# Probability Theory

(30%) 1. Let X and Y be i.i.d. random variables with continuous distribution function F

(a) Show that $\{ X = Y \}$ is a measurable set.

(b) Show that $P( X = Y ) = 0$.

(c) Give an example of two dependent random variables X,Y with the same distribution F, $P( X = Y ) = 0$.

(d) Give an example of two independent random variables X, Y with different distributions, $P( X = Y ) = 0$.

(30%) 2. Suppose X is a non-negative random variable

Show that

$$\lim_{n\to\infty} nE(\frac{1}{X}1_{[X>n]}) = 0 \text{ , and}$$

$$\lim_{n\to\infty} n^{-1}E(\frac{1}{X}1_{[X>n^{-1}]}) = 0$$

(20%) 3. Suppose that for every $\varepsilon > 0$, $\sum_n \sup_{m\geq n} P(|X_m - X_n| \geq \varepsilon) < \infty$

. Then $\{ X_n \}$ converges almost surely.

(20%) 4. (a) Suppose $\{ X_n \}$ is a sequence of random variables such that $X_n - X_m$ is $\boldsymbol{n}(0, \sigma_{mn}^2)$ distributed, where $\sigma_{mn}^2 = | m - n | / mn$. Show that $\{ X_n \}$ converges almost surely.

(b) Let X be a random variable which is independent of itself. Show that X is constant, with probability 1.