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NAVIER–STOKES EQUATIONS INTERACTING WITH A NONLINEAR ELASTIC BIOFLUID SHELL*

C. H. ARTHUR CHENG†, DANIEL COUTAND†, AND STEVE SHKOLLER‡

Abstract. We study a moving boundary value problem consisting of a viscous incompressible fluid moving and interacting with a nonlinear elastic fluid shell. The fluid motion is governed by the Navier–Stokes equations, while the fluid shell is modeled by a bending energy which extremizes the Willmore functional and a membrane energy with density given by a convex function of the local area ratio. The fluid flow and shell deformation are coupled together by continuity of displacements and tractions (stresses) along the moving surface defining the shell. We prove the existence and uniqueness of solutions in Sobolev spaces for a short time.

Key words. Navier–Stokes equations, free boundary problems, shell theory, biofluids, Willmore energy

AMS subject classifications. 74F10, 35Q30, 74K25, 35Q72, 74B20, 74H20, 74H25, 76D05

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1. Introduction.

1.1. The problem statement and background. We are concerned with establishing the existence and uniqueness of solutions to the time-dependent incompressible Navier–Stokes equations interacting with a nonlinear elastic fluid shell (biomembrane) for a short time. Recently, there have been many experimental and analytic studies on the configurations and deformations of elastic biomembranes (see, for example, [3], [11], [13], [16], [17], [18], [19], and [21]), but the basic analysis of the coupled fluid-structure interaction remains open. The fundamental difficulties arise from the degenerate elliptic operators that arise as the shell tractions. As we detail below, the bending energy of the shell is the well-known Willmore function, the integral over the moving surface of the square of the mean curvature. The degenerate elliptic operator arising from the first variation of this functional is a fourth order nonlinear operator that smoothes only in the direction which is normal to the moving domain. Our analysis will provide a well-posedness theorem and explain the interesting interaction between the shape of the shell and the flow of the fluid.

Fluid-structure interaction problems involving moving material interfaces have been the focus of active research since the 1990s. The first problem solved in this area was for the case of a rigid body moving in a viscous fluid (see [9], [14], and the early works of [22] and [20] for a rigid body moving in a Stokes flow in the full space). The case of an elastic body moving in a viscous fluid was considerably more challenging because of some apparent regularity incompatibilities between the parabolic fluid phase and the hyperbolic solid phase. The first existence results in this area were for regularized elasticity laws, such as in [10] for a *finite* number of

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†Department of Mathematics, University of California, Davis, CA 95616 (cchsiao@math.ucdavis.edu, coutand@math.ucdavis.edu).
‡Corresponding author. Department of Mathematics, University of California, Davis, CA 95616 (shkoller@math.ucdavis.edu).

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elastic modes, or in [1], [4], and [2] for hyperviscous elasticity laws, or in [15] in which a phase-field regularization "fattens" the sharp interface via a diffuse-interface model.

The treatment of classical elasticity laws for the solid phase, without any regularizing term, was considered only recently in [7] for the three-dimensional linear St. Venant–Kirchhoff constitutive law and in [8] for quasi-linear elastodynamics coupled to the Navier–Stokes equations. Some of the basic new ideas introduced in those works concerned a functional framework that scales in a hyperbolic fashion (and is therefore driven by the solid phase), the introduction of approximate problems either penalized with respect to the divergence-free constraint in the moving fluid domain or smoothed by an appropriate parabolic artificial viscosity in the solid phase (chosen in an asymptotically convergent and consistent fashion), and the tracking of the motion of the interface by difference quotient techniques.

In our companion paper [5], we study the interaction of the Navier–Stokes equations with an elastic solid shell. Herein, we treat the case of a fluid shell or biomembrane. This is a moving boundary problem that models the motion of a viscous incompressible Newtonian fluid inside of a deformable elastic fluid structure.

Let $\Omega \subset \mathbb{R}^3$ denote an open bounded domain with boundary $\Gamma := \partial\Omega$. For each $t \in (0,T]$, we wish to find the domain $\Omega(t)$, a divergence-free velocity field $u(t,\cdot)$, a pressure function $p(t,\cdot)$ on $\Omega(t)$, and a volume-preserving transformation $\eta(t,\cdot) : \Omega \to \mathbb{R}^3$ such that

$$\Omega(t) = \eta(t,\Omega), \tag{1.1a}$$

$$\eta_t(t,x) = u(t,\eta(t,x)), \tag{1.1b}$$

$$u_t + \nabla_u u - \nu\Delta u = -\nabla p + f \quad \text{in } \Omega(t), \tag{1.1c}$$

$$\operatorname{div} u = 0 \quad \text{in } \Omega(t), \tag{1.1d}$$

$$(\nu \operatorname{Def} u - p\,\mathrm{Id})n = \mathfrak{t}_{shell} \quad \text{on } \Gamma(t), \tag{1.1e}$$

$$u(0,x) = u_0(x) \quad \forall\, x \in \Omega, \tag{1.1f}$$

$$\eta(0,x) = x \quad \forall\, x \in \Omega, \tag{1.1g}$$

where ν is the kinematic viscosity, $n(t,\cdot)$ is the outward pointing unit normal to $\Gamma(t)$, $\Gamma(t) := \partial\Omega(t)$ denotes the boundary of $\Omega(t)$, $\operatorname{Def} u$ is twice the rate of deformation tensor of u, given in coordinates by $u^i_{,j} + u^j_{,i}$, and $\mathfrak{t}_{shell}$ is the traction imparted onto the fluid by the elastic shell, which we describe next.

We shall consider a thin elastic shell modeled by the nonlinear Saint Venant–Kirchhoff constitutive law. With ε denoting the thickness of the shell, the hyperelastic stored energy function has the asymptotic expansion

$$E_{shell} = \varepsilon E_{mem} + \varepsilon^3 E_{ben} + \mathcal{O}(\varepsilon^4).$$

The membrane energy satisfies

$$E_{mem} = \int_\Gamma \mathcal{P}(\mathcal{J})dS, \tag{1.2}$$

where $\mathcal{J}$ is the local area ratio and $\mathcal{P}$ is a convex function attaining its minimum at $\mathcal{J} = 1$, while the bending energy E_{ben} is given by

$$E_{ben} = \int_{\Gamma(t)} (\sigma H^2 - \sigma_1 K)dS, \tag{1.3}$$

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where H and K denote the mean and Gauss curvatures on $\Gamma(t)$, respectively, and where σ and σ_1 are positive constants. The traction vector

$$\mathfrak{t}_{shell} = \varepsilon \mathfrak{t}_{mem} + \varepsilon^3 \mathfrak{t}_{ben} + \mathcal{O}(\varepsilon^4)$$

is computed from the first variation of the energy function E_{shell}; the traction vector associated with the membrane energy is

$$\mathfrak{t}_{\text{mem}} = \Big[\mathcal{J}\mathcal{P}''(\mathcal{J}) + 2\mathcal{P}'(\mathcal{J})\Big]\mathcal{J}_{,\beta} g^{\alpha\beta}\eta_{,\alpha} + \Big[\mathcal{J}\mathcal{P}'(\mathcal{J}) + \mathcal{P}(\mathcal{J})\Big]Hn, \tag{1.4}$$

while the traction associated with the bending energy has a simple intrinsic characterization given by

$$\mathfrak{t}_{ben} = \sigma(\Delta_g H - 2HK + 2H^3)n, \tag{1.5}$$

where Δ_g denotes the Laplacian with respect to the induced metric g on $\Gamma(t)$:

$$\Delta_g f = \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x^\alpha}\bigg(\sqrt{\det(g)} g^{\alpha\beta} \frac{\partial f}{\partial x^\beta}\bigg).$$

In this paper, we ignore the inertia of the shell and focus our analysis on the difficulties associated with the degenerate elliptic operators in $\mathfrak{t}_{shell}$.

1.2. Outline of the paper. In section 2, in addition to the use of Lagrangian variables, we introduce a new coordinate system near the boundary (shell) and three new maps, η^ν, η^τ, and h, which facilitate the computation of the membrane and bending tractions $\mathfrak{t}_{\text{mem}}$ and $\mathfrak{t}_{\text{ben}}$. A key observation is the symmetry relation (2.7) which reduces the derivative count on the tangential reparameterization map η^τ.

The space of solutions (to the problem $\mathfrak{t}_{mem} = 0$) is introduced in section 3, and the main theorem is stated in section 4. Section 5 defines our notation, and section 6 provides some useful lemmas.

In section 7, we introduce the linearized and regularized problems. The regularization requires smoothing certain variables as well as the introduction of a certain artificial viscosity term on the boundary of the fluid domain. Weak solutions of this linear problem are established via a penalization scheme which approximates the incompressibility of the fluid.

In section 8, we establish a regularity theory for our weak solution using energy estimates for the mollified problem with constants that depend on the mollification parameters. In section 9, we improve these estimates so that the constants are independent of the artificial viscosity as well as other regularization parameters. This requires an elliptic estimate, arising from the boundary condition (1.1e), which provides additional regularity for the shape of the boundary.

In section 10, the Tychonoff fixed-point theorem is used to prove the existence of solutions to the original nonlinear problem (1.1). Uniqueness, following required compatibility conditions, is established in sections 4 and 10.

In section 11, we consider the inclusion of the lower order membrane traction into the problem formulation so that the full problem is solved.

The inclusion of the inertial term $\epsilon_1 \eta_{tt}$ into the membrane traction $\mathfrak{t}_{\text{mem}}$ will be studied in a future publication.

2. Lagrangian formulation.

2.1. A new coordinate system near the shell. Consider the isometric immersion $\eta_0 : (\Gamma, g_0) \to (\mathbb{R}^3, \text{Id})$. Let $\mathcal{B} = \Gamma \times (-\epsilon_1, \epsilon_1)$, where ϵ_1 is chosen sufficiently

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small so that the map

$$B : \mathcal{B} \to \mathbb{R}^3 : (y, z) \mapsto y + zN(y)$$

is itself an immersion, defining a tubular neighborhood of Γ in $\mathbb{R}^3$. We can choose a coordinate system $\frac{\partial}{\partial y^\alpha}$, $\alpha = 1, 2$, and $\frac{\partial}{\partial z}$ on $\mathcal{B}$, where $\frac{\partial}{\partial y^\alpha}$ denotes the tangential derivative and $\frac{\partial}{\partial z}$ denotes the normal derivative.

Let $G = B^*(\mathrm{Id})$ denote the induced metric on $\mathcal{B}$ from $\mathbb{R}^3$ so that

$$G(y, z) = G_z(y) + dz \otimes dz,$$

where G_z is the metric on the surface $\Gamma \times \{z\}$; note that $G_0 = g_0$.

REMARK 1. *By assumption,* $g_{0\alpha\beta} = \frac{\partial}{\partial y^\alpha} \cdot \frac{\partial}{\partial y^\beta}$, *where* $\cdot$ *denotes the usual Cartesian inner product on* $\mathbb{R}^n$. *Let* $C_{\alpha\beta}$ *denote the covariant components of the second fundamental form of the base manifold* Γ *so that* $C_{\alpha\beta} = -N_{,\alpha} \cdot \frac{\partial}{\partial y^\beta}$. *Then* G_z *is given by*

$$(G_z)_{\alpha\beta} = (g_0)_{\alpha\beta} - 2zC_{\alpha\beta} + z^2 g_0^{\gamma\delta} C_{\alpha\gamma} C_{\beta\delta}.$$

Let $h : \Gamma \to (-\epsilon_1, \epsilon_1)$ be a smooth height function and consider the graph of h in $\mathcal{B}$, parameterized by $\phi : \Gamma \to \mathcal{B} : y \mapsto (y, h(y))$. The tangent space to graph(h), considered as a submanifold of $\mathcal{B}$, is spanned at a point $\phi(x)$ by the vectors

$$\phi_* \left(\frac{\partial}{\partial y^\alpha} \right) = \frac{\partial \phi}{\partial y^\alpha} = \frac{\partial}{\partial y^\alpha} + \frac{\partial h}{\partial y^\alpha} \frac{\partial}{\partial z},$$

and the normal to graph(h) is given by

$$n(y) = J_h^{-1}(y) \left(-G_{h(y)}^{\alpha\beta} \frac{\partial h}{\partial y^\alpha} \frac{\partial}{\partial y^\beta} + \frac{\partial}{\partial z} \right), \tag{2.1}$$

where $J_h = (1 + h_{,\alpha} G_{h(y)}^{\alpha\beta} h_{,\beta})^{1/2}$. The mean curvature H of graph(h) is defined to be the trace of ∇n, where

$$(\nabla n)_{ij} = G \left(\nabla^{\mathcal{B}}_{\frac{\partial}{\partial w^i}} n, \frac{\partial}{\partial w^j} \right) \quad \text{for } i, j = 1, 2, 3,$$

where $\frac{\partial}{\partial w^\alpha} = \frac{\partial}{\partial y^\alpha}$ for $\alpha = 1, 2$ and $\frac{\partial}{\partial w^3} = \frac{\partial}{\partial z}$, and $\nabla^{\mathcal{B}}$ denotes the covariant derivative. Using (2.1),

$$\begin{aligned}
(\nabla n)_{\alpha\beta} &= G \left(\nabla^{\mathcal{B}}_{\frac{\partial}{\partial y^\alpha}} \left[-J_h^{-1} G_h^{\gamma\delta} h_{,\gamma} \frac{\partial}{\partial y^\delta} + J_h^{-1} \frac{\partial}{\partial z} \right], \frac{\partial}{\partial y^\beta} \right) \\
&= -(G_h)_{\delta\beta} \left[(J_h^{-1} G_h^{\gamma\delta} h_{,\gamma})_{,\alpha} + J_h^{-1} (-G_h^{\gamma\sigma} h_{,\gamma} \Gamma^\delta_{\alpha\sigma} + \Gamma^\delta_{\alpha 3}) \right]; \\
(\nabla n)_{33} &= G \left(\nabla^{\mathcal{B}}_{\frac{\partial}{\partial z}} \left[-J_h^{-1} G_h^{\gamma\delta} h_{,\gamma} \frac{\partial}{\partial y^\delta} + J_h^{-1} \frac{\partial}{\partial z} \right], \frac{\partial}{\partial z} \right) \\
&= J_h^{-1} (-G_h^{\gamma\delta} h_{,\gamma} \Gamma^3_{3\delta} + \Gamma^3_{33}),
\end{aligned}$$

where Γ^k_{ij} denotes the Christoffel symbols with respect to the metric G. It follows that the curvature of graph(h) (in the divergence form) is

$$H = -(J_h^{-1} G_h^{\gamma\delta} h_{,\gamma})_{,\delta} + J_h^{-1} (-G_h^{\gamma\delta} h_{,\gamma} \Gamma^j_{j\delta} + \Gamma^j_{j3}), \tag{2.2}$$

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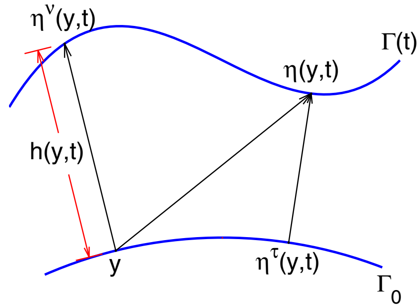

Fig. 1. *The maps η^τ and η^ν.*

or (in the quasi-linear form)

$$H = -J_h^{-1} G_h^{\alpha\beta}\Big[\delta_{\beta\gamma} - J_h^{-2} G_h^{\gamma\delta} h_{,\beta} h_{,\delta}\Big] h_{,\alpha\gamma} + G_h^{\alpha\beta} F_{\alpha\beta}(y, h, \nabla h), \tag{2.3}$$

where $F_{\alpha\beta}$ denotes a smooth generic function of y, h, and ∇h.

Remark 2. *Note that G_h denotes the metric $G_{z=h(y)}$ and not the metric on the submanifold graph(h).*

Remark 3. *If the initial height function is zero, i.e., $h(0) = 0$, then $H(0) = \Gamma^j_{j3}(0)$ which is the mean curvature of the base manifold Γ as required.*

2.2. Tangential reparameterization symmetry. Let $\mathcal{N}$ denote the normal bundle to Γ so that for each $y \in \Gamma$ we have the Whitney sum $\mathbb{R}^3 = T_y\Gamma \oplus \mathcal{N}_y$.

Given a signed height function $h : \Gamma \times [0, T) \to \mathbb{R}$, for each $t \in [0, T)$, define the *normal* map (see Figure 1)

$$\eta^\nu : \Gamma \times [0, T) \to \Gamma(t), \quad (y, t) \mapsto y + h(y, t)N(y), \quad N(y) \in \mathcal{N}_y.$$

Then there exists a unique *tangential* map $\eta^\tau : \Gamma \times [0, T) \to \Gamma$ (a diffeomorphism as long as h remains a graph) such that the diffeomorphism $\eta(t)$ has the decomposition

$$\eta(\cdot, t) = \eta^\nu(\cdot, t) \circ \eta^\tau(\cdot, t), \quad \eta(y, t) = \eta^\tau(y, t) + h(\eta^\tau(y, t), t)N(\eta^\tau(y, t)).$$

The tangent vector $\eta_{,\alpha}$ to $\Gamma(t)$ can be decomposed with respect to the Whitney sum as $\eta_{,\alpha}(y,t) = \eta^\kappa_{,\alpha}(y,t)\frac{\partial}{\partial y^\kappa} + h_{,\kappa}(\eta^\tau(y,t),t)\eta^\kappa_{,\alpha}\frac{\partial}{\partial z}$, and hence the induced metric $g_{\alpha\beta} = \eta_{,\alpha} \cdot \eta_{,\beta}$ may be expressed as

$$g_{\alpha\beta} = \Big[\Big((G_h)_{\kappa\sigma} + h_{,\kappa} h_{,\sigma}\Big) \circ \eta^\tau\Big] \eta^\kappa_{,\alpha} \eta^\sigma_{,\beta} := \Big[\mathcal{G}_{\kappa\sigma} \circ \eta^\tau\Big] \eta^\kappa_{,\alpha} \eta^\sigma_{,\beta}. \tag{2.4}$$

Note that $\mathcal{G}_{\kappa\sigma}$ is the induced metric with respect to the *normal* map η^ν. Furthermore, we have the following useful relationship between the determinant of the two induced metrics:

$$\det(g) = \det(\nabla_0 \eta^\tau)^2 \Big[\det(G_h) J_h^2\Big] \circ \eta^\tau = \det(\nabla_0 \eta^\tau)^2 \Big[\det(\mathcal{G})\Big] \circ \eta^\tau, \tag{2.5}$$

where ∇_0 denotes the surface gradient.

Remark 4. *The identity* (2.4) *can also be read as* $(\eta^\tau)^* g = \mathcal{G}$.

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Let y and $\tilde{y} = \varphi(y)$ denote two different coordinate systems on Γ with associated metrics

$$g_{\alpha\beta} = \frac{\partial \eta^i}{\partial y^\alpha}\frac{\partial \eta^i}{\partial y^\beta}, \qquad \tilde{g}_{\alpha\beta} = \frac{\partial \eta^i}{\partial \tilde{y}^\alpha}\frac{\partial \eta^i}{\partial \tilde{y}^\beta}.$$

It follows that $\varphi^*\tilde{g} = g$. Let H, $\tilde{H}$, K, $\tilde{K}$, n, and $\tilde{n}$ denote the mean curvature, Gauss curvature, and the unit normal vector computed with respect to y and $\tilde{y}$, respectively. Since H, K, and n depend only on the shape of $\Gamma(t)$, these geometric quantities are invariant to tangential reparameterization; thus, we have the identity

$$\tilde{H} = H \circ \varphi, \quad \tilde{K} = K \circ \varphi, \quad \tilde{n} = n \circ \varphi. \tag{2.6}$$

Similarly, computing the first variation of $\int_{\Gamma(t)} H^2 dS$ in our two coordinate systems yields

$$\Big[\Big(\Delta_g H + H(H^2 - K)\Big)n\Big](y) = \Big[\Big(\Delta_{\tilde{g}}\tilde{H} + \tilde{H}(\tilde{H}^2 - \tilde{K})\Big)\tilde{n}\Big](\tilde{y}) \qquad \forall\ \tilde{y} = \varphi(y).$$

By (2.6), we have the following important identity:

$$\Big[\Delta_{\varphi^*\tilde{g}} H\Big](y) = \Big[\Delta_{\tilde{g}}(H \circ \varphi)\Big](\tilde{y}) \qquad \forall\ \tilde{y} = \varphi(y), \tag{2.7}$$

and hence

$$[\Delta_{\mathcal{G}}(H \circ \eta^{-\tau})] \circ \eta^\tau = \Delta_g H, \tag{2.8}$$

where by (2.3),

$$H \circ \eta^{-\tau} = -J_h^{-1} G_h^{\alpha\beta}\Big[\delta_{\beta\gamma} - J_h^{-2} G_h^{\gamma\delta} h_{,\beta} h_{,\delta}\Big] h_{,\alpha\gamma} + G_h^{\alpha\beta} F_{\alpha\beta}(y, h, \nabla h). \tag{2.9}$$

2.3. Bounds on η^τ. Let u^τ denote the tangential velocity defined by $\eta^\tau_t = u^\tau \circ \eta^\tau$. Time differentiating the relation $\eta = \eta^\nu \circ \eta^\tau$ and using the definition of η^ν, we find that

$$u^\tau = (\nabla_0 \eta^\nu)^{-1}\Big[u \circ \eta^\nu - h_t \frac{\partial}{\partial z}\Big]. \tag{2.10}$$

From the trace theorem, it follows that

$$\|u^\tau\|_{H^{2.5}(\Gamma)} \le C\mathcal{P}(\|h\|_{H^{3.5}(\Gamma)}, \|\eta\|_{H^3(\Omega)})\Big[\|v\|_{H^3(\Omega)} + \|h_t\|_{H^{2.5}(\Gamma)}\Big] \tag{2.11}$$

for some polynomial $\mathcal{P}$. Since $\eta^\tau(y,t) = y + \int_0^t (u^\tau \circ \eta^\tau)(y,s)ds$, it follows that

$$\|\nabla_0 \eta^\tau(y,t)\|_{H^{1.5}(\Gamma)} \le C\Big[1 + \int_0^t \|u^\tau\|_{H^{2.5}(\Gamma)}\Big(1 + \|\nabla_0\eta^\tau\|_{H^{1.5}(\Gamma)}\Big)^4 ds\Big],$$

and hence by Gronwall's inequality,

$$\|\nabla_0 \eta^\tau(y,t)\|_{H^{1.5}(\Gamma)} \le C\Big[1 + \int_0^t \|u^\tau\|_{H^{2.5}(\Gamma)} ds\Big] \tag{2.12}$$

for $t \in [0,T]$ sufficiently small. Furthermore, we also have

$$\|\eta^\tau_t(y,t)\|_{H^{2.5}(\Gamma)} \le C\|u^\tau\|_{H^{2.5}(\Gamma)}\Big[1 + \|\nabla_0\eta^\tau\|_{H^{1.5}(\Gamma)}\Big]^4. \tag{2.13}$$

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2.4. An expression for $\mathbf{t_{ben}}$ in terms of h and η^τ. Now we can compute $\mathfrak{t}_{ben}$ in terms of h and η^τ: the highest order term of $\Delta_g H$ is

$$\left\{ \frac{1}{\sqrt{\det(\mathcal{G})}} \frac{\partial}{\partial y^\gamma} \left[\sqrt{\det(\mathcal{G})} \mathcal{G}^{\gamma\delta} \frac{\partial}{\partial y^\delta} \Big(J_h^{-1} (G_h^{\alpha\beta} - J_h^{-2} G_h^{\alpha\kappa} G_h^{\beta\sigma} h_{,\kappa} h_{,\sigma}) h_{,\alpha\beta} \Big) \right] \right\} \circ \eta^\tau .$$

Since $\mathcal{G}_{\alpha\beta} = (G_h)_{\alpha\beta} + h_{,\alpha} h_{,\beta}$, the inverse of $\mathcal{G}_{\gamma\delta}$ is

$$\frac{1}{\det(\mathcal{G})} \begin{bmatrix} (G_h)_{22} + h_{,2}^2 & -(G_h)_{12} - h_{,1} h_{,2} \\ -(G_h)_{12} - h_{,1} h_{,2} & (G_h)_{11} + h_{,1}^2 \end{bmatrix},$$

which can also be written as

$$\mathcal{G}^{\alpha\beta} = J_h^{-2} \Big[G_h^{\alpha\beta} - (-1)^{\kappa+\sigma} \det(G_h)^{-1} (1 - \delta_{\alpha\kappa})(1 - \delta_{\beta\sigma}) h_{,\kappa} h_{,\sigma} \Big].$$

Therefore, the highest order term of $\Delta_g H$ can be written as

$$\frac{1}{\sqrt{\det(g_0)}} \Big[\sqrt{\det(g_0)} A^{\alpha\beta\gamma\delta} h_{,\alpha\beta} \Big]_{,\gamma\delta} \circ \eta^\tau ,$$

where

$$\begin{aligned} A^{\alpha\beta\gamma\delta} = {} & J_h^{-3} \Big[G_h^{\alpha\gamma} - (-1)^{\kappa+\sigma} \det(G_h)^{-1} (1 - \delta_{\alpha\kappa})(1 - \delta_{\gamma\sigma}) h_{,\kappa} h_{,\sigma} \Big] \\ & \times (G_h^{\beta\delta} - J_h^{-2} G_h^{\beta\kappa} G_h^{\delta\sigma} h_{,\kappa} h_{,\sigma}) \end{aligned} \tag{2.14}$$

is a fourth-rank tensor.

2.5. Lagrangian formulation of the problem. Let $\eta(t,x) = x + \int_0^t u(s,x) ds$ denote the Lagrangian particle placement field, a volume-preserving embedding of Ω onto $\Omega(t) \subset \mathbb{R}^3$, and denote the cofactor matrix of $\nabla\eta(x,t)$ by

$$a(x,t) = [\nabla\eta(x,t)]^{-1}. \tag{2.15}$$

Let $v = u \circ \eta$ denote the Lagrangian or material velocity field, $q = p \circ \eta$ the Lagrangian pressure function, and $F = f \circ \eta$ the forcing function in the material frame. In the following discussion, we also set $\varepsilon = 1$. Then the system (1.1) can be reformulated as

$$\eta_t = v \quad \text{in } (0,T) \times \Omega, \tag{2.16a}$$

$$v_t^i - \nu (a_\ell^j D_\eta(v)_\ell^i)_{,j} = -(a_i^k q)_{,k} + F^i \quad \text{in } (0,T) \times \Omega, \tag{2.16b}$$

$$a_i^k v_{,k}^i = 0 \quad \text{in } (0,T) \times \Omega, \tag{2.16c}$$

$$(\nu D_\eta(v)_\ell^i - q\delta_\ell^i) a_\ell^j N_j = \sigma\Theta \Big[L(h) B_* (-G_h^{\alpha\beta} h_{,\alpha}, 1) \Big] \circ \eta^\tau \quad \text{on } (0,T) \times \Gamma, \tag{2.16d}$$

$$h_t = B_*((-G_h^{\alpha\beta} h_{,\alpha}, 1)) \cdot (v \circ \eta^{-\tau}) \quad \text{on } (0,T) \times \Gamma, \tag{2.16e}$$

$$v = u_0 \quad \text{on } \{t = 0\} \times \Omega, \tag{2.16f}$$

$$h = 0 \quad \text{on } \{t = 0\} \times \Gamma, \tag{2.16g}$$

$$\eta = \mathrm{Id} \quad \text{on } \{t = 0\} \times \Omega, \tag{2.16h}$$

where $D_\eta(v)_\ell^i := (a_\ell^k v_{,k}^i + a_i^k v_{,k}^\ell)$, N denotes the outward-pointing unit normal to Γ, Θ is defined in Remark 5, and B_* is the pushforward of B defined as

$$B_*(\gamma'(0)) = (B \circ \gamma)'(0) \qquad \forall\ \gamma(t) \subset \Gamma.$$

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$L(h)$ is the representation of $\mathfrak{t}_{shell} \cdot n$ using the height function h. It is defined as follows:

$$L(h) = \frac{1}{\sqrt{\det(g_0)}}\left[\sqrt{\det(g_0)}A^{\alpha\beta\gamma\delta}h_{,\alpha\beta}\right]_{,\gamma\delta} + L_1^{\alpha\beta\gamma}(y, h, Dh, D^2h)h_{,\alpha\beta\gamma}$$
$$+ L_2(y, h, Dh, D^2h),$$

where L_1 and L_2 are polynomials of their variables with $L_1(y, 0) = 0$, and g_0 is the metric tensor on Γ. Note that $\mathfrak{t}_{mem}$ is included in L_2, since it is a second order operator of h.

