Exercises for §6-1

4. Let $f: \mathbb{R}^n \to \mathbb{R}^m$ and suppose there is a constant M such that for $x \in \mathbb{R}^n$, $\|f(x)\| \le M\|x\|^2$.
Prove that f is differentiable at $x_0 = 0$ and that $Df(x_0) = 0$

Pf: $\because \|f(x)\| \le M\|x\|^2 \Rightarrow f(0) = 0$

Let $Df(0) = 0$

$$\lim_{x \to 0} \frac{\|f(x) - f(0) - Df(0)(x-0)\|}{\|x - 0\|} = \lim_{x \to 0} \frac{\|f(x)\|}{\|x\|} \le \lim_{x \to 0} M\|x\| = 0$$

By definition and 6.1.2 $\Rightarrow Df(0) = 0$

5 If $f: \mathbb{R} \to \mathbb{R}$ is differentiable and $|f(x)| \le |x|$, must $Df(0) = 0$

Ans No Let $f(x) = x \Rightarrow |f(x)| \le |x|$ $Df(0) = 1$

Exercises for §6.4

1 Use Theorem 6.4.1 to show that $f(x,y)$ defined by

$$f(x,y) = \frac{(xy)^2}{\sqrt{x^2+y^2}}, \quad (x,y) \neq (0,0)$$

and $f(x,y) = 0 \quad (x,y) = (0,0)$

is differentiable at $(0,0)$

Pf: Check $\frac{\partial f(x,y)}{\partial x}$ and $\frac{\partial f(x,y)}{\partial y}$ are continuous at $(0,0)$

$$\frac{\partial f(x,y)}{\partial x} = \frac{(\sqrt{x^2+y^2})2xy^2 - (xy)^2 \frac{1}{2}\frac{1}{\sqrt{x^2+y^2}} 2x}{x^2+y^2} = \frac{\frac{(x^2+y^2)2xy^2 - x(xy)^2}{\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{(x^2+y^2)2xy^2 - x(xy)^2}{\sqrt{x^2+y^2}}$$

$$= \sqrt{x^2+y^2}\, 2xy^2 - \frac{x^3y^2}{\sqrt{x^2+y^2}}$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{\partial f(x,y)}{\partial x} = 0$$

$\frac{\partial f(x,y)}{\partial y}$ Similar By 6.4.1 $\Rightarrow f(x,y)$ is differentiable at $(0,0)$

2 Investigate the differentiability of

$$f(x,y)=\frac{xy}{\sqrt{x^2+y^2}}$$

at $(0,0)$ if $f(0,0)=0$

Ans If $Df(x,y)$ exist

$$\frac{\partial f(0,0)}{\partial x}=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=0$$

$$\frac{\partial f(0,0)}{\partial y}=\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y}=0$$

$\Rightarrow Df(0,0)=(0,0)$

$\Rightarrow Df(0,0)(e_1,e_2)=0 \quad e_1\ e_2\neq 0$

$\lim_{t\to 0}\frac{1}{t}f(te_1,te_2)=\lim\frac{e_1e_2}{\sqrt{e_1^2+e_2^2}}\neq 0 \quad e_1\ e_2\neq 0 \quad \Rightarrow f$ is not differentiable

5 Find a function $f:\mathbb{R}^2\to\mathbb{R}$ that is differentiable at each point but whose partials are not continuous at $(0,0)$

Ans: Let $f(x,y)=x^2\sin\frac{1}{x}$ if $x\neq 0$

$f(x,y)=0$ if $x=0$

$\Rightarrow |f(x,y)|\leq |x|^2\leq x^2+y^2$

By exercises for §6.1 4

$Df(0,0)$ exist and $Df(0,0)=0$

$\frac{\partial f(x,y)}{\partial y}\equiv 0 \quad \frac{\partial f(x,y)}{\partial x}=2x\sin\frac{1}{x}-\cos\frac{1}{x}$ is continuous when $x\neq 0$

$\Rightarrow Df(x,y)$ exist

But $\lim_{x\to 0}2x\sin\frac{1}{x}-\cos\frac{1}{x}$ 不存在 $\Rightarrow \frac{\partial f(x,y)}{\partial x}$ is not continuous

Exercises for § 6.5

4 Write out the chain rule releting rectangular coordinates to spherical coordinates in three dimensions.

Ans: $f(x,y,z) = f(r\sin\varphi\cos\theta,\ r\sin\varphi\sin\theta\ ,\ r\cos\varphi)$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial f}{\partial \varphi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \varphi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \varphi} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \varphi}$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} + 0$$

5 Let $f: \mathbb{R} \to \mathbb{R}$ and $F: \mathbb{R}^2 \to \mathbb{R}$ be differentiable and satisfy $F(x, f(x)) = 0$ and $\partial F/\partial y \neq 0$. Prove that $f'(x) = -\dfrac{\partial F/\partial x}{\partial F/\partial y}$ where $y = f(x)$

pf:

$$0 = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} \qquad y = f(x)$$

$$= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} f'(x)$$

$$\Rightarrow f'(x) = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

Exercises for § 6-6

1 Prove that

$$\frac{d}{dt} f(x_0 + th)\Big|_{t=0} = Df(x_0)\cdot h$$

by using the chain rule, where $f: \mathbb{R}^m \to \mathbb{R}^n$

pf: $D f\circ g\,(x_0) = D f(g(x_0)) \circ Dg(x_0)$

Let $g(t) = x_0 + th$ $\Rightarrow \frac{d}{dt} f(x_0 + th)\Big|_{t=0} = Df(x_0)\circ \frac{d}{dt}(x_0 + th) = Df(x_0)\cdot h$

