NAME:____________ ID NO.:_____________________ CLASS: __________

Problem 1: (15 points) Show that $f(z) = \frac{\mathrm{Log}(z+5i)}{z^2+3z+2}$ is analytic everywhere except at the points $-1, -2$, and on the ray $\{(x, y)|x \le 0, y = -5\}$.

Proof. First, the function $\frac{1}{z^2+3z+2}$ is analytic everywhere except at -1, and -2. Second, the function $\mathrm{Log}(z + 5i)$ is analytic except at those points where $z + 5i = 0$, or where $z + 5i$ lies on the negative real axis, i.e. $\{(x; y)|x \le 0, y = -5\}$. Hence the function $f(z) = \frac{\mathrm{Log}(z+5i)}{z^2+3z+2}$ is analytic everywhere except at the points $-1, -2$, and on the ray $\{(x, y)|x \le 0, y = -5\}$. □

Problem 2:

(1) **(10 points)** Show that $\tanh^{-1} z = \frac{1}{2} \log\left(\frac{1+z}{1-z}\right)$.

Hint. Consider $z = \tanh w = \frac{(e^w - e^{-w})}{(e^w + e^{-w})}$. Then write w as a function of z. □

(2) **(5 points)** Compute $\tanh^{-1}(1 + 2i)$.

Solution. $\frac{1}{4} \ln 2 + i(\frac{3}{8} + n)\pi$, where n is an integer. □

Problem 3:(10 points) Determine whether $\log(i^2)$ and $2 \log i$ are equal or not when the branch $\log z = \ln r + i\theta (r > 0, \frac{3\pi}{4} < \theta < \frac{11\pi}{4})$ is used. Show your work.

Solution. $\log(i^2) = \log(-1) = i\pi$ and $2 \log i = 2(i\frac{10\pi}{4}) = i\frac{5\pi}{2}$ when the stated branch of the logarithmic function is used. Hence $\log(i^2) \neq 2 \log i$. □

Problem 3: (15 points) Consider $I = \int_C \frac{1}{z^3(z+4)} dz$.

(1) Evaluate the integral I when the contour C is the positively oriented circle $|z| = 2$.

Hint. First we notice that $\frac{1}{z+4}$ is analytic inside and on C. Then we use the Extension of Cauchy Integral Formula. $\cdots$ The solution is $\frac{\pi i}{32}$. □

(2) Evaluate the integral I when the contour C is the positively oriented circle $|z+2| = 3$.

Hint. The singularities 0 and -4 of $\frac{1}{z^3(z+4)}$ are inside the contour C. We consider the contour C_1 the positively oriented circle $|z| = \frac{1}{2}$ and the contour C_2 the positively oriented circle $|z + 4| = \frac{1}{2}$. Then C_1 and C_2 are inside C. So we can apply the Cauchy-Goursat theorem for multiply connected domain. $\cdots$ The solution is 0 □

Problem 4:(10 points) Consider the function $f(z) = (z-2)^4$ and the closed rectangular region R with vertices at $0, -1, -1+4i, 4i$. Find the points on or inside R at which $|f(z)|$ attains its maximum and minimum values.

Solution. Observe that $|z-2| = d$ is the distance from z to 2 and $|(z-2)^4| = |z-2|^4 = d^4$. The maximum (minimum) modulus principle implies that the maximum and minimum values occur at the boundary. From the geometry, we can see that the maximum and minimum values of d, and therefore $|f(z)|$, occur at the boundary points, namely $-1+4i$ and 0. Hence $\max |f(z)|$ occurs at $z = -1+4i$ and $\min |f(z)|$ occurs at $z = 0$. □

Problem 5:(10 points) Let f be an entire function with the property that $|f(z)| \geq 1$ for all z. Show that f is constant.

Hint. Liouville's theorem. □

Problem 6: (15 points) Compute $\int_C \frac{1}{z^2-z}dz$, where C is a line segment from 2 to $2+i$.

Proof. $\frac{1}{z^2-z} = -\frac{1}{z} + \frac{1}{z-1}$ is analytic everywhere except at 0 and 1. Let $D = \mathbb{C}\backslash\{(x,y)|x \leq 1, y = 0\}$. Then $\frac{1}{z^2-z} = -\frac{1}{z} + \frac{1}{z-1}$ is analytic in D and $-\operatorname{Log} z + \operatorname{Log}(z-1)$ is an antiderivative of $\frac{1}{z^2-z} = -\frac{1}{z} + \frac{1}{z-1}$. By the extension of fundamental theorem of calculus, $\int_C \frac{1}{z^2-z}da = \int_2^{2+i} \frac{1}{z^2-z}dz = (-\operatorname{Log} z + \operatorname{Log}(z-1))|_2^{2+i} = -\frac{1}{2}\ln 5 + \frac{3}{2}\ln 2 + i(\arctan\frac{1}{2} + \frac{1}{4}\pi)$. □

Problem 7: (10 points)

Let C_R denote the upper half of the circle $|z| = R (R > 2)$, taken in the counterclockwise direction. Use ML bounds to find an upper bound on the modulus of the following contour integral $\int_{C_R} \frac{2z^2-1}{z^4+5z^2+4}dz$.

Hint. $M = \frac{(2R^2+1)}{(R^2-1)(R^2-4)}$ and $L = \pi R$. The upper bound is $\frac{\pi R(2R^2+1)}{(R^2-1)(R^2-4)}$ □