Remark 5. *For a point $\eta(y, t) \in \Gamma(t)$, there are two ways of defining the unit normal n to $\Gamma(t)$:*

1. *Let $n = \sqrt{g}^{-1}a^T N$, where N is the unit normal to Γ.*
2. *Let $n = [J_h^{-1}(-G_h^{\alpha\beta}h_{,\alpha}\frac{\partial}{\partial y^\beta} + \frac{\partial}{\partial z})] \circ \eta^\tau$ (denoted by $[J_h^{-1}(-\nabla_0 h, 1)] \circ \eta^\tau$).*

The function Θ is defined by

$$\Theta(-\nabla_0 h \circ \eta^\tau, 1) = a^T N.$$

Equating the modulus of both sides, by (2.5) *we must have*

$$\Theta = \sqrt{\det(g)}[(J_h^{-1}) \circ \eta^\tau] = \det(\nabla_0\eta^\tau)\sqrt{\det(G_h) \circ \eta^\tau}.$$

Remark 6. *An equivalent form of* (2.16e) *is given by*

$$h_t = -h_{,\alpha}(v \circ \eta^{-\tau})_\alpha + (v \circ \eta^{-\tau})_z.$$

This equation states that the shape of the boundary moves with the normal velocity of the fluid.

Remark 7. *For many of the nonlinear estimates that appear later, it is important that $L(h)$ is linear in the third derivative $h_{,\alpha\beta\gamma}$.*

Remark 8. *Without using the symmetry* (2.8)*, we can still compute $\Delta_g H$ in terms of h and η^τ by using* (2.4) *and* (2.5)*; however, L_1 would then depend on $\nabla_0^2\eta^\tau$ and thus lose one derivative of regularity, preventing the closure of our energy estimate.*

3. Notation and conventions.

For $T > 0$, we set

$$\mathcal{V}^1(T) = \left\{v \in L^2(0, T; H^1(\Omega)) \,\middle|\, v_t \in L^2(0, T; H^1(\Omega)')\right\};$$
$$\mathcal{V}^2(T) = \left\{v \in L^2(0, T; H^2(\Omega)) \,\middle|\, v_t \in L^2(0, T; L^2(\Omega))\right\};$$
$$\mathcal{V}^k(T) = \left\{v \in L^2(0, T; H^k(\Omega)) \,\middle|\, v_t \in L^2(0, T; H^{k-2}(\Omega))\right\} \quad \text{for } k \geq 3;$$
$$\mathcal{H}(T) = \left\{h \in L^2(0, T; H^{5.5}(\Gamma)) \,\middle|\, h_t \in L^2(0, T; H^{2.5}(\Gamma)),\, h_{tt} \in L^2(0, T; H^{0.5}(\Gamma))\right\}$$

with norms

$$\|v\|^2_{\mathcal{V}^1(T)} = \|v\|^2_{L^2(0,T;H^1(\Omega))} + \|v_t\|^2_{L^2(0,T;H^1(\Omega)')};$$
$$\|v\|^2_{\mathcal{V}^2(T)} = \|v\|^2_{L^2(0,T;H^2(\Omega))} + \|v_t\|^2_{L^2(0,T;L^2(\Omega))};$$
$$\|v\|^2_{\mathcal{V}^k(T)} = \|v\|^2_{L^2(0,T;H^k(\Omega))} + \|v_t\|^2_{L^2(0,T;H^{k-2}(\Omega))} \quad \text{for } k \geq 3;$$
$$\|h\|^2_{\mathcal{H}(T)} = \|h\|^2_{L^2(0,T;H^{5.5}(\Gamma))} + \|h_t\|^2_{L^2(0,T;H^{2.5}(\Gamma))} + \|h_{tt}\|^2_{L^2(0,T;H^{0.5}(\Gamma))}.$$

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We then introduce the space (of "divergence-free" vector fields)

$$\mathcal{V}_v = \left\{ w \in H^1(\Omega) \;\middle|\; a_i^j(t) w^i_{,j} = 0 \ \forall\, t \in [0,T] \right\}$$

and

$$\mathcal{V}_v(T) = \left\{ w \in L^2(0,T;H^1(\Omega)) \;\middle|\; a_i^j(t) w^i_{,j} = 0 \ \forall\, t \in [0,T] \right\},$$

where the cofactor matrix a is defined by (2.15). We use X_T to denote the space $\mathcal{V}^3(T) \times \mathcal{H}(T)$ with norm

$$\|(v,h)\|^2_{X_T} = \|v\|^2_{\mathcal{V}^3(T)} + \|h\|^2_{\mathcal{H}(T)}$$

and use Y_T, a subspace of X_T, to denote the space

$$Y_T = \left\{ (v,h) \in \mathcal{V}^3(T) \times \mathcal{H}(T) \;\middle|\; h_t \in L^\infty(0,T;H^2(\Gamma)) \right\}$$

with norm

$$\begin{aligned} \|(v,h)\|^2_{Y_T} = \|(v,h)\|^2_{X_T} &+ \|v\|^2_{L^\infty(0,T;H^2(\Omega))} + \|h\|^2_{L^\infty(0,T;H^4(\Gamma))} \\ &+ \|h_t\|^2_{L^\infty(0,T;H^2(\Gamma))}. \end{aligned}$$

We will solve (2.16) by a fixed-point method in an appropriate subset of Y_T.

4. The main theorem. Before stating the main theorem, we define the following quantities. Let q_0 be defined by

$$\Delta q_0 = -\nabla u_0 : (\nabla u_0)^T + \nu [a^k_\ell D_\eta (u_0)^i_\ell]_{,ki}(0) + \operatorname{div} F(0) \quad \text{in} \quad \Omega, \tag{4.1a}$$

$$q_0 = \nu(\operatorname{Def} u_0 \cdot N) \cdot N - \sigma L(0) \quad \text{on} \quad \Gamma \tag{4.1b}$$

and

$$u_1 = \nu \Delta u_0 - \nabla q_0 + F(0). \tag{4.2}$$

We also define the projection operator $\mathcal{P}_{ij}(x) : \mathbb{R}^3 \to T_{\eta(x,t)}\Gamma(t)$ by

$$\mathcal{P}_{ij}(x) = [\delta_{ij} - (J_h^{-2} \circ \eta^\tau) a^k_i a^\ell_j N_k(x) N_\ell(x)] = \left[\delta_{ij} - \frac{a^k_i N_k(x)}{|a^k_i N_k(x)|} \frac{a^\ell_j N_\ell(x)}{|a^\ell_j N_\ell(x)|} \right].$$

THEOREM 4.1. *Let $\nu > 0$, $\sigma > 0$ be given, and*

$$F \in L^2(0,T;H^2(\Omega)), \quad F_t \in L^2(0,T;L^2(\Omega)), \quad F(0) \in H^1(\Omega).$$

Suppose that the shell traction satisfies the compatibility condition

$$[\operatorname{Def} u_0 \cdot N]_{tan} = 0. \tag{4.3}$$

There exists $T > 0$ depending on u_0 and F such that there exists a solution $(v,h) \in Y_T$ of problem (2.16). Moreover, if $u_0 \in H^{5.5}(\Omega) \cap H^{7.5}(\Gamma)$ and the associated u_1, q_0 also satisfy the compatibility condition

$$\begin{aligned} CP := & \left[g_0^{ki} u^j_{0,k} N_j N_\ell + g_0^{k\ell} u^j_{0,k} N_j N_i \right] \left[\nu (\operatorname{Def} u_0)^j_i - q_0 \delta^j_i \right] N_j \\ & + \nu(\delta_{i\ell} - N_i N_\ell) \left[(\operatorname{Def} u_1)^j_i - \left((\nabla u_0 \nabla u_0) + (\nabla u_0 \nabla u_0)^T \right)^j_i \right] N_j \\ & - (\delta_{i\ell} - N_i N_\ell) \left[\nu (\operatorname{Def} u_0)^j_i - q_0 \delta^j_i \right] u^k_{0,j} N_k = 0, \end{aligned} \tag{4.4}$$

then the solution $(v,h) \in Y_T$ is unique.

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5. A bounded convex closed set of Y_T.

Definition 5.1. *Given $M > 0$, let $C_T(M)$ denote the subset of Y_T consisting of elements of (v, h) in Y_T such that*

$$\|(v,h)\|^2_{Y_T} \le M \tag{5.1}$$

and such that $v(0) = u_0$, $h(0) = 0$, and $h_t(0) = (B_0)_*((0,1)) \cdot u_0$.

Remark 9. *For $(v, h) \in C_T(M)$, define u^τ by (2.10) and let η^τ be the associated flow map. Also define v^τ as $u^\tau \circ \eta^\tau$. By (2.12) and (2.13), we have*

$$\sup_{t\in[0,T]} \|\nabla_0 \eta^\tau(t)\|_{H^{1.5}(\Gamma)} + \|v^\tau\|^2_{L^2(0,T;H^{2.5}(\Gamma))} \le C(M) \tag{5.2}$$

for some constant $C(M)$.

We will make use of the following lemmas (proved in [7]).

Lemma 5.2. *There exists $T_0 \in (0, T)$ such that for all $T \in (0, T_0)$ and for all $v \in C_T(M)$, the matrix a is well defined (by (2.15)) with the estimate (independent of $v \in C_T(M)$)*

$$\begin{aligned} &\|a\|_{L^\infty(0,T;H^2(\Omega))} + \|a_t\|_{L^\infty(0,T;H^1(\Omega))} + \|a_t\|_{L^2(0,T;H^2(\Omega))} \\ &+ \|a_{tt}\|_{L^\infty(0,T;L^2(\Omega))} + \|a_{tt}\|_{L^2(0,T;H^1(\Omega))} \le C(M). \end{aligned} \tag{5.3}$$

Lemma 5.3. *There exist $T_1 \in (0, T)$ and a constant C (independent of M) such that for all $T \in (0, T_1)$ and $v \in C_T(M)$, for all $\phi \in H^1(\Omega)$ and $t \in [0, T]$*

$$C\|\phi\|^2_{H^1(\Omega)} \le \int_\Omega \Big[|v|^2 + |D_\eta(v)|^2\Big] dx, \tag{5.4}$$

where

$$|D_\eta(v)|^2 := D_\eta(v)^i_j D_\eta(v)^i_j = (a^k_j v^i_{,k} + a^k_j v^i_{,k})(a^\ell_j v^i_{,\ell} + a^\ell_i v^j_{,\ell}).$$

In the remainder of the paper, we will assume that

$$0 < T < \min\{T_0, T_1, \bar{T}\}$$

for some fixed $\bar{T}$ where the forcing F is defined on the time interval $[0, \bar{T}]$.

6. Preliminary results.

6.1. Pressure as a Lagrange multiplier. In the following discussion, we use $H^{1;2}(\Omega; \Gamma)$ to denote the space $H^1(\Omega) \cap H^2(\Gamma)$ with norm

$$\|u\|^2_{H^{1;2}(\Omega;\Gamma)} = \|u\|^2_{H^1(\Omega)} + \|u\|^2_{H^2(\Gamma)}$$

and $\bar{\mathcal{V}}_{\bar{v}}$ ($\bar{\mathcal{V}}_{\bar{v}}(T)$) to denote the space

$$\Big\{v \in \mathcal{V}_{\bar{v}} \;\Big|\; v \in H^2(\Gamma)\Big\}\Big(\Big\{v \in \mathcal{V}_{\bar{v}}(T) \;\Big|\; v \in L^2(0,T;H^2(\Gamma))\Big\}\Big).$$

Lemma 6.1. *For all $p \in L^2(\Omega)$, $t \in [0, T]$, there exist a constant $C > 0$ and $\phi \in H^{1;2}(\Omega; \Gamma)$ such that $a^j_i(t)\phi^i_{,j} = p$ and*

$$\|\phi\|_{H^{1;2}(\Omega;\Gamma)} \le C\|p\|_{L^2(\Omega)}. \tag{6.1}$$

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Proof. We solve the following problem on the time-dependent domain $\Omega(t)$:

$$\operatorname{div}(\phi \circ \eta(t)^{-1}) = p \circ \eta(t)^{-1} \quad \text{in } \eta(t,\Omega) := \Omega(t).$$

The solution to this problem can be written as the sum of the solutions to the following two problems:

$$\operatorname{div}(\phi \circ \eta(t)^{-1}) = p \circ \eta(t)^{-1} - \bar{p}(t) \qquad \text{in} \quad \eta(t,\Omega), \tag{6.2}$$

$$\operatorname{div}(\phi \circ \eta(t)^{-1}) = \bar{p}(t) \qquad \text{in} \quad \eta(t,\Omega), \tag{6.3}$$

where $\bar{p}(t) = \frac{1}{|\Omega|}\int_\Omega p(t,x)dx$. The existence of the solution to problem (6.2) with zero boundary condition is standard (see, for example, [12, Chapter 3]), and the solution to problem (6.3) can be chosen as a linear function (linear in x), for example, $\bar{p}(t)x_1$. The estimate (6.1) follows from the estimates of the solutions to (6.2). ☐

Define the linear functional on $H^{1;2}(\Omega;\Gamma)$ by $(p, a_i^j(t)\varphi^i_{,j})_{L^2(\Omega)}$, where $\varphi \in H^{1;2}(\Omega;\Gamma)$. By the Riesz representation theorem, there is a bounded linear operator $Q(t) : L^2(\Omega) \to H^{1;2}(\Omega;\Gamma)$ such that for all $\varphi \in H^{1;2}(\Omega;\Gamma)$,

$$(p, a_i^j(t)\varphi^i_{,j})_{L^2(\Omega)} = (Q(t)p, \varphi)_{H^{1;2}(\Omega;\Gamma)} := (Q(t)p, \varphi)_{H^1(\Omega)} + (Q(t)p, \varphi)_{H^2(\Gamma)}.$$

Letting $\varphi = Q(t)p$ shows that

$$\|Q(t)p\|_{H^{1;2}(\Omega;\Gamma)} \le C\|p\|_{L^2(\Omega)}$$

for some constant $C > 0$. By Lemma 6.1,

$$\|p\|^2_{L^2(\Omega)} \le \|Q(t)p\|_{H^{1;2}(\Omega;\Gamma)}\|\varphi\|_{H^{1;2}(\Omega;\Gamma)} \le C\|Q(t)p\|_{H^{1;2}(\Omega;\Gamma)}\|p\|_{L^2(\Omega)},$$

which shows that $R(Q(t))$ is closed in $H^{1;2}(\Omega;\Gamma)$. Since $\bar{\mathcal{V}}_v(t) \subset R(Q(t))^\perp$ and $R(Q(t))^\perp \subset \bar{\mathcal{V}}_v(t)$, it follows that

$$H^{1;2}(\Omega;\Gamma)(t) = R(Q(t)) \oplus_{H^{1;2}(\Omega;\Gamma)} \bar{\mathcal{V}}_v(t). \tag{6.4}$$

We can now introduce our Lagrange multiplier.

LEMMA 6.2. *Let $\mathcal{L}(t) \in H^{1;2}(\Omega;\Gamma)'$ be such that $\mathcal{L}(t)\varphi = 0$ for any $\varphi \in \bar{\mathcal{V}}_v(t)$. Then there exists a unique $q(t) \in L^2(\Omega)$, which is termed the pressure function, satisfying*

$$\forall\ \varphi \in H^{1;2}(\Omega;\Gamma), \qquad \mathcal{L}(t)(\varphi) = (q(t), a_i^j(t)\varphi^i_{,j})_{L^2(\Omega)}.$$

Moreover, there is a $C > 0$ (which does not depend on $t \in [0,T]$ and ϵ_1 and on the choice of $v \in C_T(M)$) such that

$$\|q(t)\|_{L^2(\Omega)} \le C\|\mathcal{L}(t)\|_{H^{1;2}(\Omega;\Gamma)'}.$$

Proof. By the decomposition (6.4), for given $\tilde{a}$, let $\varphi = v_1 + v_2$, where $v_1 \in \mathcal{V}_v(t)$ and $v_2 \in R(Q(t)$. It follows that

$$\mathcal{L}(t)(\varphi) = \mathcal{L}(t)(v_2) = (\psi(t), v_2)_{H^{1;2}(\Omega;\Gamma)} = (\psi(t), \varphi)_{H^{1;2}(\Omega;\Gamma)}$$

for a unique $\psi(t) \in R(Q(t))$.

From the definition of $Q(t)$ we then get the existence of a unique $q(t) \in L^2(\Omega)$ such that

$$\forall\ \varphi \in H^{1;2}(\Omega;\Gamma), \qquad \mathcal{L}(t)(\varphi) = (q(t), a_i^j(t)\varphi^i_{,j})_{L^2(\Omega)}.$$

The estimate stated in the lemma is then a simple consequence of (6.1). ☐

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6.2. Estimates for a and h. We make use of near-identity transformations. The following lemmas can be found in [7].

LEMMA 6.3. *There exist $K > 0$ and $T_0 > 0$ such that if $0 < t \le T_0$, then, for any $(\tilde{v}, \tilde{h}) \in C_{T_0}(M)$,*

$$\|\tilde{a}^T - Id\|_{L^\infty(0,T;\mathcal{C}^0(\overline{\Omega}_0))} \le K\sqrt{t}, \tag{6.5a}$$

$$\|\tilde{a} - Id\|_{L^\infty(0,T;H^2(\Omega))} \le K\sqrt{t}, \tag{6.5b}$$

$$\|\tilde{a}_t - \tilde{a}_t(0)\|_{L^\infty(0,T;H^1(\Omega))} \le C(M)t, \tag{6.5c}$$

$$\|\tilde{a}_t\|_{L^\infty(0,T;H^1(\Omega))} \le K. \tag{6.5d}$$

We also need the following lemma.

LEMMA 6.4. *For any $(\tilde{v}, \tilde{h}) \in C_{T_0}(M)$,*

$$\|\tilde{h}\|_{H^{3.5}(\Gamma)} \le CMt^{1/4} \tag{6.6}$$

for all $0 < t \le T_0$.

Proof. For $(\tilde{v}, \tilde{h}) \in C_T(M)$, $\|\tilde{h}\|^2_{H^4(\Gamma)} + \|\tilde{h}_t\|^2_{H^2(\Gamma)} \le M$. By $\tilde{h}(0) = 0$,

$$\|\tilde{h}(t)\|_{H^2(\Gamma)} \le \int_0^t \|\tilde{h}_t\|_{H^2(\Gamma)} ds \le \sqrt{M}t.$$

Finally, the interpolation inequality

$$\|\nabla_0^2 f(t)\|_{H^{1.5}(\Gamma)} \le C\|\nabla_0^4 f\|^{3/4}_{L^2(\Gamma)} \|\nabla_0^2 f\|^{1/4}_{L^2(\Gamma)} \tag{6.7}$$

implies

$$\|\tilde{h}\|_{H^{3.5}(\Gamma)} \le C\|\tilde{h}\|^{3/4}_{H^4(\Gamma)} \|\tilde{h}\|^{1/4}_{H^2(\Gamma)} \le CMt^{1/4}. \qquad \square$$

COROLLARY 6.5. *$\|L_1(t)\|_{H^{1.5}(\Gamma)}$ and $\|L_2(t)\|_{H^{1.5}(\Gamma)}$ converge to zero as $t \to 0$, uniformly in $(v, h) \in C_{T_0}(M)$. Furthermore, for $t \le 1$,*

$$\|L_1(t)\|_{H^{1.5}(\Gamma)} + \|L_2(t)\|_{H^{1.5}(\Gamma)} \le C(M)t^{1/4}.$$

By the fact that $\|\tilde{h}_t\|^2_{H^2(\Gamma)} \le M$ and $\|\tilde{h}_{tt}\|^2_{L^2(0,T;H^{0.5}(\Gamma))} \le M$ if $(\tilde{v}, \tilde{h}) \in C_T(M)$, similar computations lead to the following lemma.

LEMMA 6.6. *For all $(\tilde{v}, \tilde{h}) \in C_T(M)$,*

$$\|\tilde{h}_t(t)\|_{H^{1.5}(\Gamma)} \le CMt^{1/8} \tag{6.8}$$

for all $0 < t \le T$.

7. The linearized problem. Suppose that $(\tilde{v}, \tilde{h}) \in C_T(M)$ is given. Let $\tilde{\eta}(t) = \text{Id} + \int_0^t \tilde{v}(s)ds$ and $\tilde{a} = (\nabla\tilde{\eta})^{-1}$. We are concerned with the following time-dependent linear problem, whose fixed point $v = \tilde{v}$ provides a solution to (2.16):

$$v_t^i - \nu[\tilde{a}_\ell^k D_{\tilde{\eta}}(v)_\ell^i]_{,k} = -(\tilde{a}_i^k q)_{,k} + F^i \quad \text{in } (0,T) \times \Omega, \tag{7.1a}$$

$$\tilde{a}_i^j v_{,j}^i = 0 \quad \text{in } (0,T) \times \Omega, \tag{7.1b}$$

$$\begin{aligned}[\nu D_{\tilde{\eta}}(v)_i^j - q\delta_i^j]\tilde{a}_j^\ell N_\ell = {} & \sigma\tilde{\Theta}\Big[\mathcal{L}_{\tilde{h}}(h)(-\nabla_0\tilde{h}, 1)\Big] \circ \tilde{\eta}^\tau \quad \text{on } (0,T) \times \Gamma, \\ & + \sigma\tilde{\Theta}\Big[[\mathcal{M}(\tilde{h})(-\nabla_0\tilde{h}, 1)] \circ \tilde{\eta}^\tau\Big]\end{aligned} \tag{7.1c}$$

$$h_t \circ \tilde{\eta}^\tau = [\tilde{h}_{,\alpha} \circ \tilde{\eta}^\tau] v_\alpha - v_z \quad \text{on } (0,T) \times \Gamma, \tag{7.1d}$$

$$v = u_0 \quad \text{on } \{t = 0\} \times \Omega, \tag{7.1e}$$

$$h = 0 \quad \text{on } \{t = 0\} \times \Gamma, \tag{7.1f}$$

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where $D_{\tilde{\eta}}(v)^j_i = \tilde{a}^k_i v^j_{,k} + \tilde{a}^k_j v^i_{,k}$, $\tilde{\Theta} = \det(\nabla_0 \tilde{\eta}^\tau)$, and

$$\mathcal{L}_{\tilde{h}}(h) = \frac{1}{\sqrt{\det(g_0)}} \Big[\sqrt{\det(g_0)} \tilde{A}^{\alpha\beta\gamma\delta} h_{,\alpha\beta} \Big]_{,\gamma\delta}$$

with

$$\tilde{A}^{\alpha\beta\gamma\delta} = J_{\tilde{h}}^{-3} \sqrt{\det(G_{\tilde{h}})} \Big[G^{\alpha\gamma}_{\tilde{h}} - (-1)^{\kappa+\sigma} \det(G_{\tilde{h}})^{-1} (1-\delta_{\alpha\kappa})(1-\delta_{\gamma\sigma}) \tilde{h}_{,\kappa} \tilde{h}_{,\sigma} \Big] \\ \times (G^{\beta\delta}_{\tilde{h}} - J^{-2}_{\tilde{h}} G^{\beta\mu}_{\tilde{h}} G^{\delta\nu}_{\tilde{h}} \tilde{h}_{,\mu} \tilde{h}_{,\nu})$$

and

$$\mathcal{M}(\tilde{h}) = \sqrt{\det(G_{\tilde{h}}) \circ \tilde{\eta}^\tau} \Big[L_1^{\alpha\beta\gamma}(y, \tilde{h}, D\tilde{h}, D^2\tilde{h}) \tilde{h}_{,\alpha\beta\gamma} + L_2(y, \tilde{h}, D\tilde{h}, D^2\tilde{h}) \Big].$$

Here the thickness ϵ_1 is assumed to be 1.

We will also use $L_{\tilde{h}}(h)$ to denote $\mathcal{L}_{\tilde{h}}(h) + \mathcal{M}(\tilde{h})$.

REMARK 10. *$\mathcal{L}_{\tilde{h}}$ is a coercive fourth order operator for small $\tilde{h} \le \delta$. Actually, it is easy to see that $\mathcal{L}_{\tilde{h}}$ is coercive at time $t = 0$, and the coercivity of $\mathcal{L}_{\tilde{h}}$ for $t > 0$ (but sufficiently small) follows from the continuity of $\tilde{h}$ in time into the space $H^2(\Gamma)$. Moreover, by Lemma* 6.4*, we have the following corollary.*

COROLLARY 7.1. *There exist $\nu_1 > 0$ and $0 < T \le T_0$ such that for all $0 < t \le T$,*

$$\nu_1 \|\nabla_0^2 f(t)\|^2_{L^2(\Gamma)} \le \int_\Gamma \tilde{A}^{\alpha\beta\gamma\delta} f_{,\alpha\beta}(t) f_{,\gamma\delta}(t) dS$$

for all $0 < t \le T$. Later we will denote the right-hand side quantity of this inequality by $E_{\bar{h}}(f)$, where the subscript $\bar{h}$ indicates that $\bar{A}$ is a function of $\bar{h}$.

REMARK 11. *Given $(\tilde{v}, \tilde{h}) \in \mathcal{V}^3(T) \times \mathcal{H}(T)$, for the corresponding $\tilde{\eta}^\tau$, we have*

$$\|\tilde{\eta}^\tau\|^2_{L^\infty(0,T;H^{2.5}(\Omega))} + \|\tilde{\eta}^\tau_t\|^2_{L^2(0,T;H^{2.5}(\Gamma))} \le C(M),$$

where (2.13) *and* (2.12) *are used to obtain this estimate.*

The solution of (7.1) is found as a weak limit of a sequence of regularized problems.

DEFINITION 7.2 (mollifiers on Γ). *For $\epsilon_1 > 0$, let*

$$K^p_{\epsilon_1} := (1 - \epsilon_1 \Delta_0)^{-\frac{p}{2}} : H^s(\Gamma) \to H^{s+p}(\Gamma)$$

denote the usual self-adjoint Friedrich mollifier on the compact manifold Γ, where Δ_0 is the surface Laplacian defined on Γ.

By the Sobolev extension theorem, there exist bounded extension operators

$$E_s : H^s(\Omega) \to H^s(\mathbb{R}^n), \qquad s \ge 1.$$

For fixed (but small) ϵ_1 and $\epsilon_{1 1} > 0$, let ρ_{ϵ_1} be a (positive) smooth mollifier on $\mathbb{R}^n$. Set $\bar{v} = \rho_{\epsilon_1} * E_1(\tilde{v})$, $\tilde{F} = \rho_{\epsilon_1} * E_2(F)$, $\tilde{u}_0 = \rho_{\epsilon_1} * E_3(u_0)$, where $*$ denotes the convolution in space, and $\bar{h} = K^m_{\epsilon_1}(\tilde{h})$ for large enough m. Define $\bar{\eta}$ and $\bar{a}$ in the same fashion as $\tilde{\eta}$ and $\tilde{a}$. Note that $\bar{v} \to \tilde{v} \in V(T)$, $\tilde{F} \to F$ in $\mathcal{V}^2(T)$, $\tilde{u_0} \to u_0$ in $H^{2.5}(\Omega)$, and $\bar{h} \to \tilde{h}$ in $\mathcal{H}(T)$ as $\epsilon_1 \to 0$.

