Problem 2. Let (M, d) and (N, ρ) be metric spaces, $A \subseteq M$, and $f_k : A \to N$ be uniformly continuous functions, and $\{f_k\}_{k=1}^{\infty}$ converges uniformly to $f : A \to N$ on A. Show that f is uniformly continuous on A.

Proof. Let $\varepsilon > 0$ be given. Since $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f on A, there exists $N > 0$ such that

$$\sup_{x \in A} \big|f_k(x) - f(x)\big| < \frac{\varepsilon}{3} \qquad \forall\, k \geqslant N\,.$$

Since f_N is uniformly continuous on A, there exists $\delta > 0$ such that

$$\big|f_N(x) - f_N(y)\big| < \frac{\varepsilon}{3} \qquad \forall\, |x - y| < \delta \text{ and } x, y \in A\,.$$

Therefore, if $|x - y| < \delta$ and $x, y \in A$,

$$\big|f(x) - f(y)\big| \leqslant \big|f(x) - f_N(x)\big| + \big|f_N(x) - f_N(y)\big| + \big|f_N(y) - f(y)\big| < \varepsilon$$

which suggests that f is uniformly continuous on A. □

Problem 4. Complete the following.

(a) Suppose that $f_k, f, g : [0, \infty) \to \mathbb{R}$ are functions such that

1. $\forall\, R > 0$, f_k and g are Riemann integrable on $[0, R]$;
2. $|f_k(x)| \leqslant g(x)$ for all $k \in \mathbb{N}$ and $x \in [0, \infty)$;
3. $\forall\, R > 0$, $\{f_k\}_{k=1}^{\infty}$ converges to f uniformly on $[0, R]$;
4. $\displaystyle\int_0^{\infty} g(x)dx \equiv \lim_{R\to\infty} \int_0^R g(x)dx < \infty$.

Show that $\displaystyle\lim_{k\to\infty} \int_0^{\infty} f_k(x)dx = \int_0^{\infty} f(x)dx$; that is,

$$\lim_{k\to\infty} \lim_{R\to\infty} \int_0^R f_k(x)dx = \lim_{R\to\infty} \lim_{k\to\infty} \int_0^R f_k(x)dx\,.$$

(b) Let $f_k(x)$ be given by $f_k(x) = \begin{cases} 1 & \text{if } k-1 \leqslant x < k\,, \\ 0 & \text{otherwise.} \end{cases}$ Find the (pointwise) limit f of the sequence $\{f_k\}_{k=1}^{\infty}$, and check whether $\displaystyle\lim_{k\to\infty} \int_0^{\infty} f_k(x)dx = \int_0^{\infty} f(x)dx$ or not. Briefly explain why one can or cannot apply (a).

(c) Let $f_k : [0, \infty) \to \mathbb{R}$ be given by $f_k(x) = \dfrac{x}{1 + kx^4}$. Find $\displaystyle\lim_{k\to\infty} \int_0^{\infty} f_k(x)dx$.

Proof. 1. First we note that since $|f_k(x)| \leqslant g(x)$ for all $x \in [0, \infty)$, $|f(x)| \leqslant g(x)$ for all $x \in [0, \infty)$. Let $\varepsilon > 0$ be given. Since $\displaystyle\int_0^{\infty} g(x)dx < \infty$, there exists $L > 0$ such that

$$\int_L^{\infty} g(x)dx \equiv \int_0^{\infty} g(x)dx - \int_0^L g(x)dx = \lim_{R\to\infty} \int_L^R g(x)dx < \frac{\varepsilon}{3}\,.$$

Let $I_n = \displaystyle\int_0^n g(x)dx$ and $F_n \equiv \displaystyle\int_0^n f(x)dx$. Since $\{I_n\}_{n=1}^\infty$ is increasing and convergent, $\{I_n\}_{n=1}^\infty$ is Cauchy; thus $\{F_n\}_{n=1}^\infty$ is also Cauchy since

$$|F_n - F_m| \leqslant |I_n - I_m| \,.$$

Therefore, $\displaystyle\lim_{n\to\infty}\int_0^n f(x)dx$ exists. Denote the limit by F. We then have

$$|F_n - F| = \lim_{m\to\infty} |F_n - F_m| \leqslant \frac{\varepsilon}{3} \qquad \text{if } \ n \geqslant L.$$

Claim: $\displaystyle\lim_{R\to\infty}\int_0^R f(x)dx = F$.