Exercises for §6.7

2. Prove the following (weak version of) L'Hôpital's rule: If f', g' exist at x_0, $g'(x_0) \neq 0$ and $f(x_0) = 0 = g(x_0)$, then $\lim_{x \to x_0} [f(x)/g(x)] = f'(x_0)/g'(x_0)$

Prove $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \lim_{x \to x_0} \frac{\frac{f(x)-f(x_0)}{x-x_0}}{\frac{g(x)-g(x_0)}{x-x_0}}$

$\because f'(x_0), g'(x_0)$ exist and $g'(x_0) \neq 0$

$$= \frac{\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}}{\lim_{x \to x_0} \frac{g(x)-g(x_0)}{x-x_0}} = \frac{f'(x_0)}{g'(x_0)}$$

5. Let $f: A \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable with A convex and suppose $\|\text{grad} f(x)\| \leq M$ for $x \in A$. Prove $|f(x) - f(y)| \leq M\|x-y\|$ for $x, y \in A$. Do you think this is true if A is not convex?

Pf: (a) By 6.7.1 $f(x) - f(y) = Df(c) \cdot (x-y)$

$\Rightarrow |f(x)-f(y)| = |Df(c) \cdot (x-y)|$

$\leq \|Df(c)\| \cdot \|x-y\|$

$\leq M\|x-y\|$

(b) No. Let $A = \mathbb{R} \setminus \{0\}$

$f(x) = 1 \quad x > 0$

$f(x) = 0 \quad x < 0$

$f'(x) = 0 \quad \forall x \in A$

$\Rightarrow \underset{x>0 \quad y<0}{|f(x) - f(y)|} = 1 \geq \underset{0}{\underset{\shortparallel}{f'(x)}} |x-y|$

6 Let $f: \mathbb{R} \to \mathbb{R}$ be differentiable. Assume that for all $x \in \mathbb{R}$, $0 \le f'(x) \le f(x)$. Show that $g(x) = e^{-x} f(x)$ is decreasing. If f vanishes at some point, conclude that f is zero

Prove (a) $g'(x) = -e^{-x} f(x) + e^{-x} f'(x)$

$= e^{-x} (f'(x) - f(x))$

$\because$ $0 \le f'(x) \le f(x)$

$\Rightarrow$ $f'(x) - f(x) \le 0$

$\Rightarrow$ $e^{-x} (f'(x) - f(x)) < 0$

$\Rightarrow$ g is decreasing

(b) Let f vanishes at some point y

$f(y) = 0$ $f(x) > 0$ $x < y$

$0 > f(y) - f(x) = f'(c)(y - x) > 0$ $\rightarrow\leftarrow$

$\Rightarrow$ $f(x) = 0$ $\forall x \in \mathbb{R}$

Exercise for § 6-2

3 Let L be a linear map of $\mathbb{R}^n \to \mathbb{R}^m$, Let $g: \mathbb{R}^n \to \mathbb{R}^m$ be such that $\|g(x)\| \le M \|x\|^2$, and let $f(x) = L(x) + g(x)$. Prove that $Df(0) = L$

pf: By § 6-1 exercise 4 $\Rightarrow Dg(0) = 0$

By 6.2.4 example $DL = L$

$Df(0) = DL(0) + Dg(0) = L + 0 = L$

4 Discuss the possibility of defining Df for f a mapping from one normed space to another

Ans Let $f: A \subset M \to N$ $M \cdot N$ are norm space

f is said to be differentiable at $x \in A$ if there is a linear function denoted $Df(x_0) = M \to N$ and called the derivative of f at x_0, such that

$$\lim_{x \to x_0} \frac{\| f(x) - f(x_0) - Df(x_0)(x - x_0) \|_N}{\| x - x_0 \|_M} = 0$$

If $M = \mathbb{R}^n$ $N = \mathbb{R}^m$

$Df(x_0)$ is continuous

But $M \cdot N$ are infinite dimensional space

$Df(x_0)$ maybe not continuous

Exercise for § 6.3

1 Let $f(x)=x^2$ if x is irrational and let $f(x)=0$ if x is rational, Is f continuous at 0?

Is it differentiable at 0?

pf = Check f is continuous at 0

$\forall \varepsilon > 0$ Let $\delta = \sqrt{\varepsilon}$ $|f(x)-f(0)| = |x^2| < \varepsilon$ if $|x|<\delta$

check f is differentiable at 0

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{x^2}{x} = 0 \quad \text{if } x\in \mathbb{Q}^c$$

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{0}{x} = 0 \quad \text{if } x\in \mathbb{Q}$$

$\Rightarrow$ f is differentiable at 0 and $Df(0)=0$

2 Is the local Lipschitz condition in Theorem 6.3.1 enough to guarantee differentiablity

Ans No Let $f(x)=|x|$ $x\in\mathbb{R}$

$|f(x)-f(y)| \leq 1\,|x-y|$

f not differentiable at 0

Exercise for § 6.8

3 Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = x^2 \sin\left(\frac{1}{x}\right) \quad \text{if } x \in (-1\ \ 1)\quad x \neq 0$$

and

$$f(x) = 0 \quad \text{if } x = 0$$

Investigate the validity of Taylor's theorem for f about the point $x=0$

Ans:

