Problem 50

1° $$\overline{f\cdot 1_B}(x) = \begin{cases} f\cdot 1_B(x) & \text{if } x\in A \\ 0 & \text{o.w} \end{cases}$$

$$= \begin{cases} f(x) & \text{if } x\in B \\ 0 & \text{o.w.} \end{cases}$$

Let $f^{(B)}(x)$ be a function defined on B s.t $f^{(B)}(x) = f(x)\ \forall x\in B$

then $\overline{f^{(B)}(x)} = \overline{f\cdot 1_B(x)}$

Thus the set of discontinuity of $\overline{f\cdot 1_B(x)}$

equals to the set of discontinuity of $\overline{f^{(B)}(x)}$

Since $f 1_B$ is Riemann integrable over A,

the set of discontinuity of $\overline{f\cdot 1_B(x)}$ has

measure zero.

so the set of discontinuity of $\overline{f^{(B)}(x)}$ has

measure zero.

Then f is integrable over B.

2° $\forall\ \varepsilon>0$, choose a partition P on B s.t

$$U(f^{(B)}, P) < \int_B f^{(B)}(x)\,dx + \frac{\varepsilon}{2}$$

Let P' be a partition of A s.t. $P'\supseteq P$

then $\int_A f\cdot 1_B(x)\,dx \le U(f\cdot 1_B, P') = U(f\cdot 1_B, P)$
$= U(f^{(B)}, P) < \int_B f^{(B)}(x)\,dx + \frac{\varepsilon}{2}$ ①

Problem 50

p2

On the other hand,

$$\int_B f^{(B)}(x)\,dx - \frac{\varepsilon}{2} < L(f^{(B)}, P) = L(f \cdot 1_B, P)$$

$$= L(f \cdot 1_B, P')$$

$$\leq \int_A f \cdot 1_B(x)\,dx$$

Thus

$$\int_B f^{(B)}(x)\,dx - \frac{\varepsilon}{2} < \int_A f \cdot 1_B(x)\,dx < \int_B f^{(B)}(x)\,dx + \frac{\varepsilon}{2}$$

Since ε is arbitrary,

$$\int_B f^{(B)}(x)\,dx = \int_A f \cdot 1_B(x)\,dx$$

Q.E.D.

Problem 51

Let $\varphi(x) = \underline{\int_c^d} f(x,y)\,dy$

$\psi(x) = \overline{\int_c^d} f(x,y)\,dy$

$\because$ f is Riemann integrable

$\therefore$ $\varphi(x)$ and $\psi(x)$ are Riemann integrable on $[a,b]$

$\because$ $\psi(x) - \varphi(x) \geqslant 0$ and $\int_a^b \varphi(x)\,dx = \int_a^b \psi(x)\,dx$

(By Fubini's Theorem)

$\therefore$ $\left\{ x \in [a,b] \,\middle|\, \underline{\int_c^d} f(x,y)\,dy \neq \overline{\int_c^d} f(x,y)\,dy \right\}$

have measure zero

Similarly, $\left\{ y \in [c,d] \,\middle|\, \underline{\int_a^b} f(x,y)\,dx \neq \overline{\int_a^b} f(x,y)\,dx \right\}$

have measure zero.

Problem 53.

1° Fix $x \in [0,1]$

① If $x \neq \frac{k}{2^n}$ for some $n, k \in \mathbb{N}$, $k < 2^n$

then $f_x(y) = 0$, so $f_x(y)$ is integrable

② If $x = \frac{k}{2^n}$ for some $n, k \in \mathbb{N}$, $k < 2^n$

then $f_x(y) = \begin{cases} 1 & \text{if } y = \frac{l}{2^n} \text{ for some } l \in \mathbb{N} \text{ with } l < 2^n \\ 0 & \text{o.w.} \end{cases}$

Since $f_x(y)$ is discontinuous at finite number

$y = \frac{1}{2^n}, \frac{2}{2^n}, \dots, \frac{2^n - 1}{2^n}$, then

$f_x(y)$ is integrable and $\int_0^1 f_x(y)\,dy = 0$

Similarly, $\int_0^1 f_y(x)\,dx = 0 \quad \forall y \in [0,1]$

Thus $\int_0^1 \int_0^1 f(x,y)\,dx\,dy = \int_0^1 \int_0^1 f(x,y)\,dy\,dx = 0$

2° Claim: f is not Riemann integrable

It suffices to show $D := \{(x,y) \mid f(x,y) = 1\}$ is dense in $[0,1] \times [0,1]$

$\forall (a,b) \in [0,1] \times [0,1]$, $\forall \varepsilon > 0$, $\exists n > 0 \ni \frac{1}{2^n} < \frac{\varepsilon}{\sqrt{2}}$

$\because [0,1] = \bigcup_{i=0}^{2^n - 1} [\frac{i}{2^n}, \frac{i+1}{2^n}]$ $\therefore a \in [\frac{k}{2^n}, \frac{k+1}{2^n}]$ and

