Exercises for §5.8

2. Suppose that p_n is a sequence of polynomials converging uniformly to f on $[0,1]$, and f is not a polynomial. Prove that the degree of the p_n are not bounded. [Hint: An N-th degree polynomial p is uniquely determined by its values at $N+1$ points $x_0, \cdots, x_N$ via **Lagrange's interpolation formula**

$$p(x) = \sum_{i=0}^{N} \pi_i(x) \frac{p(x_i)}{\pi_x(x_i)} \,,$$

where $\pi_i(x) = (x-x_0)(x-x_1)\cdot(x-x_N)/(x-x_i)$.]

Proof. Suppose that the degree of the polynomials p_n (converging to f) is bounded by N. Let $A_n = (a_{N,n}, a_{N-1,n}, \cdots, a_{0,n})$ be the coefficients of the polynomial p_n; that is,

$$p_n(x) = \sum_{k=0}^{N} a_{k,n} x^k \,.$$

Since p_n converges to f uniformly, p_n converges at points of the form $\frac{\ell}{N}$ for all $\ell = 0, 1, \cdots, N$. This implies that $a_{0,n}$ is a Cauchy sequence, and

$$\lim_{n,m\to\infty} \begin{bmatrix} \frac{1}{N} & \frac{1}{N^2} & \frac{1}{N^3} & \cdots & \frac{1}{N^N} \\ \frac{2}{N} & \frac{2^2}{N^2} & \frac{2^3}{N^3} & \cdots & \frac{2^N}{N^N} \\ \frac{3}{N} & \frac{3^2}{N^2} & \frac{3^3}{N^3} & \cdots & \frac{3^N}{N^N} \\ \vdots & & \ddots & \vdots & \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_{1,n} - a_{1,m} \\ a_{2,n} - a_{2,m} \\ a_{3,n} - a_{3,m} \\ \vdots \\ a_{N,n} - a_{N,m} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} .$$

By the Vandermonde determinant,

$$\det\left(\begin{bmatrix} \frac{1}{N} & \frac{1}{N^2} & \frac{1}{N^3} & \cdots & \frac{1}{N^N} \\ \frac{2}{N} & \frac{2^2}{N^2} & \frac{2^3}{N^3} & \cdots & \frac{2^N}{N^N} \\ \frac{3}{N} & \frac{3^2}{N^2} & \frac{3^3}{N^3} & \cdots & \frac{3^N}{N^N} \\ \vdots & & \ddots & \vdots & \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \right) = \frac{1}{N^{(1+N)/2}} \prod_{1\le i<j\le N} (j-i) \neq 0 \,.$$

Therefore, we also know that $a_{k,n}$ forms a Cauchy sequence (in $\mathbb{R}$) for all $k = 1, \cdots, N$. By the completeness of $\mathbb{R}$,

$$a_{k,n} \to c_k \qquad \text{as} \quad n \to \infty$$

for some $c_k \in \mathbb{R}$. This implies that for all $x \in [0,1]$,

$$\lim_{n\to\infty} p_n(x) = c_N x^N + c_{N-1}x^{N-1} + \cdots c_1 x + c_0 \equiv p(x)\,.$$

In other words, p_n converges pointwise to p. Since we assume that $p_n \to f$ uniformly, f must be equal to p which is a contradiction. $\square$