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0 \quad \left(\because \left|\sin\left(\tfrac{1}{x}\right)\right| \leq 1\right)$$

$$f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\frac{1}{x}$$

$\lim_{x \to 0} f'(x)$ does not exist ($\because \cos\frac{1}{x}$ oscillates between $+1$ and -1)

$\Rightarrow f'(x)$ is not continuous at 0

5 Compute the second-order Taylor formula for $f(x, y) = e^x \cos y$ around $(0, 0)$

Ans:

$$\frac{\partial f}{\partial x} = e^x \cos y \qquad f(0,0) = 1$$

$$\frac{\partial f}{\partial y} = -e^x \sin y \qquad \frac{\partial f}{\partial x}(0,0) = 1$$

$$\frac{\partial f}{\partial y}(0\ 0) = 0$$

$$\frac{\partial^2 f}{\partial x^2} = e^x \cos y \qquad \frac{\partial^2 f}{\partial x^2}(0,0) = 1$$

$$\frac{\partial^2 f}{\partial y^2} = -e^x \cos y \qquad \frac{\partial^2 f}{\partial y^2}(0,0) = -1$$

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \frac{\partial^2 f}{\partial y \partial x}(0,0) = 0$$

$$\frac{\partial^2 f}{\partial x \partial y} = -e^x \sin y$$

$$\frac{\partial^2 f}{\partial y \partial x} = -e^x \sin y$$

$$f(h, k) = 1 + (1, 0) \cdot (h, k) + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} \cdot (h, k) + R_2(h, k, 0)$$

$$= 1 + h + \frac{1}{2}(h^2 - k^2) + R_2((h, k), 0)$$

where $R_2((h, k), 0) / \|(h, k)\|^2 \to 0$ as $(h, k) \to (0, 0)$

Exercise for § 6.9

1 Prove that

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix}$$

is negative definite iff $a<0$ and $ad-b^2>0$

pf: "$\Rightarrow$" Negative definite means

$$(x \cdot y)\begin{pmatrix} a & b \\ b & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} < 0 \quad \text{if } (x \cdot y) \neq (0 \cdot 0)$$

$$\Leftrightarrow ax^2+2bxy+dy^2<0$$

Let $(x \cdot y) = (1 \cdot 0) \Rightarrow a<0$

Let $y=1$ $\therefore$ $ax^2+2bx+d<0$ for all x

The function has maximum at $2ax+2b=0$

$$\Rightarrow x=-b/a$$

$$\Rightarrow a\left(-\frac{b}{a}\right)^2+2b\left(-\frac{b}{a}\right)+d<0$$

$$\frac{b^2}{a}-\frac{2b^2}{a}+d<0$$

$$ab-b^2<0$$

"$\Leftarrow$" same way

4 (This exercise assumes a knowledge of linear algebra) Let A be a symmetric matrix. Show that A is positive definite if and only if the eigenvalues of A (which exist and are real, since A is symmetric) are positive. Is this true if A is not symmetric

(1)

Pf: $\because A$ is symmetric

$$A = U^* \lambda U \qquad U^*U = UU^* = I$$

$$\lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$\langle Ax, x\rangle = \langle U^* \lambda U x, x\rangle$$

$$= \langle \lambda Ux, Ux\rangle > 0 \quad \text{if } \lambda_1, \cdots \lambda_n > 0 \quad x \neq 0$$

$\Rightarrow A$ is positive definite

(2) $A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ $\qquad \langle A(1,1), (1,1)\rangle = -1$

$$[1 \ 1]\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = -1$$

$(1, 1) \cdot (1 \ -2)$

5 Check that the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

has $\Delta_i \geq 0$ yet the matrix is not semidefinite

Ans Let $e_1 = (1 \cdot 0 \cdot 0)$ $e_3 = (0 \cdot 0 \cdot 1)$

Then $\langle Ae_1, e_1 \rangle = 1$

but $\langle Ae_3 \;\; e_3 \rangle = -1$ so A is not semidefinite

Exercises for Chapter 6

2 Let $f: A \subset \mathbb{R} \to \mathbb{R}^m$ and assume df_i/dx exist for $i=1, \cdots, m$. Show that Df exist

pf Fix $x \in A$

we need to show for any $\varepsilon > 0$ $\exists\, \delta > 0$ s.t $|y-x| < \delta$ $y \in A$ implies

$$\| f(y) - f(x) - Df(x)(y-x) \| < \varepsilon |y-x|$$

$\because \frac{df_i}{dx}$ exist $\Rightarrow$ for any $\varepsilon > 0$ $\exists\, \delta_i > 0$ s.t $|y-x| < \delta_i$ $y \in A$ implies

$$\left| f_i(y) - f_i(x) - \frac{df_i(x)}{dx}(y-x) \right| < \frac{\varepsilon}{\sqrt{m}} |y-x|$$

let $Df(x)(y-x) = \begin{bmatrix} \frac{df_1}{dx}(x) \\ \vdots \\ \frac{df_m}{dx}(x) \end{bmatrix} (y-x) = \begin{bmatrix} \frac{df_1}{dx}(x)(y-x) \\ \vdots \\ \frac{df_m}{dx}(x)(y-x) \end{bmatrix}$

