Qualifying Exam.: Ordinary Differential Equations

Department of Mathematics, National Central University, August, 2015.

1.(15%) Let $u(t)$ and $f(t)$ be nonnegative continuous functions in a real interval $I = [a, b]$. Suppose $k(t, s)$ and $k_t(t, s)$ are continuous functions such that $k(t, s) \geq 0$ and $k_t(t, s) \leq 0$ for $t, s \in I$. Suppose

$$u(t) \leq c + \int_a^t f(s)u(s)ds + \int_a^t f(s)(\int_a^s k(s,\tau)u(\tau)d\tau)ds,\ a \leq \tau \leq s \leq t \leq b,$$

where c is a nonnegative constant. Prove that

$$u(t) \leq c[1 + \int_a^t f(s)\exp\{\int_a^s \big(f(\tau) + k(\tau,\tau)\big)d\tau\}ds],\ t \in I.$$

2. (20%) For the nonlinear system:

$$\begin{cases} x' = x - y + x^2 - xy, \\ y' = -y + x^2. \end{cases}$$

(1) determine the critical points for the equation;

(2) determine the linearized system for each critical point and discuss whether it can be used to approximate the behaviour of the non-linear system;

(3) what is the type and stability of each critical point?

3. (15%) Show that the nonlinear system

$$\begin{cases} x' = 3x - y - 4x^3 - 7xy^2, \\ y' = x + 3y - 4x^2y - 7y^3. \end{cases}$$

has at least one periodic orbit in the annulus $\sqrt{3/7} < x^2 + y^2 < \sqrt{3}/2$.

4. (20%) (1) (10%) Consider the equation:

$$(2-t)y''' + (2t-3)y'' - ty' + y = 0, \quad t < 2.$$

Find the general solution of the equation by using the particular solution e^t.

(2) (10%) Solve the following inhomogeneous initial value problem explicitly:

$$\mathbf{X}' = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} t \\ 1 \end{pmatrix}, \quad \mathbf{X}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

5. (15%) Consider the system:

$$\begin{cases} x' = x(1 - \frac{x}{2}) - 3xy, \\ y' = 2y(1 - \frac{y}{3}) - 4xy, \end{cases} \qquad x(0),\ y(0) > 0.$$

Prove that there is no periodic solution in the first quadrant.

6. (15%) Consider the system:

$$\begin{cases} x' = x - y - x^3 - xy^2, \\ y' = x + y - x^2 y - y^3, \\ z' = 2z, \end{cases}$$

Compute the Poincaré map and the Floquet's (characteristic) exponent for the periodic solution $\gamma(t) := (\cos t, \sin t, 0)$.