Advanced Calculus MA-2046 Midterm 1

National Central University, Apr. 2, 2013

Name: ______________________________ **Class:** _________

Problem 1. (15%) Express the sum of the series $\sum\limits_{n=1}^{\infty} x^{n+1}/n(n+1)$ as an integral to show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} = 2\log 2 - 1\,.$$

Proof. Since

$$\frac{d^2}{dx^2}\sum_{n=1}^{N} \frac{x^{n+1}}{n(n+1)} = \sum_{n=1}^{N} x^{n-1} = \frac{1-x^N}{1-x}\,;$$

we find that

$$\begin{aligned}\sum_{n=1}^{N} \frac{x^{n+1}}{n(n+1)} &= \int_0^x\int_0^y \frac{1-z^N}{1-z}\,dzdy = -\int_0^x \log(1-y)dy - \int_0^x\int_0^y \frac{z^N}{1-z}\,dzdy \\ &= (1-x)\log(1-x) + x - \int_0^x\int_0^y \frac{z^N}{1-z}dzdy\,,\end{aligned}$$

where we use the fact that

$$\int \log x dx = x\log x - x + C$$

to derive the anti-derivative of $\log(1-y)$.

In particular, when $x=-1$ we have

$$\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n(n+1)} = 2\log 2 - 1 - \int_0^{-1}\int_0^y \frac{z^N}{1-z}dzdy\,;$$

thus

$$\begin{aligned}\Big|\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n(n+1)} - (2\log 2 - 1)\Big| &\le \int_{-1}^0\int_{-1}^0 \frac{|z|^N}{1-z}\,dzdy \\ &\le \int_{-1}^0 |z|^N dz = \frac{1}{N+1} \to 0 \qquad \text{as } N\to\infty\,.\end{aligned}$$

□

Problem 2. (10%) Let the function $f_n : [0,1] \to \mathbb{R}$ be given by

$$f_n(x) = \begin{cases} \sin(x^2) & \text{if } \ x \in \Big[\dfrac{k}{2^n}, \dfrac{k+1}{2^n}\Big) \text{ for some even number } k\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

and set

$$F_n(x) = \int_0^x f_n(t)dt\,, \qquad 0 \le x \le 1\,.$$

Prove that F_n has a uniformly convergent subsequence.

Proof. By Corollary 5.6.3, it suffices to show that the family $\{F_n\}_{n=1}^{\infty}$ is pointwise bounded and equi-continuous.

1. Since $|f_n(x)| \le 1$ for all $x \in [0,1]$,
$$|F_n(x)| = \Big|\int_0^x f_n(t)dt\Big| \le \int_0^1 1dt = 1 \qquad \forall\, x \in [0,1]$$
which implies that $\{F_n\}_{n=1}^{\infty}$ is pointwise bounded.

2. Since $|f_n(x)| \le 1$ for all $x \in [0,1]$, by the property of integrals,
$$|F_n(x) - F_n(y)| = \Big|\int_x^y f_n(t)dt\Big| \le \Big|\int_x^y 1dt\Big| = |x-y|$$
which implies that $\{F_n\}_{n=1}^{\infty}$ is equi-continuous. □

Problem 3. (15%) In the book you are asked to show the following theorem:

Theorem. *Let $g : [0,\infty) \to \mathbb{R}$ be a continuous function such that*
$$\int_0^\infty g(x)dx = \lim_{R\to\infty} \int_0^R g(x)dx < \infty\,,$$
and $f_n : [0,\infty) \to \mathbb{R}$ be a sequence of continuous functions satisfying

(a) *$|f_n(x)| \le g(x)$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$.*

(b) *For each $R > 0$, f_n converges to f uniformly on $[0,R]$.*

Then $\lim_{n\to\infty} \int_0^\infty f_n(x)dx = \int_0^\infty f(x)dx\,.$

Use this theorem to show that the sequence
$$f_n(x) = \begin{cases} \dfrac{\sin x}{1+x^2} & x \in [0, 2n\pi] \\ 0 & \text{otherwise} \end{cases}$$
satisfies
$$\lim_{n\to\infty} \int_0^\infty f_n(x)\,dx = \int_0^\infty \frac{\sin x}{1+x^2} dx\,.$$

Sol: Let $g(x) = \dfrac{1}{1+x^2}$. Then

(a) $\displaystyle\int_0^\infty g(x)dx = \int_0^\infty \frac{1}{1+x^2}dx = \tan^{-1} x\Big|_{x=0}^{x=\infty} = \frac{\pi}{2}\,;$

(b) $|f_n(x)| \le g(x)$ for all $x \in [0, \mathbb{R}^+)\,;$

(c) For any fixed $R > 0$, if $2N\pi \ge R$, then $f_n(x) = f(x)$ in $[0,R]$ for all $n \ge N$. In other words, $f_n \to f$ uniformly on $[0,R]$ for any $R > 0$.