$$\| f(y) - f(x) - Df(x)(y-x) \| = \| ((f_1(y) - f_1(x)), (f_2(y) - f_2(x)) \cdots (f_m(y) - f_m(x))) -$$

$$\left(\frac{df_1}{dx}(x)(y-x), \frac{df_2}{dx}(x)(y-x) \cdots \frac{df_m}{dx}(x)(y-x) \right)$$

$$= \left\| \left(\left(f_1(y) - f_1(x) - \frac{df_1}{dx}(x)(y-x) \right) \cdots\cdots f_m(y) - f_m(x) - \frac{df_m}{dx}(x)(y-x) \right\|$$

Let $\delta = \min\{\delta_i\}$ $i=1 \cdots m$ $= \sqrt{\left(\left(f_1(y) - f_1(x) - \frac{df_1}{dx}(x)(y-x)\right)\right)^2 + \cdots + \left(f_m(y) - f_m(x) - \frac{df_m}{dx}(x)(y-x)\right)^2}$

$$\leq \varepsilon |y-x|$$

$\Rightarrow$ Df exist

6 a If $f: A \subset \mathbb{R}^n \to \mathbb{R}^m$ and $g: B \subseteq \mathbb{R}^m \to \mathbb{R}^p$ are twice differentiable and $f(A) \subset B$.
then for $x_0 \in A$, $x, y \in \mathbb{R}^n$; show that

$$D^2(g \circ f)(x_0)(x, y) = D^2(g(f(x_0)))(Df(x_0) \cdot x, Df(x_0) \cdot y)$$
$$+ Dg(f(x_0)) \cdot D^2 f(x_0)(x, y)$$

b: If $p: \mathbb{R}^n \to \mathbb{R}^m$ is a linear map plus some constant and $f: A \subset \mathbb{R}^m \to \mathbb{R}^s$ is k times differentiable, prove that

$$D^k(f \circ p)(x_0)(x_1 \cdots x_k) = D^k f(p(x_0))(Dp(x_0)(x_1) \cdots Dp(x_0)(x_k))$$

If $p=1$

Pf: a $D^2(g \circ f)(x_0) = \sum_{i_1 i_2 = 1}^{n} \left(\frac{\partial^2 (g \circ f)}{\partial x_{i_1} \partial x_{i_2}} \right) = \sum_{i_1 i_2 = 1}^{n} \left(\frac{\partial}{\partial x_{i_1}} \left(\frac{\partial (g \circ f)}{\partial x_{i_2}} \right) \right)$ $\qquad y_i = f_i(x_0)$

$$= \sum_{i_1 i_2 = 1}^{n} \left(\frac{\partial}{\partial x_{i_1}} \left(\sum_{i=1}^{m} \frac{\partial g(f(x_0))}{\partial y_i} \cdot \frac{\partial f(x_0)}{\partial x_{i_2}} \right) \right)$$

$$= \sum_{i_1 i_2 = 1}^{n} \left(\sum_{i_1 i_2 = 1}^{m} \frac{\partial^2 g(f(x_0))}{\partial y_{i_1} \partial y_{i_2}} \frac{\partial f(x_0)}{\partial x_{i_1}} \cdot \frac{\partial f(x_0)}{\partial x_{i_2}} + \frac{\partial g(f(x_0))}{\partial y_i} \cdot \frac{\partial^2 f(x_0)}{\partial x_{i_1} \partial x_{i_2}} \right)$$

$$= D^2(g(f(x_0)))(Df(x_0) \cdot x, Df(x_0) y)$$
$$+ Dg(f(x_0)) \cdot D^2 f(x_0)(x, y)$$

If $p \neq 1$ Let $(g \circ f)_l$ $l = 1 \cdots p$

b $\because P = Ax + B$ $\qquad \frac{\partial^k P}{\partial x_{i_1} \cdots \partial x_{i_k}} = 0 \qquad k \geq 2$

$$\Rightarrow D^k(g \circ p)(x_0) = D^k f(p(x_0))(Dp(x_0)(x_1) \cdots Dp(x_0)(x_k))$$

8 Show that if $f: A \subset \mathbb{R}^2 \to \mathbb{R}$ has a critical point $x_0 \in A$ and we let

$$\Delta = \frac{\partial^2 f}{\partial x_1 \partial x_1} \cdot \frac{\partial^2 f}{\partial x_2 \partial x_2} - \left(\frac{\partial^2 f}{\partial x_1 \partial x_2}\right)^2$$

be evaluated at x_0, then

a $\Delta > 0$ and $\frac{\partial^2 f}{\partial x_1 \partial x_1} > 0$ imply f has a local minimum at x_0

b $\Delta > 0$ and $\frac{\partial^2 f}{\partial x_1 \partial x_1} < 0$ imply f has a local maximum at x_0

c $\Delta < 0$ implies x_0 is a saddle point of f

pf: a

$\because \underset{\Delta_2}{\underset{\parallel}{\Delta > 0}}$ and $\underset{\Delta_1}{\underset{\parallel}{\frac{\partial^2 f}{\partial x_1 \partial x_2} > 0}}$ $\Rightarrow H_{x_0}(f)$ is postive definite