Proof of claim: Since

$$\int_0^R f(x)dx = \int_0^{[R]} f(x)dx + \int_{[R]}^R f(x)dx \,,$$

we find that for $R \geqslant L$,

$$\begin{aligned}
\Big|\int_0^R f(x)dx - F\Big| &= \Big|\int_0^R f(x)dx - \int_0^{[R]} f(x)dx + \int_0^{[R]} f(x)dx - F\Big| \\
&\leqslant \int_{[R]}^R \big|f(x)\big|dx + \Big|\int_0^{[R]} f(x)dx - F\Big| \\
&\leqslant \int_{[R]}^R \big|g(x)\big|dx + \big|F_{[R]} - F\big| \leqslant \frac{2}{3}\varepsilon < \varepsilon \,.
\end{aligned}$$

Therefore, $\displaystyle\lim_{R\to\infty}\int_0^R f(x)dx = F$. Similarly, $\displaystyle\lim_{R\to\infty}\int_0^R f_k(x)dx$ exists.

For this particular L, since $\{f_k\}_{k=1}^\infty$ converges uniformly to f on $[0, L]$,

$$\lim_{k\to\infty}\int_0^L f_k(x)dx = \int_0^L f(x)dx \,;$$

thus there exists $N > 0$ such that

$$\Big|\int_0^L \big(f_k(x) - f(x)\big)dx\Big| < \frac{\varepsilon}{3} \qquad \forall\, k \geqslant N \,.$$

As a consequence, if $k \geqslant N$,

$$\begin{aligned}
\Big|\int_0^\infty \big(f_k(x) - f(x)\big)dx\Big| &= \lim_{R\to\infty}\Big|\int_0^R \big(f_k(x) - f(x)\big)dx\Big| \\
&\leqslant \Big|\int_0^L \big(f_k(x) - f(x)\big)dx\Big| + \lim_{R\to\infty}\int_L^R \big(|f_k(x)| + |f(x)|\big)dx \\
&\leqslant \Big|\int_0^L \big(f_k(x) - f(x)\big)dx\Big| + 2\int_L^\infty g(x)dx < \varepsilon \,.
\end{aligned}$$

2. We show that the pointwise limit of $\{f_k\}_{k=1}^\infty$ is zero. Let $a \in \mathbb{R}$. Then $a < N - 1$ for some $N = N(a) \in \mathbb{N}$. Therefore,

$$\big|f_k(a) - 0\big| = 0 \qquad \forall\, k \geqslant N \,.$$

Clearly this convergence is not uniform (since the number N depends crucially on a). However, for any $R > 0$, $\{f_k\}_{k=1}^{\infty}$ converges uniformly to 0 on $[0, R]$.

Moreover,

$$\int_0^{\infty} f_k(x)dx = 1 \quad \text{but} \quad \int_0^{\infty} f(x)dx = 0\,;$$

thus $\lim\limits_{k\to\infty} \int_0^{\infty} f_k(x)dx \neq \int_0^{\infty} f(x)dx$.

We cannot apply (a) since we cannot find a function g satisfying 2 and 4 simultaneously.

3. Let $x_k \in [0,\infty)$ be such that $f_k(x_k) = \sup\limits_{x\in[0,\infty)} f_k(x)$. In fact, since

$$f_k'(x) = \frac{1 + kx^4 - 4kx^4}{(1+kx^4)^2} = \frac{1 - 3kx^4}{(1+kx^4)^2} \quad \text{and} \quad \lim_{x\to\infty} f_k(x) = 0\,;$$

$x_k = (3k)^{-\frac{1}{4}}$. Therefore,

$$0 \leqslant f_k(x) \leqslant f_k(x_k) = \frac{3}{4}(3k)^{-\frac{1}{4}}$$

which suggests that $\{f_k\}_{k=1}^{\infty}$ converges uniformly to 0 on $[0,\infty)$. Moreover, letting $g(x) = \dfrac{x}{1+x^4}$, we have $\big|f_k(x)\big| \leqslant g(x)$ for all $k \in \mathbb{N}$ and $x \in [0,\infty)$, and if $R > 1$,

$$\begin{aligned} I_R \equiv \int_0^R g(x)dx &\leqslant \int_0^1 g(x)dx + \int_1^R g(x)dx \leqslant \int_0^1 g(x)dx + \int_1^R \frac{1}{x^3}dx \\ &= \int_0^1 g(x)dx - \frac{1}{2}R^{-2} + \frac{1}{2} \leqslant \int_0^1 g(x)dx + \frac{1}{2} < \infty\,. \end{aligned}$$

Since I_R is increasing in R and bounded above, $\lim\limits_{R\to\infty} I_R$ must exist which ensures that $\int_0^{\infty} g(x)dx < \infty$. Therefore, we can apply (a) to conclude that

$$\lim_{k\to\infty} \int_0^{\infty} f_k(x)dx = \int_0^{\infty} 0dx = 0\,.$$

□

Problem 6. Let (M, d) be a metric space, and $K \subseteq M$ be a compact subset.

