Exercises for §5.1

1. Let $f_n(x) = (x - 1/n)^2$, $0 \le x \le 1$. Does f_n converge uniformly?

Sol. We note that the pointwise limit of $f_n(x)$ is x^2 since $1/n \to 0$ as $n \to \infty$. Let $f(x) = x^2$, then

$$\sup_{x\in[0,1]} \big|f_n(x) - f(x)\big| = \sup_{x\in[0,1]} \Big| -\frac{2x}{n} + \frac{1}{n^2}\Big| \le \frac{2}{n} + \frac{1}{n^2} \to 0 \quad \text{as} \quad n \to \infty\,.$$

which implies that the convergence is uniform.

3. Let $f_n : \mathbb{R} \to \mathbb{R}$ be uniformly continuous and let f_n converge uniformly to f. Do you think that f must be uniformly continuous? Discuss.

Sol. Let $\epsilon > 0$ be given. Since $f_n \to f$ uniformly, there exists $N > 0$ such that

$$\sup_{x\in\mathbb{R}} \big|f_n(x) - f(x)\big| < \frac{\epsilon}{3} \quad \text{whenever} \quad n \ge N.$$

Since f_N is assumed to be uniformly continuous, there exists $\delta > 0$ such that

$$\big|f_N(x) - f_N(y)\big| < \frac{\epsilon}{3} \quad \text{whenever} \quad |x - y| < \delta.$$

Therefore,

$$\begin{aligned} \big|f(x) - f(y)\big| &\le \big|f(x) - f_N(x)\big| + \big|f_N(x) - f_N(y)\big| + \big|f_N(y) - f(y)\big| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \qquad \text{whenever} \quad |x - y| < \delta\,. \end{aligned}$$

This implies that f is uniformly continuous on $\mathbb{R}$. □

Exercises for §5.2

1. Discuss the convergence and uniform convergence of

a. $f_n(x) = x^n/(n + x^n)$, $x \ge 0$, $n = 1, 2, \cdots$

b. $f_n(x) = e^{-x^2/n}$, $x \in \mathbb{R}$, $n = 1, 2, \cdots$

Sol. We determine the pointwise limit first, and then see if the convergence is uniform.

a. For each $x \in [0, 1]$, $f_n(x) \to 0$, while for each $x > 1$, $f_n(x) \to 1$. Therefore, the pointwise limit of $f_n(x)$ is $f(x) = \begin{cases} 0 & \text{if } x \in [0,1], \\ 1 & \text{if } x > 1. \end{cases}$ Moreover,

$$\frac{d}{dx} f_n(x) = \frac{nx^{n-1}(n + x^n) - nx^{n-1}x^n}{(n + x^n)^2} = \frac{n^2 x^{n-1}}{(n + x^n)^2} \ge 0$$

which implies that f_n is an increasing function. Therefore,

$$\sup_{x\geq 0}\left|f_n(x)-f(x)\right| = \max\left\{\sup_{x\in[0,1]}\left|f_n(x)\right|, \sup_{x>1}\left|f_n(x)-1\right|\right\}$$
$$= \max\left\{\left|f_n(1)\right|, \left|f_n(0)-1\right|\right\} = \frac{n}{n+1} \not\to 0 \quad \text{as} \quad n\to\infty\,;$$

thus the convergence is not uniform.

b. For each $x \geq 0$, $f_n(x) \to 1$ (why?). However, the convergence is not uniform since

$$\sup_{x\in\mathbb{R}}\left|e^{-x^2/n}-1\right| = 1 \not\to 0 \quad \text{as } n\to\infty\,. \qquad \square$$

4. Discuss the uniform convergence of $\sum_{n=1}^{\infty} 1/(x^2+n^2)$.

Sol. Let $g_n(x) = 1/(x^2+n^2)$ and $M_n = \frac{1}{n^2}$. Then $\left|g_n(x)\right| \leq M_n$ for all $x \in \mathbb{R}$, and $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-series test or improper integral test. Therefore, the Weierstrass M-test implies that the series

$$\sum_{n=1}^{\infty} \frac{1}{x^2+n^2}$$

converges uniformly. $\square$

Exercises for Chapter 5

2. Determine which of the following *sequences* converge (pointwise or uniformly) as $k\to\infty$. Check the continuity of the limit in each case.

