Exercises for §6.1

4. Let $f: \mathbb{R}^n \to \mathbb{R}^m$ and suppose there is a constant M such that for $x \in \mathbb{R}^n$, $\|f(x)\| \le M\|x\|^2$.
Prove that f is differentiable at $x_0 = 0$ and that $Df(x_0) = 0$

Pf: $\because \|f(x)\| \le M\|x\|^2 \Rightarrow f(x) = 0$

Let $Df(0) = 0$

$$\lim_{x\to 0} \frac{\|f(x) - f(0) - Df(0)(x-0)\|}{\|x-0\|} = \lim_{x\to 0} \frac{\|f(x)\|}{\|x\|} \le \lim_{x \to 0} M\|x\| = 0$$

By definition and 6.1.2 $\Rightarrow Df(0) = 0$

5 If $f: \mathbb{R} \to \mathbb{R}$ is differentiable and $|f(x)| \le |x|$, must $Df(0) = 0$

Ans No Let $f(x) = x \Rightarrow |f(x)| \le |x|$ $Df(0) = 1$

Exercises for §6.4

1 Use Theorem 6.4.1 to show that $f(x, y)$ defined by

$$f(x, y) = \frac{(xy)^2}{\sqrt{x^2+y^2}} \quad , (x, y) \ne (0, 0)$$

and $f(x, y) = 0$ $(x, y) = (0, 0)$

is differentiable at $(0, 0)$

Pf: Check $\frac{\partial f(x,y)}{\partial x}$ and $\frac{\partial f(x,y)}{\partial y}$ are continuous at $(0, 0)$

$$\frac{\partial f(x,y)}{\partial x} = \frac{(\sqrt{x^2+y^2})2xy^2 - (xy)^2 \frac{1}{2}\frac{1}{\sqrt{x^2+y^2}} 2x}{x^2+y^2} = \frac{\frac{(x^2+y^2)2xy^2 - x(xy)^2}{\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{(x^2+y^2)2xy^2 - x(xy)^2}{\sqrt{x^2+y^2}}$$

$$= \sqrt{x^2+y^2}\, 2xy^2 - \frac{x^3y^2}{\sqrt{x^2+y^2}}$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{\partial f(x,y)}{\partial x} = 0$$

$\frac{\partial f(x,y)}{\partial y}$ Similar By 6.4.1 $\Rightarrow f(x, y)$ is differentiable at $(0, 0)$

2 Investigate the differentiability of

$$f(x,y)=\frac{xy}{\sqrt{x^2+y^2}}$$

at (0, 0) if $f(0,0)=0$

Ans If $Df(x,y)$ exist

$$\frac{\partial f(0,0)}{\partial x}=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=0$$

$$\frac{\partial f(0,0)}{\partial y}=\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y}=0$$

$\Rightarrow Df(0,0)=(0,0)$

$\Rightarrow Df(0,0)(e_1,e_2)=0 \quad e_1 e_2\neq 0$

$\lim_{t\to 0}\frac{1}{t}f(te_1,te_2)=\lim\frac{e_1e_2}{\sqrt{e_1^2+e_2^2}}\neq 0 \quad e_1 e_2\neq 0 \Rightarrow f$ is not differentiable

5 Find a function $f:\mathbb{R}^2\to\mathbb{R}$ that is differentiable at each point but whose partials are not continuous at (0, 0)

Ans: Let $f(x,y)=x^2\sin\frac{1}{x}$ if $x\neq 0$

$f(x,y)=0$ if $x=0$

$\Rightarrow |f(x,y)|\leq|x|^2\leq x^2+y^2$

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$Df(0,0)$ exist and $Df(0,0)=0$

$\frac{\partial f(x,y)}{\partial y}=0 \quad \frac{\partial f(x,y)}{\partial x}=2x\sin\frac{1}{x}-\cos\frac{1}{x}$ is continuous when $x\neq 0$

$\Rightarrow Df(x,y)$ exist

But $\lim_{x\to 0}2x\sin\frac{1}{x}-\cos\frac{1}{x}$ 不存在 $\Rightarrow \frac{\partial f(x,y)}{\partial x}$ is not continuous