1. Show that the set $U = \big\{f \in \mathscr{C}(K;\mathbb{R}) \,\big|\, a < f(x) < b \text{ for all } x \in K\big\}$ is open in $\big(\mathscr{C}(K;\mathbb{R}), \|\cdot\|_\infty\big)$ for all $a, b \in \mathbb{R}$.

2. Show that the set $F = \big\{f \in \mathscr{C}(K;\mathbb{R}) \,\big|\, a \leqslant f(x) \leqslant b \text{ for all } x \in K\big\}$ is closed in $\big(\mathscr{C}(K;\mathbb{R}), \|\cdot\|_\infty\big)$ for all $a, b \in \mathbb{R}$.

3. Let $A \subseteq M$ be a subset, not necessarily compact. Prove or disprove that the set $B = \big\{f \in \mathscr{C}_b(A;\mathbb{R}) \,\big|\, f(x) > 0 \text{ for all } x \in A\big\}$ is open in $\big(\mathscr{C}_b(A;\mathbb{R}), \|\cdot\|_\infty\big)$.

Proof. 1. 參考共筆 Example 5.50.

2. 參考共筆 Example 5.51.

3. 鄭老師班上課筆記。 □

Problem 8. Let (M,d) be a metric space, $(\mathcal{V},\|\cdot\|)$ be a normed space, and $A \subseteq M$ be a subset. Suppose that $B \subseteq \mathscr{C}_b(A;\mathcal{V})$ be equi-continuous. Prove or disprove that $\text{cl}(B)$ is equi-continuous.

Proof. We prove that $\text{cl}(B)$ is equi-continuous. Let $\varepsilon > 0$ be given. Since B is equi-continuous, there exists $\delta > 0$ such that

$$\|f(x) - f(y)\| < \frac{\varepsilon}{3} \qquad \text{if } d(x,y) < \delta,\ x,y \in A \text{ and } f \in B.$$

If $g \in \text{cl}(B)$, there exists $f \in B$ such that $\|g - f\|_\infty < \dfrac{\varepsilon}{3}$; thus if $g \in \text{cl}(B)$ and $d(x,y) < \delta$ with $x,y \in A$,

$$\|g(x) - g(y)\| \leqslant \|g(x) - f(x)\| + \|f(x) - f(y)\| + \|f(y) - g(y)\| < \varepsilon\,.$$

Therefore, $\text{cl}(B)$ is equi-continuous. □

Problem 10. Assume that $\{f_k\}_{k=1}^\infty$ is a sequence of monotone increasing functions on $\mathbb{R}$ with $0 \leqslant f_k(x) \leqslant 1$ for all x and $k \in \mathbb{N}$.

1. Show that there is a subsequence $\{f_{k_j}\}_{j=1}^\infty$ which converges **pointwise** to a function f (This is usually called the Helly selection theorem).
2. If the limit f is continuous, show that $\{f_{k_j}\}_{j=1}^\infty$ converges uniformly to f on any compact set of $\mathbb{R}$.

Proof. 1. We divide the proof into the following 7 steps.

(a) By the diagonal process (Lemma 5.63 in the new lecture note), we can find a subsequence $\{f_{k_j}\}_{j=1}^\infty$ that converges pointwise to some $\widetilde{f}$ on $\mathbb{Q}$.

(b) Since $f_{k_j}(x) \leqslant f_{k_j}(y)$ if $x \leqslant y$, passing j to the limit we obtain that

$$\widetilde{f}(x) \leqslant \widetilde{f}(y) \quad \text{if } x \leqslant y \text{ and } x,y \in \mathbb{Q}.$$

In other words, $\widetilde{f}$ is an increasing function on $\mathbb{Q}$.

