NAME:______________ ID NO.:________________________ CLASS: ____________

Problem 1: (10 points) Determine all of the values of z at which the following function is analytic $f(z) = \frac{\text{Log}(z+3)}{z^2+i}$.

Solution. First, the function $\frac{1}{z^2+i}$ is analytic everywhere except at $\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$ and $-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$. Second, the function $\text{Log}(z+3)$ is analytic except at those points where $z+3=0$, or where $z+3$ lies on the negative real axis, i.e. $\{(x;y)|x \le -3, y=0\}$. Hence the function $f(z) = \frac{\text{Log}(z+3)}{z^2+i}$ is analytic everywhere except at the points $\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$ and on the ray $\{(x,y)|x \le -3, y = -0\}$. □

Problem 2: (10 points) Solve the equation $\cos z = \sqrt{2}$ for z.

Mathod 1. By definition

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \sqrt{2}.$$

By solving the equation

$$e^{2iz} - 2\sqrt{2}e^{iz} + 1 = 0,$$

we get

$$e^{iz} = \sqrt{2} \pm 1.$$

This implies that

$$iz = \log(\sqrt{2} \pm 1) = \ln(\sqrt{2} \pm 1) + i2n\pi, n \in \mathbb{Z}.$$

Hence the solution is

$$-i\ln(\sqrt{2} \pm 1) + 2n\pi, n \in \mathbb{Z} \quad \text{or} \quad \pm i\ln(\sqrt{2}+1) + 2n\pi, n \in \mathbb{Z}.$$

□

Method 2.

$$\cos z = \cos x \,\text{coshy} - i\sin x \sinh y = \sqrt{2}.$$

This implies

$$\cos x \,\text{coshy} = \sqrt{2} \quad \text{and} \quad \sin x \sinh y = 0.$$

Case 1: $\sin x = 0$.

$$\sin x = 0 \quad \text{and} \quad \cos x > 0 \Rightarrow x = 2n\pi, n \in \mathbb{Z}.$$

$$\left(\text{since} \quad \cos x \,\text{coshy} = \sqrt{2} \quad \text{and} \quad \cosh y = \frac{e^y + e^{-y}}{2} > 0 \Rightarrow \cos x > 0.\right)$$

$$\Rightarrow \cos x = 1 \quad \text{and} \quad \cosh y = \frac{e^y - e^{-y}}{2} = \sqrt{2} \Rightarrow e^y = \sqrt{2} \pm 1 \Rightarrow y = \ln(\sqrt{2} \pm 1).$$

Case 2: $\sinh y = 0$.

$$\sinh y = \frac{e^y - e^{-y}}{2} = 0 \Rightarrow y = 0 \Rightarrow \cosh y = \frac{e^y + e^{-y}}{2} = 1$$

$$\Rightarrow \cos x = \sqrt{2} > 1 \rightarrow\leftarrow$$

Therefore, the solution is

$$z = x + iy = 2n\pi + i\ln(\sqrt{2} \pm 1), n \in \mathbb{Z}.$$

$\square$

Problem 3:

(1) **(15 points)** Show that $\log(i^{1/2}) = \frac{1}{2}\log i$.

(2) **(5 points)** Find the principal value of $[\frac{e}{2}(-1 - \sqrt{3}i)]^{3\pi i}$.

Solution. (1) Homework 4 (Exercise 5 of section 33).

(2)

$$P.V.[\frac{e}{2}(-1 - \sqrt{3}i)]^{3\pi i} = \exp[3\pi i \operatorname{Log}\frac{e}{2}(-1 - \sqrt{3}i)] = \exp[3\pi i \operatorname{Log}\frac{e}{2}(2e^{i(-\frac{2}{3}\pi)})]$$

$$= \exp[3\pi i(\ln e - i\frac{2}{3}\pi)] = \exp(2\pi^2 + i3\pi) = -e^{2\pi^2}.$$

$\square$

Problem 4: (15 points) Consider $I = \int_C \frac{\cos z}{(z+\pi)^5} dz$.

(1) Evaluate the integral I when the contour C is the square whose edges lie along the lines $x = \pm 4$ and $y = \pm 4$ with positive orientation.

Solution. Since $-\pi$ is in the interior of the contour C, by the extension of Cauchy integral formula, we have

$$I = \int_C \frac{\cos z}{(z+\pi)^5} dz = \frac{2\pi i}{4!}\left(\frac{d^4}{dz^4}\cos z\right)\Big|_{z=-\pi} = -\frac{\pi i}{12}.$$

$\square$

(2) Evaluate the integral I when the contour C is the square whose edges lie along the lines $x = \pm 1$ and $y = \pm 1$ with positive orientation.

Solution. Since there is no singularity, by Cauchy-Goursat theorem, we have

$$I = \int_C \frac{\cos z}{(z+\pi)^5} dz = 0.$$

□

Problem 5: (10 points) Find the maximum and minimum moduli of $z^2 - z$ in the disc: $|z| \leq 1$.