Therefore, by the theorem we conclude that

$$\lim_{n\to\infty} \int_0^\infty f_n(x)\,dx = \int_0^\infty \frac{\sin x}{1+x^2}\,dx\,. \qquad \square$$

Problem 4. (20%) Let $\mathcal{B}$ be the collection of polynomials of the form

$$\sum_{k=0}^{m} a_k x^{2k}\,.$$

Show that $\mathcal{B}$ is dense in $\mathcal{C}([0,1],\mathbb{R})$. Is $\mathcal{B}$ also dense in $\mathcal{C}([-1,1],\mathbb{R})$?

Proof. We check the following properties:

1. Suppose that $f, g \in \mathcal{B}$; that is, $f(x) = \sum\limits_{k=0}^{m} a_k x^{2k}$ and $g(x) = \sum\limits_{j=0}^{n} b_j x^{2j}$. Then
   (a) $(f\cdot g)(x) = \sum\limits_{k=0}^{m}\sum\limits_{j=0}^{n} a_k b_j x^{2(k+j)} = \sum\limits_{\ell=0}^{m+n} c_\ell x^{2\ell}$, where
   $$c_\ell = \sum_{\substack{k+j=\ell \\ 0\le k\le m, 0\le j\le n}} a_k b_j\,;$$
   thus $f\cdot g \in \mathcal{B}$.
   (b) Define $a_k = 0$ if $k > m$ and $b_j = 0$ if $j > n$. Then
   $$(f+g)(x) = \sum_{k=0}^{\ell} c_k x^{2k}$$
   with $\ell = \max\{m,n\}$ and $c_k = a_k + b_k$. Therefore, $f+g \in \mathcal{B}$.
   (c) $(\alpha f)(x) = \sum\limits_{k=0}^{m} (\alpha a_k) x^{2k}$; thus $\alpha f \in \mathcal{B}$.
2. It is easy to see that the constant function $f = 1$ belongs to $\mathcal{B}$. In fact, $1 = 1 * x^0$.
3. Finally, if $x_1 \neq x_2$ and $x_1, x_2 \in [0,1]$, the function $f(x) = x^2$ satisfies $f(x_1) \neq f(x_2)$.

As a consequence,

1. $\mathcal{B}$ is an algebra;
2. $\mathcal{B}$ separates points;
3. the constant function $f = 1$ belongs to $\mathcal{B}$.

thus the Stone-Weierstrass theorem implies that $\mathcal{B}$ is dense in $\mathcal{C}([0,1],\mathbb{R})$.

However, $\mathcal{B}$ is **not** dense in $\mathcal{C}([-1,1],\mathbb{R})$ since, for example, the function $f(x)=x$ cannot be approximated by sequence of even functions in the uniform norm. In fact, suppose that $f_k \in \mathcal{B}$ and $f_k \to f$ in $\mathcal{C}([-1,1],\mathbb{R})$. Then $f_k(1) \to 1$ and $f_k(-1) \to -1$. However, by the definition of $\mathcal{B}$, $f_k(1) = f_k(-1)$; thus it is impossible that $f_k(1) \to 1$ and $f_k(-1) \to -1$ simultaneously. □

Problem 5. (40%) Prove or disprove the following statements:

1. Let f_k be a convergence sequence in $\mathcal{C}_b(A,\mathbb{R}^m)$, then $\{f_k\}_{k=1}^{\infty}$ is a closed set in $\mathcal{C}_b(A,\mathbb{R}^m)$.

2. Let M be a metric space (**not** necessary complete) and $\Phi : M \to M$ be such that
$$d\big(\Phi(x),\Phi(y)\big) < d(x,y)$$
for all $x,y \in M$, $x \neq y$. Then Φ has **at most one** fixed point.