(c) For each $x \in \mathbb{R}$, define $f(x) = \sup\limits_{r \leqslant x, r \in \mathbb{Q}} \widetilde{f}(r) = \lim\limits_{\substack{r \to x^- \\ r \in \mathbb{Q}}} \widetilde{f}(r)$. Then due to the monotonicity of $\widetilde{f}$, $f = \widetilde{f}$ on $\mathbb{Q}$, and f is also monotone increasing. Let S be the collection of all discontinuity of f; that is,

$$S = \big\{x \in \mathbb{R} \,\big|\, f \text{ is not continuous at } x\big\}\,.$$

Then for each $x \in S$, $a_x \equiv \lim\limits_{y \to x^-} f(y) = \sup\limits_{y<x} f(y)$ and $b_x \equiv \lim\limits_{y \to x^+} f(y) = \inf\limits_{y>x} f(y)$ exist and $a_x < b_x$. In other words, each discontinuity x of f corresponds to an interval $[a_x, b_x]$, and by the monotonicity of f, $[a_x,b_x] \cap [a_y,b_y] = \varnothing$ if $x,y \in S$ and $x \neq y$. Since each $[a_x,b_x]$ contains at least one rational number q_x and $q_x \neq q_y$ if $x,y \in X$ and $x \neq y$, S must be countable.

(d) Since $\mathbb{Q}\cup S$ is countable, by the diagonal process again there exists a subsequence $\{f_{k_{j_\ell}}\}_{\ell=1}^{\infty}$ that converges pointwise to some $\bar{f}$ on $\mathbb{Q}\cup S$.

(e) For each $x\in\mathbb{R}$, define $g(x)=\sup\limits_{s\leqslant x, s\in\mathbb{Q}\cup S}\bar{f}(s)=\lim\limits_{\substack{s\to x^-\\ s\in\mathbb{Q}\cup S}}\bar{f}(s)$. As in (c), $g=\bar{f}$ on $\mathbb{Q}\cup S$, and g is monotone increasing. Our goal is showing that $\{f_{k_{j_\ell}}\}_{\ell=1}^{\infty}$ converges pointwise to g on $\mathbb{R}$. However, since $\mathbb{R}=(\mathbb{Q}\cup S)\cup(\mathbb{Q}\cup S)^{\complement}$, $\{f_{k_{j_\ell}}\}_{\ell=1}^{\infty}$ converges to $\bar{f}$ on $\mathbb{Q}\cup S$, and $g=\bar{f}$ on $\mathbb{Q}\cup S$, it suffices to show that $\{f_{k_{j_\ell}}(a)\}_{\ell=1}^{\infty}$ converges to $g(a)$ if f is continuous at a.

(f) Claim: if f is continuous at a, then $g(a)=f(a)$.

Proof of claim: It should be clear that $g(x)\geqslant f(x)$ for all $x\in\mathbb{R}$ since $\bar{f}=\tilde{f}$ on $\mathbb{Q}$. Now suppose that $\sup\limits_{r\leqslant a, r\in\mathbb{Q}}\tilde{f}(r)=f(a)<g(a)=\sup\limits_{s\leqslant a, s\in\mathbb{Q}\cup S}\bar{f}(s)$ for some $a\notin S$. Then $\exists\, s_n\in S$, $s_n\nearrow a$ (this means $s_n<a$ and $s_n\to a$ as $n\to\infty$) such that $\lim\limits_{\ell\to\infty}f_{k_{j_\ell}}(s_n)=\bar{f}(s_n)>g(a)-\dfrac{1}{n}$. We note that since $a\notin S$ and $s_n\in S$, $s_n\neq a$ for all $n\in\mathbb{N}$. Now, for each fixed $s_n<a$, for all $r\in(s_n,a]\cap\mathbb{Q}$, by the monotonicity of $f_{k_{j_\ell}}$, $f_{k_{j_\ell}}(r)\geqslant f_{k_{j_\ell}}(s_n)$. Therefore,

$$\tilde{f}(r)=\lim_{\ell\to\infty}f_{k_{j_\ell}}(r)\geqslant\lim_{\ell\to\infty}f_{k_{j_\ell}}(s_n)>g(a)-\frac{1}{n};$$

thus $f(a)=\sup_{r\leqslant a, r\in\mathbb{Q}}\tilde{f}(r)\geqslant g(a)-\dfrac{1}{n}$. Since this holds for all $n\in\mathbb{N}$, we must have $f(a)\geqslant g(a)$ which is a contradiction.