$b \in [\frac{l}{2^n}, \frac{l+1}{2^n}]$

for some $l, k \in \mathbb{N} \cup \{0\}$

Choose $(x,y)=\left(\frac{k}{2^n},\frac{l}{2^n}\right)$

$$\left|(a,b)-(x,y)\right|$$

$$=\left|\left(a-\frac{k}{2^n},\ b-\frac{l}{2^n}\right)\right|$$

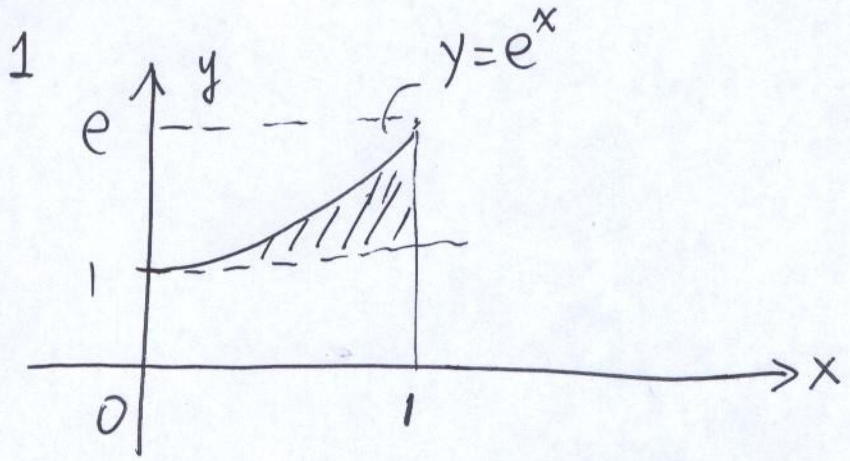
$$=\sqrt{\left(a-\frac{k}{2^n}\right)^2+\left(b-\frac{l}{2^n}\right)^2}<\sqrt{\frac{\varepsilon^2}{2}+\frac{\varepsilon^2}{2}}=\varepsilon$$

Hence f is nowhere continuous on $[0,1]\times[0,1]$,

Thus f is not Riemann integrable.

Problems4

1



2.

$$Rt=\int_1^e\int_{\ln y}^1 (x+y)\,dx\,dy$$

$u=\ln y$
$\Rightarrow du=\frac{1}{y}\,dy$

$$=\int_1^e \left(\frac{x^2}{2}+xy\right)\Big|_{\ln y}^1\,dy$$

$$=\int_1^e\left[\left(\frac{1}{2}+y\right)-\left(\frac{1}{2}(\ln y)^2+y\ln y\right)\right]dy$$

$$=\int_0^1\left[\frac{1}{2}+e^u-\frac{1}{2}u^2-e^u\cdot u\right]\cdot e^u\,du$$

$$=\int_0^1\left(\frac{1}{2}e^u+e^{2u}-\frac{1}{2}u^2e^u-e^{2u}\cdot u\right)du$$

$$=\frac{1}{2}\int_0^1 e^u\,du+\int_0^1 e^{2u}\,du-\frac{1}{2}\int_0^1 u^2e^u\,du-\int_0^1 e^{2u}\cdot u\,du$$

$$=\frac{1}{2}e^u\Big|_0^1+\frac{1}{2}e^{2u}\Big|_0^1-\frac{1}{2}\left[u^2e^u\Big|_0^1-2\underbrace{\int_0^1 e^u\cdot u\,du}_{1}\right]-\underbrace{\int_0^1 e^{2u}\cdot u\,du}_{\frac{1}{4}(e^2+1)}$$

$$=\frac{1}{2}(e-1)+\frac{1}{2}(e^2-1)-\frac{1}{2}\left[e-2\right]-\frac{1}{4}(e^2+1)$$

$$=\frac{1}{4}e^2-\frac{1}{4}=\frac{1}{4}(e^2-1)$$ ※

Problem 54

2 p2

Problem 54

2.

$$\int_0^1\left(\int_1^{e^x}(x+y)\,dy\right)dx$$

$$=\int_0^1 \left(xy+\frac{1}{2}y^2\right)\Big|_1^{e^x} dx$$

$$=\int_0^1\left(xe^x+\frac{1}{2}e^{2x}-x-\frac{1}{2}\right)dx$$

$$=\int_0^1 xe^x\,dx+\frac{1}{2}\int_0^1 e^{2x}\,dx-\int_0^1\left(x+\frac{1}{2}\right)dx$$

$$=xe^x\Big|_0^1-\int_0^1 e^x\,dx+\frac{1}{4}e^{2x}\Big|_0^1-\left(\frac{x^2}{2}+\frac{1}{2}x\right)\Big|_0^1$$

$$=e-(e-1)+\frac{1}{4}(e^2-1)-\left(\frac{1}{2}+\frac{1}{2}\right)$$

$$=1+\frac{1}{4}(e^2-1)-1=\frac{1}{4}(e^2-1)$$