3. Let $\mathcal{P}(\mathbb{R},\mathbb{R})$ denote the collection of all polynomials on $\mathbb{R}$. Then $\mathcal{P}(\mathbb{R},\mathbb{R})$ is closed in $\mathcal{C}(\mathbb{R},\mathbb{R})$ with respect to the uniform norm defined by
$$\|f\| = \sup_{x\in\mathbb{R}} |f(x)|\,.$$

4. Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of continuous functions, and $f_n \to f$ uniformly on $[0,1]$. Then the family $\big\{f_n\big\}_{n=1}^{\infty}$ is equi-continuous.

Sol:

1. **False**. For example, let $m=1$, and $f_k : A \to \mathbb{R}$ be constant function $f_k(x) \equiv \dfrac{1}{k}$. Then $f_k \to 0$ uniformly, but the sequence $\{f_k\}_{k=1}^{\infty}$ is not closed since it does not contain its limit function $f=0$.

2. **True**. Suppose that $x \neq y$ are two fixed-points. Then $\Phi(x)=x$ and $\Phi(y)=y$. Moreover,
$$d(x,y) = d(\Phi(x),\Phi(y)) < d(x,y)$$
which is impossible since $d(x,y)$ is a real number. Therefore, there is at most one fixed-point of Φ.

3. **True**. *This problem is another way of stating Exercise problem 50 in Chapter 5.* Now suppose that $p_n \in \mathcal{P}(\mathbb{R},\mathbb{R})$ and $p_n \to f$ uniformly. Then p_n is a Cauchy sequence (with respect to the uniform norm); thus for any given $\epsilon > 0$, there exists $N > 0$ such that
$$\max|p_n(x) - p_m(x)| \equiv \|p_n - p_m\| < \epsilon \qquad \text{whenever } n,m \geq N\,.$$

Since $p_n - p_m$ is also a polynomial, the maximum of $p_n - p_m$ cannot be finite unless $p_n - p_m$ is a constant. In other words, $p_n - p_m = c_{n,m}$ for some constant $c_{n,m}$ converging to 0 as $n, m \to \infty$. This implies that if $p_N(x) = \sum\limits_{k=0}^{m} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$, then for all $n \geq N$,

$$p_n(x) = \sum_{k=1}^{m} a_k x^k + b_n = b_n + a_1 x + a_2 x^2 + \cdots + a_m x^m \qquad (\star)$$

for some sequence b_n. We also note that $b_n - b_m = c_{n,m}$, so b_n is a Cauchy sequence in $\mathbb{R}$; thus convergent. We emphasize that the coefficients a_k in $(\star)$ does not depends on n if $k \geq 1$. Therefore, for each $x \in \mathbb{R}$,

$$p_n(x) \to p(x) \equiv \sum_{k=1}^{m} a_k x^k + b$$

for some constant b. This implies that p_n converges pointwise to p. On the other hand, $p_n \to f$ uniformly; thus $f = p$.

4. **True**. Let $\epsilon > 0$ be given. Since $f_n \to f$ uniformly, f_n is a Cauchy sequence in $C([0,1], \mathbb{R})$; thus there exists $N > 0$ such that

$$\max_{x \in [0,1]} |f_n(x) - f_m(x)| \equiv \|f_n - f_m\| < \frac{\epsilon}{3} \quad \text{whenever } n, m \geq N\,.$$

Since $f_1, f_2, \cdots, f_N$ are continuous functions on a compact set $[0,1]$, they are uniformly continuous; thus there exists $\delta_1, \delta_2, \cdots, \delta_N$ such that

$$|f_k(x) - f_k(y)| < \frac{\epsilon}{3} \quad \text{whenever } |x - y| < \delta_k,\ k = 1, 2, \cdots N\,.$$

Let $\delta = \min\{\delta_1, \cdots, \delta_N\}$. Then

$$|f_k(x) - f_k(y)| < \epsilon \quad \text{whenever } |x - y| < \delta,\ k = 1, 2, \cdots N\,, \qquad (\star\star)$$

and for $k > N$,

$$\begin{aligned} |f_k(x) - f_k(y)| &\leq |f_k(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_k(y)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon\,. \end{aligned} \qquad (\star\star\star)$$

Inequalities $(\star\star)$ and $(\star\star\star)$ together suggest that

$$|f_k(x) - f_k(y)| < \epsilon \quad \text{whenever } |x - y| < \delta\,;$$

so $\{f_n\}_{n=1}^{\infty}$ is equi-continuous.