The regularized problem takes the form

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$$v_t^i - \nu[\bar{a}_\ell^k D_{\bar{\eta}}(v)_\ell^i]_{,k} = -(\bar{a}_i^k q)_{,k} + \tilde{F}^i \quad \text{in } (0,T)\times\Omega, \tag{7.2a}$$

$$\bar{a}_i^j v_{,j}^i = 0 \quad \text{in } (0,T)\times\Omega, \tag{7.2b}$$

$$\begin{aligned}[\nu D_{\bar{\eta}}(v)_i^j - q\delta_i^j]\bar{a}_j^\ell N_\ell &= \sigma \mathcal{L}_{\bar{h}}^{\epsilon_2}(h^{\epsilon_2})(-\nabla_0 \bar{h}\circ\bar{\eta}^\tau, 1)\\ &\quad + \sigma \mathcal{M}_{\bar{h}}^{\epsilon_2}(-\nabla_0\bar{h}\circ\bar{\eta}^\tau,1) + \kappa\Delta_0^2 v \quad \text{on } (0,T)\times\Gamma,\end{aligned} \tag{7.2c}$$

$$h_t \circ \bar{\eta}^\tau = [(\bar{h}_{,\alpha})\circ\bar{\eta}^\tau]v_\alpha - v_z \quad \text{on } (0,T)\times\Gamma, \tag{7.2d}$$

$$v = \tilde{u}_0 \quad \text{on } \{t=0\}\times\Omega, \tag{7.2e}$$

$$h = 0 \quad \text{on } \{t=0\}\times\Gamma, \tag{7.2f}$$

where

$$\begin{aligned}\bar{\mathcal{L}}_{\bar{h}}^{\epsilon_2}(f) &= \frac{\bar{\Theta}}{\sqrt{\det(g_0)}}\Big[\Big(\sqrt{\det(g_0)}\bar{A}^{\alpha\beta\gamma\delta}f_{,\alpha\beta}\Big)_{,\gamma\delta}\Big]^{\epsilon_2}\circ\bar{\eta}^\tau,\\ \bar{\mathcal{M}}_{\bar{h}}^{\epsilon_2} &= \bar{\Theta}\Big[\Big(L_1^{\alpha\beta\gamma}(\cdot,\bar{h},D\bar{h},D^2\bar{h})\bar{h}_{,\alpha\beta\gamma} + L_2(\cdot,\bar{h},D\bar{h})\Big)^{\epsilon_2}\Big]^{\epsilon_2}\circ\bar{\eta}^\tau(y,t).\end{aligned}$$

Note that $\bar{\mathcal{L}}_{\bar{h}}^{\epsilon_2}(f) + \bar{\mathcal{M}}_{\bar{h}}^{\epsilon_2} = \bar{\Theta}\big[L_{\bar{h}}(f)\big]^{\epsilon_2}\circ\bar{\eta}^\tau$.

7.1. Weak solutions.

DEFINITION 7.3. *A vector $v \in \bar{\mathcal{V}}_{\bar{v}}(T)$ with $v_t \in \bar{\mathcal{V}}_{\bar{v}}(T)'$ for almost all (a.a.) $t\in(0,T)$ is a weak solution of* (7.2)*, provided that*

$$\begin{aligned}&\text{(i) } \langle v_t,\varphi\rangle + \frac{\nu}{2}\int_\Omega D_{\bar{\eta}}v : D_{\bar{\eta}}\varphi dx + \sigma\int_\Gamma \bar{A}^{\alpha\beta\gamma\delta}h_{,\alpha\beta}^{\epsilon_2}\Big[-\bar{h}_{,\sigma}(\varphi^\sigma\circ\bar{\eta}^{-\tau})\\ &\quad + (\varphi^z\circ\bar{\eta}^{-\tau})\Big]_{,\gamma\delta}^{\epsilon_2} dS + \kappa\int_\Gamma \Delta_0 v\cdot\Delta_0\varphi dS = \langle\tilde{F},\varphi\rangle - \sigma\langle\mathcal{M}_{\bar{h}}^{\epsilon_2},\varphi\rangle_\Gamma,\end{aligned} \tag{7.3a}$$

$$\text{(ii) } v(0,\cdot) = \tilde{u}_0 \tag{7.3b}$$

for a.a. $t\in[0,T]$, where $\langle\cdot,\cdot\rangle$ denotes the duality product between $\bar{\mathcal{V}}_v(t)$ and its dual $\bar{\mathcal{V}}_v(t)'$, and h is given by the evolution equation (7.2d) *and the initial condition* (7.2f)*:*

$$h(y,t) = \int_0^t \Big[-\bar{h}_{,\alpha}(y,s)v^\alpha(\bar{\eta}^{-\tau}(y,s),0,s) + v^z(\bar{\eta}^{-\tau}(y,s),0,s)\Big]ds. \tag{7.4}$$

7.2. Penalized problems. Letting $\theta > 0$ denote the penalized parameter, we define w_θ (also with ϵ_1 and ϵ_{11} dependence in mind) to be the "unique" solution of the problem (whose existence can be obtained via a modified Galerkin method which will be presented in the following sections):

$$\begin{aligned}&\text{(i)}\quad \langle w_{\theta t},\varphi\rangle + \frac{\nu}{2}\int_\Omega D_{\bar{\eta}}w_\theta : D_{\bar{\eta}}\varphi dx + \sigma\int_\Gamma \bar{A}^{\alpha\beta\gamma\delta}h_{,\alpha\beta}^{\epsilon_2}\Big[-\bar{h}_{,\sigma}(\varphi^\sigma\circ\bar{\eta}^{-\tau})\\ &\quad + (\varphi^z\circ\bar{\eta}^{-\tau})\Big]_{,\gamma\delta}^{\epsilon_2}dS + \kappa\int_\Gamma\Delta_0 v\cdot\Delta_0\varphi dS + \left(\frac{1}{\theta}\bar{a}_i^j v_{,j}^i, \bar{a}_k^\ell\varphi_{,\ell}^k\right)_{L^2(\Omega)}\\ &= \langle\tilde{F},\varphi\rangle - \sigma\langle\bar{\mathcal{M}}_{\bar{h}}^{\epsilon_2}(-\nabla_0\bar{h}\circ\bar{\eta}^\tau,1),\varphi\rangle_\Gamma,\end{aligned} \tag{7.5a}$$

$$\text{(ii) } v(0,\cdot) = \tilde{u}_0, \tag{7.5b}$$

where $\langle\cdot,\cdot\rangle$ denotes the pairing between $H^1(\Omega)$ and its dual, and h in (7.5a) satisfies (7.4) with v replaced by w_θ.

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7.3. Weak solutions for the penalized problem. The goal of this section is to establish the existence of v to the problem (7.2) (or the weak formulation (7.3)), as well as the energy inequality satisfied by v and v_t. Before proceeding, we introduce variables $\tilde{q}_0$ and $\tilde{w}_1$ as follows: let $\tilde{q}_0$ be the solution of the Laplace equation

$$\Delta\tilde{q}_0 = \nabla\tilde{u}_0 : (\nabla\tilde{u}_0)^t - \operatorname{div}\tilde{F}(0) \quad \text{in } \Omega, \tag{7.6a}$$

$$\tilde{q}_0 = \nu(\operatorname{Def}\tilde{u}_0)^j_i N_i N_j - \sigma\mathcal{M}_0^{\epsilon_2}(0) + \kappa\Delta_0^2\tilde{u}_0\cdot N \quad \text{on } \Gamma \tag{7.6b}$$

and $\tilde{w}_1$ be defined by

$$\tilde{w}_1 = \nu\Delta\tilde{u}_0 - \nabla\tilde{q}_0 + \tilde{F}(0). \tag{7.7}$$

By elliptic regularity,

$$\begin{aligned}\|\tilde{q}_0\|^2_{H^1(\Omega)} &\le C\Big[\|\tilde{u}_0\|^2_{H^2(\Omega)} + \|\tilde{F}(0)\|^2_{L^2(\Omega)} + \|\mathcal{M}_0^{\epsilon_2}(0)\|^2_{H^{0.5}(\Gamma)} + \|\Delta_0^2\tilde{u}_0\|^2_{H^{0.5}(\Gamma)}\Big]\\ &\le C(M)\Big[\|\tilde{u}_0\|^2_{H^2(\Omega)} + \|\tilde{u}_0\|^2_{H^{4.5}(\Gamma)} + \|\tilde{F}(0)\|^2_{L^2(\Omega)} + 1\Big],\end{aligned}$$

and hence

$$\|\tilde{w}_1\|^2_{L^2(\Omega)} \le C(M)\Big[\|\tilde{u}_0\|^2_{H^2(\Omega)} + \|\tilde{u}_0\|^2_{H^{4.5}(\Gamma)} + \|\tilde{F}(0)\|^2_{L^2(\Omega)} + 1\Big].$$

REMARK 12. *By* (6.6)*, the constant $C(M)$ in the estimates above can also be refined as a constant independent of M if T is chosen small enough.*

By introducing a (smooth) basis $(e_\ell)_{\ell=1}^\infty$ of $H^{1;2}(\Omega;\Gamma)$, taking the approximation at rank $m \ge 2$ under the form $w_\ell(t,x) = \sum_{k=1}^\ell d_k(t)e_k(x)$ with

$$h_\ell(y,t) = \int_0^t \Big[-\bar{h}_{,\alpha}(y,s)w_\ell^\alpha(\bar{\eta}^{-\tau}(y,s),0,s) + w_\ell^z(\bar{\eta}^{-\tau}(y,s),0,s)\Big]ds, \tag{7.8}$$

and satisfying on $[0,T]$,

(7.9a)

$$\begin{aligned}
&\text{(i)}\quad (w_{\ell tt},\varphi)_{L^2(\Omega)} + \nu(\bar{a}^j_i w_{\ell t,j}, \bar{a}^k_i\varphi_{,k})_{L^2(\Omega)} + \nu((\bar{a}^j_i\bar{a}^k_i)_t w_\ell, \varphi_{,k})_{L^2(\Omega)}\\
&+\nu\int_\Omega\Big[\bar{a}^j_r\bar{a}^k_i w^i_{\ell t,j} + (\bar{a}^j_r\bar{a}^k_i)_t w^i_{\ell,j}\Big]\varphi^r_{,k}dx + \kappa\int_\Gamma \Delta_0 w_{\ell t}\cdot\Delta_0\varphi dS - ((\bar{a}^j_i q_\ell)_t, \varphi^i_{,j})_{L^2(\Omega)}\\
&+\sigma\int_\Gamma \bar{A}^{\alpha\beta\gamma\delta}[-\bar{h}_{,\sigma}(w_\ell^\sigma\circ\bar{\eta}^{-\tau}) + w_\ell^z\circ\bar{\eta}^{-\tau}]^{\epsilon_2}_{,\alpha\beta}[-\bar{h}_{,\sigma}(\varphi^\sigma\circ\bar{\eta}^{-\tau}) + \varphi^z\circ\bar{\eta}^{-\tau}]^{\epsilon_2}_{,\gamma\delta}dS\\
&+\sigma\int_\Gamma (\bar{A}^{\alpha\beta\gamma\delta})_t h^{\epsilon_2}_{\ell,\alpha\beta}[-\bar{h}_{,\sigma}(\varphi^\sigma\circ\bar{\eta}^{-\tau}) + \varphi^z\circ\bar{\eta}^{-\tau}]^{\epsilon_2}_{,\gamma\delta}dS\\
&+\sigma\int_\Gamma \bar{A}^{\alpha\beta\gamma\delta}h^{\epsilon_2}_{\ell,\alpha\beta}[-\bar{h}_{t,\sigma}(\varphi^\sigma\circ\bar{\eta}^{-\tau}) + \bar{h}_{,\sigma}\bar{v}^\kappa(\varphi^\sigma_{,\kappa}\circ\bar{\eta}^{-\tau}) + \bar{v}^\kappa(\varphi^z_{,\kappa}\circ\bar{\eta}^{-\tau})]^{\epsilon_2}_{,\gamma\delta}dS\\
&= \langle\tilde{F}_t,\varphi\rangle - \sigma\int_\Gamma\Big[L_1^{\alpha\beta\gamma}\bar{h}_{,\alpha\beta\gamma} + L_2\Big]^{\epsilon_2}_t\Big[\bar{h}_{,\sigma}(\varphi^\sigma\circ\bar{\eta}^{-\tau}) - \varphi^z\circ\bar{\eta}^{-\tau}\Big]^{\epsilon_2}dS\\
&-\sigma\int_\Gamma\Big[L_1^{\alpha\beta\gamma}\bar{h}_{,\alpha\beta\gamma} + L_2\Big]^{\epsilon_2}\Big[\bar{h}_{t,\sigma}(\varphi^\sigma\circ\bar{\eta}^{-\tau}) - \bar{h}_{,\sigma}\bar{v}^\kappa(\varphi^\sigma_{,\kappa}\circ\bar{\eta}^{-\tau}) - \bar{v}^\kappa(\varphi^z_{,\kappa}\circ\bar{\eta}^{-\tau})\Big]^{\epsilon_2}dS\\
&\forall\,\varphi\in\operatorname{span}(e_1,\dots,e_\ell),
\end{aligned}$$

(7.9b)

$$\text{(ii)}\quad w_{\ell t}(0) = (w_1)_\ell,\ w_\ell(0) = (u_0)_\ell \quad \text{in } \Omega,$$

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where $q_\ell = \tilde{q}_0 - \frac{1}{\theta}\bar{a}_i^j w^i_{\ell,j}$, and $(\tilde{u}_0)_\ell$ denotes the respective $H^{1;2}(\Omega;\Gamma)$ projections of u_0 on $\text{span}(e_1, e_2, \dots, e_\ell)$.

REMARK 13. *The existence of w_k follows from the solution of*

$$d''_k(t) + d'_\ell(t)A_{k\ell}(t) + d_\ell(t)B_{k\ell}(t) + \int_0^t d_\ell(s)C_{k\ell}(s,t)ds = F(t)$$

for functions A, B, C, and F; however, the existence of the solution d_k does not immediately follow from the fundamental theorem of ODE due to the presence of the time integral. A straightforward fixed-point argument can be implemented, whose details we leave to the interested reader.

The use of the test function $\varphi = w_{\ell t}$ in this system of ODE gives us, in turn, the energy law

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w_{\ell t}\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|D_{\bar\eta}(w_{\ell t})\|^2_{L^2(\Omega)} + \frac{\sigma}{2}\frac{d}{dt}E_{\bar h}(h^{\epsilon_2}_{\ell t,\alpha\beta}) + \theta\|q_{\ell t}\|^2_{L^2(\Omega)}\\
&+ \nu((\bar{a}_i^j\bar{a}_i^k)_t w_{\ell,j}, w_{\ell t,k})_{L^2(\Omega)} + \nu\int_\Omega (\bar{a}_r^j\bar{a}_i^k)_t w^i_{\ell,j} w^r_{\ell t,k}dx + \kappa\|\Delta_0 w_{\ell t}\|^2_{L^2(\Gamma)}\\
&+ (q_{\ell t}, \bar{a}^j_{it} w^i_{\ell,j})_{L^2(\Omega)} - (q_\ell, \bar{a}^j_{it} w^i_{\ell t,j})_{L^2(\Omega)} - \frac{\sigma}{2}\int_\Gamma (\bar{A}^{\alpha\beta\gamma\delta})_t h^{\epsilon_2}_{\ell t,\alpha\beta} h^{\epsilon_2}_{\ell t,\gamma\delta}dS\\
&- \sigma\int_\Gamma (\bar{A}^{\alpha\beta\gamma\delta})_t h^{\epsilon_2}_{\ell,\alpha\beta}\Big[h_{\ell tt} + \bar{h}_{t,\sigma}(w^\sigma_{\ell t}\circ\bar\eta^{-\tau})\Big]^{\epsilon_2}_{,\gamma\delta}dS + \sigma\int_\Gamma \bar{A}^{\alpha\beta\gamma\delta}h^{\epsilon_2}_{\ell,\alpha\beta}\\
&\times\Big[-\bar{h}_{t,\sigma}(w^\sigma_{\ell t}\circ\bar\eta^{-\tau}) + \bar{h}_{,\sigma}\bar{v}^\kappa(w^\sigma_{\ell t,\kappa}\circ\bar\eta^{-\tau}) + \bar{v}^\kappa(w^z_{\ell t,\kappa}\circ\bar\eta^{-\tau})\Big]^{\epsilon_2}_{,\gamma\delta}dS\\
=\;& \langle\tilde{F}_t, w_{\ell t}\rangle - \sigma\int_\Gamma \Big[(L_1^{\alpha\beta\gamma}\bar{h}_{,\alpha\beta\gamma} + L_2)(-\nabla_0\bar{h}, 1)\Big]_t\cdot(w_{\ell t}\circ\bar\eta^{-\tau})dS\\
&- \sigma\int_\Gamma (L_1^{\alpha\beta\gamma}\bar{h}_{,\alpha\beta\gamma} + L_2)\bar{v}^\kappa\Big[-\bar{h}_{,\sigma}(w^\sigma_{\ell t,\kappa}\circ\bar\eta^{-\tau}) + (w^z_{\ell t,\kappa}\circ\bar\eta^{-\tau})\Big]dS.
\end{aligned}\tag{7.10}$$

For the tenth term (the integral with $\frac{\sigma}{2}$ as its coefficient), we have

$$\left|\int_\Gamma (\bar{A}^{\alpha\beta\gamma\delta})_t h^{\epsilon_2}_{\ell t,\alpha\beta} h^{\epsilon_2}_{\ell t,\gamma\delta}dS\right| \le C(M)\|\bar{h}_t\|_{H^{2.5}(\Gamma)}\|\nabla_0^2 h_{\ell t}\|^2_{L^2(\Gamma)}.$$

By ϵ_2-regularization and the identity

$$\begin{aligned}
\int_\Gamma (\bar{A}^{\alpha\beta\gamma\delta})_t h^{\epsilon_2}_{\ell,\alpha\beta} h^{\epsilon_2}_{\ell tt,\gamma\delta}dS &= \int_\Gamma \frac{1}{\sqrt{\det(g_0)}}\Big[\sqrt{\det(g_0)}(\bar{A}^{\alpha\beta\gamma\delta})_t\Big]_{,\gamma\delta} h^{\epsilon_2}_{\ell,\alpha\beta} h^{\epsilon_2}_{\ell tt}dS\\
&+ \int_\Gamma \frac{2}{\sqrt{\det(g_0)}}\Big[\sqrt{\det(g_0)}(\bar{A}^{\alpha\beta\gamma\delta})_t\Big]_{,\gamma} h^{\epsilon_2}_{\ell,\alpha\beta\delta} h^{\epsilon_2}_{\ell tt}dS\\
&+ \int_\Gamma (\bar{A}^{\alpha\beta\gamma\delta})_t h^{\epsilon_2}_{\ell,\alpha\beta\gamma\delta} h^{\epsilon_2}_{\ell tt}dS,
\end{aligned}$$

we find that

$$\begin{aligned}
&\left|\int_\Gamma (\bar{A}^{\alpha\beta\gamma\delta})_t h^{\epsilon_2}_{\ell,\alpha\beta} h^{\epsilon_2}_{\ell tt,\gamma\delta}dS\right|\\
&\le C(\epsilon_2)\Big[1 + \|\bar{h}_t\|_{H^{2.5}(\Gamma)}\Big]\|\nabla_0^2 h_\ell\|_{L^2(\Gamma)}\Big[\|w_\ell\|_{H^1(\Omega)} + \|w_{\ell t}\|_{H^1(\Omega)}\Big].
\end{aligned}$$

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Similarly, the second part of the eleventh term and the last term of the left-hand side can be bounded by

$$C(\epsilon_2)\|\bar{h}_t\|_{H^{2.5}(\Gamma)}\|\nabla_0^2 h_\ell\|_{L^2(\Gamma)}\|w_{\ell t}\|_{H^1(\Omega)},$$

where we also use the ϵ_2-regularization to control $\nabla_0^3 w_{\ell t}$. It also follows that the last two terms on the right-hand side can be bounded by

$$C(M)\Big[1+\|\bar{h}_t\|_{H^{2.5}(\Gamma)}\Big]\|w_{\ell t}\|_{H^1(\Omega)}.$$

With positive θ, the fourth term of the left-hand side involving the square of $q_{\ell t}$ acts as a viscous energy term. Integrating (7.10) in time from 0 to t, we then get

(7.11)
$$\begin{aligned}
&\|w_{\ell t}\|^2_{L^2(\Omega)}+\|\nabla_0^2 h_{\ell t}\|^2_{L^2(\Gamma)}+\int_0^t\Big[\|\nabla w_{\ell t}\|^2_{L^2(\Omega)}+\kappa\|w_{\ell t}\|^2_{H^2(\Gamma)}+\theta\|q_{\ell t}\|^2_{L^2(\Omega)}\Big]ds\\
&\le C(M)\Big[\|w_{\ell t}(0)\|^2_{L^2(\Omega)}+\|w_\ell(0)\|^2_{H^1(\Omega)}+\|q_\ell(0)\|^2_{H^{0.5}(\Omega)}\Big]\\
&\quad+C(\epsilon_2)\int_0^t\Big[1+\|\bar{h}_t(s)\|^2_{H^{2.5}(\Gamma)}\Big]\|\nabla_0^2 h_{\ell t}(s)\|^2_{L^2(\Gamma)}ds\\
&\quad+C(\theta)\int_0^t\|\bar{v}(t')\|^2_{H^3(\Omega)}\int_0^{t'}\Big[\|\nabla w_{\ell t}(s)\|^2_{L^2(\Omega)}+\|q_{\ell t}(s)\|^2_{L^2(\Omega)}\Big]ds\,dt',
\end{aligned}$$

where $C(\epsilon_2), C(\theta)\to\infty$ as $\epsilon_2,\theta\to 0$, and we use

$$\|f(t)\|_X\le\|f(0)\|_X+\int_0^t\|f_t(s)\|_X ds\le\|f(0)\|_X+\sqrt{t}\int_0^t\|f_t(s)\|^2_X ds$$

for $f=w_\ell$, $f=h_\ell$, and $f=g_\ell$ to obtain (7.11).

REMARK 14. *The θ-dependence follows from estimating the terms $(q_{\ell t},\bar{a}^j_{it}w^i_{\ell,j})_{L^2(\Omega)}$:*

$$\begin{aligned}
&\Big|(q_{\ell t},\bar{a}^j_{it}w^i_{\ell,j})_{L^2(\Omega)}\Big|\le\frac{\theta}{2}\|q_{\ell t}\|^2_{L^2(\Omega)}+\frac{1}{2\theta}\|\bar{a}^j_{it}\|^2_{L^\infty(\Omega)}\|w^i_{\ell,j}\|^2_{L^2(\Omega)}\\
&\le\frac{\theta}{2}\|q_{\ell t}\|^2_{L^2(\Omega)}+\frac{C(M)}{\theta}\Big[\|\nabla w_\ell(0)\|^2_{L^2(\Omega)}+t\int_0^t\|\nabla w_{\ell t}\|^2_{L^2(\Omega)}(s)ds\Big].
\end{aligned}$$

By the Gronwall inequality, for $0\le t\le T$,

$$\begin{aligned}
&\|w_{\ell t}(t)\|^2_{L^2(\Omega)}+\|\nabla_0^2 h_{\ell t}(t)\|^2_{L^2(\Gamma)}\\
&+\int_0^t\Big[\|\nabla w_{\ell t}\|^2_{L^2(\Omega)}+\kappa\|w_{\ell t}\|^2_{H^2(\Gamma)}+\theta\|q_{\ell t}\|^2_{L^2(\Omega)}\Big]ds\le C(\epsilon_2,\theta)N_0(u_0,F),
\end{aligned}\tag{7.12}$$

where

$$N_0(u_0,F):=\|u_0\|^2_{H^{2.5}(\Omega)}+\|u_0\|^2_{H^{4.5}(\Gamma)}+\|F_t\|^2_{L^2(0,T;H^1(\Omega)')}+\|F(0)\|^2_{H^{0.5}(\Omega)}+1.$$

We can then infer that w_ℓ is defined on $[0,T]$, and that there is a subsequence, still denoted with the subscript ℓ, satisfying

(7.13a) $\qquad w_\ell\rightharpoonup w_\theta \quad$ in $L^2(0,T;H^{1;2}(\Omega;\Gamma))$,

(7.13b) $\qquad w_{\ell t}\rightharpoonup w_{\theta t} \quad$ in $L^2(0,T;H^{1;2}(\Omega;\Gamma))$,

(7.13c) $\qquad \nabla_0^2 h_\ell\rightharpoonup\nabla_0^2 h_\theta \quad$ in $L^2(0,T;L^2(\Gamma))$,

(7.13d) $\qquad \nabla_0^2 h_{\ell t}\rightharpoonup\nabla_0^2 h_{\theta t} \quad$ in $L^2(0,T;L^2(\Gamma))$,

(7.13e) $\qquad q_{\ell t}\rightharpoonup q_{\theta t} \quad$ in $L^2(0,T;L^2(\Omega))$,

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where

$$q_\theta = \tilde{q}_0 - \frac{1}{\theta}\bar{a}_i^j w^i_{\theta,j}.$$

From the standard procedure for weak solutions, we can now infer from these weak convergences and the definition of w_ℓ that $w_{\ell tt} \in L^2(0,T;H^1(\Omega)')$. In turn, $w_{\ell t} \in \mathcal{C}^0([0,T];H^1(\Omega)')$, $w_\ell \in \mathcal{C}^0([0,T];L^2(\Omega))$ with $w_\theta(0) = u_0$, $w_{\theta t}(0) = w_1$.