By 6.9.4 i f has a local minimum at x_0

b $\because \Delta > 0$ and $\frac{\partial^2 f}{\partial x_1 \partial x_2} < 0 \Rightarrow H_{x_0}(f)$ is negative definite

$\Rightarrow f$ has a local maximum at x_0

c By 6-9-4 (ii) If f has a local $\overset{(\text{minimum})}{\text{maximum}}$ at x_0, then $H_{x_0}(f)$ is $\overset{(\text{postive})}{\text{negative}}$ semidefinite

$\Rightarrow \overset{(\Delta \leq 0)}{\Delta \geq 0}$

But $\Delta < 0$ $\therefore$ f can have neither a maximum nor a minimum at x_0

$\Rightarrow$ x_0 must be a saddle point of f

9 Consider the following two possible properties for a subset X of $\mathbb{R}^n$

1 There is a point $x_0 \in X$ such that every other point x in X can be joined to x_0 by a straight line in X

2 There is a point $x_0 \in X$ such that every other point x in X can be joined to x_0 by a differentiable path in X

a. Give examples of each kind of set that are not convex

b. Show that if X is open set in $\mathbb{R}^n$ satisfying either of these conditions and $f: X \to \mathbb{R}$ is differentiable function with zero derivative. then f is constant

c Show that if X is an open subset of $\mathbb{R}^n$, then the following are equivalent:

i. Condition 2 above

ii. Path connectedness of X

iii Connectedness of X

Ans: a. Let $X = \{(x, 0) \in \mathbb{R}^2 \mid -1 \le x \le 1\} \cup \{(0, y) \in \mathbb{R}^2 \mid -1 \le y \le 1\}$

b. If 1 hold By 6.7.2 $\Rightarrow$ $f(x) = f(x_0)$ $\forall x \in X$

If 2 hold let $\gamma(t)$ be a path from x_0 to x

Let $h(t) = f(\gamma(t))$ $\Rightarrow$ $h'(t) = Df(\gamma(t)) \circ D\gamma(t) = 0$ $\because D\gamma(t) = 0$

$h: \mathbb{R}^1 \to \mathbb{R}^1$ By Mean Value $f(x) = f(x_0)$ $\forall x \in X$

c i $\Rightarrow$ ii by definition of path connectedness ($\because$ differentiable path is continuous path)

ii $\Rightarrow$ iii Theorem 3.5.2

Chapter 6

\# 12. A function $f: \mathbb{R}^n \to \mathbb{R}$ is called homogeneous of degree m if $f(tx) = t^m f(x)$ for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$.
If f is differentiable, show that for $x \in \mathbb{R}^n$,
$Df(x)x = mf(x)$, i.e. $\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = mf(x)$.

Pf: Let $g(t) = f(tx) = t^m f(x)$

對t微分: $g'(t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{d(tx_i)}{dt} = mt^{m-1} f(x)$

chain rule.

$$= \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = mt^{m-1} f(x)$$

Let $t=1$

$$\Rightarrow g'(1) = \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = mf(x).$$

that is $\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = mf(x)$ ※.

Show that maps multilinear in k variables give rise to homogeneous functions of degree k. Give other examples.

Pf: Let $f: \mathbb{R}^k \to \mathbb{R}$ be multilinear in the variables $x_1, x_2, \dots, x_k$
that is $f(\dots, \alpha x_i + \beta, \dots) = \alpha f(\dots, x_i, \dots) + f(\dots, \beta, \dots)$

Thus $f(tx) = f(tx_1, tx_2, \dots, tx_k)$

$$= t^k f(x_1, x_2, \dots, x_k)$$

$$= t^k f(x)$$

$\Rightarrow f$ is homogeneous of degree k.

16 If $f: \mathbb{R}^n \to \mathbb{R}^m$ is a differentiable and Df is a constant, show that f is a linear term plus a constant and the linear part of f is the constant value of Df

pf Let $Df = A$

$g(x) = f(x) - Ax$

$C = \begin{bmatrix} c \cdots c & c & c \\ c \cdots c & c & c \\ c & \vdots \cdots & c & c \end{bmatrix}$

$\Rightarrow Dg = Df - A = 0 \Rightarrow g(x)$ is constant $\Rightarrow f = g + Ax$

Let $f = (f_1(x_1 \cdots x_n)\ f_2(x_1 \cdots x_n) \cdots f_m(x_1 \cdots x_n))$

$Df = 0 \Rightarrow Df_1 = 0$

By Mean Value Theorem

$f_1(x) - f_1(y) = 0(x-y) = 0 \Rightarrow f_1(x) = f_1(y)$

f_i similar

17 If $f: A \subset \mathbb{R}^n \to \mathbb{R}$ is of class C^r and $Df(x_0)=0$ $D^2f(x_0)=0$ $\cdots$ $D^{r-1}f(x_0)=0$

But $D^rf(x_0)(x \cdots x) < 0$ for all $x \in \mathbb{R}^n$. $x \neq 0$, then prove that f has a local maximum at x_0

pf By Taylor's Theorem

$$f(y) - f(x_0) = \sum_{k=1}^{r-1} \frac{1}{k!} D^k f(x_0)(y-x_0 \cdots y-x_0) + \frac{1}{r!} D^r f(c)(y-x \cdots y-x)$$

