\leqslant \int_{g^{-1}(\Delta)} (|\mathbf{J}_g(\widetilde{x})| + m\varepsilon)dx \\ &\leqslant (|\mathbf{J}_g(\widetilde{x})| + m\varepsilon)\nu\big(g^{-1}(\Delta)\big) \leqslant \frac{(1+\varepsilon)^n(|\mathbf{J}_g(\widetilde{x})| + m\varepsilon)}{|\mathbf{J}_g(\widetilde{x})|}\nu(\Delta) \leqslant (1+\varepsilon)^{n+1}\nu(\Delta)\,.\end{aligned}$$

A similar argument provides a lower bounded of the left-hand side, and we conclude that

$$(1-\varepsilon)^{n+1}\nu(\Delta) \leqslant \int_{g^{-1}(\Delta)} |\mathbf{J}_g(x)|dx \leqslant (1+\varepsilon)^{n+1}\nu(\Delta) \qquad \forall\, \Delta \in \mathcal{P}\,.$$

Summing over all $\Delta \in \mathcal{P}$, we find that

$$(1-\varepsilon)^{n+1}\nu(R) \leqslant \sum_{\Delta\in\mathcal{P}} \int_{g^{-1}(\Delta)} |\mathbf{J}_g(x)|dx \leqslant (1+\varepsilon)^{n+1}\nu(R)\,.$$

Identity (8.6.1) is then concluded since $\sum\limits_{\Delta\in\mathcal{P}} \int_{g^{-1}(\Delta)} |\mathbf{J}_g(x)|dx = \int_{g^{-1}(R)} |\mathbf{J}_g(x)|dx$ and $\varepsilon > 0$ is arbitrary. □

Example 8.63. Let A be the triangular region with vertices $(0,0)$, $(4,0)$, $(4,2)$, and $f : A \to \mathbb{R}$ be given by

$$f(x,y) = y\sqrt{x-2y}\,.$$

Let $(u,v) = (x, x-2y)$. Then $(x,y) = g(u,v) = \big(u, \dfrac{u-v}{2}\big)$; thus

$$\mathbf{J}_g(u,v) = \begin{vmatrix} 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}\,.$$

Define E as the triangle with vertices $(0,0)$, $(4,0)$, $(4,4)$. Then $A = g(E)$.

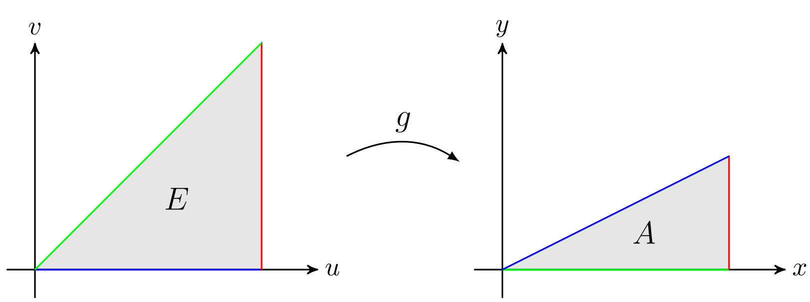


Figure 8.4: The image of E under g

Therefore,

$$\begin{aligned}\int_A f(x,y)d(x,y) &= \int_{g(E)} f(x,y)d(x,y) = \frac{1}{2}\int_E f\big(g(u,v)\big)d(u,v)\\ &= \frac{1}{4}\int_0^4\int_0^u (u-v)\sqrt{v}dvdu = \frac{1}{4}\int_0^4 \Big[\frac{2}{3}uv^{\frac{3}{2}} - \frac{2}{5}v^{\frac{5}{2}}\Big]\Big|_{v=0}^{v=u} du\\ &= \frac{1}{4}\int_0^4 (\frac{2}{3}-\frac{2}{5})u^{\frac{5}{2}}du = \frac{1}{15}\times\frac{2}{7}u^{\frac{7}{2}}\Big|_{u=0}^{u=4} = \frac{256}{105}\,.\end{aligned}$$

Example 8.64. Suppose that $f:[0,1]\to\mathbb{R}$ is Riemann integrable and $\int_0^1 (1-x)f(x)dx = 5$ (note that the function $g(x)=(1-x)f(x)$ is Riemann integrable over $[0,1]$ because of the Lebesgue theorem). We would like to evaluate the iterated integral $\int_0^1\int_0^x f(x-y)dydx$.

