9.6 A CONTINUOUS NOWHERE DIFFERENTIABLE FUNCTION

Weierstrass presented the first example of a *continuous* function $f : \mathbb{R} \to \mathbb{R}$ that has the remarkable property that there is no point at which it is differentiable: Such a function is said to be *nowhere differentiable.* We will analyze such an example, where f is defined by an expansion

$$f(x) = \sum_{k=0}^{\infty} h_k(x) \qquad \text{for all } x$$

and the function f inherits all the nondifferentiability possessed by the individual h_k's.

We first prove a preliminary proposition regarding the construction of continuous functions as series of continuous functions.

Proposition 9.43 Suppose that $\sum_{k=0}^{\infty} c_k$ is a convergent sequence of nonnegative numbers. For each nonnegative integer k, let $h_k : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$|h_k(x)| \leq c_k \qquad \text{for all } x. \tag{9.40}$$

Define

$$f(x) = \lim_{n\to\infty} \left[\sum_{k=0}^{n} h_k(x) \right] = \sum_{k=0}^{\infty} h_k(x) \qquad \text{for all } x. \tag{9.41}$$

Then the function $f : \mathbb{R} \to \mathbb{R}$ is continuous.

Proof

The proof rests on the Cauchy Convergence Criterion for the convergence of sequences of numbers and the Weierstrass Uniform Convergence Criterion for the uniform convergence of sequences of functions.

For each number x and each natural number n, define

$$f_n(x) = \sum_{k=0}^{n} h_k(x).$$

Since each function h_k is continuous, each function f_n is also continuous. We will prove that the sequence of functions $\{f_n\}$ is uniformly Cauchy on $\mathbb{R}$. Once this is proven, it follows from the Weierstrass Uniform Convergence Criterion that $\{f_n\}$ converges uniformly on $\mathbb{R}$. Then, by Theorem 9.31, we can conclude that the limit function f, being the uniform limit of a sequence of continuous functions, is continuous.

By assumption (9.40) and the Triangle Inequality, for each index n, natural number k, and any number x,

$$\begin{aligned} |f_{n+k}(x) - f_n(x)| &= |h_{n+k}(x) + \cdots + h_{n+1}(x)| \\ &\leq |h_{n+k}(x)| + \cdots + |h_{n+1}(x)| \\ &\leq c_{n+k} + \cdots + c_{n+1}. \end{aligned} \tag{9.42}$$

The Cauchy Convergence Criterion for sequences of numbers, applied to the sequence of partial sums of the series $\sum_{k=0}^{\infty} c_k$, asserts that the sequence of partial sums is a Cauchy sequence. Thus, from the estimate (9.42) we conclude that the sequence of functions $\{f_n\}$ is uniformly Cauchy on $\mathbb{R}$. ■

We will now make a particular choice of the h_k's so that the function f defined by (9.41) fails, at each point, to be differentiable.

Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is said to be periodic, with period p, provided that

$$f(x+p) = f(x) \qquad \text{for all } x$$

Observe that if a function has period p and k is any integer, then it also has period kp.

It is convenient to introduce the following descriptive terminology: For a positive number ℓ, we define the *tent function of base length* 2ℓ to be the periodic function of period 2ℓ, $h : \mathbb{R} \to \mathbb{R}$ whose values on the interval $[-\ell, \ell]$ are defined by

$$h(x) = |x| \qquad \text{for } -\ell \leq x \leq \ell.$$

For an integer m, we call the interval $[m\ell, (m+1)\ell]$ an *interval of monotonicity* for this tent function.

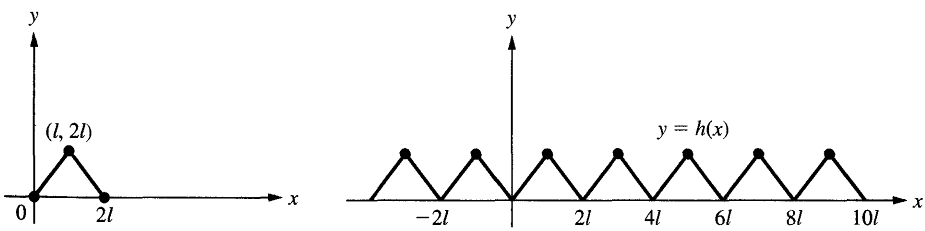

FIGURE 9.6 The tent function of base length 2ℓ.

Lemma 9.44 For $\ell > 0$, let $h : \mathbb{R} \to \mathbb{R}$ be the tent function of base length 2ℓ. Let x_0 be any number. Then either the interval $[x_0, x_0 + \ell/2]$ or the interval $[x_0 - \ell/2, x_0]$ is contained in an interval of monotonicity for the function h.

