Problem 43.

1°

$$0 = F(x, y),$$

$$0 = F_x(x, y) + F_y(x, y) \cdot f'(x) \Rightarrow f'(x) = -\frac{F_x}{F_y}$$

$$0 = F_{xx}(x, y) + F_{xy} f'(x) + \left(F_{yx} + F_{yy} f'(x)\right) f'(x)$$

$$+ F_y(x, y) \cdot f''(x)$$

$$= F_{xx} + F_{xy}\left(-\frac{F_x}{F_y}\right) + \left(F_{yx} - F_{yy}\frac{F_x}{F_y}\right)\left(-\frac{F_x}{F_y}\right)$$

$$+ F_y \cdot f''(x)$$

$$\Rightarrow f''(x) = \frac{-F_{xx}}{F_y} + \frac{F_x F_{xy}}{F_y^2} + \frac{F_x F_{yx}}{F_y^2} - \frac{F_x^2 F_{yy}}{F_y^3}$$

2°

$$0 = F(x_1, x_2, y)$$

$$\frac{\partial}{\partial x_1}: \quad F_{x_1} + F_y \frac{\partial y}{\partial x_1} = 0 \Rightarrow y_{x_1} = -\frac{F_{x_1}}{F_y}$$

$$\frac{\partial}{\partial x_2}: \quad F_{x_2} + F_y \frac{\partial y}{\partial x_2} = 0 \Rightarrow y_{x_2} = -\frac{F_{x_2}}{F_y}$$

$$\frac{\partial^2}{\partial x_1^2}: \quad \left(F_{x_1 x_1} + F_{x_1 y}\frac{\partial y}{\partial x_1}\right) + \left(F_{y x_1} + F_{yy}\, y_{x_1}\right) \cdot y_{x_1}$$

$$+ F_y\, y_{x_1 x_1} = 0$$

$$\Rightarrow F_{x_1 x_1} - \frac{F_{x_1} F_{x_1 y}}{F_y} + \left(F_{y x_1} - \frac{F_{x_1} F_{yy}}{F_y}\right) \cdot \left(-\frac{F_{x_1}}{F_y}\right) + F_y\, y_{x_1 x_1} = 0$$

$$y_{x_1x_1} = -\frac{F_{x_1x_1}}{F_y} + \frac{F_{x_1}F_{x_1y}}{F_y^2} + \left(F_{yx_1} - \frac{F_{x_1}F_{yy}}{F_y}\right)\cdot\frac{F_{x_1}}{F_y^2}$$

$$\frac{\partial^2}{\partial x_2^2}: \left(F_{x_2x_2} + F_{x_2y}\,y_{x_2}\right) + \left(F_{yx_2} + F_{yy}\,y_{x_2}\right)\cdot y_{x_2} + F_y\,y_{x_2x_2} = 0$$

$$\Rightarrow y_{x_2x_2} = -\frac{F_{x_2x_2}}{F_y} - \frac{F_{x_2y}}{F_y}\cdot\left(-\frac{F_{x_2}}{F_y}\right) - F_{yx_2}\cdot\left(-\frac{F_{x_2}}{F_y}\right)$$

$$- F_{yy}\cdot\left(-\frac{F_{x_2}}{F_y}\right)^2$$

$$= -\frac{F_{x_2x_2}}{F_y} + \frac{F_{x_2y}F_{x_2}}{F_y^2} + \frac{F_{x_2}F_{yx_2}}{F_y} - \frac{F_{yy}F_{x_2}^2}{F_y^2}$$

$$\frac{\partial^2}{\partial x_1\partial x_2}: F_{x_1x_2} + F_{x_1y}\,y_{x_2} + \left(F_{yx_2} + F_{yy}\cdot y_{x_2}\right)y_{x_1} + F_y\,y_{x_1x_2} = 0$$

$$\Rightarrow F_{x_1x_2} + F_{x_1y}\left(-\frac{F_{x_2}}{F_y}\right) + \left(F_{yx_2} - \frac{F_{x_2}F_{yy}}{F_y}\right)\left(-\frac{F_{x_1}}{F_y}\right)$$

$$+ F_y\,y_{x_1x_2} = 0$$

$$\Rightarrow y_{x_1x_2} = -\frac{F_{x_1x_2}}{F_y} + \frac{F_{x_1y}F_{x_2}}{F_y^2} + \frac{F_{yx_2}F_{x_1}}{F_y^2} - \frac{F_{x_2}F_{x_1}F_{yy}}{F_y^3}$$

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Problem 44

1. $$L(f,P) = \sum_{\substack{0 \le i \le m-1 \\ 0 \le j \le n-1}} \inf_{(x,y) \in \Delta_{ij}} \tilde{f}(x,y)\, \mathbb{A}(\Delta_{ij})$$

$$\le \sum_{\substack{0 \le i \le m-1 \\ 0 \le j \le n-1}} \sup_{(x,y) \in \Delta_{ij}} \tilde{f}(x,y)\, \mathbb{A}(\Delta_{ij})$$

$$\le M \sum_{\substack{0 \le i \le m-1 \\ 0 \le j \le n-1}} \mathbb{A}(\Delta_{ij}) = M\nu(A)$$

and $$L(f,P) \geq \sum_{\substack{0 \le i \le m-1 \\ 0 \le j \le n-1}} m\, \mathbb{A}(\Delta_{ij}) = m\nu(A)$$

Thus $m\nu(A) \le L(f,\mathcal{P}) \le U(f,\mathcal{P}) \le M\nu(A)$

Problem 45

1. A is a set of volume zero

$\Rightarrow$ A is a set of measure zero

By Proposition 8.11, if A is a set of volume zero, $\forall \varepsilon > 0$, $\exists$ finite (open) rectangles $S_1, S_2, \dots, S_N$ such that

$$A \subseteq \bigcup_{k=1}^{N} S_k \quad \text{and} \quad \sum_{k=1}^{N} V(S_k) < \varepsilon$$

Hence A is also a set of measure zero by definition.

A is a set of measure zero

$\Rightarrow$ A is a set of volume zero

The statement is false, for example the real line $\mathbb{R} \times \{0\}$ on $\mathbb{R}^2$ is a set of measure zero, but is not of volume zero.

Problem45

Problem46

Problem 45

3. Suppose $[a,b]\setminus A$ has measure zero,

then $[a,b] = A \cup ([a,b]\setminus A)$ has measure zero $\rightarrow\leftarrow$

Problem 46

False, for example $A = [0,1] \cap \mathbb{Q}$

$$f(x) = 1 \text{ on } A$$

Clearly $f(x)$ is bounded and continuous on A,

$\bar{f}(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{o.w.} \end{cases}$, the set of

discontinuity of $\bar{f}$ is not measure zero,

then f is not Riemann integrable.