It is nature to consider the change of variable $(u,v)=(x-y,x)$ or $(u,v)=(x-y,y)$. Suppose the later case. Then $(x,y)=g(u,v)=(u+v,u)$; thus

$$\mathbf{J}_g(u,v) = \begin{vmatrix}1 & 1\\ 1 & 0\end{vmatrix} = -1\,.$$

Moreover, the region of integration is the triangle A with vertices $(0,0)$, $(1,0)$, $(1,1)$, and three sides $y=0$, $x=1$, $x=y$ correspond to $u=0$, $u+v=1$ and $v=0$. Therefore, if E denotes the triangle enclosed by $u=0$, $v=0$ and $u+v=1$ on the (u,v)-plane, then $g(E)=A$, and

$$\begin{aligned}\int_0^1\int_0^x f(x-y)dydx &= \int_A f(x-y)d(x,y) = \int_{g(E)} f(x-y)d(x,y)\\ &= \int_E (f\circ g)(u,v)|\mathbf{J}_g(u,v)|d(u,v) = \int_0^1\int_0^{1-u} f(u)dvdu\\ &= \int_0^1 (1-u)f(u)du = 5\,.\end{aligned}$$

Example 8.65. Let A be the region in the first quadrant of the plane bounded by the curves $xy-x+y=0$ and $x-y=1$, and $f:A\to\mathbb{R}$ be given by

$$f(x,y) = x^2y^2(x+y)e^{-(x-y)^2}\,.$$

We would like to evaluate the integral $\int_A f(x,y)d(x,y)$.

Let $(u,v)=(xy-x+y,x-y)$. Unlike the previous two examples we do not want to solve for (x,y) in terms of (u,v) but still assume that $(x,y)=g(u,v)$. By the inverse function theorem,

$$\mathbf{J}_g(u,v)\Big|_{(u,v)=g^{-1}(x,y)} = \left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1} = \begin{vmatrix} y-1 & x+1 \\ 1 & -1 \end{vmatrix}^{-1} = \frac{1}{-y+1-x-1} = -\frac{1}{x+y}\,.$$

Moreover, the curve $xy - x + y = 0$ corresponds to $u = 0$, while the lines $x - y = 1$ and $y = 0$ correspond to $v = 1$ and $u + v = 0$, respectively; thus if E is the region enclosed by $u = 0$, $v = 1$ and $u + v = 0$, then $A = g(E)$.

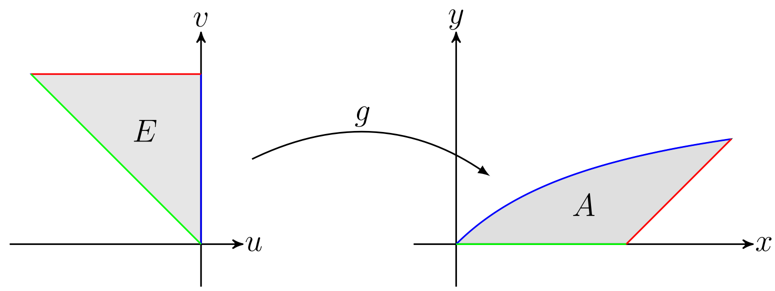


Figure 8.5: The image of E under g

Therefore,

$$\begin{aligned}\int_A f(x,y)d(x,y) &= \int_{g(E)} f(x,y)d(x,y) = \int_E (f\circ g)(u,v)|\mathbf{J}_g(u,v)|d(u,v)\\ &= \int_0^1\int_{-v}^0 (u+v)^2 e^{-v^2}dudv = \frac{1}{3}\int_0^1 v^3 e^{-v^2}dv\\ &= \frac{1}{6}\int_0^1 we^{-w}dw = -\frac{1}{6}(w+1)e^{-w}\Big|_{w=0}^{w=1} = -\frac{1}{6}\Big(\frac{2}{e}-1\Big)\,.\end{aligned}$$

Example 8.66 (Polar coordinates)**.** (Not yet finished!!!)

Example 8.67 (Cylindrical coordinates)**.** (Not yet finished!!!)

Example 8.68 (Polar coordinates)**.** (Not yet finished!!!)

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