Moreover, (7.13) implies that w_θ satisfies

(7.14a)

$$\begin{aligned}
\text{(i)}\quad & \int_0^T \Big[(w_{\theta tt},\varphi)_{L^2(\Omega)} + \nu(\bar{a}_i^j w_{\theta t,j}, \bar{a}_i^k\varphi_{,k})_{L^2(\Omega)} + \nu((\bar{a}_i^j\bar{a}_i^k)_t w_\theta, \varphi_{,k})_{L^2(\Omega)}\Big]dt \\
& + \nu\int_0^T \left[\int_\Omega \bar{a}_r^j\bar{a}_i^k w^i_{\theta t,j}\varphi^r_{,k}dx + \nu\int_\Omega (\bar{a}_r^j\bar{a}_i^k)_t w^i_{\theta,j}\varphi^r_{,k}dx\right]dt + \sigma\int_0^T\int_\Gamma \bar{A}^{\alpha\beta\gamma\delta} \\
& \quad \times[-\bar{h}_{,\sigma}(w_\theta^\sigma\circ\bar{\eta}^{-\tau}) + w_\theta^z\circ\bar{\eta}^{-\tau}]^{\epsilon_2}_{,\alpha\beta}[-\bar{h}_{,\sigma}(\varphi^\sigma\circ\bar{\eta}^{-\tau}) + \varphi^z\circ\bar{\eta}^{-\tau}]^{\epsilon_2}_{,\gamma\delta}dSdt \\
& + \sigma\int_0^T\int_\Gamma (\bar{A}^{\alpha\beta\gamma\delta})_t h^{\epsilon_2}_{\theta,\alpha\beta}[-\bar{h}_{,\sigma}(\varphi^\sigma\circ\bar{\eta}^{-\tau}) + \varphi^z\circ\bar{\eta}^{-\tau}]^{\epsilon_2}_{,\gamma\delta}dSdt \\
& + \sigma\int_0^T\int_\Gamma \bar{A}^{\alpha\beta\gamma\delta}h^{\epsilon_2}_{\theta,\alpha\beta}[-\bar{h}_{t,\sigma}(\varphi^\sigma\circ\bar{\eta}^{-\tau}) + \bar{h}_{,\sigma}\bar{v}^\kappa(\varphi^\sigma_{,\kappa}\circ\bar{\eta}^{-\tau}) + \bar{v}^\kappa(\varphi^z_{,\kappa}\circ\bar{\eta}^{-\tau})]^{\epsilon_2}_{,\gamma\delta}dSdt \\
& + \kappa\int_0^T\int_\Gamma \Delta_0 w_{\theta t}\cdot\Delta_0\varphi dSdt - \int_0^T ((\bar{a}_i^j q_\theta)_t, \varphi^i_{,j})_{L^2(\Omega)}dt \\
& = \int_0^T \Bigg\{\langle\tilde{F}_t,\varphi\rangle - \sigma\int_\Gamma \Big[L_1^{\alpha\beta\gamma}\bar{h}_{,\alpha\beta\gamma} + L_2\Big]^{\epsilon_2}_t\Big[\bar{h}_{,\sigma}(\varphi^\sigma\circ\bar{\eta}^{-\tau}) - \varphi^z\circ\bar{\eta}^{-\tau}\Big]^{\epsilon_2}dS \\
& \qquad - \sigma\int_\Gamma \Big[L_1^{\alpha\beta\gamma}\bar{h}_{,\alpha\beta\gamma} + L_2\Big]^{\epsilon_2}\Big[\bar{h}_{t,\sigma}(\varphi^\sigma\circ\bar{\eta}^{-\tau}) - \bar{h}_{,\sigma}\bar{v}^\kappa(\varphi^\sigma_{,\kappa}\circ\bar{\eta}^{-\tau}) \\
& \qquad\qquad - \bar{v}^\kappa(\varphi^z_{,\kappa}\circ\bar{\eta}^{-\tau})\Big]^{\epsilon_2}dS\Bigg\}dt,
\end{aligned}$$

(7.14b)

$$\text{(ii)}\quad w_{\theta t}(0) = \tilde{w}_1,\ w_\theta(0) = \tilde{u}_0 \quad \text{in } \Omega$$

for all $\varphi \in L^2(0,T;H^{1;2}(\Omega;\Gamma))$. Choosing φ to be independent of time, we find that for all $t\in[0,T]$,

$$\begin{aligned}
& (w_{\theta t},\varphi)_{L^2(\Omega)} + \frac{\nu}{2}\int_\Omega D_{\bar{\eta}}(w_\theta) : D_{\bar{\eta}}(\varphi)dx + \kappa\int_\Gamma \Delta_0 w_\theta\cdot\Delta_0\varphi dS \\
& + \sigma\int_\Gamma \bar{A}^{\alpha\beta\gamma\delta}h^{\epsilon_2}_{\theta,\alpha\beta}[-\bar{h}_{,\sigma}(\varphi^\sigma\circ\bar{\eta}^{-\tau}) + \varphi^z\circ\bar{\eta}^{-\tau}]^{\epsilon_2}_{,\gamma\delta}dS - (\bar{a}_i^j q_\theta, \varphi^i_{,j})_{L^2(\Omega)} \\
= {} & \langle\tilde{F},\varphi\rangle + \sigma\int_\Gamma \Big[L_1^{\alpha\beta\gamma\delta}\bar{h}_{,\alpha\beta\gamma} + L_2\Big]^{\epsilon_2}\Big[-\bar{h}_{,\sigma}\varphi^\sigma\circ\bar{\eta}^{-\tau} + \varphi^z\circ\bar{\eta}^{-\tau}\Big]^{\epsilon_2}dS + c(\varphi)
\end{aligned}$$

for all $\varphi \in H^{1;2}(\Omega;\Gamma)$, where $c(\varphi)\in\mathbb{R}$ is given by

$$\begin{aligned}
c(\varphi) = {} & (\tilde{w}_1,\varphi)_{L^2(\Omega)} + \frac{\nu}{2}\int_\Omega \mathrm{Def}(\tilde{u}_0) : \mathrm{Def}\,\varphi dx - \left(\tilde{q}_0 - \frac{1}{\theta}\operatorname{div}\tilde{u}_0, \operatorname{div}\varphi\right)_{L^2(\Omega)} \\
& - (\tilde{F}(0),\varphi)_{L^2(\Omega)} - \sigma(\bar{\mathcal{M}}_0^{\epsilon_2}(0)(0,1),\varphi)_{L^2(\Gamma)} + \kappa(\Delta_0\tilde{u}_0, \Delta_0\varphi)_{L^2(\Gamma)}.
\end{aligned}$$

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By compatibility conditions (7.6) and (7.7), $c(\varphi) = 0$. Therefore, the weak limit (w_θ, h_θ) satisfies, for all $t \in [0,T]$,

$$\begin{aligned}
&(w_{\theta t}, \varphi)_{L^2(\Omega)} + \frac{\nu}{2}\int_\Omega D_{\bar\eta}(w_\theta) : D_{\bar\eta}(\varphi)dx + \kappa\int_\Gamma \Delta_0 w_\theta \cdot \Delta_0\varphi dS \\
&- (\bar a_i^j q_\theta, \varphi^i_{,j})_{L^2(\Omega)} + \sigma\int_\Gamma \bar A^{\alpha\beta\gamma\delta} h^{\epsilon_2}_{\theta,\alpha\beta}[-\bar h_{,\sigma}(\varphi^\sigma\circ\bar\eta^{-\tau}) + \varphi^z\circ\bar\eta^{-\tau}]^{\epsilon_2}_{,\gamma\delta} dS \\
&= \langle\tilde F, \varphi\rangle - \sigma\int_\Gamma \left[L_1^{\alpha\beta\gamma\delta}\bar h_{,\alpha\beta\gamma} + L_2\right]^{\epsilon_2}\left[-\bar h_{,\sigma}\varphi^\sigma\circ\bar\eta^{-\tau} + \varphi^z\circ\bar\eta^{-\tau}\right]^{\epsilon_2} dS
\end{aligned} \tag{7.15}$$

for all $\varphi \in H^{1;2}(\Omega;\Gamma)$.

Since $w_\theta \in L^2(0,T;H^{1;2}(\Omega;\Gamma))$, we can use it as a test function in (7.15) and obtain (after time integration)

$$\begin{aligned}
&\frac{1}{2}\|w_\theta\|^2_{L^2(\Omega)} + \frac{\sigma}{2}E_{\bar h}(h_\theta^{\epsilon_2}) + \int_0^t \Big[\frac{\nu}{2}\|D_{\bar\eta}w_\theta\|^2_{L^2(\Omega)} + \kappa\|\Delta_0 w_\theta\|^2_{L^2(\Gamma)} \\
&+ \theta\|q_\theta\|^2_{L^2(\Omega)}\Big]ds - \theta\int_0^t (q_\theta, \tilde q_0)dt - \frac{\sigma}{2}\int_0^t\int_\Gamma (\bar A^{\alpha\beta\gamma\delta})_t h^{\epsilon_2}_{\theta,\alpha\beta} h^{\epsilon_2}_{\theta,\gamma\delta} dS ds \\
&= \frac{1}{2}\|\tilde u_0\|^2_{L^2(\Omega)} + \int_0^t \langle\tilde F, \varphi\rangle + \sigma\langle\bar{\mathcal M}^{\epsilon_2}_{\bar h}(-\nabla_0\bar h\circ\bar\eta^\tau, 1), \varphi\rangle_\Gamma dt.
\end{aligned} \tag{7.16}$$

Consequently,

$$\begin{aligned}
&\left[\|w_\theta(t)\|^2_{L^2(\Omega)} + \|\nabla_0^2 h_\theta^{\epsilon_2}(t)\|^2_{L^2(\Gamma)}\right] + \int_0^t \|\nabla w_\theta\|^2_{L^2(\Omega)}ds + \kappa\int_0^t \|w_\theta\|^2_{H^2(\Gamma)}ds \\
&+ \theta\int_0^t \|q_\theta\|^2_{L^2(\Omega)}ds \\
&\le C(M)\Big[\|\tilde u_0\|^2_{L^2(\Omega)} + \theta\|\tilde q_0\|^2_{L^2(\Omega)} + \|\tilde F\|^2_{H^1(\Omega)'} + \|\bar{\mathcal M}^{\epsilon_2}_{\bar h}(-\nabla_0\bar h\circ\bar\eta^\tau, 1)\|^2_{L^2(\Gamma)}\Big] \\
&+ C(M)\int_0^t \|\bar h_t\|_{H^{2.5}(\Gamma)}\|\nabla_0^2 h_\theta^{\epsilon_2}\|^2_{L^2(\Gamma)}ds \\
&\le C(M)\left[N_1(u_0,F) + \int_0^t \|\bar h_t\|_{H^{2.5}(\Gamma)}\|\nabla_0^2 h_\theta^{\epsilon_2}\|^2_{L^2(\Gamma)}ds\right],
\end{aligned}$$

where

$$\begin{aligned}
N_1(u_0,F) = {}& \|u_0\|^2_{H^2(\Omega)} + \|u_0\|^2_{H^{4.5}(\Gamma)} + \|F\|^2_{L^2(0,T;H^1(\Omega)')} + \|F_t\|^2_{L^2(0,T;H^1(\Omega)')} \\
&+ \|F(0)\|^2_{H^1(\Omega)} + 1.
\end{aligned}$$

By the Gronwall inequality,

$$\begin{aligned}
&\sup_{0\le t\le T}\left[\|w_\theta(t)\|^2_{L^2(\Omega)} + \|\nabla_0^2 h_\theta^{\epsilon_2}(t)\|^2_{L^2(\Gamma)}\right] + \int_0^T \left[\|\nabla w_\theta\|^2_{L^2(\Omega)} + \theta\|q_\theta\|^2_{L^2(\Omega)}\right]ds \\
&\le C(M)N_1(u_0,F).
\end{aligned} \tag{7.17}$$

7.4. Improved pressure estimates. By ϵ_2-regularization, we can rewrite (7.15) as, for a.a. $t \in [0,T]$,

$$\begin{aligned}
&(w_{\theta t}, \varphi)_{L^2(\Omega)} + \frac{\nu}{2}\int_\Omega D_{\bar\eta}(w_\theta) : D_{\bar\eta}(\varphi)dx + \kappa(\Delta_0 w_\theta, \Delta_0\varphi)_{L^2(\Gamma)} - (\bar a_i^j q_\theta, \varphi^i_{,j})_{L^2(\Omega)} \\
&+ \sigma\int_\Gamma \bar{\mathcal L}^{\epsilon_2}_{\bar h}(h_\theta^{\epsilon_2})\left[-\bar h_{,\sigma}\circ\bar\eta^\tau\varphi^\sigma + \varphi^z\right]dS = \langle\tilde F, \varphi\rangle + \sigma\langle\bar{\mathcal M}^{\epsilon_2}_{\bar h}(-\nabla_0\bar h\circ\bar\eta^\tau, 1), \varphi\rangle_\Gamma.
\end{aligned}$$

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Therefore, by the Lagrange multiplier lemma, we conclude that

$$\|q_\theta\|^2_{L^2(\Omega)} \le C(M)\Big[\|w_{\theta t}\|^2_{H^1(\Omega)'} + \|\nabla w_\theta\|^2_{L^2(\Omega)} + \|\tilde F\|^2_{H^1(\Omega)'} + \kappa\|\Delta_0^2 w_\theta\|^2_{H^{-2}(\Gamma)} + \|[\bar{\mathcal{L}}^{\epsilon_2}_{\bar h}(h^{\epsilon_2}_\theta) + \bar{\mathcal{M}}^{\epsilon_2}_{\bar h}](-\nabla_0 \bar h \circ \bar\eta^\tau, 1)\|^2_{H^{-2}(\Gamma)}\Big],$$

and hence

$$\|q_\theta\|^2_{L^2(\Omega)} \le C(M)\Big[\|w_{\theta t}\|^2_{L^2(\Omega)} + \|\nabla w_\theta\|^2_{L^2(\Omega)} + \kappa\|w_\theta\|^2_{H^2(\Gamma)} + \|\nabla_0^2 h_\theta\|^2_{L^2(\Gamma)} + \|F\|^2_{H^1(\Omega)'} + 1\Big]. \tag{7.18}$$

7.5. Weak limits as $\theta \to 0$. Since $w_{\theta t} \in L^2(0,T;H^{1;2}(\Omega;\Gamma))$, we can use it as a test function in (7.14). Similar to the way we obtain (7.11), we find that

$$\begin{aligned}
&\frac{1}{2}\|w_{\theta t}\|^2_{L^2(\Omega)} + \frac{\nu}{2}\int_0^t \|D_{\bar\eta} w_{\theta t}\|^2_{L^2(\Omega)}ds + \frac{\sigma}{2}E_{\bar h}(h^{\epsilon_2}_{\theta t}) + \kappa\int_0^t \|\Delta_0^2 w_{\theta t}\|^2_{L^2(\Gamma)}ds\\
&\quad + \theta\int_0^t \|q_{\theta t}\|^2_{L^2(\Omega)}ds + \int_0^t (q_{\theta t}, \bar a^j_{it} w^i_{\theta,j})_{L^2(\Omega)}ds - \int_0^t (q_\theta, \bar a^j_i w^i_{\theta t,j})ds\\
&\le C(M)N_0(u_0,F) + C(M)\int_0^t \|\bar v(t')\|^2_{H^3(\Omega)}\int_0^{t'} \|\nabla w_{\theta t}(s)\|^2_{L^2(\Omega)}ds dt'\\
&\quad + C(\epsilon_2)\int_0^t \Big[1 + \|\bar h_t\|_{H^{2.5}(\Gamma)}\Big]\|\nabla_0^2 h^{\epsilon_2}_{\theta t}\|^2_{L^2(\Gamma)}ds.
\end{aligned}$$

By (7.18),

$$\begin{aligned}
&\left|\int_0^t (q_\theta, \bar a^j_i w^i_{\theta t,j})ds\right| \le C(M,\delta)\int_0^t \|q_\theta\|^2_{L^2(\Omega)}ds + \delta\int_0^t \|\nabla w_{\theta t}\|^2_{L^2(\Omega)}ds\\
&\le C(M)\Big[N_1(u_0,F) + \int_0^t \Big(\|w_{\theta t}\|^2_{L^2(\Omega)} + \kappa\|w_\theta\|^2_{H^2(\Gamma)} + \|\nabla_0^2 h_\theta\|^2_{L^2(\Gamma)}\Big)ds\Big]\\
&\quad + \delta\int_0^t \|\nabla w_{\theta t}\|^2_{L^2(\Omega)}ds,
\end{aligned} \tag{7.19}$$

where (7.17) is used to bound $\|\nabla w_\theta\|^2_{L^2(0,T;L^2(\Omega))}$.

Integrating by parts,

$$\begin{aligned}
\int_0^t (q_{\theta t}, \bar a^j_{it} w^i_{\theta,j})_{L^2(\Omega)}ds &= (q_\theta, \bar a^j_{it} w^i_{\theta,j})_{L^2(\Omega)}(t) + (\tilde q_0, \tilde u^j_{0,i}\tilde u^i_{0,j})_{L^2(\Omega)}\\
&\quad - \int_0^t (q_\theta, \bar a^j_{itt} w^i_{\theta,j})_{L^2(\Omega)}ds - \int_0^t (q_\theta, \bar a^j_{it} w^i_{\theta t,j})_{L^2(\Omega)}ds.
\end{aligned}$$

By ϵ_1-regularization, the last two terms can be bounded by

$$C(M)\int_0^t \|q_\theta\|_{L^2(\Omega)}\Big[C(\epsilon_1)\|\nabla w_\theta\|_{L^2(\Omega)} + \|\nabla w_{\theta t}\|_{L^2(\Omega)}\Big]ds,$$

and hence

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$$\left|\int_0^t (q_\theta, \bar{a}^j_{itt} w^i_{\theta,j})_{L^2(\Omega)} ds\right| + \left|\int_0^t (q_\theta, \bar{a}^j_{it} w^i_{\theta t,j})_{L^2(\Omega)} ds\right|$$
$$\leq C(M,\delta)\int_0^t \|q_\theta\|^2_{L^2(\Omega)} ds + C(\epsilon_1)\int_0^t \|\nabla w_\theta\|^2_{L^2(\Omega)} ds + \delta\int_0^t \|\nabla w_{\theta t}\|^2_{L^2(\Omega)} ds$$
$$\leq C(\epsilon_1,\delta)N_1(u_0,F) + C(M,\delta)\int_0^t \|w_{\theta t}\|^2_{L^2(\Omega)} ds + C(\epsilon_2)\int_0^t \|\nabla_0^2 h_\theta\|^2_{L^2(\Gamma)} ds$$
$$+\,\delta\int_0^t \|\nabla w_{\theta t}\|^2_{L^2(\Omega)} ds. \tag{7.20}$$

For $(q_\theta, \bar{a}^j_{it} w^i_{\theta,j})_{L^2(\Omega)}(t)$, it is easy to see that

$$\left|(q_\theta, \bar{a}^j_{it} w^i_{\theta,j})_{L^2(\Omega)}(t)\right| \leq \delta_1 \|w_{\theta t}\|^2_{L^2(\Omega)} + C(\epsilon_1,\delta_1)\|\nabla w_\theta\|^2_{L^2(\Omega)}$$
$$\leq C(\epsilon_1,\delta_1)\|\nabla w_\theta\|^2_{L^2(\Omega)} + \delta_1 C(\epsilon_2)\|\nabla_0^2 h_\theta\|^2_{L^2(\Gamma)} + \delta_1\Big[\|w_{\theta t}\|^2_{L^2(\Omega)} + \|F\|_{L^2(\Omega)} + 1\Big],$$

while for $(\tilde{q}_0, \tilde{u}^j_{0,i}\tilde{u}^i_{0,j})_{L^2(\Omega)}$, it is bounded by $C(M)N_1(u_0,F)$. Combining (7.19), (7.20), and the estimates above, by choosing $\delta > 0$ and $\delta_1 > 0$ small enough,

$$\|w_{\theta t}\|^2_{L^2(\Omega)} + \|\nabla_0^2 h_{\theta t}\|^2_{L^2(\Gamma)} + \int_0^t \Big[\|\nabla w_{\theta t}\|^2_{L^2(\Omega)} + \kappa\|w_{\theta t}\|^2_{H^2(\Gamma)} + \theta\|q_{\theta t}\|^2_{L^2(\Omega)}\Big] ds$$
$$\leq C(\epsilon_2,\epsilon_1)\Bigg[N_2(u_0,F) + \int_0^t \Bigg(\|w_{\theta t}\|^2_{L^2(\Omega)} + (1 + \|\bar{h}_t\|_{H^{2.5}(\Gamma)})\|\nabla_0^2 h_{\theta t}\|^2_{L^2(\Gamma)}$$
$$+ \|\bar{v}\|^2_{H^3(\Omega)}\int_0^s \|\nabla w_{\theta t}\|^2_{L^2(\Omega)} dt'\Bigg) ds\Bigg] + C_1(\epsilon_2,\epsilon_1)\|\nabla w_\theta\|^2_{L^2(\Omega)},$$

where $N_2(u_0,F) = N_1(u_0,F) + \|F\|^2_{L^\infty(0,T;L^2(\Omega))}$. By the Gronwall inequality,

$$\|w_{\theta t}\|^2_{L^2(\Omega)} + \|\nabla_0^2 h_{\theta t}\|^2_{L^2(\Gamma)} + \int_0^t \Big[\|\nabla w_{\theta t}\|^2_{L^2(\Omega)} + \kappa\|w_{\theta t}\|^2_{H^2(\Gamma)}\Big] ds$$
$$\leq C(\epsilon_2,\epsilon_1)N_2(u_0,F) + C_1(\epsilon_2,\epsilon_1)\|\nabla w_\theta\|^2_{L^2(\Omega)}. \tag{7.21}$$

By using $w_\theta(t) = \tilde{u}_0 + \int_0^t w_{\theta t} ds$, we see that

$$\|w_{\theta t}\|^2_{L^2(\Omega)} + \|\nabla_0^2 h_{\theta t}\|^2_{L^2(\Gamma)} + \int_0^t \Big[\|\nabla w_{\theta t}\|^2_{L^2(\Omega)} + \kappa\|w_{\theta t}\|^2_{H^2(\Gamma)}\Big] ds$$
$$\leq C(\epsilon_2,\epsilon_1)N_2(u_0,F) + C_1(\epsilon_2,\epsilon_1)t\int_0^t \|\nabla w_{\theta t}\|^2_{L^2(\Omega)} ds.$$

Therefore, for any $0 \leq t \leq t_1 = \min\{T, \frac{1}{2C_1}\}$, we have

$$\|w_{\theta t}\|^2_{L^2(\Omega)} + \|\nabla_0^2 h_{\theta t}\|^2_{L^2(\Gamma)} + \frac{1}{2}\int_0^t \Big[\|\nabla w_{\theta t}\|^2_{L^2(\Omega)} + \kappa\|w_{\theta t}\|^2_{H^2(\Gamma)}\Big] ds$$
$$\leq C(\epsilon_2,\epsilon_1)N_2(u_0,F).$$

By $w_\theta(t_1) = \tilde{u}_0 + \int_0^{t_1} w_{\theta t} ds$, we also have

$$\|\nabla w_\theta(t_1)\|^2_{L^2(\Omega)} \leq C(\epsilon_2,\epsilon_1)N_2(u_0,F). \tag{7.22}$$

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For $t \geq t_1$, since $w_\theta(t) = w_\theta(t_1) + \int_{t_1}^t w_{\theta t} ds$, we have from (7.21) and (7.22) that

$$
\begin{aligned}
&\|w_{\theta t}\|^2_{L^2(\Omega)} + \|\nabla_0^2 h_{\theta t}\|^2_{L^2(\Gamma)} + \int_0^t \left[\|\nabla w_{\theta t}\|^2_{L^2(\Omega)} + \kappa \|w_{\theta t}\|^2_{H^2(\Gamma)}\right] ds \\
&\leq C(\epsilon_2, \epsilon_1) N_2(u_0, F) + C_1(\epsilon_2, \epsilon_1)\left[\|w_\theta(t_1)\|^2_{L^2(\Omega)} + (t - t_1)\int_{t_1}^t \|\nabla_0 w_{\theta t}\|^2_{L^2(\Omega)} ds\right] \\
&\leq C(\epsilon_2, \epsilon_1) N_2(u_0, F) + C_1(\epsilon_2, \epsilon_1)(t - t_1)\int_{t_1}^t \|\nabla_0 w_{\theta t}\|^2_{L^2(\Omega)} ds.
\end{aligned}
$$

Therefore, for any $t_1 \leq t \leq 2t_1$, we also have

$$
\begin{aligned}
&\|w_{\theta t}\|^2_{L^2(\Omega)} + \|\nabla_0^2 h_{\theta t}\|^2_{L^2(\Gamma)} + \frac{1}{2}\int_0^t \left[\|\nabla w_{\theta t}\|^2_{L^2(\Omega)} + \kappa \|w_{\theta t}\|^2_{H^2(\Gamma)}\right] ds \\
&\leq C(\epsilon_2, \epsilon_1) N_2(u_0, F),
\end{aligned}
$$

which with $w_\theta(2t_1) = \tilde{u}_0 + \int_0^{2t_1} w_{\theta t} ds$ gives

$$
\|\nabla w_\theta(2t_1)\|^2_{L^2(\Omega)} \leq C(\epsilon_2, \epsilon_1) N_2(u_0, F).
$$

By induction, for any $t \in [0, T]$,

$$
\begin{aligned}
&\|w_{\theta t}\|^2_{L^2(\Omega)} + \|\nabla_0^2 h_{\theta t}\|^2_{L^2(\Gamma)} + \frac{1}{2}\int_0^t \left[\|\nabla w_{\theta t}\|^2_{L^2(\Omega)} + \kappa \|w_{\theta t}\|^2_{H^2(\Gamma)}\right] ds \\
&\leq C(\epsilon_2, \epsilon_1) N_2(u_0, F).
\end{aligned} \tag{7.23}
$$

We also get a θ-independent bound for $\|q_\theta\|^2_{L^2(0,T;L^2(\Omega))}$ by (7.18):

$$
\|q_\theta\|^2_{L^2(0,T;L^2(\Omega))} \leq C(\epsilon_2, \epsilon_1) N_2(u_0, F). \tag{7.24}
$$

Let $\theta = \frac{1}{m}$. Energy inequalities (7.17), (7.23), and (7.24) show that there exists a subsequence $w_{\frac{1}{m_\ell}}$ such that

$$
w_{\frac{1}{m_\ell}} \rightharpoonup \mathfrak{v} \quad \text{in} \quad L^2(0, T; H^{1;2}(\Omega; \Gamma)), \tag{7.25a}
$$

$$
w_{\frac{1}{m_\ell} t} \rightharpoonup \mathfrak{v}_t \quad \text{in} \quad L^2(0, T; H^{1;2}(\Omega; \Gamma)), \tag{7.25b}
$$

$$
\nabla_0^2 h_{\frac{1}{m_\ell}} \rightharpoonup \nabla_0^2 \mathfrak{h} \quad \text{in} \quad L^2(0, T; L^2(\Omega)), \tag{7.25c}
$$

$$
\nabla_0^2 h_{\frac{1}{m_\ell} t} \rightharpoonup \nabla_0^2 \mathfrak{h}_t \quad \text{in} \quad L^2(0, T; L^2(\Omega)), \tag{7.25d}
$$

$$
q_{\frac{1}{m_\ell}} \rightharpoonup \mathfrak{q} \quad \text{in} \quad L^2(0, T; L^2(\Omega)). \tag{7.25e}
$$

Moreover, (7.17) also shows that $\|\bar{a}_i^j w^i_{\frac{1}{m},j}\|_{L^2(0,T;L^2(\Omega))} \to 0$ as $m \to \infty$. Therefore, the weak limit $\mathfrak{v}$ satisfies the "divergence-free" condition (7.2b), i.e.,

$$
\mathfrak{v} \in \mathcal{V}_{\bar{v}}(T). \tag{7.26}
$$

Since (7.17) is independent of θ and ϵ_2, by the property of lower semicontinuity of norms,

$$
\begin{aligned}
&\sup_{0 \leq t \leq T}\left[\|\mathfrak{v}(t)\|^2_{L^2(\Omega)} + \|\nabla_0^2 \mathfrak{h}(t)\|^2_{L^2(\Gamma)}\right] + \|\nabla \mathfrak{v}\|^2_{L^2(0,T;L^2(\Omega))} + \kappa \|\mathfrak{v}\|^2_{H^2(\Gamma)} \\
&\leq C(M) N_1(u_0, F).
\end{aligned} \tag{7.27}
$$

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By (7.25) and ϵ_2-regularization, the weak limit $(\mathfrak{v}, \mathfrak{h}, \mathfrak{q})$ satisfies, for all $\varphi \in L^2(0,T;H^{1;2}(\Omega;\Gamma))$,

$$\int_0^T (\mathfrak{v}_t, \varphi)_{L^2(\Omega)} dt + \frac{\nu}{2}\int_0^T\int_\Omega D_{\bar\eta}(\mathfrak{v}) : D_{\bar\eta}(\varphi) dx dt + \kappa \int_0^T \int_\Gamma \Delta_0 \mathfrak{v} \cdot \Delta_0 \varphi dS dt$$
$$- \int_0^T (\bar a_i^j \mathfrak{q}, \varphi^i_{,j})_{L^2(\Omega)} dt + \sigma \int_0^T \int_\Gamma \bar A^{\alpha\beta\gamma\delta} \mathfrak{h}^{\epsilon_2}_{,\alpha\beta}[-\bar h_{,\sigma}(\varphi^\sigma \circ \bar\eta^{-\tau}) + \varphi^z \circ \bar\eta^{-\tau}]^{\epsilon_2}_{,\gamma\delta} dS dt$$
$$= \int_0^T \left\{ \langle \tilde F, \varphi \rangle - \sigma \int_\Gamma \left[L_1^{\alpha\beta\gamma\delta} \bar h_{,\alpha\beta\gamma} + L_2 \right]^{\epsilon_2} \left[-\bar h_{,\sigma} \varphi^\sigma \circ \bar\eta^{-\tau} + \varphi^z \circ \bar\eta^{-\tau} \right]^{\epsilon_2} dS \right\} dt.$$

By the density argument, we find that for a.a. $t \in [0,T]$, $\varphi \in H^{1;2}(\Omega;\Gamma)$,

$$(\mathfrak{v}_t, \varphi)_{L^2(\Omega)} + \frac{\nu}{2}\int_\Omega D_{\bar\eta}(\mathfrak{v}) : D_{\bar\eta}(\varphi) dx + \kappa \int_\Gamma \Delta_0 \mathfrak{v} \cdot \Delta_0 \varphi dS - (\bar a_i^j \mathfrak{q}, \varphi^i_{,j})_{L^2(\Omega)}$$
$$+ \sigma \int_\Gamma \bar A^{\alpha\beta\gamma\delta} \mathfrak{h}^{\epsilon_2}_{,\alpha\beta}[-\bar h_{,\sigma}(\varphi^\sigma \circ \bar\eta^{-\tau}) + \varphi^z \circ \bar\eta^{-\tau}]^{\epsilon_2}_{,\gamma\delta} dS \tag{7.28}$$
$$= \langle \tilde F, \varphi \rangle - \sigma \int_\Gamma \left[L_1^{\alpha\beta\gamma\delta} \bar h_{,\alpha\beta\gamma} + L_2 \right]^{\epsilon_2} \left[-\bar h_{,\sigma} \varphi^\sigma \circ \bar\eta^{-\tau} + \varphi^z \circ \bar\eta^{-\tau} \right]^{\epsilon_2} dS,$$

or after a change of variable $y' = \bar\eta^\tau(y,t)$,

$$(\mathfrak{v}_t, \varphi)_{L^2(\Omega)} + \frac{\nu}{2}(D_{\bar\eta}\mathfrak{v}, D_{\bar\eta}\varphi)_{L^2(\Omega)} + \kappa \int_\Gamma \Delta_0 \mathfrak{v} \cdot \Delta_0 \varphi dS - (\bar a_i^j \mathfrak{q}, \varphi^i_{,j})_{L^2(\Omega)} \tag{7.29}$$
$$+ \sigma \int_\Gamma \mathcal{L}^{\epsilon_2}_{\bar h}(\mathfrak{h})(-\nabla_0 \bar h \circ \bar\eta^\tau, 1) \cdot \varphi dS = \langle \tilde F, \varphi \rangle - \sigma \int_\Gamma \bar{\mathcal{M}}^{\epsilon_2}_{\bar h}(-\nabla_0 \bar h \circ \bar\eta^\tau, 1) \cdot \varphi dS.$$

Furthermore, if $\varphi \in \mathcal{V}_{\bar v}$, then

$$(\mathfrak{v}_t, \varphi)_{L^2(\Omega)} + \frac{\nu}{2}(D_{\bar\eta}\mathfrak{v}, D_{\bar\eta}\varphi)_{L^2(\Omega)} + \kappa \int_\Gamma \Delta_0 \mathfrak{v} \cdot \Delta_0 \varphi dS$$
$$+ \sigma \int_\Gamma \mathcal{L}^{\epsilon_2}_{\bar h}(\mathfrak{h})(-\nabla_0 \bar h \circ \bar\eta^\tau, 1) \cdot \varphi dS = \langle \tilde F, \varphi \rangle - \sigma \int_\Gamma \bar{\mathcal{M}}^{\epsilon_2}_{\bar h}(-\nabla_0 \bar h \circ \bar\eta^\tau, 1) \cdot \varphi^{\epsilon_2} dS$$

for a.a. $t \in [0,T]$. In other words, $(\mathfrak{v}, \mathfrak{h}, \mathfrak{q})$ is a weak solution of (7.2).