a. $(\sin x)/k$ on $\mathbb{R}$

b. $1/(kx+1)$ on $]0,1[$

c. $x/(kx+1)$ on $]0,1[$

d. $x/(1+kx^2)$ on $\mathbb{R}$

e. $(1,(\cos x)/k^2)$, a sequence of functions from $\mathbb{R}$ to $\mathbb{R}^2$

Sol.

a. The sequence of functions $(\sin x)/k$ on $\mathbb{R}$ converges to 0 uniformly since

$$\sup_{x\in\mathbb{R}}\left|\frac{\sin x}{k}-0\right| \leq \frac{1}{k} \to 0 \quad \text{as} \quad k\to\infty.$$

b. The pointwise limit of the sequence $1/(kx+1)$ on $]0,1[$ is 0; however, the convergence is not uniform since

$$\sup_{x\in]0,1[} \left|\frac{1}{kx+1} - 0\right| \geq \frac{1}{k\cdot\frac{1}{k}+1} = \frac{1}{2} \not\to 0 \quad \text{as} \quad k\to\infty.$$

c. The pointwise limit of the sequence $x/(kx+1)$ on $]0,1[$ is 0 (why?). Moreover, the fact that

$$\frac{d}{dx}\frac{x}{kx+1} = \frac{1}{(kx+1)^2} > 0$$

implies that the function $\dfrac{x}{kx+1}$ is increasing; thus

$$\sup_{x\in]0,1[} \left|\frac{x}{kx+1} - 0\right| \leq \frac{1}{k+1} \to 0 \quad \text{as} \quad k\to\infty$$

which implies that the convergence is uniform.

d. The pointwise limit of the sequence $x/(1+kx^2)$ on $\mathbb{R}$ is 0. As the previous case, we compute the derivative and find that

$$\frac{d}{dx}\frac{x}{1+kx^2} = \frac{1-kx^2}{(1+kx^2)};$$

which implies that the maximum and the minimum of the function $x/(1+kx^2)$ occurs at $x=-1/\sqrt{k}$ and $x=1/\sqrt{k}$, respectively (why?). Therefore,

$$\sup_{x\in\mathbb{R}} \left|\frac{x}{1+kx^2} - 0\right| = \frac{1}{2\sqrt{k}} \to 0 \quad \text{as} \quad k\to\infty$$

which implies that the convergence is also uniform.

e. The pointwise limit of the sequence $(1, (\cos x)/k^2)$ is $(1,0)$, and the convergence is uniform since

$$\sup_{x\in\mathbb{R}} \left|\left(1, \frac{\cos x}{k^2}\right) - (1,0)\right| \leq \frac{1}{k^2} \quad \text{as} \quad k\to\infty. \qquad \square$$

3. Determine which of the following real *series* $\sum_{k=1}^{\infty} g_k$ converge (pointwise or uniformly). Check the continuity of the limit in each case.

a. $g_k(x) = \begin{cases} 0, & x\leq k \\ (-1)^k, & x>k. \end{cases}$

b. $g_k(x) = \begin{cases} 1/k^2, & |x|\leq k \\ 1/x^2, & |x|>k \end{cases}$

c. $g_k(x) = \left(\frac{(-1)^k}{\sqrt{k}}\right)\cos(kx)$ on $\mathbb{R}$.

d. $g_k(x) = x^k$ on $]0,1[$.

Sol. We first determine the sum of the series (if converges), and then determine the type of convergence and the continuity of the limit.

a. If $x \le 1$ $g_k(x) = 0$ for all k; thus the partial sum $\sum_{k=1}^{\mathrm{n}} g_k(x)$ vanishes. If $x > 1$, then $g_k(x) \neq 0$ only when $k \le [x]$ (where $[x]$ denotes the largest integer which is not greater than x); thus

$$\sum_{k=1}^{\infty} g_k(x) = \sum_{k=1}^{[x]} g_k(x)\,.$$

Therefore,

$$\sum_{k=1}^{\infty} g_k(x) = \begin{cases} 0 & x \le 1 \\ \sum\limits_{k=1}^{[x]} g_k(x) & x > 1. \end{cases}$$

To see if the convergence is uniform, we check the supremum of the difference between the partial sum and the limit, and find that

$$\sup_{x\in\mathbb{R}} \left| \sum_{k=1}^{n} g_k(x) - \sum_{k=1}^{[x]} g_k(x) \right| = 1 \quad \text{if} \quad n = [x] + 2k - 1 \text{ for some } k \in \mathbb{N}.$$

As a consequence, the convergence is not uniform.