$\because$ $Df(x_0)=0$ $D^2f(x_0)=0$ $\cdots$ $D^{r-1}f(x_0)=0$

$$\Rightarrow f(y) - f(x_0) = \frac{1}{r!} D^r f(c)(y-x \cdots y-x_0)$$

$\because$ $D^rf(x_0)$ is continuous and $D^rf(x_0) < 0$

$\exists\, \delta > 0$ $\ni$ $D^rf(x) < 0$ when $\|x-x_0\| < \delta$

Let $\|y-x_0\| < \delta$ $\Rightarrow$ $c \in D(x_0, \delta)$ $\Rightarrow$ $D^rf(c) < 0$

$\Rightarrow$ $f(y)-f(x_0) < 0$ $\Rightarrow$ $f(y) < f(x_0)$ for all $y \in D(y_0, \delta)$ $y \neq x_0$

$\Rightarrow$ f has a local maximum at x_0

18 Prove that the equation $x^3+bx+c=0$, where $b>0$ has exactly one solution $x\in\mathbb{R}$

Pf Let $f(x)=x^3+bx+c$

$\Rightarrow \lim_{x\to\infty} f(x)=\infty$

$\lim_{x\to-\infty} f(x)=-\infty$

By Intermediate Value Theorem

$\exists\, c\in\mathbb{R} \quad f(c)=0$

If $\exists\, b\neq c \quad f(b)=0$

$\Rightarrow b<c$ or $b>c$ assume $b<c$

By Mean Value Theorem

$\exists\, d\in(b,c) \qquad f(c)-f(b)=f'(d)(c-b)$

$\Rightarrow f'(d)=0$

$f'(x)=3x^2+b$

$3d^2+b=0$

$b=-3d^2$

$\Rightarrow b\leq 0$ $\times$

24 Let $f(x,y)$ be a real-valued function on $\mathbb{R}^2$. Show that if f is of class C^1 and $\partial^2 f/\partial x\partial y$ exist and is continuous, then $\partial^2 f/\partial y\partial x$ exists and $\partial^2 f/\partial x\partial y = \partial^2 f/\partial y\partial x$ (this is weaker that saying that f is of class C^2). Generalize

pf $\because f \in C^1 \Rightarrow \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exist and continuous

Fix $(x,y) \in A$

let $S_{h,k} = [f(x+h, y+k) - f(x+h, y)] - [f(x, y+k) - f(x,y)]$

let $g_k(u) = f(u, y+k) - f(u,y)$

$\Rightarrow S_{h,k} = g_k(x+h) - g_k(x)$

By Mean Value Theorem

$S_{h,k} = g_k'(c_{h,k}) \cdot h \quad c_{h,k} \in (x, x+h)$

$S_{h,k} = \left(\frac{\partial f}{\partial x}(c_{h,k}, y+k) - \frac{\partial f}{\partial x}(c_{h,k}, y) \right) \cdot h$

$\because S_{h,k} = [f(x+h, y+k) - f(x, y+k)] - [f(x+h, y) - f(x,y)]$

let $g_h(u) = f(x+h, u) - f(x,u)$

$\Rightarrow S_{h,k} = g_h(y+k) - g_h(y)$

By Mean Value Theorem $\quad d_{h,k} \in (y, y+k)$

$S_{h,k} = \left(\frac{\partial f}{\partial y}(x+h, d_{h,k}) - \frac{\partial f}{\partial y}(x, d_{h,k}) \right) \cdot k$

$\because \frac{\partial^2 f}{\partial x \partial y}$ exist. By Mean Value Theorem

$\Rightarrow S_{h,k} = \frac{\partial^2 f}{\partial x \partial y}(e_{h,k}, d_{h,k})\, hk$

$\because \frac{\partial^2 f}{\partial x \partial y}$ exist and continuous

Let $h \to 0 \quad S_{h,k} \to \frac{\partial^2 f}{\partial x \partial y}(x, d_{h,k}) k = \frac{\partial f}{\partial x}(x, y+k) - \frac{\partial f}{\partial x}(x,y)$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(x,y)\right)=\lim_{k\to 0}\frac{\frac{\partial f}{\partial x}(x,y+k)-\frac{\partial f}{\partial x}(x,y)}{k}=\lim_{k\to 0}\frac{\frac{\partial^2 f}{\partial x\partial y}(x,d_{nk})\,k}{k}$$

$$=\lim_{k\to 0}\frac{\partial^2 f}{\partial x\partial y}(x,d_{nk})$$

$$=\frac{\partial^2 f}{\partial x\partial y}(x,y)\quad \because \frac{\partial^2 f}{\partial x\partial y}\text{ is continuous}$$

25 Let $f: \mathbb{R}^n \to \mathbb{R}$ and suppose that the partial derivative $\partial f / \partial x_i$ $i=1, \cdots, n$, exist and that $\partial f / \partial x_i$ $i=1, \cdots, n-1$, are continuous. Prove that f is differentiable

pf Fix $x \in \mathbb{R}^n$

We need to show for any $\varepsilon > 0$ $\exists \delta > 0$ s.t. $\|y-x\| < \delta$ $y \in \mathbb{R}^n$ implies

$$|f(y) - f(x) - Df(x)(y-x)| < \varepsilon \|x-y\|$$

$$f(y) - f(x) = f(y_1 \cdots y_n) - f(x_1, y_2, \cdots y_n) + f(x_1, y_2 \cdots y_n) - f(x_1, x_2, y_3 \cdots y_n) + f(x_1, x_2, y_3 \cdots y_n)$$

$$- f(x_1, x_2, x_3, y_4 \cdots y_n) + \cdots + f(x_1 \cdots x_{n-1}, y_n) - f(x_1 \cdots x_n)$$