8. Estimates independent of ϵ_2.

8.1. Partition of unity. Since Ω is compact, by partition of unity, we can choose two nonnegative smooth functions ζ_0 and ζ_1 so that

$$\zeta_0 + \zeta_1 = 1 \quad \text{in } \Omega,$$
$$\text{supp}(\zeta_0) \subset\subset \Omega,$$
$$\text{supp}(\zeta_1) \subset\subset \Gamma \times (-\epsilon_1, \epsilon_1) := \Omega_1.$$

We will assume that $\zeta_1 = 1$ inside the region $\Omega_1' \subset \Omega_1$ and $\zeta_0 = 1$ inside the region $\Omega' \subset \Omega$. Note that then $\zeta_1 = 1$, while $\zeta_0 = 0$ on Γ.

8.2. Higher regularity.

8.2.1. ϵ_2-independent bounds for $\mathfrak{q}$. Similar to (7.18), we have

$$\|\mathfrak{q}\|^2_{L^2(\Omega)} \le C(M)\Big[\|\mathfrak{v}_t\|^2_{L^2(\Omega)} + \|\nabla \mathfrak{v}\|^2_{L^2(\Omega)} + \kappa \|\mathfrak{v}\|^2_{H^2(\Gamma)} + \|\nabla_0^2 \mathfrak{h}^{\epsilon_2}\|^2_{L^2(\Gamma)}$$
$$+ \|F\|^2_{L^2(\Omega)} + 1\Big]. \tag{8.1}$$

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8.2.2. Interior regularity. Converting the fluid equation (7.2) into Eulerian variables by composing with $\bar\eta^{-1}$, we obtain a Stokes problem in the domain $\bar\eta(\Omega)$:

$$-\nu\Delta\mathfrak{u} + \nabla\mathfrak{p} = \tilde F\circ\bar\eta^{-1} - \mathfrak{v}_t\circ\bar\eta^{-1} + \nu\bar a^j_{\ell,j}\circ\bar\eta^{-1}\mathfrak{u}_{,\ell} - \mathfrak{p}\bar a^j_{i,j}\circ\bar\eta^{-1}, \tag{8.2a}$$

$$\operatorname{div}\mathfrak{u} = 0, \tag{8.2b}$$

where $\mathfrak{u} = \mathfrak{v}\circ\bar\eta^{-1}$ and $\mathfrak{p} = \mathfrak{q}\circ\bar\eta^{-1}$. By the regularity results for the Stokes problem,

$$\begin{aligned}&\|\mathfrak{u}\|^2_{H^2(\bar\eta(\Omega))} + \|\mathfrak{p}\|^2_{H^1(\bar\eta(\Omega))}\\ &\le C\Big[\|\tilde F\circ\bar\eta^{-1}\|^2_{L^2(\bar\eta(\Omega))} + \|\mathfrak{v}_t\circ\bar\eta^{-1}\|^2_{L^2(\bar\eta(\Omega))} + \|\nabla\mathfrak{u}\|^2_{L^2(\bar\eta(\Omega))} + \|\mathfrak{p}\|^2_{L^2(\bar\eta(\Omega))}\\ &\qquad + \|\mathfrak{u}\|^2_{H^{1.5}(\Gamma)}\Big]\end{aligned}$$

or

$$\begin{aligned}\|\mathfrak{v}\|^2_{H^2(\Omega)} + \|\mathfrak{q}\|^2_{H^1(\Omega)} &\le C\Big[\|F\|^2_{L^2(\Omega)} + \|\mathfrak{v}_t\|^2_{L^2(\Omega)} + \|\mathfrak{v}\|^2_{H^{1.5}(\Gamma)}\Big]\\ &\quad + C(M)\Big[\|\nabla\mathfrak{v}\|^2_{L^2(\Omega)} + \|\mathfrak{q}\|^2_{L^2(\Omega)}\Big]\end{aligned}$$

for some constant C independent of M and ϵ_1. By (8.1),

$$\begin{aligned}\|\mathfrak{v}\|^2_{H^2(\Omega)} + \|\mathfrak{q}\|^2_{H^1(\Omega)} &\le C(M)\Big[\|\mathfrak{v}_t\|^2_{L^2(\Omega)} + \|\nabla\mathfrak{v}\|^2_{L^2(\Omega)} + \|\mathfrak{v}\|^2_{H^2(\Gamma)}\\ &\quad + \|\nabla_0^2\mathfrak{h}^{\epsilon_2}\|^2_{L^2(\Gamma)} + \|F\|^2_{L^2(\Omega)} + 1\Big].\end{aligned} \tag{8.3}$$

Similarly,

$$\begin{aligned}\|\mathfrak{v}\|^2_{H^3(\Omega)} + \|\mathfrak{q}\|^2_{H^2(\Omega)} &\le C\Big[\|F\|^2_{H^1(\Omega)} + \|\mathfrak{v}_t\|^2_{H^1(\Omega)} + \|\mathfrak{v}\|^2_{H^{2.5}(\Gamma)}\Big]\\ &\quad + C(M)\Big[\|\nabla\mathfrak{v}\|^2_{H^1(\Omega)} + \|\mathfrak{q}\|^2_{H^1(\Omega)}\Big],\end{aligned}$$

and therefore by (8.1) and (8.3),

$$\begin{aligned}\|\mathfrak{v}\|^2_{H^3(\Omega)} + \|\mathfrak{q}\|^2_{H^2(\Omega)} &\le C(M)\Big[\|\mathfrak{v}_t\|^2_{H^1(\Omega)} + \|\nabla\mathfrak{v}\|^2_{L^2(\Omega)} + \|\nabla_0^2\mathfrak{v}\|^2_{H^1(\Omega_1)}\\ &\quad + \|\nabla_0^2\mathfrak{h}^{\epsilon_2}\|^2_{L^2(\Gamma)} + \|F\|^2_{H^1(\Omega)} + 1\Big].\end{aligned} \tag{8.4}$$

For the regularized problem, because the ϵ_1-regularization ensures that the forcing and the initial data are smooth, while the ϵ_2-regularization ensures that the right-hand side of (7.2c) is smooth, by the standard difference quotient technique, it is also easy to see that

$$\nabla_0^k\mathfrak{v} \in L^2(0,T;H^1(\Omega_1)\cap H^2(\Gamma)) \quad \text{for } k = 1,2,3,4. \tag{8.5}$$

Since (7.25b) implies that $\mathfrak{v}_t \in L^2(0,T;H^1(\Omega))$, by ϵ_2-regularization and (8.4) we conclude that

$$\mathfrak{v}\in L^2(0,T;H^3(\Omega)), \qquad \mathfrak{q}\in L^2(0,T;H^2(\Omega)). \tag{8.6}$$

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8.3. Estimates for $\mathfrak{v}_t(0)$ and $\mathfrak{q}(0)$. By (8.6) and ϵ_2-regularization, $(\mathfrak{v}, \mathfrak{h}, \mathfrak{q})$ satisfies the strong form (7.2). Taking the "divergence" of (7.2a) and then making use of condition (7.2b), we find that

$$-\bar{a}_{it}^k \mathfrak{v}^i_{,k} - \nu \bar{a}_i^k [\bar{a}_\ell^j D_{\bar{\eta}}(\mathfrak{v})^i_\ell]_{,jk} = -\bar{a}_i^k (\bar{a}_i^j \mathfrak{q})_{,jk} + \bar{a}_i^k \tilde{F}^i_{,k}. \tag{8.7}$$

Let $t = 0$; by the identity $\bar{a}_{kt}^\ell = -\bar{a}_k^i \bar{v}^j_{,i} \bar{a}_j^\ell$,

$$\Delta \mathfrak{q}(0) = \nabla \tilde{u}_0 : (\nabla \tilde{u}_0)^T - \operatorname{div}(\tilde{F}(0)) \quad \text{in } \Omega$$

with

$$\mathfrak{q}(0) = \nu (\operatorname{Def} \tilde{u}_0)_i^j N_i N_j - \sigma \mathcal{M}_0^{\epsilon_2}(0) + \kappa \Delta_0^2 \tilde{u}_0 \quad \text{on } \Gamma,$$

while (7.2a) gives us

$$\mathfrak{v}_t(0) = \nu \Delta \tilde{u}_0 - \nabla \mathfrak{q}(0) + \tilde{F}(0) \quad \text{in } \Omega.$$

By standard elliptic regularity result,

$$\|\mathfrak{v}_t(0)\|^2_{L^2(\Omega)} + \|\mathfrak{q}(0)\|^2_{H^1(\Omega)} \le C N_0(u_0, F) \tag{8.8}$$

for some constant independent of M, ϵ_1, and ϵ_2.

8.4. $L^2_t L^2_x$-estimates for $\mathfrak{v}_t$. Since $\mathfrak{v}_t \in L^2(0,T; H^1(\Omega))$, we can use it as a test function in (7.29). By (7.26), we find that

$$\begin{aligned}
&\|\mathfrak{v}_t\|^2_{L^2(\Omega)} + \frac{\nu}{4}\frac{d}{dt}\int_\Omega |D_{\bar{\eta}}\mathfrak{v}|^2 dx - \frac{\nu}{2}\int_\Omega (D_{\bar{\eta}}\mathfrak{v})^j_i \bar{a}^k_{jt} \mathfrak{v}^i_{,k} dx + \kappa \int_\Gamma \Delta_0 \mathfrak{v} \cdot \Delta_0 \varphi dS \\
&\quad + \int_\Omega \mathfrak{q} \bar{a}^\ell_{kt} \mathfrak{v}^k_{,\ell} dx + \sigma \int_\Gamma \mathcal{L}^{\epsilon_2}_{\bar{h}}(\mathfrak{h})(-\nabla_0 \bar{h} \circ \bar{\eta}^\tau, 1) \cdot \mathfrak{v}_t dS \\
&= \langle \tilde{F}, \mathfrak{v}_t \rangle - \sigma \int_\Gamma \mathcal{M}^{\epsilon_2}_{\bar{h}}(-\nabla_0 \bar{h} \circ \bar{\eta}^\tau, 1) \cdot \mathfrak{v}_t dS.
\end{aligned}$$

By (5.3),

$$\int_\Omega (D_{\bar{\eta}}\mathfrak{v})^j_i \bar{a}^k_{jt} \mathfrak{v}^i_{,k} dx \le C(M) C(\delta) \|\nabla \mathfrak{v}\|^2_{L^2(\Omega)} + \delta \|\mathfrak{v}\|^2_{H^2(\Omega)},$$

and by (8.1) and the interpolation inequality,

$$\begin{aligned}
\left| \int_\Omega \mathfrak{q} \bar{a}^\ell_{kt} \mathfrak{v}^k_{,\ell} dx \right| &\le C(M) C(\delta) \Big[\|\nabla \mathfrak{v}\|^2_{L^2(\Omega)} + \|\nabla_0^4 \mathfrak{h}^{\epsilon_2}\|^2_{L^2(\Gamma)} + \|F\|^2_{L^2(\Omega)} + 1 \Big] \\
&\quad + \delta \|\mathfrak{v}\|^2_{H^2(\Omega)} + \frac{1}{2} \|\mathfrak{v}_t\|^2_{L^2(\Omega)}
\end{aligned}$$

for some $C(\delta)$. Also, the last term on the left-hand side is bounded by

$$\begin{aligned}
&C(M) \Big[\|\nabla_0^4 \mathfrak{h}^{\epsilon_2}\|_{L^2(\Gamma)} + 1 \Big] \|\mathfrak{v}_t\|_{H^1(\Omega)} \\
&\le C(M) C(\delta_1) \Big[\|\nabla_0^4 \mathfrak{h}^{\epsilon_2}\|^2_{L^2(\Gamma)} + 1 \Big] + \delta_1 \|\mathfrak{v}_t\|^2_{H^1(\Omega)}.
\end{aligned}$$

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Combining all the estimates above,

$$\frac{1}{2}\|\mathfrak{v}_t\|^2_{L^2(\Omega)} + \frac{\nu}{4}\frac{d}{dt}\int_\Omega |D_{\bar\eta}\mathfrak{v}|^2 dx + \frac{\kappa}{2}\frac{d}{dt}\int_\Gamma |\Delta_0 \mathfrak{v}|^2 dS$$
$$\le C\Big[\|\nabla \mathfrak{v}\|^2_{L^2(\Omega)} + \|\nabla_0^4 \mathfrak{h}^{\epsilon_2}\|^2_{L^2(\Gamma)} + \|F\|^2_{L^2(\Omega)} + 1\Big] + \delta\|\mathfrak{v}\|^2_{H^2(\Omega)} + \delta_1\|\mathfrak{v}_t\|^2_{H^1(\Omega)}$$

for some constant C depending on M, δ, and δ_1. Therefore, by (7.27),

$$\text{(8.9)}\quad \int_0^t \|\mathfrak{v}_t\|^2_{L^2(\Omega)} ds + \|\nabla \mathfrak{v}(t)\|^2_{L^2(\Omega)} + \kappa\|\mathfrak{v}\|^2_{H^2(\Gamma)}$$
$$\le C\Big[N_2(u_0, F) + \int_0^t \|\nabla_0^4 \mathfrak{h}^{\epsilon_2}\|^2_{L^2(\Gamma)} ds\Big] + \delta\int_0^t \|\mathfrak{v}\|^2_{H^2(\Omega)} ds + \delta_1 \int_0^t \|\mathfrak{v}_t\|^2_{H^1(\Omega)} ds.$$

8.5. Energy estimates for $\nabla_0^2 v$ near the boundary. Because of (8.5), $\nabla_0^2(\zeta_1^2\nabla_0^2\mathfrak{v})$ in (7.28) can be used as a test function in (7.29). It follows that

$$\Big|\int_\Gamma \Big[\bar{\mathcal{L}}^{\epsilon_2}_{\bar h}(\mathfrak{h}^{\epsilon_2}) + \bar{\mathcal{M}}^{\epsilon_2}_{\bar h}\Big](-\nabla_0 \bar h \circ \bar\eta^\tau, 1)\cdot \nabla_0^4 \mathfrak{v} dS\Big|$$
$$\le C(M)\Big[\|\nabla_0^2 \mathfrak{h}^{\epsilon_2}\|_{H^2(\Gamma)} + 1\Big]\|\mathfrak{v}\|_{H^4(\Gamma)}$$
$$\le C(M,\delta_3)\Big[1 + \|\mathfrak{h}\|^2_{H^4(\Gamma)}\Big] + \delta_3\|\mathfrak{v}\|^2_{H^4(\Gamma)}.$$

By (7.4), we find that

$$\|\mathfrak{h}\|^2_{H^4(\Gamma)} \le C(\epsilon_1)\Big[\int_0^t \|\bar h\|_{H^5(\Gamma)}\|\mathfrak{v}\|_{H^4(\Gamma)} ds\Big]^2 \le C(\epsilon_1)\int_0^t \|\mathfrak{v}\|^2_{H^4(\Gamma)} ds,$$

and hence

$$\Big|\int_\Gamma \Big[\bar{\mathcal{L}}^{\epsilon_2}_{\bar h}(\mathfrak{h}^{\epsilon_2}) + \bar{\mathcal{M}}^{\epsilon_2}_{\bar h}\Big](-\nabla_0 \bar h \circ \bar\eta^\tau, 1)\cdot \nabla_0^4 \mathfrak{v} dS\Big|$$
$$\le \bar C\Big[1 + \int_0^t \|\mathfrak{v}\|^2_{H^4(\Gamma)}\Big] + \delta_3\|\mathfrak{v}\|^2_{H^4(\Gamma)}$$

for some constant $\bar C$ depending on M, ϵ_1, and δ_3. Since

$$\Delta_0 f = \frac{1}{\sqrt{\det(g_0)}}\frac{\partial}{\partial y^\alpha}\Big[\sqrt{\det(g_0)}g_0^{\alpha\beta}\frac{\partial}{\partial y^\beta} f\Big],$$

by the regularity on Γ (and hence on g_0),

$$\int_\Gamma |\Delta_0 \nabla_0^2 \mathfrak{v}|^2 dS \le \int_\Gamma \Delta_0^2 \mathfrak{v}\cdot(\nabla_0^4 v) dS + C\|\mathfrak{v}\|_{H^3(\Gamma)}\|\mathfrak{v}\|_{H^4(\Gamma)}$$
$$\le \int_\Gamma \Delta_0^2 v\cdot(\nabla_0^4 v) dS + C(\delta)\|\mathfrak{v}\|^2_{H^1(\Omega)} + \delta\|\mathfrak{v}\|^2_{H^4(\Gamma)},$$

which implies, by choosing $\delta > 0$ small enough, that

$$\nu_2\|v\|^2_{H^4(\Gamma)} \le \int_\Gamma \Delta_0^2 v\cdot(\nabla_0^4 v) dS + C\|v\|^2_{H^1(\Omega)}.$$

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By the identity

$$
\begin{aligned}
&(\mathfrak{q}, \bar a_k^\ell \nabla_0^2(\zeta_1^2 \nabla_0^2 \mathfrak{v}^k)_{,\ell}) \\
&= (\mathfrak{q}, \nabla_0^2 \bar a_k^\ell (\zeta_1^2 \nabla_0^2 \mathfrak{v}^k)_{,\ell}) + 4(\zeta_1 \nabla_0 \mathfrak{q}, \nabla_0 \bar a_k^\ell \zeta_{1,\ell} \nabla_0^2 \mathfrak{v}^k) + 2(\nabla_0 \mathfrak{q}, \zeta_1^2 \nabla_0 \bar a_k^\ell \nabla_0^2 \mathfrak{v}^k_{,\ell}) \\
&\quad - 2(\zeta_1 \nabla_0 \mathfrak{q}, \nabla_0(\bar a_k^\ell \zeta_{1,\ell} \nabla_0^2 \mathfrak{v}^k)) + 2(\mathfrak{q}, \nabla_0(\bar a_k^\ell \zeta_{1,\ell} \nabla_0 \zeta_1 \nabla_0^2 \mathfrak{v}^k)) \\
&\quad + (\nabla_0 \mathfrak{q}, \nabla_0(\zeta_1^2 \nabla_0 \bar a_k^\ell \nabla_0 \mathfrak{v}^k_{,\ell})),
\end{aligned} \tag{8.10}
$$

(5.3) and (8.3) imply that

$$
\begin{aligned}
&(\mathfrak{q}, \bar a_k^\ell \nabla_0'^2 (\zeta_1^2 \nabla_0^2 \mathfrak{v}^k)_{,\ell}) \le C(M) \|\mathfrak{q}\|_{H^1(\Omega)} \|\mathfrak{v}\|_{H^3(\Omega)} \\
&\le C(M)C(\delta)\Big[\|\mathfrak{v}_t\|^2_{L^2(\Omega)} + \|\nabla \mathfrak{v}\|^2_{L^2(\Omega)} + \|\nabla\nabla_0 \mathfrak{v}\|^2_{L^2(\Omega_1)} + \kappa \|\mathfrak{v}\|^2_{H^2(\Gamma)} \\
&\qquad + \|\nabla_0^2 \mathfrak{h}^{\epsilon_2}\|^2_{L^2(\Gamma)} + \|F\|^2_{L^2(\Omega)} + 1\Big] + \delta \|\mathfrak{v}\|^2_{H^3(\Omega)}.
\end{aligned}
$$

For the viscosity term,

$$
\begin{aligned}
&\int_\Omega D_{\bar\eta} \mathfrak{v} : D_{\bar\eta}(\nabla_0^2(\zeta_1^2 \nabla_0^2 \mathfrak{v}))dx \\
&= \|\zeta_1 D_{\bar\eta} \nabla_0^2 \mathfrak{v}\|^2_{L^2(\Omega)} + \frac{1}{2}\int_\Omega \Big[\nabla_0^2(\bar a_i^k \bar a_i^\ell) \mathfrak{v}^j_{,\ell} + \nabla_0^2(\bar a_i^k \bar a_j^\ell) \mathfrak{v}^i_{,\ell}\Big](\zeta_1^2 \nabla_0^2 \mathfrak{v}^j)_{,k} dx \\
&\quad + \int_\Omega \Big[\nabla_0(\bar a_i^k \bar a_i^\ell) \nabla_0 \mathfrak{v}^j_{,\ell} + \nabla_0(\bar a_i^k \bar a_j^\ell) \nabla_0 \mathfrak{v}^i_{,\ell}\Big](\zeta_1^2 \nabla_0^2 \mathfrak{v}^j)_{,k} dx \\
&\quad + \int_\Omega D_{\bar\eta}(\nabla_0^2 \mathfrak{v})_i^j \bar a_i^k \zeta_1 \zeta_{1,k} \nabla_0^2 \mathfrak{v}^j dx,
\end{aligned}
$$

and hence by interpolation

$$
\begin{aligned}
\frac{1}{2}\|\zeta_1 D_{\bar\eta} \nabla_0'^2 \mathfrak{v}\|^2_{L^2(\Omega)} &\le \int_\Omega D_{\bar\eta} \mathfrak{v} : D_{\bar\eta}(\nabla_0^2(\zeta_1^2 \nabla_0^2 \mathfrak{v}))dx \\
&\quad + C(M)C(\delta)\Big[\|\nabla \mathfrak{v}\|^2_{L^2(\Omega)} + \|\nabla\nabla_0 \mathfrak{v}\|^2_{L^2(\Omega_1')}\Big] + \delta\|\mathfrak{v}\|^2_{H^3(\Omega)}.
\end{aligned}
$$

Summing all the estimates, by letting $\delta_3 = \frac{\nu_2 \kappa}{2}$, we conclude that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\zeta_1 \nabla_0^2 \mathfrak{v}\|^2_{L^2(\Omega)} + \frac{\nu}{4}\|\zeta_1 D_{\bar\eta} \nabla_0^2 \mathfrak{v}\|^2_{L^2(\Omega)} + \frac{\nu_2 \kappa}{2}\|\mathfrak{v}\|^2_{H^4(\Gamma)} \\
&\le \bar C\Big[\|\mathfrak{v}_t\|^2_{L^2(\Omega)} + \|\mathfrak{v}\|^2_{H^1(\Omega)} + \|\nabla\nabla_0 \mathfrak{v}\|^2_{L^2(\Omega_1')} + \|\mathfrak{v}\|^2_{H^2(\Gamma)} + \|\nabla_0^2 \mathfrak{h}^{\epsilon_2}\|^2_{L^2(\Gamma)} \\
&\qquad + \|F\|^2_{H^1(\Omega)} + 1\Big] + \bar C \int_0^t \|\mathfrak{v}\|^2_{H^4(\Gamma)} ds + \delta \|\mathfrak{v}\|^2_{H^3(\Omega)}
\end{aligned}
$$

for some constant $\bar C$ depending on M, κ, ϵ_1, and δ. Integrating the inequality above in time from 0 to t, by (7.27) we find that

$$
\begin{aligned}
&\|\nabla_0^2 \mathfrak{v}(t)\|^2_{L^2(\Omega_1)} + \int_0^t \Big[\|\nabla\nabla_0^2 \mathfrak{v}\|^2_{L^2(\Omega_1)} + \kappa\|\mathfrak{v}\|^2_{H^4(\Gamma)}\Big] ds \\
&\le \bar C N_2(u_0, F) + \bar C \int_0^t \Big[\|\mathfrak{v}_t\|^2_{L^2(\Omega)} + \|\nabla\nabla_0 \mathfrak{v}\|^2_{L^2(\Omega_1')} + \|\mathfrak{v}\|^2_{H^2(\Gamma)}\Big] ds \\
&\quad + \bar C \int_0^t \int_0^s \|\mathfrak{v}(r)\|^2_{H^4(\Gamma)} dr + \delta \int_0^t \|\mathfrak{v}\|^2_{H^3(\Omega)} ds.
\end{aligned} \tag{8.11}
$$

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By using $\nabla_0(\zeta_1^2\nabla_0\mathfrak{v})$ as a testing function in (7.29), similar computations lead to

$$\begin{aligned}
&\|\nabla_0\mathfrak{v}(t)\|^2_{L^2(\Omega_1)} + \int_0^t \Big[\|\nabla\nabla_0\mathfrak{v}\|^2_{L^2(\Omega_1)} + \kappa\|\mathfrak{v}\|^2_{H^3(\Gamma)}\Big]ds \\
&\le C(M)N_2(u_0,F) + C(M,\delta)\int_0^t \Big[\|\mathfrak{v}_t\|^2_{L^2(\Omega)} + \kappa\|\mathfrak{v}\|^2_{H^2(\Gamma)}\Big]ds \\
&\quad + C(M)\int_0^t\int_0^s \|\mathfrak{v}(r)\|^2_{H^4(\Gamma)}drds + \delta\int_0^t \|\mathfrak{v}\|^2_{H^3(\Omega)}ds.
\end{aligned} \tag{8.12}$$

8.6. Energy estimates for v_t: $L^2_tH^1_x$-estimates. In this section, we time differentiate (7.29) and then use $\mathfrak{v}_t$ as a test function to obtain

$$\begin{aligned}
&\langle \mathfrak{v}_{tt}, \mathfrak{v}_t\rangle + \nu\int_\Omega \Big[\bar a^k_\ell (D_{\bar\eta}\mathfrak{v})^i_{\ell,k}\Big]_t \mathfrak{v}^i_t dx + \sigma\int_\Gamma \Big[\bar{\mathcal{L}}^{\epsilon_2}_{\bar h}(\mathfrak{h}^{\epsilon_2})(-\nabla_0\bar h\circ\bar\eta^\tau, 1)\Big]_t\cdot \mathfrak{v}_t dS \\
&+\kappa\int_\Gamma |\Delta_0\mathfrak{v}_t|^2 dS - \int_\Omega (\bar a^\ell_k\mathfrak{q})_t\mathfrak{v}^k_{t,\ell}dx = \langle F_t,\mathfrak{v}_t\rangle - \sigma\int_\Gamma \Big[\bar{\mathcal{M}}^{\epsilon_2}_{\bar h}(-\nabla_0\bar h\circ\bar\eta^\tau,1)\Big]_t\cdot\mathfrak{v}_t dS.
\end{aligned}$$

By the chain rule,

$$\begin{aligned}
&\int_\Gamma \Big[(\bar{\mathcal{L}}^{\epsilon_2}_{\bar h}(\mathfrak{h}^{\epsilon_2}) + \bar{\mathcal{M}}^{\epsilon_2}_{\bar h})(-\nabla_0\bar h\circ\bar\eta^\tau,1)\Big]_t\cdot\mathfrak{v}_t dS \\
&= \int_\Gamma \bar\Theta_t\Big[L_{\bar h}(\mathfrak{h}^{\epsilon_2})\Big]^{\epsilon_2}\circ\bar\eta^\tau(-\nabla_0\bar h\circ\bar\eta^\tau,1)\cdot\mathfrak{v}_t dS \\
&\quad + \int_\Gamma \bar\Theta\bar\eta^\tau_t\cdot\Big[\nabla_0[L_{\bar h}(\mathfrak{h}^{\epsilon_2})]^{\epsilon_2}(-\nabla_0\bar h,1)\Big]\circ\bar\eta^\tau\cdot\mathfrak{v}_t dS \\
&\quad + \int_\Gamma \bar\Theta\Big[[L_{\bar h}(\mathfrak{h}^{\epsilon_2})]^{\epsilon_2}(\nabla_0\bar h,-1)\Big]_t\circ\bar\eta^\tau\cdot\mathfrak{v}_t dS.
\end{aligned}$$

By using the $H^2(\Gamma)$-$H^{-2}(\Gamma)$ duality pairing with ϵ_1-regularization on $\bar\Theta$ and $\bar v$, it follows that

$$\begin{aligned}
&\left|\int_\Gamma \Big[(\bar{\mathcal{L}}^{\epsilon_2}_{\bar h}(\mathfrak{h}^{\epsilon_2}) + \bar{\mathcal{M}}^{\epsilon_2}_{\bar h})(-\nabla_0\bar h\circ\bar\eta^\tau,1)\Big]_t\cdot\mathfrak{v}_t dS\right| \\
&\le C(\epsilon_1)\Big[\|\nabla_0^3\mathfrak{h}\|_{L^2(\Gamma)} + \|\nabla_0^2\mathfrak{h}_t\|_{L^2(\Gamma)} + 1\Big]\|\mathfrak{v}_t\|_{H^2(\Gamma)} \\
&\le C(\epsilon_1,\delta_3)\left[\int_0^t \|\mathfrak{v}\|^2_{H^4(\Gamma)}ds + \|\mathfrak{v}\|^2_{H^2(\Gamma)} + 1\right] + \delta_3\|\mathfrak{v}_t\|^2_{H^2(\Gamma)} \\
&\le \bar C\left[\int_0^t \|\mathfrak{v}\|^2_{H^4(\Gamma)}ds + \|\mathfrak{v}\|^2_{H^1(\Omega)} + 1\right] + \delta\|\mathfrak{v}\|^2_{H^3(\Omega)} + \delta_3\|\mathfrak{v}_t\|^2_{H^2(\Gamma)}
\end{aligned}$$

for some constant $\bar C$ depending on M, ϵ_1, δ, and δ_3, where we estimate $\|\mathfrak{v}\|^2_{H^2(\Gamma)}$ by interpolation.