b. Since $g_k(x) \le \frac{1}{k^2}$ and $\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$, the Weierstrass M-test implies that the series $\sum_{k=1}^{\infty} g_k(x)$ converges uniformly (thus the convergence is also pointwise). Moreover, it is clear that g_k is continuous on $\mathbb{R}$; thus the partial sum $\sum_{k=1}^{\mathrm{n}} g_k$ is also continuous. By Proposition 5.1.4, the limit is continuous.

c. If $x = \pi$, $\cos(kx) = (-1)^k$; thus $\sum_{k=1}^{\infty} \left(\frac{(-1)^k}{\sqrt{k}}\right)\cos(kx)$ diverges at $x = \pi$. Therefore, the series $\sum_{k=1}^{\infty} g_k(x)$ does not converge pointwise.

d. This is Example 5.1.9 (c). □

5. Suppose that $f_k \to f$ uniformly, where $f_k : A \subset \mathbb{R}^n \to \mathbb{R}$; $g_k \to g$ uniformly, where $g_k : A \to \mathbb{R}^m$; there is a constant M_1 such that $\|g(x)\| \le M_1$ for all x; and there is a constant M_2 such that $|f(x)| \le M_2$ for all x. Show that $f_k g_k \to fg$ uniformly. Find a counter example if M_1 or M_2 does not exist. Are M_1 and M_2 necessary for pointwise convergence?

Sol. Since $f_k \to f$ uniformly, there exists $N > 0$ such that

$$\sup_{x\in A} \big|f_k(x) - f(x)\big| < 1 \quad \text{whenever } k \geq N.$$

Therefore,

$$\sup_{x\in A} \big|f_k(x)\big| \leq \sup_{x\in A} |f(x)| + 1 \leq M_2 + 1 \quad \text{whenever } k \geq N.$$

For $k \geq N$,

$$\begin{aligned} \big|f_k(x)g_k(x) - f(x)g(x)\big| &\leq \big|f_k(x) - f(x)\big|\big|g(x)\big| + \big|f_k(x)\big|\big|g_k(x) - g(x)\big| \quad (0.1)\\ &\leq M_1\big|f_k(x) - f(x)\big| + (M_2 + 1)\big|g_k(x) - g(x)\big|\,; \end{aligned}$$

thus for $k \geq N$

$$\begin{aligned} &\sup_{x\in A} \big|f_k(x)g_k(x) - f(x)g(x)\big| \\ &\qquad \leq M_1 \sup_{x\in A} \big|f_k(x) - f(x)\big| + (M_2 + 1) \sup_{x\in A} \big|g_k(x) - g(x)\big| \end{aligned}$$

which converges to 0 as $k \to \infty$ since $f_k \to f$ and $g_k \to g$ uniformly (why?). Therefore, $f_k g_k$ converges uniformly to fg.

If f and g both are not bounded functions, the limit on the right-hand side of (0.1) might not vanish. Let $f_k(x) = g_k(x) = x + \frac{1}{k}$, $f(x) = g(x) = x$, and $A = \mathbb{R}$. Then $f_k \to f$ and $g_k \to g$ uniformly on A (why?), but

$$\big|f_k(x)g_k(x) - f(x)g(x)\big| = \Big(x + \frac{1}{k}\Big)^2 - x^2 = \frac{2x}{k} + \frac{1}{k^2}\,;$$

thus

$$\sup_{x\in\mathbb{R}} \big|f_k(x)g_k(x) - f(x)g(x)\big| = \infty$$

which implies that the convergence cannot be uniform.

For pointwise convergence, on the other hand, does not require that f and g are bounded. In fact, if $f_k \to f$ and $g_k \to g$ pointwise, $f_k g_k \to fg$ pointwise. □

8. Does pointwise convergence of continuous functions on a compact set to a continuous limit imply uniform convergence on that set?

Sol. Let $K = [0, 1]$, and $f_k : K \to \mathbb{R}$ be defined by

$$f_k(x) = \begin{cases} kx & 0 \leq x \leq \frac{1}{k} \\ 2 - kx & \frac{1}{k} < x \leq \frac{2}{k} \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_k \to 0$ pointwise (why?), but the convergence is not uniform since

$$\sup_{x\in[0,1]} \big|f_k(x) - 0\big| = 1 \not\to 0 \quad \text{as} \quad k \to \infty\,. \qquad \square$$

19. Prove that

$$\sum_{n=1}^{\infty} \Big(\frac{\sin nx}{n^2}\Big) x^3$$

defines a continuous function on all of $\mathbb{R}$.