Exercises for § 6.5

4 Write out the chain rule releting rectangular coordinates to spherical coordinates in three dimensions.

Ans: $f(x, y, z) = f(r\sin\varphi\cos\theta,\ r\sin\varphi\sin\theta,\ r\cos\varphi)$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial f}{\partial \varphi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \varphi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \varphi} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \varphi}$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} + 0$$

5 Let $f: \mathbb{R} \to \mathbb{R}$ and $F: \mathbb{R}^2 \to \mathbb{R}$ be differentiable and satisfy $F(x, f(x)) = 0$ and $\partial F/\partial y \neq 0$. Prove that $f'(x) = -\dfrac{\partial F/\partial x}{\partial F/\partial y}$ where $y = f(x)$

pf: $y = f(x)$

$$0 = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x}$$

$$= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} f'(x)$$

$$\Rightarrow f'(x) = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

Exercises for § 6-6

1 Prove that

$$\frac{d}{dt} f(x_0 + th)\Big|_{t=0} = Df(x_0)\cdot h$$

by using the chain rule, where $f: \mathbb{R}^n \to \mathbb{R}^n$

pf: $D f\circ g\,(x_0) = D f(g(x_0)) \circ Dg(x_0)$

Let $g(t) = x_0 + th$ $\Rightarrow \frac{d}{dt} f(x_0 + th)\Big|_{t=0} = Df(x_0)\circ \frac{d}{dt}(x_0 + th) = Df(x_0)\cdot h$

Exercises for §6.7

2. Prove the following (weak version of) L'Hôpital's rule: If f', g' exist at x_0 $g'(x_0) \neq 0$ and $f(x_0) = 0 = g(x_0)$, then $\lim_{x \to x_0} [f(x)/g(x)] = f'(x_0)/g'(x_0)$

Prove $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \lim_{x \to x_0} \frac{\frac{f(x)-f(x_0)}{x-x_0}}{\frac{g(x)-g(x_0)}{x-x_0}}$

$\because f'(x_0)\ g'(x_0)$ exist and $g'(x_0) \neq 0$

$$= \frac{\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}}{\lim_{x \to x_0} \frac{g(x)-g(x_0)}{x-x_0}} = \frac{f'(x_0)}{g'(x_0)}$$

5. Let $f: A \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable with A convex and suppose $\|\text{grad} f(x)\| \leq M$ for $x \in A$. Prove $|f(x) - f(y)| \leq M\|x-y\|$ for $x, y \in A$. Do you think this is true if A is not convex?

Pf: (a) By 6.7.1 $f(x) - f(y) = Df(c) \cdot (x-y)$

$\Rightarrow |f(x)-f(y)| = |Df(c) \cdot (x-y)|$

$\leq \|Df(c)\| \cdot \|x-y\|$

$\leq M\|x-y\|$

(b) No. Let $A = \mathbb{R} \setminus \{0\}$

$f(x) = 1 \quad x > 0$

$f(x) = 0 \quad x < 0$

$f'(x) = 0 \quad \forall x \in A$

$\Rightarrow \underset{x>0 \quad y<0}{|f(x) - f(y)|} = 1 \geq \underset{0}{f'(x)}|x-y|$

6 Let $f: \mathbb{R} \to \mathbb{R}$ be differentiable. Assume that for all $x \in \mathbb{R}$, $0 \leq f'(x) \leq f(x)$. Show that $g(x) = e^{-x} f(x)$ is decreasing. If f vanishes at some point, conclude that f is zero

Prove (a) $g'(x) = -e^{-x} f(x) + e^{-x} f'(x)$

$$= e^{-x}(f'(x) - f(x))$$

$\because$ $0 \leq f'(x) \leq f(x)$

$\Rightarrow f'(x) - f(x) \leq 0$

$\Rightarrow e^{-x}(f'(x) - f(x)) \leq 0$

$\Rightarrow$ g is decreasing

(b) Let f vanishes at some point y

$f(y) = 0 \quad f(x) > 0 \quad x < y$

$0 > f(y) - f(x) = f'(c)(y-x) > 0 \quad \rightarrow\leftarrow$

$\Rightarrow f(x) = 0 \quad \forall x \in \mathbb{R}$