國立中央大學數學系105學年度博士班入學考試題〔分析〕

(1) Assume f is Lebesgue integrable on $\mathbb{R}$. Prove that $g(y) \equiv \int_{-\infty}^{\infty} f(x)e^{-(x^2y^2)}dx$ is a bounded, continuous function on $\mathbb{R}$. (10%)

(2) Let $\{f_n\}$ and f be real-valued measurable functions on a measurable set E.

(a) Give precise definitions for the following types of convergences:

$f_n \to f$ a.e., $f_n \to f$ almost uniformly, $f_n \to f$ in measure. (6%)

(b) For each pair of convergences, say, type I and type II, indicate whether type I implies type II. (6%)

(3) (a) Let $1 \leq p < q < \infty$. Determine whether one of the two spaces $L^p([0,1])$ and $L^q([0,1])$ is contained in the other, prove your answer. (10%)

(b) Do the same for ℓ^1 and ℓ^2. (10%)

(4) Determine whether the set $C([0,1]) \equiv \{f : [0,1] \to \mathbb{R} : f \text{ is continuous on } [0,1]\}$ with the metric

$$\rho(f,g) \equiv \int_0^1 |f(x) - g(x)|dx$$

is a complete metric space. Give your reasons. (10%)

(5) Let $\{f_n\}$ be a sequence of functions in $L^p([0,1])$, $1 \leq p < \infty$, which converge almost everywhere to a function f in $L^p([0,1])$. Show that $\{f_n\}$ converges to f in $L^p([0,1])$ if and only if $\|f_n\|_p \to \|f\|_p$. (15%)

(6) Let $F : (\ell^2, \|\cdot\|_2) \to \mathbb{R}$ be a bounded linear functional on ℓ^2. Find the unique element $(a_1, a_2, a_3, \cdots)$ in ℓ^2 such that

$$F((x_1, x_2, \cdots)) = \sum_{n=1}^{\infty} a_n x_n \quad \text{for any } (x_1, x_2, \cdots) \in \ell^2,$$

and prove that $\|(a_1, a_2, a_3, \cdots)\|_2 = \|F\|$. (10%)

(7) Let $C = C[0,1]$ be the normed space of all continuous real-valued functions on $[0,1]$ with the norm $\|f\|_\infty = \sup\{|f(x)| : 0 \leq x \leq 1\}$. Let

$U = \{f \in C : \|f\|_\infty \leq 1\}$;

$V = \{f \in C : f(x) > 0 \text{ for all } x \in [0,1]\}$;

$W = \{f \in C : 1 < \int_0^1 f(x)dx < 2\}$;

$X = \{f_0, f_1, f_2, f_3, \cdots\}$, where $f_n \in C$ for all n and $f_n \to f_0$ uniformly on $[0,1]$;

$Y =$ the closure of $A \equiv \{f \in C : f \text{ is differentiable on } (0,1), f(0) = 1 \text{ and } |f'| \leq 2\}$ in C;

$Z =$ the closure of $B \equiv \{p \in C : p \text{ is a polynomial}\}$ in C.

(a) Which sets are open in C? (Don't need to prove them.) (5%)

(b) Which sets are complete in C? (Don't need to prove them.) (5%)

(c) Which sets are compact in C? If the set is compact, give your proof. (13%)