Proof. We only need to show that the series is continuous at each point $a \in \mathbb{R}$. To see this, let $f_n(x) = \sum_{k=1}^{n} \Big(\frac{\sin nx}{n^2}\Big) x^3$ be the partial sum. We treat f_n as a sequence of functions defined on the interval $\big[-2|a|, 2|a|\big]$ and show that f_n converges uniformly. If the convergence is indeed uniform, then the limit $f(x) = \sum_{n=1}^{\infty} \Big(\frac{\sin nx}{n^2}\Big) x^3$ must be continuous on $\big[-2|a|, 2|a|\big]$ by Proposition 5.1.4.

Nevertheless, on $\big[-2|a|, 2|a|\big]$ we find that $\Big|\frac{\sin nx}{n^2} x^3\Big| \le \frac{8|a|^3}{n^2}$, and it is clear that $\sum_{n=1}^{\infty} \frac{8|a|^3}{n^2} < \infty$. Therefore, the Weierstrass M-test implies that the series

$$\sum_{n=1}^{\infty} \Big(\frac{\sin nx}{n^2}\Big) x^3$$

converges uniformly on $\big[-2|a|, 2|a|\big]$ (which is equivalent to that $f_n \to f$ uniformly). $\square$

29. Discuss the uniform continuity of the following:

- **a.** $f(x) = x^2$, $x \in]-1, 1[$.
- **b.** $f(x) = x^{1/3}$, $x \in [0, \infty[$.
- **c.** $f(x) = e^{-x}$, $x \in [0, \infty[$.
- **d.** $f(x) = x \sin(1/x)$, $0 < x \le 1$, $f(0) = 0$.
- **e.** $f(x) = \sin[\ln(1 + x^3)]$, $-1 < x \le 1$, $f(-1) = 0$.

Sol.

- **a.** We may extend the domain of $f(x)$ by letting $f(\pm 1) = 1$. Then this extension, still denoted by f, is continuous on $[-1, 1]$. Since $[-1, 1]$ is compact, by Theorem 4.6.2, f is uniformly continuous on $[-1,1]$; thus f is uniformly continuous on $]-1,1[$.

b. The function $f(x) = x^{1/3}$ is uniformly Hölder continuous on $x \in [0,\infty[$. In fact,

$$\frac{\left|f(x)-f(y)\right|}{|x-y|^{1/3}} = \frac{|x-y|^{2/3}}{x^{2/3}+x^{1/3}y^{1/3}+y^{2/3}} \le 1 \quad \forall x,y>0.$$

Therefore, $\left|f(x)-f(y)\right| \le |x-y|^{1/3}$ for all $x,y \ge 0$ (why?). This implies that f is uniformly continuous on $[0,\infty)$.

c. The function $f(x) = e^{-x}$ is uniformly Lipschitz continuous on $[0,\infty[$ since the mean value theorem implies that

$$\left|f(x)-f(y)\right| = \left|f'(\xi)\right||x-y| \le |x-y|$$

since $|f'| \le 1$ on $[0,\infty[$. Therefore, for any given $\epsilon > 0$, $\delta = \epsilon$ will provide us the δ in the definition of uniform continuity. Therefore, f is uniformly continuous on $[0,\infty[$.

d. The function $f : [0,1] \to \mathbb{R}$ is continuous since by the squeeze theorem

$$\lim_{x\to 0} f(x) = 0 = f(0)$$

and it is obvious that f is continuous at point $x \neq 0$. By Theorem 4.6.2, f is uniformly continuous.

e. We note that f is not continuous at -1 since the limit of f as $x \to -1$ does not exist. This implies that f cannot be uniformly continuous on $[-1,1]$. □

33. Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of increasing functions on $[0,1]$, and suppose that $f_n \to 0$ pointwise. Must f_n converge uniformly? What if f_n just converges pointwise to some limit f?

Sol. The convergence is uniform since

$$\sup_{x\in[0,1]} \left|f_n(x)-0\right| \overset{\text{(why?)}}{\le} \left|f_n(0)\right| + \left|f_n(1)\right| \to 0 \quad \text{as} \quad n\to\infty.$$

Without the condition that f_n is increasing or $f=0$, then f_n might not converges uniformly. For example,

1. The sequence f_n in Exercise 8 converges pointwise to 0 but the convergence is not uniformly.

2. $f_n(x) = x^n$ which converges pointwise to $f = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$ but the convergence is not uniform. $\square$