Also by interpolation,

$$\begin{aligned}
\int_\Omega |D_{\bar\eta}\mathfrak{v}_t|^2dx &= 2\int_\Omega \Big[\bar a^k_i D_{\bar\eta}(\mathfrak{v})^j_i\Big]_t \mathfrak{v}^j_{t,k}dx - 2\int_\Omega \Big[(\bar a^k_i\bar a^\ell_i)_t\mathfrak{v}^j_{,\ell} + (\bar a^k_i\bar a^\ell_j)_t\mathfrak{v}^i_{,\ell}\Big]\mathfrak{v}^j_{t,k}dx \\
&\le 2\int_\Omega \Big[\bar a^k_i D_{\bar\eta}(\mathfrak{v})^j_i\Big]_t\mathfrak{v}^j_{t,k}dx + C(M)C(\delta,\delta_1)\|\nabla\mathfrak{v}\|^2_{L^2(\Omega)} \\
&\quad - \int_\Omega (\bar a^\ell_k\mathfrak{q})_t\mathfrak{v}^k_{t,\ell}dx + \delta\|\mathfrak{v}\|^2_{H^2(\Omega)} + \delta_1\|\mathfrak{v}_t\|^2_{H^1(\Omega)}.
\end{aligned}$$

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Note that

$$\langle F_t, \mathfrak{v}_t\rangle \le C\|F_t\|_{H^1(\Omega)'}\|\mathfrak{v}_t\|_{H^1(\Omega)} \le C(\delta_1)\|F_t\|^2_{H^1(\Omega)'} + \delta_1\|\mathfrak{v}_t\|^2_{H^1(\Omega)}.$$

Summing all the estimates above,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\mathfrak{v}_t\|^2_{L^2(\Omega)} + \frac{\nu}{4}\|\nabla\mathfrak{v}_t\|^2_{L^2(\Omega)} + \kappa\|\Delta_0\mathfrak{v}_t\|^2_{L^2(\Gamma)}\\ &\le \bar{C}\left[\int_0^t \|\mathfrak{v}\|^2_{H^4(\Gamma)}ds + \|\mathfrak{v}\|^2_{H^1(\Omega)} + 1\right] + C(\delta_1)\|F_t\|^2_{H^1(\Omega)'}\\ &\quad + \delta\|\mathfrak{v}\|^2_{H^3(\Omega)} + \delta_1\|\mathfrak{v}_t\|^2_{H^1(\Omega)} + \delta_3\|\mathfrak{v}_t\|^2_{H^2(\Gamma)} + \int_\Omega (\bar{a}^\ell_k\mathfrak{q})_t\mathfrak{v}^k_{t,\ell}dx\end{aligned} \tag{8.13}$$

for some constant $\bar{C}$ depending on M, κ, δ, and δ_1. As in [7] and [8], the integral involving the pressure q has the following estimate:

$$\int_0^t\int_\Omega (\bar{a}^\ell_k\mathfrak{q})_t\mathfrak{v}^k_{t,\ell}dxds \le C(M)C(\delta,\delta_1)N_3(u_0,F) + \delta\int_0^t \|\mathfrak{v}\|^2_{H^3(\Omega)}ds + \delta_1\int_0^t \|\mathfrak{v}_t\|^2_{H^1(\Omega)}ds,$$

where

$$N_3(u_0,F) := \|u_0\|^2_{H^{2.5}(\Omega)} + \|u_0\|^2_{H^{4.5}(\Gamma)} + \|F\|^2_{L^2(0,T;H^1(\Omega))} + \|F_t\|^2_{L^2(0,T;H^1(\Omega)')} + \|F(0)\|^2_{H^1(\Omega)} + 1.$$

Integrating (8.13) in time from 0 to t and choosing $\delta_1, \delta_3 > 0$ small enough, (7.27) and (8.9) imply that, for all $t \in [0,T]$,

$$\begin{aligned}&\|\mathfrak{v}_t(t)\|^2_{L^2(\Omega)} + \int_0^t\left[\|\nabla\mathfrak{v}_t\|^2_{L^2(\Omega)} + \kappa\|\mathfrak{v}_t\|^2_{H^2(\Gamma)}\right]ds\\ &\le \bar{C}N_3(u_0,F) + \bar{C}\int_0^t\int_0^s \|\mathfrak{v}(r)\|^2_{H^4(\Gamma)}drds + \delta\int_0^t \|\mathfrak{v}\|^2_{H^3(\Omega)}ds\end{aligned} \tag{8.14}$$

for some constant $\bar{C}$ depending on M, κ, δ, and δ_2. In (8.14), (8.8) is used to bound $\|v_t(0)\|^2_{L^2(\Omega)}$.

8.7. ϵ_2-independent estimates. Integrating (8.3) in time from 0 to t, (7.27), (8.9), and (8.12) imply that

$$\begin{aligned}&\int_0^t\left[\|\mathfrak{v}\|^2_{H^2(\Omega)} + \|\mathfrak{q}\|^2_{H^1(\Omega)}\right]ds\\ &\le C(M)N_1(u_0,F) + \int_0^t\left[\|\mathfrak{v}_t\|^2_{L^2(\Omega)} + \|\mathfrak{v}\|^2_{H^2(\Gamma)}\right]ds\\ &\le \bar{C}N_3(u_0,F) + \bar{C}\int_0^t\int_0^s \|\mathfrak{v}(r)\|^2_{H^4(\Gamma)}drds + \delta\int_0^t \|\mathfrak{v}\|^2_{H^3(\Omega)}ds\end{aligned} \tag{8.15}$$

for some constant $\bar{C}$ depending on M, κ, and δ. Integrating (8.4) in time from 0 to t, making use of (8.11), (8.12), (8.14), and (8.15), and then choosing $\delta > 0$ small enough and T even smaller, we find that

$$\int_0^t\left[\|\mathfrak{v}\|^2_{H^3(\Omega)} + \|\mathfrak{q}\|^2_{H^2(\Omega)}\right]ds \le \bar{C}N_3(u_0,F) + \bar{C}\int_0^t\int_0^s \|\mathfrak{v}(r)\|^2_{H^4(\Gamma)}drds \tag{8.16}$$

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for some constant $\bar{C}$ depending on M, κ, and ϵ_1.

Having (8.16), by choosing $\delta_2 > 0$ small enough, the estimates (8.11) can be rewritten as

$$\begin{aligned} &\|\nabla_0^2 \mathfrak{v}(t)\|^2_{L^2(\Omega_1)} + \int_0^t \Big[\|\nabla\nabla_0^2 \mathfrak{v}\|^2_{L^2(\Omega_1)} + \kappa\|\mathfrak{v}\|^2_{H^4(\Gamma)}\Big] ds \\ &\le \bar{C} N_3(u_0, F) + \bar{C}\int_0^t\int_0^s \|\mathfrak{v}(r)\|^2_{H^4(\Gamma)} dr ds \end{aligned} \tag{8.17}$$

for some constant $\bar{C}$ depending on M, κ, and ϵ_1. Therefore,

$$X(t) \le \bar{C}\Big[\int_0^t X(s) ds + N_3(u_0, F)\Big],$$

where

$$X(t) = \int_0^t \|\mathfrak{v}\|^2_{H^4(\Gamma)} ds.$$

By the Gronwall inequality,

$$\int_0^t \int_0^s \|\mathfrak{v}(r)\|^2_{H^4(\Gamma)} dr ds \le \bar{C} N_3(u_0, F) \tag{8.18}$$

for all $t \in [0, T]$ for some constant $\bar{C}$ depending on M, κ, and ϵ_1. Having (8.18), estimates (8.9), (8.14), (8.16), and (8.17) along with the standard embedding theorem lead to

$$\begin{aligned} &\sup_{0\le t\le T}\Big[\|\mathfrak{v}(t)\|^2_{H^2(\Omega)} + \|\mathfrak{v}_t(t)\|^2_{L^2(\Omega)}\Big] + \|\mathfrak{v}\|^2_{\mathcal{V}^3(T)} + \|\mathfrak{q}\|^2_{L^2(0,T;H^2(\Omega))} \\ &+ \kappa\|\mathfrak{v}\|^2_{L^2(0,T;H^4(\Gamma))} \le \bar{C} N_3(u_0, F) \end{aligned} \tag{8.19}$$

for some constant $\bar{C}$ depending on M, κ, and ϵ_1.

8.8. Weak limits as $\epsilon_2 \to 0$. Since the estimate (8.19) is independent of ϵ_2, the weak limit as $\epsilon_2 \to 0$ of the sequence $(\mathfrak{v}, \mathfrak{h}, \mathfrak{q})$ exists. We will denote the weak limit of $(\mathfrak{v}, \mathfrak{h}, \mathfrak{q})$ by $(v_\kappa, h_\kappa, q_\kappa)$. By lower semicontinuity, (8.8) and thus (8.19) hold for the weak limit $(v_\kappa, h_\kappa, q_\kappa)$. Furthermore,

$$\begin{aligned} &\langle v_{\kappa t}, \varphi\rangle + \frac{\nu}{2}\int_\Omega D_{\bar{\eta}} v_\kappa : D_{\bar{\eta}}\varphi dx + \sigma\int_\Gamma \bar{\Theta}\Big[[\mathcal{L}_{\bar{h}}(h_\kappa)(-\nabla_0\bar{h}, 1)]\circ\bar{\eta}^\tau\Big]\cdot\varphi dS \\ &+ \kappa\int_\Gamma \Delta_0 v_\kappa\cdot\Delta_0\varphi dS - (q_\kappa, \bar{a}^\ell_k \varphi^k_{,\ell})_{L^2(\Omega)} \\ &= \langle F, \varphi\rangle - \sigma\int_\Gamma \bar{\Theta}\Big[[\mathcal{M}(\bar{h})(-\nabla_0\bar{h}, 1)]\circ\bar{\eta}^\tau\Big]\cdot\varphi dS \end{aligned} \tag{8.20}$$

for all $\varphi \in H^{1;2}(\Omega;\Gamma)$ and a.a. $t \in [0, T]$.

9. Estimates independent of κ and ϵ_1.

9.1. Energy estimates which are independent of κ. Although (8.19) does not imply that $h_\kappa \in H^4(\Gamma)$, h_κ is indeed in $H^4(\Gamma)$ by (7.4). Therefore, we have that

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$(v_\kappa, h_\kappa, q_\kappa)$ satisfies

$$v_{\kappa}{}_t^i - \nu[\bar a^k_\ell D_{\bar\eta}(v_\kappa)^i_\ell]_{,k} = -(\bar a^k_i q_\kappa)_{,k} + \tilde F^i \quad \text{in } (0,T)\times\Omega, \tag{9.1a}$$

$$\bar a^j_i v_{\kappa}{}^i_{,j} = 0 \quad \text{in } (0,T)\times\Omega, \tag{9.1b}$$

$$[\nu D_{\bar\eta}(v_\kappa)^j_i - q_\kappa\delta^j_i]\bar a^\ell_j N_\ell = \sigma\bar\Theta[\mathcal{L}_{\bar h}(h_\kappa)(-\nabla_0\bar h,1)]\circ\bar\eta^\tau + \sigma\bar\Theta[\mathcal{M}_{\bar h}(-\nabla_0\bar h,1)]\circ\bar\eta^\tau + \kappa\Delta_0^2 v_\kappa \quad \text{on } (0,T)\times\Gamma, \tag{9.1c}$$

$$h_t\circ\bar\eta^\tau = [(\bar h_{,\alpha})\circ\bar\eta^\tau]v_\alpha - v_z \quad \text{on } (0,T)\times\Gamma, \tag{9.1d}$$

$$v = \tilde u_0 \quad \text{on } \{t=0\}\times\Omega, \tag{9.1e}$$

$$h = 0 \quad \text{on } \{t=0\}\times\Gamma. \tag{9.1f}$$

Having (9.1c), (A.7) in Appendix A implies that h_κ is in $H^5(\Gamma)$ for a.a. $t\in[0,T]$ with estimate

$$\int_0^t \|\nabla_0^2 h_\kappa\|^2_{H^3(\Gamma)}ds \le C(\epsilon_1)\int_0^t \Big[\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma)} + \|v_\kappa\|^2_{H^3(\Omega)} + \|q_\kappa\|^2_{H^2(\Omega)} + 1\Big]ds,$$

where the forcing f in (A.7) is given by

$$[\nu D_{\bar\eta}(v_\kappa)^j_i - q_\kappa\delta^j_i]\bar a^\ell_j N_\ell - \sigma\bar\Theta[\mathcal{M}_{\bar h}(-\nabla_0\bar h,1)]\circ\bar\eta^\tau.$$

By the same argument, (7.18) holds with all θ replaced by κ. Therefore, by (8.4) (which follows from (7.18)),

$$\int_0^t \|\nabla_0^2 h_\kappa\|^2_{H^3(\Gamma)}ds \le C(\epsilon_1)\int_0^t \Big[\|v_{\kappa t}\|^2_{H^1(\Omega)} + \|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma)} + \|\nabla_0^2 v_\kappa\|^2_{H^1(\Omega_1)}\Big]ds + C(\epsilon_1)N_2(u_0,F). \tag{9.2}$$

With this extra regularity of h_κ, the energy estimate (8.19) can be made independent of κ. In section B.2 in Appendix B, we prove that

$$\begin{aligned}\frac{\nu_1}{2}\|\nabla_0^4 h_\kappa(t)\|^2_{L^2(\Gamma)} &\le \int_0^t\int_\Gamma \bar\Theta\Big[[L_{\bar h}(h_\kappa)(-\nabla_0\bar h,1)]\circ\bar\eta^\tau\Big]\cdot\nabla_0^2(\zeta_1^2\nabla_0^2 v_\kappa)dSds\\ &+ C'\int_0^t \Big[1 + \|\tilde v\|^2_{H^3(\Omega)} + \|\tilde h_t\|^2_{H^{2.5}(\Gamma)} + \|\tilde h\|^2_{H^5(\Gamma)}\Big]\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma)}ds\\ &+ C'\int_0^t \Big[\|\tilde h\|^2_{H^5(\Gamma)} + 1\Big]ds + \delta\int_0^t \|v_\kappa\|^2_{H^3(\Omega)}ds + \delta_1\int_0^t \|\nabla_0^2 h_\kappa\|^2_{H^3(\Gamma)}ds\end{aligned}$$

for some constant C' depending on M, ϵ_1, δ, and δ_1. By (9.2),

$$\begin{aligned}\frac{\nu_1}{2}\|\nabla_0^4 h_\kappa(t)\|^2_{L^2(\Gamma)} &\le \int_0^t\int_\Gamma \bar\Theta\Big[[L_{\bar h}(h_\kappa)(-\nabla_0\bar h,1)]\circ\bar\eta^\tau\Big]\cdot\nabla_0^2(\zeta_1^2\nabla_0^2 v_\kappa)dSds\\ &+ C'N_2(u_0,F) + C'\int_0^t \Big[\|\nabla_0^2 v_\kappa\|^2_{H^1(\Omega_1)} + K(s)\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma)}\Big]ds\\ &+ \delta\int_0^t \|v_\kappa\|^2_{H^3(\Omega)}ds + \delta_1\int_0^t \|v_{\kappa t}\|^2_{H^1(\Omega)}ds,\end{aligned} \tag{9.3}$$

where

$$K(s) := 1 + \|\tilde v\|^2_{H^3(\Omega)} + \|\tilde h_t\|^2_{H^{2.5}(\Gamma)} + \|\tilde h\|^2_{H^5(\Gamma)}.$$

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With (9.3), (8.11) now is replaced by

$$\Big[\|\nabla_0^2 v_\kappa(t)\|^2_{L^2(\Omega_1)} + \|\nabla_0^4 h_\kappa(t)\|^2_{L^2(\Gamma)}\Big] + \int_0^t \Big[\|\nabla\nabla_0^2 v_\kappa\|^2_{L^2(\Omega_1)} + \kappa\|v_\kappa\|^2_{H^4(\Gamma)}\Big]ds$$
$$\leq C' N_2(u_0,F) + C'\int_0^t \Big[\|v_{\kappa t}\|^2_{L^2(\Omega)} + \|\nabla_0^2 v_\kappa\|^2_{H^1(\Omega_1)} + K(s)\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma)}\Big]ds$$
$$+\,\delta\int_0^t \|v_\kappa\|^2_{H^3(\Omega)}ds + \delta_1\int_0^t \|v_{\kappa t}\|^2_{H^1(\Omega)}ds \tag{9.4}$$

for some C' depending on M, ϵ_1, δ, and δ_1, where (A.5) is applied to bound $\kappa\|v_\kappa\|^2_{H^3(\Gamma)}$ (this is where $\|v_{\kappa t}\|^2_{L^2(\Omega)}$ comes from). Similar computations lead to

$$\Big[\|\nabla_0 v_\kappa(t)\|^2_{L^2(\Omega_1)} + \|\nabla_0^3 h_\kappa(t)\|^2_{L^2(\Gamma)}\Big] + \int_0^t \Big[\|\nabla\nabla_0 v_\kappa\|^2_{L^2(\Omega_1)} + \kappa\|v_\kappa\|^2_{H^3(\Gamma)}\Big]ds \tag{9.5}$$
$$\leq CN_2(u_0,F) + C\int_0^t \|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma)}ds + \delta\int_0^t \|v_\kappa\|^2_{H^3(\Omega)}ds$$

for some constant C depending on M and δ.

In Appendix C, we establish the following κ- and ϵ_1-independent inequality for the time-differentiated problem:

$$\int_0^t \|\nabla_0^2 h_{\kappa t}\|^2_{L^2(\Gamma)}ds \leq \int_0^t\int_\Gamma \Big[[L_{\bar h}(h_\kappa)(\nabla_0\bar h,-1)]\circ\bar\eta^\tau\Big]_t\cdot v_{\kappa t}dS$$
$$+\,CN_3(u_0,F) + C\int_0^t K(s)\Big[\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma)} + \|\nabla_0^2 h_{\kappa t}\|^2_{L^2(\Gamma)}\Big]ds$$
$$+\,(\delta + Ct^{1/2})\int_0^t \|v_\kappa\|^2_{H^3(\Omega)}ds + (\delta_1 + Ct^{1/2})\int_0^t \|v_{\kappa t}\|^2_{H^1(\Omega)}ds + \delta_2\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma)}$$

for some constant C depending on M, δ, δ_1, and δ_2. Therefore, (8.14) can be replaced by the following estimate:

$$\Big[\|v_{\kappa t}\|^2_{L^2(\Omega)} + \|\nabla_0^2 h_{\kappa t}\|^2_{L^2(\Gamma)}\Big] + \int_0^t \Big[\|\nabla v_{\kappa t}\|^2_{L^2(\Omega)} + \kappa\|\Delta_0 v_{\kappa t}\|^2_{L^2(\Gamma)}\Big]ds \tag{9.6}$$
$$\leq CN_3(u_0,F) + C\int_0^t K(s)\Big[\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma)} + \|\nabla_0^2 h_{\kappa t}\|^2_{L^2(\Gamma)}\Big]ds$$
$$+\,(\delta + Ct^{1/2})\int_0^t \|v_\kappa\|^2_{H^3(\Omega)}ds + (\delta_1 + Ct^{1/2})\int_0^t \|v_{\kappa t}\|^2_{H^1(\Omega)}ds + \delta_2\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma)}.$$

9.2. κ-independent estimates. Just as in section 8.7, we find that

$$\int_0^t \Big[\|v_\kappa\|^2_{H^3(\Omega)} + \|q_\kappa\|^2_{H^2(\Omega)}\Big]ds$$
$$\leq C(M)N_2(u_0,F) + C(M)\int_0^t \Big[\|v_{\kappa t}\|^2_{H^1(\Omega)} + \|\nabla_0^2 v_\kappa\|^2_{H^1(\Omega_1)}\Big]ds. \tag{9.7}$$

By choosing $\delta = \delta_1 = \delta_2 = 1/8$ and $T > 0$ so that $CT^{1/2} < 1/8$ in (9.6), we find that

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$$\int_0^t \Big[\|v_\kappa\|^2_{H^3(\Omega)} + \|q_\kappa\|^2_{H^2(\Omega)}\Big] ds \le C N_3(u_0,F) + \frac{1}{8}\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma)}$$
$$+ C(M)\int_0^t \Big[\|\nabla_0^2 v_\kappa\|^2_{H^1(\Omega_1)} + K(s)\Big(\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma)} + \|\nabla_0^2 h_{\kappa t}\|^2_{L^2(\Gamma)}\Big)\Big] ds. \tag{9.8}$$

Combining the estimates (7.27), (8.9), (9.4), and (9.5) with (9.6),

$$\Big[\|v_\kappa\|^2_{H^1(\Omega)} + \|\nabla_0^2 v_\kappa\|^2_{L^2(\Omega_1)} + \|\nabla_0^2 h_\kappa\|^2_{H^2(\Gamma)} + \|v_{\kappa t}\|^2_{L^2(\Omega)} + \|\nabla_0^2 h_{\kappa t}\|^2_{L^2(\Gamma)}\Big](t)$$
$$+ \int_0^t \Big[\|\nabla v_\kappa\|^2_{L^2(\Omega)} + \|\nabla\nabla_0 v_\kappa\|^2_{L^2(\Omega_1)} + \|\nabla\nabla_0^2 v_\kappa\|^2_{L^2(\Omega_1)} + \|v_{\kappa t}\|^2_{H^1(\Omega)}\Big] ds$$
$$\le C' N_3(u_0,F) + C'\int_0^t \Big[\|v_{\kappa t}\|^2_{L^2(\Omega)} + K(s)\Big(\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma)} + \|\nabla_0^2 h_{\kappa t}\|^2_{L^2(\Gamma)}\Big)\Big] ds$$

for some constant C' depending on M and ϵ_1. By the Gronwall inequality and (8.4),

$$\sup_{0\le t\le T}\Big[\|v_\kappa\|^2_{H^2(\Omega)} + \|v_{\kappa t}\|^2_{L^2(\Omega)} + \|\nabla_0^2 h_{\kappa t}\|^2_{L^2(\Gamma)} + \|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma)}$$
$$+ \|q_\kappa\|^2_{H^1(\Omega)}\Big](t) + \|v_\kappa\|^2_{\mathcal{V}^3(T)} + \|q_\kappa\|^2_{L^2(0,T;H^2(\Omega))} \le C(\epsilon_1) N_3(u_0,F).$$

9.3. Weak limits as $\kappa \to 0$. Just as in section 8.8, the weak limit $(v_{\epsilon_1}, h_{\epsilon_1}, q_{\epsilon_1})$ of $(v_\kappa, h_\kappa, q_\kappa)$ as $\kappa \to 0$ exists in $V(T) \times L^2(0,T;H^4(\Gamma)) \times L^2(0,T;H^2(\Omega))$ with estimate

$$\sup_{0\le t\le T}\Big[\|v_{\epsilon_1}(t)\|^2_{H^2(\Omega)} + \|v_{\epsilon_1 t}(t)\|^2_{L^2(\Omega)} + \|\nabla_0^2 h_{\epsilon_1 t}(t)\|^2_{L^2(\Gamma)} + \|\nabla_0^4 h_{\epsilon_1}(t)\|^2_{L^2(\Gamma)}$$
$$+ \|q_{\epsilon_1}(t)\|^2_{H^1(\Omega)}\Big] + \|v_\kappa\|^2_{\mathcal{V}^3(T)} + \|q_{\epsilon_1}\|^2_{L^2(0,T;H^2(\Omega))} \le C(\epsilon_1) N_3(u_0,F). \tag{9.9}$$

Equation (9.9) implies that for a.a. $t \in [0,T]$,

$$\|v_\kappa(t)\|_{H^{2.5}(\Gamma)} \le \bar{C}(t)$$

for some $\bar{C}(t)$ independent of κ, and therefore for a.a. $t \in [0,T]$,

$$\kappa \int_\Gamma \Delta_0 v_\kappa \cdot \Delta_0 \varphi dS \to 0$$

as $\kappa \to 0$. This observation with (8.20) shows that $(v_{\epsilon_1}, h_{\epsilon_1}, q_{\epsilon_1})$ satisfies, for a.a. $t \in [0,T]$,

$$(v_{\kappa t}, \varphi)_{L^2(\Omega)} + \frac{\nu}{2}\int_\Omega D_{\bar\eta} v_\kappa : D_{\bar\eta}(\varphi) dx + \sigma \int_\Gamma \bar\Theta \mathcal{L}_{\bar h}(h_\kappa)\Big[-\bar h_{,\sigma} \circ \bar\eta^\tau \varphi^\sigma + \varphi^z\Big] dS$$
$$- (\bar a_i^j q_\kappa, \varphi^i_{,j})_{L^2(\Omega)} = \langle \tilde F, \varphi\rangle + \sigma\langle \bar\Theta \mathcal{M}_{\bar h}(-\nabla_0 \bar h \circ \bar\eta^\tau, 1), \varphi\rangle_\Gamma \tag{9.10}$$

for all $\varphi \in H^{1;2}(\Omega;\Gamma)$. Since (9.10) also defines a linear functional on $H^1(\Omega)$, by the density argument, we have that (9.10) holds for all $\varphi \in H^1(\Omega)$. As $(v_{\epsilon_1}, h_{\epsilon_1}, q_{\epsilon_1})$ are smooth enough, we can integrate by parts and find that $(v_{\epsilon_1}, h_{\epsilon_1}, q_{\epsilon_1})$ satisfies (7.2) with (7.2c) replaced by

$$[\nu D_{\bar\eta}(v_{\epsilon_1})_i^j - q_{\epsilon_1}\delta_i^j]\bar a_j^\ell N_\ell = \sigma\Big[\bar\Theta[(\mathcal{L}_{\bar h}(h_{\epsilon_1}) + \mathcal{M}(\bar h))(\nabla_0 \bar h, -1)] \circ \bar\eta^\tau\Big] \quad \text{on } (0,T)\times\Gamma. \tag{9.11}$$

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9.4. $H^{5.5}$-regularity of h_κ. By (9.11), we have the following lemma.