Solution. Since $z^2 - z$ is analytic throughout the disc and not constant in the interior of the disc. By the maximum moduli principle, the maximum moduli of $z^2 - z$ occur at the boundary $|z| = 1$. Note that $|z^2 - z| = |z(z-1)| = |z||z-1|$. Hence it is enough to consider the maximum of $|z-1|$. It is not difficult to see that the maximum value 2 occurs at $z = -1$. Obviously the minimum value of $|z^2 - z| = |z||z-1|$ is 0 which occurs at $z = 0, 1$. Note that $z^2 - z = 0$ at $z = 0$ which is in the interior of the disc. Hence the minimum moduli principle in Exercise 2 of Section 59, where the condition $f(z) \neq 0$ is needed, does not apply here. □

Problem 6: (15 points) Let C be the positively oriented circle $\{|z| = 2\}$. Evaluate the contour integral

$$I = \int_C \frac{\cos(\pi z)}{z(z-1)} dz.$$

Solution. The singularities $z = 0, 1$ are in the interior of C. Let C_1 be the positively oriented circle $\{|z| = 1/3\}$ and C_2 be the positively oriented circle $\{|z-1| = 1/3\}$. Then C_1 and C_2 are in the interior of C and disjoint. The function $\frac{\cos(\pi z)}{z(z-1)}$ is analytic in the region that is inside of C and outside of C_1, C_2. By the Cauchy-Goursat theorem for multiply connected domain.

$$\int_C \frac{\cos(\pi z)}{z(z-1)} dz - \int_{C_1} \frac{\cos(\pi z)}{z(z-1)} dz - \int_{C_2} \frac{\cos(\pi z)}{z(z-1)} dz = 0.$$

The function $\frac{\cos z}{z-1}$ is analytic in the interior of C_1 and the function $\frac{\cos z}{z}$ is analytic in the interior of C_2. By Cauchy integral formula, we have

$$\int_{C_1} \frac{\cos(\pi z)}{z(z-1)} dz = \int_{C_1} \frac{\cos(\pi z)/(z-1)}{z} dz = 2\pi i \frac{\cos 0}{0-1} = -2\pi i$$

and

$$\int_{C_2} \frac{\cos(\pi z)}{z(z-1)} dz = \int_{C_2} \frac{\cos(\pi z)/z}{z-1} dz = 2\pi i \frac{\cos \pi}{1} = -2\pi i.$$

Therefore

$$\int_C \frac{\cos(\pi z)}{z(z-1)} dz = \int_{C_1} \frac{\cos(\pi z)}{z(z-1)} dz + \int_{C_2} \frac{\cos(\pi z)}{z(z-1)} dz = -4\pi i.$$

□

Problem 7: (10 points) Evaluate the contour integral

$$I = \int_C z^{-1+i} dz$$

where the branch is defined by $z^{-1+i} = e^{(-1+i)\log z}$ ($|z| > 0$, $0 < \arg z < 2\pi$) and C is the positively oriented unit circle $|z| = 1$.

Solution. Let $z(\theta) = e^{i\theta}, 0 < \theta < 2\pi$, then

$$z'(\theta) = ie^{i\theta}$$

and

$$z(\theta)^{-1+i} = e^{(-1+i)(\ln 1 + i\theta)} = e^{-i\theta}e^{-\theta}.$$

$$\Rightarrow I = \int_C z^{-1+i} dz = \int_0^{2\pi} e^{-i\theta}e^{-\theta} ie^{i\theta} d\theta = i\int_0^{2\pi} e^{-\theta} d\theta = -i(e^{-2\pi} - 1).$$

□

Problem 8: (10 points) Use ML inequality to show that

$$\Big| \int_C \frac{(z^2+3)e^{iz} \operatorname{Log} z}{z^2-2} dz \Big| \leq \frac{7(3\ln 2 + \pi)\pi}{9},$$

where C is the contour $\{z | z = 2e^{i\theta}, 0 \leq \theta \leq \frac{\pi}{3}\}$.

Proof. By

$$|z^2 - 2| \geq ||z|^2 - 2| = 2, \qquad |z^2 + 3| \leq |z|^2 + 3 = 7$$

and

$$|e^{iz}| = |e^{ix-y}| = e^{-y} \leq 1, \qquad |\operatorname{Log} z| = |\ln 2 + i\theta| \leq \ln 2 + \frac{\pi}{3}.$$

We obtain

$$\Big|\frac{(z^2+3)e^{iz} \operatorname{Log} z}{z^2-2}\Big| \leq \mathrm{M} = \frac{7(3\ln 2 + \pi)}{6}.$$

We also have

$$\mathrm{L} = \int_C dz = \frac{2\pi}{3}.$$

Therefore

$$\Big| \int_C \frac{(z^2+3)e^{iz} \operatorname{Log} z}{z^2-2} dz \Big| \leq \int_C \Big|\frac{(z^2+3)e^{iz} \operatorname{Log} z}{z^2-2}\Big| dz \leq \mathrm{ML} =\leq \frac{7(3\ln 2 + \pi)\pi}{9}.$$

□