(g) By (e) and (f), we only need to show that $\{f_{k_{j_\ell}}(a)\}_{\ell=1}^{\infty}$ converges to $f(a)$ if f is continuous at a. Suppose that f is continuous at a, and $\varepsilon>0$ be given. Then $\exists\,\delta>0$ such that

$$\big|f(x)-f(a)\big|<\frac{\varepsilon}{6}\qquad\text{if }\ |x-a|<\delta\,.$$

Let $r,s\in(a-\delta,a+\delta)\cap\mathbb{Q}$ such that $s<x<r$. Since $f_{k_{j_\ell}}(r)\to\bar{f}(r)$ and $f_{k_{j_\ell}}(s)\to\bar{f}(s)$ as $\ell\to\infty$, $\exists N>0$ such that

$$\big|f_{k_{j_\ell}}(r)-f(r)\big|<\frac{\varepsilon}{6}\text{ and }\big|f_{k_{j_\ell}}(s)-f(s)\big|<\frac{\varepsilon}{6}\quad\forall\,\ell\geqslant N\,.$$

Therefore, if $\ell\geqslant N$,

$$\begin{aligned}\big|f_{k_{j_\ell}}(x)-f(x)\big|&\leqslant\big|f_{k_{j_\ell}}(x)-f_{k_{j_\ell}}(r)\big|+\big|f_{k_{j_\ell}}(r)-f(r)\big|+\big|f(r)-f(x)\big|\\&\leqslant f_{k_{j_\ell}}(r)-f_{k_{j_\ell}}(s)+\big|f_{k_{j_\ell}}(r)-f(r)\big|+\big|f(r)-f(x)\big|\\&\leqslant f(r)-f(s)+|f(s)-f_{k_{j_\ell}}(s)|+2\big|f_{k_{j_\ell}}(r)-f(r)\big|+\big|f(r)-f(x)\big|\\&\leqslant\big|f(x)-f(s)\big|+|f(s)-f_{k_{j_\ell}}(s)|+2\big|f_{k_{j_\ell}}(r)-f(r)\big|+2\big|f(r)-f(x)\big|\\&<6\cdot\frac{\varepsilon}{6}=\varepsilon.\end{aligned}$$

2. Let $\varepsilon>0$ be given, and $K\subseteq\mathbb{R}$ be a compact set. Choose $a,b\in\mathbb{R}$ so that $K\subseteq[a,b]$. Since $[a,b]$ is compact and $f:[a,b]\to\mathbb{R}$ is continuous, f is uniformly continuous on

$[a,b]$. Therefore, there exists $\delta > 0$ such that

$$\big|f(x) - f(y)\big| < \frac{\varepsilon}{2} \qquad \text{if } \ |x-y| < \delta \text{ and } x, y \in [a,b].$$

Choose $n \in \mathbb{N}$ so that $\dfrac{b-a}{n} < \delta$. Divide $[a,b]$ into n sub-intervals $[x_i, x_{i+1}]$, $i = 0, \cdots, n-1$, where $x_i = a + \dfrac{i(b-a)}{n}$. Then

$$\big|f(x) - f(y)\big| < \frac{\varepsilon}{2} \qquad \text{if } \ x, y \in [x_i, x_{i+1}].$$

Moreover, since $\{f_{k_j}\}_{k=1}^{\infty}$ converges pointwise to f on $[a,b]$, there exists $N_i > 0$ such that

$$\big|f_{k_j}(x_i) - f(x_i)\big| < \frac{\varepsilon}{2} \qquad \forall\, j \geqslant N_i\,.$$

Let $N = \max\{N_1, \cdots, N_n\}$. Then for $x \in [x_i, x_{i+1}]$ and $j \geqslant N$, if $f_{k_j}(x) - f(x) \geqslant 0$,

$$\begin{aligned} \big|f_{k_j}(x) - f(x)\big| &= f_{k_j}(x) - f(x) \leqslant f_{k_j}(x_{i+1}) - f(x_i) \\ &\leqslant f_{k_j}(x_{i+1}) - f(x_{i+1}) + f(x_{i+1}) - f(x_i) < \varepsilon\,, \end{aligned}$$

while if $f_{k_j}(x) - f(x) \leqslant 0$,

$$\begin{aligned} \big|f_{k_j}(x) - f(x)\big| &= f(x) - f_{k_j}(x) \leqslant f(x_{i+1}) - f_{k_j}(x_i) \\ &\leqslant f(x_{i+1}) - f(x_i) + f(x_i) - f_{k_j}(x_i) < \varepsilon. \end{aligned}$$

As a consequence,

$$\big|f_{k_j}(x) - f(x)\big| < \varepsilon \qquad \forall\, j \geqslant N \text{ and } x \in [a,b]\,;$$

thus $\{f_{k_j}\}_{j=1}^{\infty}$ converges uniformly to f on $[a,b]$. □

Problem 12. Let (M,d) be a metric space, $K \subseteq M$ be a compact subset, and $\Phi : K \to K$ be such that $d\big(\Phi(x), \Phi(y)\big) < d(x,y)$ for all $x, y \in K$, $x \neq y$.