LEMMA 9.1. *For a.a. $t \in [0,T]$, $h_{\epsilon_1}(t) \in H^{5.5}(\Gamma)$ with*

$$\|h_{\epsilon_1}\|^2_{H^{5.5}(\Gamma)} \le C(M)\Big[\|v_{\epsilon_1 t}\|^2_{H^1(\Omega)} + \|\nabla v_{\epsilon_1}\|^2_{L^2(\Omega)} + \|\nabla_0^2 v_{\epsilon_1}\|^2_{H^1(\Omega_1)} + \|\nabla_0^4 h_{\epsilon_1}\|^2_{L^2(\Gamma)} + \|F\|^2_{H^1(\Omega)} + 1\Big], \tag{9.12}$$

and hence

$$\|h_{\epsilon_1}\|^2_{L^2(0,T;H^{5.5}(\Gamma))} \le C(M)e^{C(M)+T}N_3(u_0,F). \tag{9.13}$$

Proof. We write the boundary condition (9.11) as

$$\mathcal{L}_{\bar h}(h_{\epsilon_1}) = \frac{1}{\sigma} J_{\bar h}^{-2}(-\nabla_0 \bar h, 1)\cdot \Big\{\bar\Theta^{-1}\Big[[\nu D_{\bar\eta}(v_{\epsilon_1})^j_i - q_{\epsilon_1}\delta^j_i]\bar a^\ell_j N_\ell\Big]\Big\}\circ \bar\eta^{-\tau} - \mathcal{M}(\bar h). \tag{9.14}$$

By Corollary 7.1, $\mathcal{L}_{\bar h}$ is uniformly elliptic with the elliptic constant ν_1 which is independent of M which defines our convex subset $C_T(M)$. Since $\bar h \in \mathcal{H}(T)$, $\mathcal{M}(\bar h) \in L^2(0,T;H^{2.5}(\Gamma)) \cap L^\infty(0,T;H^1(\Gamma))$, and hence by (8.19), the right-hand side of (9.14) is bounded in $H^{1.5}(\Gamma)$. The important point is that these bounds are independent of ϵ_1. Thus, elliptic regularity of $\mathcal{L}_{\bar h}$ proves the estimate

$$\|h_{\epsilon_1}\|^2_{H^{5.5}(\Gamma)} \le C(M)\Big[\|D_{\bar\eta}(v_{\epsilon_1})\|^2_{H^{1.5}(\Gamma)} + \|q_{\epsilon_1}\|^2_{H^{1.5}(\Gamma)} + 1\Big]$$

so that with (8.4), (9.12) is proved. ∎

9.5. Energy estimates which are independent of ϵ_1. Having estimate (9.12), one can follow exactly the same procedure as in section 9.2 to show that the constant C' appearing in (9.9) is independent of ϵ_1, provided that we have an ϵ_1-independent version of (9.4). By section B.2, we indeed have such an estimate:

$$\begin{aligned}\frac{\nu_1}{2}\|\nabla_0^4 h_{\epsilon_1}(t)\|^2_{L^2(\Gamma)} &\le \int_0^t\int_\Gamma \bar\Theta\Big[[L_{\bar h}(h_{\epsilon_1})(-\nabla_0\bar h, 1)]\circ\bar\eta^\tau\Big]\cdot\nabla_0^2(\zeta_1^2\nabla_0^2 v_{\epsilon_1})dSds \\ &\quad + CN_2(u_0,F) + C\int_0^t K(s)\|\nabla_0^4 h_{\epsilon_1}\|^2_{L^2(\Gamma)}ds + (\delta + Ct^{1/2})\int_0^t \|v_{\epsilon_1}\|^2_{H^3(\Omega)}ds \\ &\quad + (\delta_1 + Ct^{1/2})\int_0^t \|v_{\epsilon_1 t}\|^2_{H^1(\Omega)}ds\end{aligned}$$

for some constant C depending on M, δ, and δ_1. Therefore, we can conclude that

$$\begin{aligned}&\sup_{0\le t\le T}\Big[\|v_{\epsilon_1}\|^2_{H^2(\Omega)} + \|v_{\epsilon_1 t}\|^2_{L^2(\Omega)} + \|\nabla_0^2 h_{\epsilon_1 t}\|^2_{L^2(\Gamma)} + \|\nabla_0^4 h_{\epsilon_1}\|^2_{L^2(\Gamma)} \\ &\quad + \|q_{\epsilon_1}\|^2_{H^1(\Omega)}\Big](t) + \|v_{\epsilon_1}\|^2_{\mathcal{V}^3(T)} + \|q_{\epsilon_1}\|^2_{L^2(0,T;H^2(\Omega))} \le C(M)e^{C(M)+T}N_3(u_0,F).\end{aligned} \tag{9.15}$$

REMARK 15. *Literally speaking, we cannot use $\nabla_0^2(\zeta_1^2\nabla_0^2 v_{\epsilon_1})$ as a test function in* (9.10)*, since it is not a function in $H^1(\Omega)$. However, since $h_{\epsilon_1} \in H^{5.5}(\Gamma)$ for a.a. $t \in [0,T]$,* (9.10) *also holds for all $\varphi \in H^1(\Omega)' \cap H^{-1.5}(\Gamma)$ and $\nabla_0^2(\zeta_1^2\nabla_0^2 v_{\epsilon_1})$ is a function of this kind.*

9.6. Weak limits as $\epsilon_1 \to 0$. The same argument leads to the fact that weak limits of $(v_{\epsilon_1}, h_{\epsilon_1}, q_{\epsilon_1})$ (denoted by (v,h,q)) as $\epsilon_1 \to 0$ exist and (v,h,q) satisfies (7.1).

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9.7. Uniqueness. In this section, we show that for a given $(\tilde{v}, \tilde{h}) \in Y_T$, the solution to (7.1) is unique in Y_T. Suppose (v_1, h_1) and (v_2, h_2) are two solutions (in Y_T) to (7.3). Let $w = v_1 - v_2$ and $g = h_1 - h_2$; then w and g satisfy

$$\begin{aligned} &\langle w_t, \varphi\rangle + \frac{\nu}{2}\int_\Omega D_{\tilde{\eta}} w : D_{\tilde{\eta}}\varphi dx + \sigma \int_\Gamma \tilde{\Theta}\Big[\tilde{L}_{\tilde{h}}\Big(\int_0^t (\tilde{h}_{,\alpha} w_\alpha - w_z) ds\Big)\Big] \circ \tilde{\eta}^\tau \\ &\quad \times (-\tilde{h}_{,\alpha} \circ \tilde{\eta}^\tau \varphi^\alpha + \varphi^z) dS = 0 \end{aligned} \tag{9.16}$$

for all $\varphi \in \mathcal{V}_v(T)$ with $w(0) = 0$, where $\tilde{L}$ equals L, except $L_1 = L_2 = 0$. Since w is in $\mathcal{V}_v(T)$, letting $w = \varphi$ in (9.16) leads to

$$\begin{aligned} &\Big[\|v\|^2_{H^1(\Omega)} + \|\nabla_0^2 v\|^2_{L^2(\Omega_1)} + \|\nabla_0^4 h\|^2_{L^2(\Gamma)} + \|v_t\|^2_{L^2(\Omega)} + \|\nabla_0^2 h_t\|^2_{L^2(\Gamma)}\Big](t) \\ &\quad + \int_0^t \Big[\|\nabla v\|^2_{L^2(\Omega)} + \|\nabla\nabla_0 v\|^2_{L^2(\Omega_1)} + \|\nabla\nabla_0^2 v\|^2_{L^2(\Omega_1)} + \|v_t\|^2_{H^1(\Omega)}\Big] ds \\ &\le C(M)\int_0^t K(s)\Big[\|\nabla_0^4 h\|^2_{L^2(\Gamma)} + \|\nabla_0^2 h_t\|^2_{L^2(\Gamma)}\Big] ds. \end{aligned}$$

Therefore, by the Gronwall inequality and the zero initial condition ($w(0) = 0$), we have that w (and hence g) is identical to zero.

10. Fixed-point argument. From previous sections, we establish a map Θ_T from Y_T into Y_T; i.e., given $(\tilde{v}, \tilde{h}) \in C_T(M)$, there exists a unique $\Theta_T(\tilde{v}, \tilde{h}) = (v, h)$ satisfying (7.1). Theorem 4.1 is then proved if this mapping Θ_T has a fixed point. We shall make use of the Tychonoff fixed-point theorem which states as follows.

THEOREM 10.1. *For a reflexive Banach space X, and $C \subset X$ a closed, convex, bounded subset, if $F : C \to C$ is weakly sequentially continuous into X, then F has at least one fixed point.*

In order to apply the Tychonoff fixed-point theorem, we need to show that $\Theta(\tilde{v}, \tilde{h}) \in C_T(M)$, and this is the case if T is small enough. In the following discussion, we will always assume T is smaller than a fixed constant (for example, say $T \le 1$) so that the right-hand side of (9.15) can be written as $C(M)N_3(u_0, F)$.

REMARK 16. *The space Y_T is not reflexive. We will treat $C_T(M)$ as a convex subset of X_T and apply the Tychonoff fixed-point theorem on the space X_T.*

Before proceeding with the fixed-point proof, we note that Lemma 6.3 implies that for a short time, the constant $C(M)$ in (8.1) and (8.4) can be chosen to be independent of M. To be more precise, for a.a. $0 < t \le T_1$,

$$\|q\|^2_{L^2(\Omega)} \le C\Big[\|v_t\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)} + \|\nabla_0^4 h\|^2_{L^2(\Gamma)} + \|F\|^2_{L^2(\Omega)} + 1\Big], \tag{10.1}$$

$$\begin{aligned} \|v\|^2_{H^3(\Omega)} + \|q\|^2_{H^2(\Omega)} \le C\Big[&\|v_t\|^2_{H^1(\Omega)} + \|\nabla v\|^2_{H^1(\Omega)} + \|\nabla_0 v\|^2_{H^1(\Omega_1)} \\ &+ \|\nabla_0^2 v\|^2_{H^1(\Omega)} + \|F\|^2_{H^1(\Omega)} + 1\Big], \end{aligned} \tag{10.2}$$

and

$$\begin{aligned} \|h\|^2_{H^{5.5}(\Gamma)} \le C\Big[&\|v_t\|^2_{H^1(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)} + \|\nabla_0^2 v\|^2_{H^1(\Omega_1)} + \|\nabla_0^4 h\|^2_{L^2(\Gamma)} \\ &+ \|F\|^2_{H^1(\Omega)} + 1\Big] \end{aligned} \tag{10.3}$$

for some constant C independent of M.

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10.1. Continuity in time of h. By the evolution equation (7.1d) and the fact that $v \in \mathcal{V}^3(T_1)$, $h_t \in L^2(0,T_1;H^{2.5}(\Gamma))$. Since $h \in L^2(0,T_1;H^{5.5}(\Gamma))$, we have that $h \in \mathcal{C}^0([0,T_1];H^4(\Gamma))$ by the standard interpolation theorem. Although there is no uniform rate that h converges to zero in $H^4(\Gamma)$, we have the following lemma.

LEMMA 10.2. *Let $(v,h) = \Theta_{T_1}(\tilde{v},\tilde{h})$. Then $\|h(t)\|_{H^{2.5}(\Gamma)}$ converges to zero as $t \to 0$, uniformly for all $(\tilde{v},\tilde{h}) \in C_{T_1}(M)$.*

Proof. By the evolution equation (7.1d),

$$\|h(t)\|_{H^{2.5}(\Gamma)} \le \int_0^t \|\tilde{h}_{,\alpha} v_\alpha - v_z\|_{H^{2.5}(\Gamma)} dS \le C(M) N_3(u_0,F)^{1/2} t^{1/2}.$$

The lemma follows directly from the inequality. ∎

By Lemma 10.2 and the interpolation inequality, we also have the following lemma.

LEMMA 10.3. *$\|\nabla_0^2 h(t)\|_{H^{1.5}(\Gamma)}$ converges to zero as $t \to 0$, uniformly for all $\tilde{h} \in C_{T_1}(M)$ with estimate*

$$\|\nabla_0^2 h(t)\|_{H^{1.5}(\Gamma)} \le C(M) N_3(u_0,F) t^{1/4} \tag{10.4}$$

for all $0 < t \le T_1$.

10.2. Improved energy estimates. In order to apply the fixed-point theorem, we have to use the fact that the forcing F is in $\mathcal{V}^2(T)$. We also define a new constant

$$N(u_0,F) := \|u_0\|^2_{H^{2.5}(\Omega)} + \|F\|^2_{\mathcal{V}^2(T_1)} + \|F\|^2_{L^\infty(0,T_1;L^2(\Omega))} + \|F(0)\|^2_{H^1(\Omega)} + 1.$$

Note that $N_3(u_0,F) \le N(u_0,F)$.

REMARK 17. *For the linearized problem* (7.1)*, we need only $F \in \mathcal{V}^1(T)$ to obtain a unique solution $(v,h) \in Y_T$.*

10.2.1. Estimates for $\nabla_0^2 v$ near the boundary. Note that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\|\zeta_1 \nabla_0^2 v\|^2_{L^2(\Omega)} + \sigma \int_\Gamma \tilde{\Theta} B \tilde{A}^{\alpha\beta\gamma\delta} \nabla_0^2 h_{,\alpha\beta} \nabla_0^2 h_{,\gamma\delta} dS\right] + \frac{\nu}{2}\|\zeta_1 D_{\tilde{\eta}}(\nabla_0^2 v)\|^2_{L^2(\Omega)} \\
&= \langle F, \nabla_0^2(\zeta_1^2 \nabla_0^2 v)\rangle - \frac{\nu}{4}\int_\Omega \left[\nabla_0^2(\tilde{a}_i^k \tilde{a}_i^\ell) v^j_{,\ell} + \nabla_0^2(\tilde{a}_i^k \tilde{a}_j^\ell) v^i_{,\ell}\right](\zeta_1^2 \nabla_0^2 v^j)_{,k} dx \\
&\quad - \frac{\nu}{2}\int_\Omega \left[\nabla_0(\tilde{a}_i^k \tilde{a}_i^\ell)\nabla_0 v^j_{,\ell} + \nabla_0(\tilde{a}_i^k \tilde{a}_j^\ell)\nabla_0 v^i_{,\ell}\right](\zeta_1^2 \nabla_0^2 v^j)_{,k} dx \\
&\quad - \frac{\nu}{2}\int_\Omega D_{\tilde{\eta}}(\nabla_0^2 v)^j_i \tilde{a}_i^k \zeta_1 \zeta_{1,k} \nabla_0^2 v^j dx + \int_\Omega q \tilde{a}_k^\ell [\nabla_0^2(\zeta_1^2 \nabla_0^2 v^k)]_{,\ell} dx - \sigma\left(\sum_{k=1}^3 I_k + \sum_{k=1}^8 J_k\right),
\end{aligned}$$

where I_k's and J_k's are defined in section B.1 (with $\bar{}$ replaced by $\tilde{}$, and no ϵ_1 and ϵ_2).

As in [7] and [8], we study the time integral of the right-hand side of the identity above in order to prove the validity of the requirement of applying the Tychonoff fixed-point theorem. By interpolation and (9.9),

$$\begin{aligned}
&\int_0^t \int_\Omega \left[\nabla_0^2(\tilde{a}_i^k \tilde{a}_i^\ell) v^j_{,\ell} + \nabla_0^2(\tilde{a}_i^k \tilde{a}_j^\ell) v^i_{,\ell}\right](\zeta_1^2 \nabla_0^2 v^j)_{,k} dx ds \\
&\le C \int_0^t \|\tilde{a}\tilde{a}\|_{H^2(\Omega)} \|\nabla v\|_{L^\infty(\Omega)} \|v\|_{H^3(\Omega)} ds
\end{aligned}$$

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$$
\begin{aligned}
&\le C(M)C(\delta)\int_0^t \|v\|^{1/2}_{H^3(\Omega)}\|v\|^{1/2}_{H^1(\Omega)}ds + \delta\|v\|^2_{L^2(0,T;H^3(\Omega))}\\
&\le C(M)C(\delta)N(u_0,F)^{1/2}\int_0^t \|v\|^{1/2}_{H^3(\Omega)}ds + \delta C(M)N(u_0,F)\\
&\le C(M)N(u_0,F)\Big[C(\delta)t^{3/4}+\delta\Big].
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\int_0^t\int_\Omega \Big[\nabla_0(\tilde a_i^k \tilde a_i^\ell)\nabla_0 v^j_{,\ell} + \nabla_0(\tilde a_i^k \tilde a_j^\ell)\nabla_0 v^i_{,\ell}\Big](\zeta_1^2\nabla_0^2 v^j)_{,k}dxds\\
&+\int_0^t\int_\Omega D_{\tilde\eta}(\nabla_0^2 v)^j_i \tilde a_i^k \zeta_1\zeta_{1,k}\nabla_0^2 v^j dxds \le C(M)N(u_0,F)\Big[t^{1/2}+C(\delta)t+\delta\Big].
\end{aligned}
$$

For the pressure term, by interpolation and (8.10),

$$
\begin{aligned}
&\int_0^t\int_\Omega q\tilde a_k^\ell[\nabla_0^2(\zeta_1^2\nabla_0^2 v^k)]_{,\ell}dxds\\
&\le C(M)\int_0^t \Big[\|q\|_{L^\infty(\Omega)}+\|q\|_{W^{1,4}(\Omega)}+\|q\|_{H^1(\Omega)}\Big]\|v\|_{H^3(\Omega)}ds\\
&\le C(M)C(\delta)\int_0^t \|q\|^2_{H^1(\Omega)}ds + \delta\Big[\|v\|^2_{L^2(0,T;H^3(\Omega)}+\|q\|^2_{L^2(0,T;H^2(\Omega))}\Big]\\
&\le C(M)N(u_0,F)\Big[C(\delta)t^{1/2}+\delta\Big].
\end{aligned}
$$

By the estimates already established in Appendix B, with the help of (6.6), it is also easy to see that

$$
\int_0^t\left(\sum_{k=1}^3 I_k + \sum_{k=1}^8 J_k\right)ds \le C(M)N(u_0,F)\Big[t^{1/4}+t^{1/2}+C(\delta)t^{2/3}+\delta\Big].
$$

Finally, for the forcing term, by the extra regularity we assume on F,

$$
\begin{aligned}
\int_0^t\langle F,\nabla_0^2(\zeta_1^2\nabla_0^2 v)\rangle ds &\le \int_0^t \|F\|_{H^2(\Omega)}\|v\|_{H^2(\Omega)}ds \le N(u_0,F)+\int_0^t\|v\|^2_{H^2(\Omega)}ds\\
&\le N(u_0,F)+C(M)N(u_0,F)t.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\Big[\|\nabla_0^2 v(t)\|^2_{L^2(\Omega_1)}+\sigma E_{\tilde h}(\nabla_0^2 h)\Big]+\nu\int_0^t\|D_{\tilde\eta}(\nabla_0^2 v)\|^2_{L^2(\Omega_1)}ds\\
&\le \|u_0\|^2_{H^2(\Omega)}+CN(u_0,F)+C(M)N(u_0,F)\Big[C(\delta)(t^{3/4}+t^{2/3}+t^{1/2}+t)+\delta\Big].
\end{aligned}
$$

By Corollary 7.1,

$$
\begin{aligned}
&\Big[\|\nabla_0^2 v(t)\|^2_{L^2(\Omega_1)}+\|\nabla_0^4 h(t)\|^2_{L^2(\Gamma)}\Big]+\int_0^t\|\nabla_0^2 v\|^2_{H^1(\Omega_1)}ds\\
&\le CN(u_0,F)+C(M)N(u_0,F)\Big[C(\delta)\mathcal{O}(t)+\delta\Big] \quad \text{as} \quad t\to 0, \qquad (10.5)
\end{aligned}
$$

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where C depends on ν, σ, ν_1, and the geometry of Γ.

By similar computations, we can also conclude (the (7.27), (8.9), and (9.5) variants) that

$$\begin{aligned}&\Big[\|v(t)\|^2_{L^2(\Omega)} + \|\nabla_0^2 h(t)\|^2_{L^2(\Gamma)}\Big] + \int_0^t \|v\|^2_{H^1(\Omega)} ds\\ &\le CN(u_0,F) + C(M)N(u_0,F)\mathcal{O}(t) \qquad \text{as} \quad t\to 0;\end{aligned} \tag{10.6}$$

$$\begin{aligned}&\Big[\|\nabla_0 v(t)\|^2_{L^2(\Omega_1)} + \|\nabla_0^3 h(t)|^2_{L^2(\Gamma)}\Big] + \int_0^t \|\nabla_0 v\|^2_{H^1(\Omega_1)} ds\\ &\le CN(u_0,F) + C(M)N(u_0,F)\mathcal{O}(t) \qquad \text{as} \quad t\to 0;\end{aligned} \tag{10.7}$$

$$\begin{aligned}&\|\nabla v(t)\|^2_{L^2(\Omega)} + \int_0^t \|v_t\|^2_{L^2(\Omega)} ds\\ &\le CN(u_0,F) + C(M)N(u_0,F)\mathcal{O}(t) \qquad \text{as} \quad t\to 0,\end{aligned} \tag{10.8}$$

where C depends on ν, σ, ν_1, and the geometry of Γ.

10.2.2. $L^2_t H^1_x$-estimate for v_t. For the time-differentiated problem, we are not able to use estimates such as those in sections 8.6 and 10.2.1, since no ϵ_1-regularization is present; nevertheless, we can obtain estimates at the ϵ_1-regularization level and then pass ϵ_1 to the limit once the estimate is found to be ϵ_1-independent. We have that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|v_t\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|D_{\bar\eta} v_t\|^2_{L^2(\Omega)} + \frac{\sigma}{2}\frac{d}{dt}\int_\Gamma \bar\Theta \bar A^{\alpha\beta\gamma\delta} h_{t,\alpha\beta} h_{t,\gamma\delta} dS\\ &= \langle F_t, v_t\rangle - \nu\int_\Omega \Big[(\bar a_i^k \bar a_j^\ell)_t v^j_{,\ell} + (\bar a_i^k \bar a_j^\ell)_t v^i_{,\ell}\Big] v^j_{t,k} dx + \int_\Omega q_t \bar a^\ell_{kt} v^k_{,\ell} dx\\ &+ \frac{1}{2}\int_\Gamma (\bar\Theta \bar A^{\alpha\beta\gamma\delta})_t h_{t,\alpha\beta} h_{t,\gamma\delta} dS - \int_\Gamma \frac{\bar\Theta}{\sqrt{\det(g_0)}}\Big[\sqrt{\det(g_0)}(\bar A^{\alpha\beta\gamma\delta})_t h_{,\alpha\beta}\Big]_{,\gamma\delta} h_{tt} dS\\ &- 2\int_\Gamma \bar\Theta_{,\gamma} \bar A^{\alpha\beta\gamma\delta} h_{t,\alpha\beta} h_{tt,\delta} dS - \int_\Gamma \bar\Theta_{,\gamma\delta} \bar A^{\alpha\beta\gamma\delta} h_{t,\alpha\beta} h_{tt} dS\\ &- \int_\Gamma \bar\Theta \Big[L_1^{\alpha\beta\gamma} \bar h_{,\alpha\beta\gamma}\Big]_t h_{tt} dS - \int_\Gamma \bar\Theta (L_2)_t h_{tt} dS + K_1 + K_3 + K_4 + K_5 + K_6,\end{aligned}$$

where K_i's are defined in Appendix C (without ϵ_2).

As in the previous section, the time integral of the right-hand side of the identity above is studied. It is easy to see that

$$\begin{aligned}&\int_0^t \Big[\langle F_t, v_t\rangle - \nu\Big((\bar a_i^k \bar a_j^\ell)_t v^j_{,\ell} + (\bar a_i^k \bar a_j^\ell)_t v^i_{,\ell}\Big) v^j_{t,k} + K_1 + K_5 + K_6\Big] ds\\ &\le C(M)N(u_0,F)\Big[t^{1/4} + t^{1/2} + C(\delta)(t^{1/2}+t) + \delta\Big],\end{aligned}$$

and by Appendix C, particularly Remark 22,

$$\begin{aligned}&\int_0^t\int_\Gamma \Big[\frac{1}{2}(\bar\Theta \bar A^{\alpha\beta\gamma\delta})_t h_{t,\alpha\beta} h_{t,\gamma\delta} - \frac{\bar\Theta}{\sqrt{\det(g_0)}}\Big[\sqrt{\det(g_0)}(\bar A^{\alpha\beta\gamma\delta})_t h_{,\alpha\beta}\Big]_{,\gamma\delta} h_{tt}\\ &\qquad - 2\bar\Theta_{,\gamma} \bar A^{\alpha\beta\gamma\delta} h_{t,\alpha\beta} h_{tt,\delta} - \bar\Theta_{,\gamma\delta} \bar A^{\alpha\beta\gamma\delta} h_{t,\alpha\beta} h_{tt}\Big] dS ds\\ &\le C(M)N(u_0,F) t^{1/2}.\end{aligned}$$

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Special treatment is needed for the rest of the terms, and we break this procedure into several steps.

Step 1. Let $B_1 = \int_0^t\int_\Omega (q\bar{a}_k^\ell)_t v^k_{t,\ell} dxds$. By the "divergence-free" condition (7.2b),

$$B_1 = \int_0^t \int_\Omega \bar{a}^\ell_{kt} q v^k_{t,\ell} dxds - \int_0^t \int_\Omega \bar{a}^\ell_{kt} q_t v^k_{,\ell} dxds.$$

By interpolation and (8.1),

$$\begin{aligned}
&\left|\int_0^t \int_\Omega \bar{a}^\ell_{kt} q v^k_{t,\ell} dxds\right| \\
&\le C(M)C(\delta)\int_0^t \|q\|^2_{L^2(\Omega)} ds + \delta\Big[\|q\|^2_{L^2(0,T;H^1(\Omega))} + \|v_t\|^2_{L^2(0,T;H^1(\Omega))}\Big] \\
&\le C(M)N(u_0,F)\Big[C(\delta)t + \delta\Big].
\end{aligned}$$

For the second integral, we have the following identity:

$$\begin{aligned}
\int_0^t \int_\Omega \bar{a}^\ell_{kt} q_t v^k_{,\ell} dxds = &\int_\Omega (\bar{a}^\ell_{kt} q v^k_{,\ell})(t) dx - \int_\Omega \bar{a}^\ell_{kt}(0) q(0) u^k_{0,\ell} dx \\
&- \int_0^t \int_\Omega (\bar{a}^\ell_{kt} v^k_{,\ell})_t q dxds.
\end{aligned}$$

By the identity $\bar{a}^\ell_{kt} = -\bar{a}^i_k \bar{v}^j_{,i} \bar{a}^\ell_j$,

$$\begin{aligned}
\left|\int_0^t \int_\Omega (\bar{a}^\ell_{kt} v^k_{,\ell})_t q dxds\right| &\le \int_0^t \int_\Omega \left|\Big[\bar{a}^\ell_{ktt} v^k_{,\ell} + \bar{a}^\ell_{kt} v^k_{t,\ell}\Big] q\right| dxds \\
&\le C(M)\int_0^t (1 + \|\bar{v}_t\|_{H^1(\Omega)}) \|\nabla v\|_{L^4(\Omega)} \|q\|_{L^4(\Omega)} ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left|\int_0^t \int_\Omega (\bar{a}^\ell_{kt} v^k_{,\ell})_t q dxds\right| \\
&\le C(M)C(\delta)N(u_0,F)\int_0^t \|q\|^{2\alpha}_{H^1(\Omega)} \|q\|^{2(1-\alpha)}_{L^2(\Omega)} ds + \delta\int_0^t (1 + \|\bar{v}_t\|^2_{H^1(\Omega)}) ds \\
&\le C(M)N(u_0,F)^2\Big[C(\delta)(t + t^{\frac{1-\alpha}{2}}) + \delta\Big],
\end{aligned}$$

where $\alpha = \frac{3}{4}$ if $n = 3$ and $\alpha = \frac{1}{2}$ if $n = 2$.