Proof

Recall Theorem 1.8, which asserts that for any number c there is a unique integer belonging to the interval $[c, c+1)$. We apply this theorem, with $c = [x_0/\ell] - 1$, to choose an integer m such that

$$[x_0/\ell] - 1 \leq m < x_0/\ell.$$

The left-hand side of this inequality yields $x_0 \leq (m+1)\ell$. This, with the right-hand side of the inequality, yields

$$m\ell < x_0 \leq (m+1)\ell.$$

Consider the midpoint z of the interval $[m\ell, (m+1)\ell]$. Since x_0 belongs to the interval $(m\ell, (m+1)\ell]$, either x_0 belongs to the left-hand interval $(m\ell, z]$ or it belongs to the right-hand interval $(z, (m+1)\ell]$. In the first case, the interval $[x_0, x_0+\ell/2]$ is contained in the interval $[m\ell, (m+1)\ell]$, while in the second case, $[x_0-\ell/2, x_0]$ is contained in the interval $[m\ell, (m+1)\ell]$. Of course, $[m\ell, (m+1)\ell]$ is an interval of monotonicity for the tent function h. ∎

We will need the following two observations regarding $h:\mathbb{R}\to\mathbb{R}$, the tent function of base length 2ℓ: If u and v belong to an interval of monotonicity for the function h, then

$$\frac{h(u)-h(v)}{u-v}=\pm 1. \tag{9.43}$$

Since h has period 2ℓ, any integer multiple of 2ℓ is also a period for h; that is, for any integer j and any number u,

$$h(u+j[2\ell])=h(u). \tag{9.44}$$

Theorem 9.45 For each nonnegative integer k, let $h_k:\mathbb{R}\to\mathbb{R}$ be the tent function of base length $2\ell_k$, where $\ell_k=(1/4)^k$. Define the function $f:\mathbb{R}\to\mathbb{R}$ by

$$f(x)=\sum_{k=1}^{\infty}h_k(x) \qquad \text{for all } x.$$

Then

i. the function f is continuous, but
ii. there is no point at which the function f is differentiable.

Proof
By the definition of tent function, for each nonnegative integer k and any number x,

$$|h_k(x)|\le \ell_k=(1/4)^k.$$

Therefore, since the Geometric Series $\sum_{k=0}^{\infty}(1/4)^k$ converges, it follows from Proposition 9.43 that $f:\mathbb{R}\to\mathbb{R}$ is continuous.

Let x_0 be any number. We will show that f is not differentiable at x_0 by choosing a sequence of numbers $\{x_n\}$, with each $x_n\ne x_0$, that converges to x_0 but for which the limit

$$\lim_{n\to\infty}\frac{f(x_n)-f(x_0)}{x_n-x_0}$$

does not exist.

Let n be a natural number. We apply Lemma 9.44, with $\ell=\ell_n$. Thus, either the interval $[x_0, x_0+\ell_n/2]$ or the interval $[x_0-\ell_n/2, x_0]$ is contained in an interval of monotonicity for the function h_n. In the first case define $x_n = x_0 - \ell_n/2$, and in the second case define $x_n = x_0 + \ell_n/2$. Hence, since the points x_0 and x_n belong to an interval of monotonicity for the function h_n, by (9.43),

$$\frac{h_n(x_n) - h_n(x_0)}{x_n - x_0} = \pm 1.$$

For $k > n$, the function $h_k : \mathbb{R} \to \mathbb{R}$ has period $2\ell_k$. Therefore, since

$$\frac{\ell_n}{2} = j[2\ell_k], \qquad \text{where } j \equiv \frac{\ell_n}{4\ell_k} = 4^{k-n-1} \text{ is a natural number,}$$

it follows from (9.44) that

$$h_k(x_0) - h_k(x_n) = h_k(x_0) - h_k(x_0 \pm j[2\ell_k]) = 0.$$

Consequently,

$$\frac{f(x_n) - f(x_0)}{x_n - x_0} = \sum_{k=0}^{\infty} \left[\frac{h_k(x_n) - h_k(x_0)}{x_n - x_0}\right] = \sum_{k=0}^{n} \left[\frac{h_k(x_n) - h_k(x_0)}{x_n - x_0}\right]. \tag{9.45}$$

On the other hand, for an integer k, $0 \le k < n$, since the ratio of base lengths, ℓ_k/ℓ_n, is a natural number, any monotonicity interval for h_n is contained in a monotonicity interval for h_k. Thus, again using (9.43),

$$\frac{h_k(x_n) - h_k(x_0)}{x_n - x_0} = \pm 1 \qquad \text{for } 0 \le k \le n.$$

We conclude that the right-hand side of (9.45) is the sum of $n + 1$ numbers each of which equals $+1$ or -1. Thus,

$$\frac{f(x_n) - f(x_0)}{x_n - x_0} = \begin{cases} \text{an odd integer} & \text{if } n \text{ is even} \\ \text{an even integer} & \text{if } n \text{ is odd.} \end{cases}$$

As a consequence, the limit

$$\lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{x_n - x_0}$$

does not exist. Thus, since the sequence $\{x_n\}$ converges to x_0, with each $x_n \neq x_0$, the function f is not differentiable at the point x_0. ■

EXERCISES FOR SECTION 9.6

1. Suppose that the function $g : \mathbb{R} \to \mathbb{R}$ has period p. Show that for each integer k, the function g also has period kp.
2. Suppose that $\{t_n\}$ is a sequence such that t_k is an odd integer if the index k is even, and an even integer if the index k is odd. Show that $\{t_n\}$ does not converge.