1. Show that Φ has a unique fixed-point.

2. Show that 1 is false if K is not compact.

Proof. 1. Let $f(x) = d(\Phi(x), x)$. Then if $x \neq y$,

$$\begin{aligned} \big|f(x) - f(y)\big| &= \big|d(\Phi(x), x) - d(\Phi(y), y)\big| \\ &\leqslant \big|d(\Phi(x), x) - d(\Phi(y), x)\big| + \big|d(\Phi(y), x) - d(\Phi(y), y)\big| \\ &\leqslant d(\Phi(x), \Phi(y)) + d(x,y) < 2d(x,y)\,; \end{aligned}$$

thus f is continuous on K. Suppose that Φ has no fixed-point on K. Then $\Phi(x) \neq x$ for all $x \in K$; thus $f(x) > 0$ for all $x \in K$. The Min-Max Theorem suggests that there exists $x_0 \in K$ such that

$$f(x_0) = \min_{x \in K} f(x)\,.$$

For this particular x_0, we also have $\Phi(x_0) \neq x_0$; thus

$$f(\Phi(x_0)) = d(\Phi(\Phi(x_0)), \Phi(x_0)) < d(\Phi(x_0), x_0) = f(x_0)$$

which implies that $f(x_0)$ is not the minimum of f on K, a contradiction.

2. 見共筆 Example 5.79.

□

Problem 14. Put $p_0 = 0$ and define

$$p_{k+1}(x) = p_k(x) + \frac{x^2 - p_k^2(x)}{2} \qquad \forall\, k \in \mathbb{N} \cup \{0\}\,.$$

Show that $\{p_k\}_{k=1}^{\infty}$ converges uniformly to $|x|$ on $[-1, 1]$.
Hint: Use the identity

$$|x| - p_{k+1}(x) = \big[|x| - p_k(x)\big]\Big[1 - \frac{|x| + p_k(x)}{2}\Big]$$

to prove that $0 \leqslant p_k(x) \leqslant p_{k+1}(x) \leqslant |x|$ if $|x| \leqslant 1$, and that

$$|x| - p_k(x) \leqslant |x|\Big(1 - \frac{|x|}{2}\Big)^k < \frac{2}{k+1}$$

if $|x| \leqslant 1$.

Proof. We first show by induction that for all $k \in \mathbb{N} \cup \{0\}$,

$$0 \leqslant p_k(x) \leqslant |x| \qquad \forall\, x \in [-1, 1]\,. \tag{$\star$}$$

It is clear that $(\star)$ holds for $k = 0$. Now suppose that $(\star)$ holds for $0 \leqslant p_n(x) \leqslant |x|$. Then $|x| - p_n(x) \geqslant 0$ and $1 - \dfrac{|x| + p_n(x)}{2} \geqslant 1 - |x| \geqslant 0$ on $[-1, 1]$; thus using the identity

$$|x| - p_{n+1}(x) = \big[|x| - p_n(x)\big]\Big[1 - \frac{|x| + p_n(x)}{2}\Big] \tag{$\star\star$}$$

we conclude that $|x| - p_{n+1}(x) \geqslant 0$. By induction, $(\star)$ holds for all $k \in \mathbb{N} \cup \{0\}$.

Now we can conclude from $(\star\star)$ that

$$\begin{aligned} 0 \leqslant |x| - p_k(x) &\leqslant \big[|x| - p_{k-1}(x)\big]\big(1 - \frac{|x|}{2}\big) \leqslant \big[|x| - p_{k-2}(x)\big]\big(1 - \frac{|x|}{2}\big)^2 \\ &\leqslant \cdots\cdots \leqslant |x|\big(1 - \frac{|x|}{2}\big)^k < \frac{2}{k+1}\,. \end{aligned}$$

As a consequence, $\{p_k\}_{k=1}^{\infty}$ converges uniformly to $p(x) = |x|$. □