By mean value theorem

$$f(y_1 \cdots y_n) - f(x_1\ y_2 \cdots y_n) = \frac{\partial f}{\partial x_1}(u_1, y_2 \cdots y_n)(y_1 - x_1) \qquad u_1 \in (x_1\ y_1) \quad \cdots 1$$

$\frac{\partial f}{\partial x_n}$ exist $\Rightarrow$ for any $\varepsilon > 0$ $\exists \delta_n > 0$ s.t. $|y_n - x_n| < \delta_n$

$$|f(x_1 \cdots x_n, y_n) - f(x_1 \cdots x_n) - \frac{\partial f}{\partial x_n}(x)(y_n - x_n)| < \frac{\varepsilon}{n}|y_n - x_n|$$

$$\Rightarrow f(y) - f(x) = \left(\frac{\partial f}{\partial x_1}(u_1\ y_2 \cdots y_n)\right)(y_1 - x_1) + \frac{\partial f}{\partial x_2}(x_1, u_2\ y_3 \cdots y_n)(y_2 - x_2) \qquad u_i \in (y_i, x_i)$$

$$+ \cdots + \left(\frac{\partial f}{\partial x_n}(x_1, x_2 \cdots x_{n-1}, u_n)\right)(y_n - x_n)$$

$$Df(x)(y-x) = \sum_{i=1}^{n} (\partial f / \partial x_i)(x_1 \cdots x_n)(y_i - x_i)$$

$$\Rightarrow |f(y) - f(x) - Df(x)(y-x)| \le \left|\frac{\partial f}{\partial x_1}(u_1\ y_2 \cdots y_n) - \frac{\partial f}{\partial x_1}(x_1 \cdots x_n)\right| |y_1 - x_1|$$

$$+ \cdots + \left|\frac{\partial f}{\partial x_{n-1}}(x_1 \cdots x_{n-2}\ u_{n-1}\ y_n) - \frac{\partial f}{\partial x_{n-1}}(x_1 \cdots x_n)\right| |y_{n-1} - x_n|$$

$$+ |f(x_1 \cdots x_{n-1}, y_n) - f(x_1 \cdots x_n) - \frac{\partial f}{\partial x_n}(x_1 \cdots y_n)(y_n - x_n)|$$

$\because \frac{\partial f}{\partial x_i}$ continuous $i = 1 \cdots n$

$\Rightarrow \forall \varepsilon > 0 \ \exists \delta_i \ni \|y-x\| < \delta_i \quad \left|\frac{\partial f}{\partial x_i}(y) - \frac{\partial f}{\partial x_i}(x)\right| < \frac{\varepsilon}{n}$

Let $\delta = \min\{\delta_1 \cdots \delta_n\}$

$\Rightarrow |f(y) - f(x) - Df(x)(y-x)| \le \frac{\varepsilon}{n}|y_1 - x_1| + \frac{\varepsilon}{n}|y_2 - x_2| + \cdots + \frac{\varepsilon}{n}|y_{n-1} - x_{n-1}| + \frac{\varepsilon}{n}|y_n - x_n|$

$\le \varepsilon \|y - x\|$

$\Rightarrow$. Df exist

29. Let $f_n(x) = xe^{-nx}$, $x \in [0, \infty)$ $n = 0, 1, 2$

a Show that $f(x) = \sum_{n=0}^{\infty} f_n(x)$ exist. Compute f explicitly

b Is f continuous

c Find a suitable set on which the convergence is uniform

d May we differentiate term by term

pf: a for $x > 0$ $\sum_{n=0}^{\infty} e^{-nx}$ is a convergent geometric series

$$\Rightarrow f(x) = x \cdot \frac{1}{1 - \frac{1}{e^x}} = x \cdot \frac{e^x}{e^x - 1} \quad \text{for } x > 0$$

b No $f(0) = 0$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{e^x + xe^x}{e^x} = 1 + 0 = 1$$

$\Rightarrow f$ is not continuous at 0

c fix $N \in \mathbb{N}$

$$\sum_{N+1}^{\infty} xe^{-nx} = x \frac{e^{-(N+1)x}}{1 - \frac{1}{e^x}} = x \frac{e^{-Nx}}{e^x - 1} = \frac{x}{e^x - 1} e^{-Nx}$$

when $x \to \infty$ $\frac{x}{e^x - 1} \to 0 \Rightarrow \frac{x}{e^x - 1} < M$ $x \in [\varepsilon, \infty) \Rightarrow \lim_{N \to \infty} \frac{x}{e^x - 1} e^{-Nx} \to 0 \Rightarrow$ uniform convergent on $[\varepsilon, \infty)$

