NAME:______________ ID NO.:______________________ CLASS: __________

There are problems on both sides of the page.

Problem 1:

(1) (10 points) Find the first three (nonzero) terms in the Laurent series in powers of z for the function

$$\cot z,$$

where $0 < |z| < \pi$.

(2) (5 points) Use Cauchy's residue theorem to evaluate the integral

$$\int_C \frac{\cot z}{z^4} dz,$$

where C is the unit circle $|z| = 1$, taken counterclockwise.

Solution. $\frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} + \cdots; \quad -\frac{2\pi i}{45}$ □

Problem 2: (15 points) Find the Laurent expansion of $f(z) = \frac{z}{(z-1)(z-2)}$ valid in the domain

(1) $0 < |z-1| < 1$,
(2) $0 < |z-2| < 1$,
(3) $1 < |z| < 2$.

Solution. For $z \neq 1, 2$, we have

$$f(z) = \frac{z}{(z-1)(z-2)} = \frac{2}{z-2} - \frac{1}{z-1}.$$

(1) For $0 < |z-1| < 1$, we have

$$f(z) = -\frac{1}{z-1} - 2\sum_{n=0}^{\infty}(z-1)^n.$$

(2) For $0 < |z-2| < 1$, we have

$$f(z) = \frac{2}{z-2} - \sum_{n=0}^{\infty}(-1)^n(z-2)^n,$$

(3) For $1 < |z| < 2$, we have

$$f(z) = -\sum_{n=0}^{\infty} \frac{z^n}{2^n} - \sum_{n=1}^{\infty} \frac{1}{z^n}.$$

□

Problem 3: (15 points) Evaluate the integral

$$\int_C \frac{1+z}{1-\cos z} dz,$$

where C is the circle $|z| = 7$, taken counterclockwise.

Solution. $12\pi i$ □

Problem 4: (10 points) Evaluate the integral

$$\int_C \exp\left(z + \frac{1}{z}\right) dz,$$

where C is the circle $|z| = 1$, taken counterclockwise.

Solution.

$$2\pi i \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!}$$

□

Problem 5: Compute

(1) (5 points)

$$\mathrm{Res}_{z=\pi} \frac{z - \sin z}{z^2 \sin z}$$

(2) (10 points)

$$\mathrm{Res}_{z=-i} \frac{\sqrt{z}}{(z^2+1)^2},$$

where the principal value is used for the square root.

Solution. $-\frac{1}{\pi}$; $\quad \frac{1+i}{16}\sqrt{2}$ □

Problem 6: (15 points) Use the residue at infinity to compute the integral

$$\int_C \frac{z^{n-1}}{(z-1)(2z-1)\cdots(nz-1)} dz,$$

where C is the circle $|z| = 2$, taken counterclockwise. Here n is a positive integer.

Solution.

$$\text{Res}_{z=0}\, \frac{1}{z^2} f(\frac{1}{z}) = \frac{1}{n!}; \qquad \frac{2\pi i}{n!}$$

□

Problem 7: (15 points) Use residues to find the value of the integral

$$\int_0^\infty \frac{x^2}{x^6 + 1} dx.$$

Clearly explain all the necessary estimates.

Solution. $\frac{\pi}{6}$ □