Complex Variables I — Practice Final Exam

Name:_____________ Id No.:________________________ Class: ___________

Problem 1: (10 points) Find the Maclaurin series expansion of the function

$$f(z) = \text{Log}(1+z)$$

by differentiating repeatedly and specify the region in which the expansion is valid.

Solution.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n \quad (|z| < 1).$$

□

Problem 2: (15 points) Find three different Laurent series in powers of z for the function

$$f(z) = \frac{-1}{(z-1)(z-2)}.$$

Solution.

$$\sum_{n=0}^{\infty} (2^{-n-1} - 1) z^n \text{ in } D_1 : |z| < 1;$$

$$\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} + \sum_{n=1}^{\infty} \frac{1}{z^n} \text{ in } D_2 : 1 < |z| < 2;$$

$$\sum_{n=1}^{\infty} \frac{1 - 2^{n-1}}{z^n} \text{ in } D_3 : 2 < |z| < \infty$$

□

Problem 3: (15 points)

(1) Find the first three terms in the Laurent series in powers of z for the function

$$\csc z$$

and specify the regions in which the expansion is valid.

(2) Evaluate the integral

$$\int_C \csc z dz,$$

where C is the circle $|z| = 1$, described in the positive sense.

Solution.

$$\csc z = \frac{1}{z} + \frac{1}{3!}z + \left[\frac{1}{(3!)^2} - \frac{1}{5!}\right] z^3 + \cdots \quad (0 < |z| < \pi); \quad 2\pi i$$

□

Problem 4: (15 points) Find and classify (according to the terms *pole, removable, essential*) the singular points of

$$f(z) = \frac{z}{1 - \cos z}.$$

For each pole, give its order and compute the residue there.

Solution. $f(z)$ has a simple pole at $z = 0$ and a pole of order 2 at $z = 2n\pi$ for $n = \pm 1, \pm 2, \cdots$.

$$\mathrm{Res}_{z=0}\, f(z) = 2; \quad \mathrm{Res}_{z=2n\pi}\, f(z) = 2 \text{ for } n = \pm 1, \pm 2, \cdots.$$

□

Problem 5: (15 points) Use the theorem involving only a single residue, to evaluate the integral

$$\int_C \frac{(3z+2)^2}{z(z-1)(2z+5)} dz$$

where C is the circle $|z| = 3$, described in the positive sense.

Solution.

$$9\pi i$$

□

Problem 6: (15 points) Compute

(1)

$$\mathrm{Res}_{z=i} \frac{\mathrm{Log}\, z}{(z^2+1)^2}$$

(2)

$$\mathrm{Res}_{z=\pi i} \frac{\exp(zt)}{\sinh z} + \mathrm{Res}_{z=-\pi i} \frac{\exp(zt)}{\sinh z}.$$

Solution.

$$\frac{\pi + 2i}{8}; \quad -2\cos(\pi t)$$

□

Problem 7: (15 points) Use residues to derive the integration formula

$$\int_0^\infty \frac{x^2}{(x^2+9)(x^2+4)^2} = \frac{\pi}{200}.$$