$x \to 0$ $\frac{x}{e^x - 1} \to 1 \Rightarrow \frac{x}{e^x - 1} e^{-Nx} \to 1$ as $x \to 0$

d

$$f_n'(x) = e^{-nx} - nxe^{-nx} = e^{-nx}(1 - nx)$$

Consider $g_n(x) = nxe^{-nx}$

$$\sum_{n=0}^{\infty} g_n = \sum_{n=0}^{\infty} nxe^{-nx} \quad \lim_{n \to \infty} \left| \frac{(n+1)xe^{-(n+1)x}}{nxe^{-nx}} \right| = \lim_{n \to \infty} \frac{x}{e^x} = 0$$

$\sum_{n=0}^{\infty} g_n$ convergent on $[\varepsilon, \infty)$ $\left(\because x < y \quad \frac{x}{e^x} > \frac{y}{e^y}\right)$

$\Rightarrow$ By 5.3.4 $\left(\sum_{n=0}^{\infty} f_n(x)\right)' = \sum_{n=0}^{\infty} f_n'(x)$ on $[\varepsilon, \infty)$

$\because \varepsilon$ is any $\Rightarrow \left(\sum_{n=0}^{\infty} f_n(x)\right)' = \sum_{n=0}^{\infty} f_n'(x)$ on $(0, \infty)$

30

"Since f is differentiable, it is continuous." → This is true.

"Hence it assume its maximum; that is, T is not empty"

This is not true.

Since $\mathbb{R}$ is not compace. ($f: \mathbb{R} \to \mathbb{R}$)

For example: Let $f(x) = \tan^{-1} x$, then f is bounded and has a continuous derivative.

But $\lim_{x\to\infty} f(x) = \frac{\pi}{2}$, $\lim_{x\to-\infty} f(x) = -\frac{\pi}{2}$, $f(x)$ does not assume its maximum. Hence T can be empty.

"$T \subset S$" is true.

"$x \in S$, $f'(x)=0$; hence f achieves either a maximum or a minimum there".

This is not true.

$x \in S$, $f'(x)=0$, but it is not necessary that f assume either maximum or minimum there.

For example: $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$; $f'(0)=0$, but 0 is ~~not~~ neither maximum nor minimum.

"$f(x_0) \geq 0$" is not true.

"$T = S \cap \{x \mid f(x) \geq 0\}$" is not true.

#30 T is really closed.

pf: Let $T = \{x_0 \in \mathbb{R} \mid f(x) \le f(x_0) \ \forall x \in \mathbb{R}\}$.

If $T = \phi \Rightarrow T$ is closed. done.

Suppose $T \neq \phi$

For $x_0 \in T$, let $M = f(x_0)$.

Then $T = \{x \in \mathbb{R} \mid f(x) = M\} = f^{-1}(\{M\})$

Since $\{M\}$ is closed and f is continuous

$\Rightarrow T$ is closed ※

31 Let $A \subset \mathbb{R}^n$ be compact, and construct the normed space $C(A, \mathbb{R})$ as in Chapter 5.

Define, for $x \in A$ $\delta_{x_0}: C(A, \mathbb{R}) \to \mathbb{R}: f \mapsto f(x_0)$. Prove the δ_{x_0} is differentiable

Prove: $\delta_{x_0}(f+g) = (f+g)(x_0) = f(x_0)+g(x_0) = \delta_{x_0}(f) + \delta_{x_0}(g)$

$\delta_{x_0}(\lambda f) = \lambda f(x_0) = \lambda \delta_{x_0}(f)$

$\Rightarrow \delta_{x_0}$ is linear

$f_n \to g$ in $C(A, \mathbb{R}) \Rightarrow \sup_{x \in A} |f_n(x) - g(x)| \to 0$

$\Rightarrow f_n(x_0) - g(x_0)$

$\Rightarrow \delta_{x_0}(f_n) \to \delta_{x_0}(g) \Rightarrow \delta_{x_0}$ is continuous

By the definition $\forall \varepsilon > 0$ $D\delta_{x_0}(f) = \delta_{x_0}$ $\forall f$ $f \in C(A, \mathbb{R})$

$\| \delta_{x_0}(g) - \delta_{x_0}(f) - D\delta_{x_0}(f)(g-f) \| = \| \delta_{x_0}(g) - \delta_{x_0}(f) - \delta_{x_0}(g) + \delta_{x_0}(f) \| = 0 < \varepsilon \|f-g\|_\infty$

$\Rightarrow \delta_{x_0}$ is differentiable

37 A C^2 function $f: \mathbb{R}^2 \to \mathbb{R}$ is called harmonic if

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

Assume that (x_0, y_0) is a strict local maximum and f is harmonic, Prove that all second derivative of f vanish at $(x_0\ y_0)$

Pf: $H_{(x_0,y_0)}(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$

$\because$ $(x_0\ y_0)$ is a strict local maximum

$\Rightarrow H_{(x_0,y_0)}(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$ is negative semidefinite at (x_0, y_0)

$$\frac{\partial^2 f}{\partial y^2},\ \frac{\partial^2 f}{\partial x^2} \leq 0 \text{ and } \frac{\partial^2 f}{\partial x_1^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \geq 0$$

$$\because \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = 0$$

$$\Rightarrow -\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \geq 0 \Rightarrow \frac{\partial^2 f}{\partial x \partial y} = 0 = \frac{\partial^2 f}{\partial y \partial x}$$

$$\Rightarrow H_{(x_0\ y_0)} f = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow$ all second derivative of f vanish at $(x_0\ y_0)$