The second integral equals $\int_\Omega \nabla u_0 : (\nabla u_0)^T q(0) dx$, which is bounded by $CN(u_0,F)$. It remains to estimate the first integral. By adding and subtracting $\int_\Omega \bar{a}^\ell_{kt}(0) q v^k_{,\ell} dx$, we find, by $\bar{a}_t(0) \in H^2(\Omega)$, that

$$\begin{aligned}
\left|\int_\Omega (\bar{a}^\ell_{kt} q v^k_{,\ell})(t) dx\right| &\le \int_\Omega \left|(\bar{a}^\ell_{kt} - \bar{a}^\ell_{kt}(0))(q v^k_{,\ell})(t)\right| dx + \int_\Omega \left|\bar{a}^\ell_{kt}(0) q v^k_{,\ell}\right| dx \\
&\le C\|\bar{a}_t(t) - \bar{a}_t(0)\|_{L^4(\Omega)} \|q\|_{L^2(\Omega)} \|\nabla v\|_{L^4(\Omega)} \\
&\quad + C(\delta_1)\|\nabla v\|^2_{L^2(\Omega)} + \delta_1 \|q\|^2_{L^2(\Omega)}.
\end{aligned}$$

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Noting that

$$\|\nabla v\|^2_{L^2(\Omega)} = \left\|\nabla u_0 + \int_0^t \nabla v_t ds\right\|^2_{L^2(\Omega)} \le \left[\|\nabla u_0\|_{L^2(\Omega)} + \int_0^t \|\nabla v_t\|_{L^2(\Omega)} ds\right]^2$$
$$\le 2\Big[\|u_0\|^2_{H^1(\Omega)} + C(M)N(u_0,F)t\Big],$$

(9.9), (6.5c), and (10.1) imply that

$$\left|\int_\Omega \bar{a}^\ell_{kt} q v^k_{,\ell}(t)dx\right| \le C(M)N(u_0,F)t^{1/2} + C(\delta_1)N(u_0,F)$$
$$+ \delta_1\Big[\|v_t\|^2_{L^2(\Omega)} + \|\nabla_0^4 h\|^2_{L^2(\Gamma)}\Big].$$

Summing all the estimates above, we find that

$$|B_1| \le C(\delta_1)N(u_0,F) + C(M)N(u_0,F)^2\Big[C(\delta)(t + t^{\frac{1-\alpha}{2}}) + \delta\Big]$$
$$+ \delta_1\Big[\|v_t\|^2_{L^2(\Omega)} + \|\nabla_0^4 h\|^2_{L^2(\Gamma)}\Big].$$

REMARK 18. *It may be tempting to use an interpolation inequality to show that $q \in \mathcal{C}([0,T];X)$ for some Banach space X by analyzing q_t via Laplace's equation. The problem, however, is that the boundary condition for q_t has low regularity $L^2(0,T;H^{-1.5}(\Gamma))$ (by the fact that $h_t \in L^2(0,T;H^{2.5}(\Gamma))$), and thus standard elliptic estimates do not provide the desired conclusion that $q_t \in L^2(0,T;H^1(\Omega)')$ (and hence by interpolation, $q \in \mathcal{C}([0,T];H^{0.5}(\Omega))$). However, suppose that $q_t \in L^2(0,T;H^1(\Omega)')$; then we can estimate $\int_0^t\int_\Omega \bar{a}^\ell_{kt} q_t v^k_{,\ell} dxds$ by the following method:*

$$\left|\int_0^t\int_\Omega \bar{a}^\ell_{kt} q_t v^k_{,\ell} dxds\right| \le \int_0^t \|\bar{a}^i_k \bar{v}^j_{,i} \bar{a}^\ell_j v^k_{,\ell}\|_{H^1(\Omega)} \|q_t\|_{H^1(\Omega)'} ds$$
$$\le C(M)N(u_0,F)\Big[t + t^{5/8}\Big].$$

Step 2. Let $B_2 = \int_0^t\int_\Gamma \tilde{\Theta}\big[[L_1^{\alpha\beta\gamma}\tilde{h}_{,\alpha\beta\gamma}]_t h_{tt} + (L_2)_t h_{tt}\big] dSds$. It is easy to see that

$$\left|\int_0^t\int_\Gamma \bar{\Theta}(L_2)_t h_{tt} dSds\right| \le C(M)\int_0^t \Big[\|v\|_{L^\infty(\Gamma)} + \|v_t\|_{L^2(\Gamma)}\Big] ds$$
$$\le C(M)N(u_0,F)^{1/2}(t + t^{3/4}).$$

For parts involving L_1, we have

$$\int_0^t\int_\Gamma \tilde{\Theta}\Big[L_1^{\alpha\beta\gamma}\tilde{h}_{,\alpha\beta\gamma}\Big]_t h_{tt} dSds = \int_0^t\int_\Gamma \tilde{\Theta}\Big[L_1^{\alpha\beta\gamma}\Big]_t \bar{h}_{,\alpha\beta\gamma} h_{tt} dSds \quad (\equiv B_2^1)$$
$$+ \int_0^t\int_\Gamma \tilde{\Theta} L_1^{\alpha\beta\gamma} \bar{h}_{t,\alpha\beta\gamma} h_{tt} dSds \quad (\equiv B_2^2).$$

By interpolation,

$$|B_2^1| \le C(M)\int_0^t \|\bar{\Theta}\|_{L^\infty(\Gamma)} \|\tilde{h}\|_{W^{1,4}(\Gamma)} \|h_{tt}\|_{L^4(\Gamma)} dSds$$
$$\le C(M)\int_0^t \Big[\|v\|_{H^2(\Omega)} + \|v_t\|_{H^1(\Omega)}\Big] ds$$
$$\le C(M)N(u_0,F)^{1/2}t^{1/2},$$

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while by (6.6) and Corollary 6.5,

$$
\begin{aligned}
|B_2^2| &\le \int_0^t \|\bar\Theta\|_{H^{1.5}(\Gamma)} \|\tilde h_t\|_{H^{2.5}(\Gamma)} \|L_1^{\alpha\beta\gamma}\|_{H^{1.5}(\Gamma)} \|h_{tt}\|_{H^{0.5}(\Gamma)} ds \\
&\le C(M) \|L_1^{\alpha\beta\gamma}\|_{H^{1.5}(\Gamma)} \int_0^t \|\tilde h\|_{H^{2.5}(\Gamma)} \Big[\|v\|_{H^2(\Omega)} + \|v_t\|_{H^1(\Omega)} \Big] ds \\
&\le C(M) N(u_0, F) t^{1/4}.
\end{aligned}
$$

Therefore,

$$
|B_2| \le C(M) N(u_0, F)(t + t^{3/4} + t^{1/4}).
$$

Step 3. Let $B_3 = \int_0^t K_3 ds = \int_0^t \int_\Gamma \bar\Theta [L_{\bar h}(h)]_t [(\bar v \circ \bar\eta^{-\tau}) \cdot (\nabla_0 h_t)] dS ds$. The L_1 and L_2 part of B_3 is bounded by

$$
C(M) \int_0^t \|\bar\Theta\|_{H^{1.5}(\Gamma)} \|\bar v\|_{H^{1.5}(\Gamma)} \|\bar h\|_{H^{3.5}(\Gamma)} \|\bar h_t\|_{H^2(\Gamma)} \|h_t\|_{H^2(\Omega)} ds,
$$

and hence

$$
\left| \int_0^t \bar\Theta \Big[L_1^{\alpha\beta\gamma} \bar h_{,\alpha\beta\gamma} + L_2 \Big]_t [(\bar v \circ \bar\eta^{-\tau}) \cdot (\nabla_0 h_t)] dS ds \right| \le C(M) N(u_0, F) t^{1/4}.
$$

By the integration by parts formula, the highest order part of B_3 can be expressed as

$$
\begin{aligned}
&\int_0^t \int_\Gamma \frac{\bar\Theta(\bar v \circ \bar\eta^{-\tau})}{\sqrt{\det(g_0)}} \Big[\sqrt{\det(g_0)} (\bar A^{\alpha\beta\gamma\delta})_t h_{,\alpha\beta} \Big]_{,\gamma\delta} \nabla_0 h_t dS ds \quad (\equiv B_3^1) \\
&+ \int_0^t \int_\Gamma \bar\Theta(\bar v \circ \bar\eta^{-\tau}) \bar A^{\alpha\beta\gamma\delta} h_{t,\alpha\beta} \nabla_0 h_{t,\gamma\delta} dS ds \quad (\equiv B_3^2) \\
&+ 2 \int_0^t \int_\Gamma [\bar\Theta(\bar v \circ \bar\eta^{-\tau})]_{,\gamma} \bar A^{\alpha\beta\gamma\delta} h_{t,\alpha\beta} \nabla_0 h_{t,\delta} dS ds \quad (\equiv B_3^3) \\
&+ \int_0^t \int_\Gamma [\bar\Theta(\bar v \circ \bar\eta^{-\tau})]_{,\gamma\delta} \bar A^{\alpha\beta\gamma\delta} h_{t,\alpha\beta} \nabla_0 h_t dS ds \quad (\equiv B_3^4).
\end{aligned}
$$

It is easy to see that

$$
\begin{aligned}
|B_3^1| &\le C(M) \int_0^t \|\bar\Theta \bar v \circ \bar\eta^{-\tau}\|_{H^{1.5}(\Gamma)} \|\bar h_t\|_{H^2(\Gamma)} \|h\|_{H^4(\Gamma)} \|h_t\|_{H^2(\Gamma)} dS \\
&\le C(M) N(u_0, F) t
\end{aligned}
$$

and

$$
\begin{aligned}
|B_3^3| &\le C(M) \int_0^t \|\bar\Theta \bar v \circ \bar\eta^{-\tau}\|_{W^{1,4}(\Gamma)} \|\bar A\|_{L^\infty(\Gamma)} \|h_t\|_{H^2(\Gamma)} \|h_t\|_{W^{2,4}(\Gamma)} dS \\
&\le C(M) N(u_0, F) t^{1/2}.
\end{aligned}
$$

For B_3^2, by the integration by parts formula,

$$
\begin{aligned}
B_3^2 &= \frac{1}{2} \int_0^t \int_\Gamma \bar\Theta(\bar v \circ \bar\eta^{-\tau}) \bar A^{\alpha\beta\gamma\delta} \nabla_0 \Big[h_{t,\alpha\beta} h_{t,\gamma\delta} \Big] dS ds \\
&= -\frac{1}{2} \int_0^t \int_\Gamma \frac{1}{\sqrt{\det(g_0)}} \nabla_0 \Big[\sqrt{\det(g_0)} \bar\Theta(\bar v \circ \bar\eta^{-\tau}) \bar A^{\alpha\beta\gamma\delta} \Big] h_{t,\alpha\beta} h_{t,\gamma\delta} dS ds,
\end{aligned}
$$

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and hence

$$
\begin{aligned}
|B_3^2| &\leq \int_0^t \Big[\|\nabla_0 \bar{\Theta}\|_{L^4(\Gamma)} \|\bar{v}\bar{A}\|_{L^\infty(\Gamma)} + \|\bar{\Theta}\|_{L^\infty(\Gamma)} \|\bar{v}\bar{A}\|_{W^{1,4}(\Gamma)} \Big] \\
&\quad \times \|h_t\|_{W^{2,4}(\Gamma)} \|h_t\|_{H^2(\Gamma)} ds \\
&\leq C(M) N(u_0, F)^{1/2} \int_0^t \|v\|_{H^3(\Omega)} ds \\
&\leq C(M) N(u_0, F) t^{1/2}.
\end{aligned}
$$

For B_3^4, noting that

$$
\begin{aligned}
\bar{\Theta}_{,\gamma\delta} &= \det(\nabla_0 \bar{\eta}^\tau)_{,\gamma\delta} \sqrt{\det(G_{\bar{h}}) \circ \bar{\eta}^\tau} + \det(\nabla_0 \bar{\eta}^\tau)_{,\gamma} \sqrt{\det(G_{\bar{h}}) \circ \bar{\eta}^\tau}_{,\delta} \\
&\quad + \det(\nabla_0 \bar{\eta}^\tau)_{,\delta} \sqrt{\det(G_{\bar{h}}) \circ \bar{\eta}^\tau}_{,\gamma} + \det(\nabla_0 \bar{\eta}^\tau) \sqrt{\det(G_{\bar{h}}) \circ \bar{\eta}^\tau}_{,\gamma\delta}
\end{aligned}
$$

and $\|\nabla_0 \det(\nabla_0 \bar{\eta}^\tau)\|_{H^{0.5}(\Gamma)} \leq C(M) t^{1/2}$, we find that

$$
\begin{aligned}
|B_3^4| &\leq C(M) \int_0^t \|\nabla_0 \det(\nabla_0 \bar{\eta}^\tau)\|_{H^{0.5}(\Gamma)} \|\nabla_0^2 h_t\|_{H^{0.5}(\Gamma)} \|\nabla_0 h_t\|_{H^{1.5}(\Gamma)} ds \\
&\quad + C(M) \int_0^t \|\det(\nabla_0 \bar{\eta}^\tau)\|_{L^\infty(\Gamma)} \|\nabla_0 \bar{\eta}^\tau\|_{L^\infty(\Gamma)}^2 \|\nabla_0^2 h_t\|_{L^2(\Gamma)} \|\nabla_0 h_t\|_{L^2(\Gamma)} ds \\
&\leq C(M) N(u_0, F) t^{1/2} + C(M) N(u_0, F)^{3/4} \int_0^t \|v\|_{H^3(\Omega)}^{1/2} ds \\
&\leq C(M) N(u_0, F) (t^{1/2} + t^{3/4}).
\end{aligned}
$$

Combining all the estimates, we find that

$$
|B_3| \leq C(M) N(u_0, F)(t + t^{1/2} + t^{3/4}).
$$

Step 4. Let $B_4 = \int_0^t K_4 ds = \int_0^t \int_\Gamma \bar{\Theta}[L_{\bar{h}}(h)]_t [(\nabla_0 \bar{h}, -1)_t \cdot (v \circ \bar{\eta}^{-\tau})] dS ds$. Integrating by parts,

$$
\begin{aligned}
B_4 = &- \int_0^t \int_\Gamma L_{\bar{h}}(h) \Big[\bar{\Theta}_t (\nabla_0 \bar{h}, -1)_t \cdot (v \circ \bar{\eta}^{-\tau}) + \bar{\Theta} (\nabla_0 \bar{h}, -1)_t \cdot (v \circ \bar{\eta}^{-\tau})_t \\
&+ \bar{\Theta} (\nabla_0 \bar{h}, -1)_{tt} \cdot (v \circ \bar{\eta}^{-\tau}) \Big] dS ds + \int_\Gamma \bar{\Theta} L_{\tilde{h}}(h) [(\nabla_0 \tilde{h}, -1)_t \cdot (v \circ \bar{\eta}^{-\tau})] dS.
\end{aligned}
$$

For the first integral, (6.8) implies that

$$
\begin{aligned}
&\left| \int_\Gamma \bar{\Theta} L_{\tilde{h}}(h) [(\nabla_0 \tilde{h}, -1)_t \cdot (v \circ \bar{\eta}^{-\tau})] dS \right| \\
&\leq \|\bar{\Theta}\|_{L^\infty(\Gamma)} \|L_{\tilde{h}}(h)\|_{L^2(\Gamma)} \|\nabla_0 \tilde{h}_t\|_{L^4(\Gamma)} \|v \circ \bar{\eta}^{-\tau}\|_{L^4(\Gamma)} \\
&\leq C(M) N(u_0, F) \|\tilde{h}_t\|_{H^{1.5}(\Gamma)} \\
&\leq C(M) N(u_0, F) t^{1/8}.
\end{aligned}
$$

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It is also easy to see that

$$\begin{aligned}&\left|\int_0^t\int_\Gamma L_{\tilde h}(h)\Big[\bar\Theta_t(\nabla_0\bar h,-1)_t\cdot(v\circ\bar\eta^{-\tau})+\bar\Theta(\nabla_0\tilde h,-1)_t\cdot(v\circ\bar\eta^{-\tau})_t\Big]dSds\right|\\
&\le C(M)\int_0^t\Big[\|v\|_{L^\infty(\Gamma)}+\|v_t\|_{L^4(\Gamma)}\Big]\|L_{\tilde h}(h)\|_{L^2(\Gamma)}\|\nabla_0\tilde h_t\|_{L^4(\Gamma)}ds\\
&\le C(M)N(u_0,F)^{1/2}\int_0^t\Big[\|v\|_{H^3(\Omega)}+\|v_t\|_{H^1(\Omega)}\Big]ds\\
&\le C(M)N(u_0,F)t^{1/2}.\end{aligned}$$

For the remaining terms, the $H^{0.5}(\Gamma)$-$H^{-0.5}(\Gamma)$ duality pairing leads to

$$\begin{aligned}&\left|\int_0^t\int_\Gamma \bar\Theta L_{\tilde h}(h)(\nabla_0\tilde h,-1)_{tt}\cdot vdSds\right|\\
&\le\int_0^t\|\bar\Theta\|_{H^{1.5}(\Gamma)}\|L_{\tilde h}(h)\|_{H^{0.5}(\Gamma)}\|v\|_{H^{1.5}(\Gamma)}\|\tilde h_{tt}\|_{H^{0.5}(\Gamma)}ds.\end{aligned}$$

By interpolation,

$$\|L_{\tilde h}(h)\|_{H^{0.5}(\Gamma)}\le C(M)\Big[\|h\|^{1/2}_{H^{5.5}(\Gamma)}\|h\|^{1/2}_{H^{3.5}(\Gamma)}+1\Big],$$

and hence

$$\begin{aligned}&\left|\int_0^t\int_\Gamma \tilde\Theta L_{\tilde h}(h)(\nabla_0\tilde h,-1)_{tt}\cdot(v\circ\bar\eta^{-\tau})dSds\right|\\
&\le C(M)N(u_0,F)\int_0^t\|\tilde h_{tt}\|_{H^{0.5}(\Gamma)}\Big[\|\nabla_0^5h\|^{1/2}_{L^2(\Gamma)}+1\Big]ds\\
&\le C(M)C(\delta)N(u_0,F)\int_0^t\Big[\|\nabla_0^5h\|_{L^2(\Gamma)}+1\Big]ds+\delta C(M)N(u_0,F)\\
&\le C(M)N(u_0,F)\Big[C(\delta)(t^{1/2}+t)+\delta\Big].\end{aligned}$$

All the inequalities above give us

$$|B_4|\le C(M)N(u_0,F)\Big[C(\delta)(t^{1/2}+t)+t^{1/8}+\delta\Big].$$

Summing all the estimates above, we find that

$$\begin{aligned}&\Big[\|v_t\|^2_{L^2(\Omega)}+\sigma\int_\Gamma\bar\Theta\bar A^{\alpha\beta\gamma\delta}h_{t,\alpha\beta}h_{t,\gamma\delta}|^2dS\Big](t)+\nu\int_0^t\|D_{\tilde\eta}v_t\|^2_{L^2(\Omega)}ds\\
&\le\|v_t(0)\|^2_{L^2(\Omega)}+\sigma\int_\Gamma|G_0^{\alpha\beta}h_{t,\alpha\beta}(0)|^2dS+(C+C(\delta_1))N(u_0,F)\\
&\quad+C(M)N(u_0,F)\Big[C(\delta)(t+t^{3/4}+t^{1/2}+t^{1/4}+t^{1/8}+t^{\frac{1-\alpha}{2}})+\delta\Big]\\
&\quad+\delta_1\Big[\|v_t\|^2_{L^2(\Omega)}+\|\nabla_0^4h\|^2_{L^2(\Gamma)}\Big],\end{aligned}$$

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and by Corollary 7.1,

$$\begin{aligned}
&\Big[\|v_t(t)\|^2_{L^2(\Omega)} + \|\nabla_0^2 h_t(t)\|^2_{L^2(\Gamma)}\Big] + \int_0^t \|v_t\|^2_{H^1(\Omega)} ds \\
&\le (C + C(\delta_1))N(u_0, F) + C(M)N(u_0, F)\Big[C(\delta)\mathcal{O}(t) + \delta\Big] \\
&\quad + \delta_1\Big[\|v_t\|^2_{L^2(\Omega)} + \|\nabla_0^4 h\|^2_{L^2(\Gamma)}\Big],
\end{aligned} \tag{10.9}$$

where C depends on ν, σ, ν_1, and the geometry of Γ. Since this estimate is independent of ϵ_1, we pass ϵ_1 to zero and conclude that the solution (v, h) to (7.1) also satisfies (10.9).

10.3. Mapping from $C_T(M)$ into $C_T(M)$. In this section, we are going to choose M so that $\Theta(\tilde{v}, \tilde{h}) \in C_T(M)$ if $(\tilde{v}, \tilde{h}) \in C_T(M)$.

Summing (10.5), (10.6), (10.7), (10.8), and (10.9), by (6.5) we find that

$$\begin{aligned}
&\Big[\|v(t)\|^2_{L^2(\Omega)} + \|\nabla_0 v(t)\|^2_{L^2(\Omega_1)} + \|\nabla_0^2 v(t)\|^2_{L^2(\Omega_1)} + \|v_t(t)\|^2_{L^2(\Omega)} \\
&+ \|\nabla_0^2 h(t)\|^2_{L^2(\Gamma)} + \|\nabla_0^3 h(t)\|^2_{L^2(\Gamma)} + \|\nabla_0^4 h(t)\|^2_{L^2(\Gamma)} + \|\nabla_0^2 h_t(t)\|^2_{L^2(\Gamma)}\Big] \\
&+ \int_0^t \Big[\|v\|^2_{H^1(\Omega)} + \|\nabla_0 v\|^2_{H^1(\Omega_1)} + \|\nabla_0^2 v\|^2_{H^1(\Omega_1)} + \|v_t\|^2_{H^1(\Omega)}\Big] ds \\
&\le (C + C(\delta_1))N(u_0, F) + C(M)N(u_0, F)\Big[C(\delta)\mathcal{O}(t) + \delta\Big] \\
&\quad + \delta_1\Big[\|v_t\|^2_{L^2(\Omega)} + \|\nabla_0^4 h\|^2_{L^2(\Gamma)}\Big],
\end{aligned}$$

where C depends on ν, σ, ν_1, and the geometry of Γ. Choosing $\delta_1 = \frac{1}{2}$,

$$\begin{aligned}
&\Big[\|v(t)\|^2_{L^2(\Omega)} + \|\nabla_0 v(t)\|^2_{L^2(\Omega_1)} + \|\nabla_0^2 v(t)\|^2_{L^2(\Omega_1)} + \|v_t(t)\|^2_{L^2(\Omega)} \\
&+ \|\nabla_0^2 h(t)\|^2_{L^2(\Gamma)} + \|\nabla_0^3 h(t)\|^2_{L^2(\Gamma)} + \|\nabla_0^4 h(t)\|^2_{L^2(\Gamma)} + \|\nabla_0^2 h_t(t)\|^2_{L^2(\Gamma)}\Big] \\
&+ \int_0^t \Big[\|v\|^2_{H^1(\Omega)} + \|\nabla_0 v\|^2_{H^1(\Omega_1)} + \|\nabla_0^2 v\|^2_{H^1(\Omega_1)} + \|v_t\|^2_{H^1(\Omega)}\Big] ds \\
&\le C_1 N(u_0, F) + C(M)N(u_0, F)^2\Big[C(\delta)\mathcal{O}(t) + \delta\Big],
\end{aligned}$$

where C_1 depends on ν, σ, μ, and the geometry of Γ. Similar to section 8.7, for a.a. $0 < t \le T$,

$$\begin{aligned}
&\Big[\|v(t)\|^2_{H^2(\Omega)} + \|v_t(t)\|^2_{L^2(\Omega)} + \|\nabla_0^2 h(t)\|^2_{H^2(\Gamma)} + \|\nabla_0^2 h_t(t)\|^2_{L^2(\Gamma)}\Big] \\
&+ \int_0^t \Big[\|v\|^2_{H^3(\Omega)} + \|v_t\|^2_{H^1(\Omega)} + \|q\|^2_{H^2(\Omega)}\Big] ds \\
&\le C_2 N(u_0, F) + C(M)N(u_0, F)^2\Big[C(\delta)\mathcal{O}(t) + \delta\Big]
\end{aligned} \tag{10.10}$$

for some constant C_2 depending on C_1.

By (6.6), (6.8), and (7.1d),

$$\begin{aligned}
\int_0^t \|h_t\|^2_{H^{2.5}(\Gamma)} ds &\le \int_0^t \Big[1 + \|\tilde{h}\|^2_{H^{3.5}(\Gamma)}\Big] \|v\|^2_{H^{2.5}(\Gamma)} ds \\
&\le C(M)N(u_0, F)t^{1/4}
\end{aligned} \tag{10.11}$$

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and

$$\int_0^t \|h_{tt}\|^2_{H^{0.5}(\Gamma)} ds \le C(M) \int_0^t \Big[\|\tilde{h}_t\|^2_{H^{1.5}(\Gamma)} \|v\|^2_{H^2(\Omega)} + \|\tilde{h}\|^2_{H^{2.5}(\Gamma)} \|v_t\|^2_{H^1(\Omega)} \Big] ds$$
$$\le C(M) N(u_0, F) \Big[t^{1/4} + t^{1/2} \Big]. \tag{10.12}$$

Also, by (10.3) and (10.10),

$$\int_0^t \|h\|^2_{H^{5.5}(\Gamma)} ds \le C \int_0^t \Big[\|v_t\|^2_{H^1(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)} + \|\nabla_0^2 v\|^2_{H^1(\Omega_1)} + \|\nabla_0^4 h\|^2_{L^2(\Gamma)}$$
$$+ \|F\|^2_{H^1(\Omega)} + 1 \Big] ds$$
$$\le C_3 N(u_0, F) + C(M) N(u_0, F)^2 \Big[C(\delta)\mathcal{O}(t) + \delta \Big] \tag{10.13}$$

for some constant C_3 depending on C_2.

Combining (10.10), (10.11), (10.12), and (10.13), we have the following inequality:

$$\Big[\|v(t)\|^2_{H^2(\Omega)} + \|v_t(t)\|^2_{L^2(\Omega)} + \|h(t)\|^2_{H^4(\Gamma)} + \|h_t(t)\|^2_{H^2(\Gamma)} \Big]$$
$$+ \int_0^t \Big[\|v\|^2_{H^3(\Omega)} + \|v_t\|^2_{H^1(\Omega)} + \|h\|^2_{H^{5.5}(\Gamma)} + \|h_t\|^2_{H^{2.5}(\Gamma)} + \|h_{tt}\|^2_{H^{0.5}(\Gamma)} \Big] ds$$
$$\le (C_2 + C_3) N(u_0, F) + C(M) N(u_0, F)^2 \Big[C(\delta)\mathcal{O}(t) + \delta \Big].$$

Let $M = 2(C_2+C_3)N(u_0,F)+1$ (and hence corresponding T_0 and T in Lemma 6.3 and Corollary 7.1 are fixed). Choose $\delta > 0$ small enough (but fixed) so that

$$C(M) N(u_0, F)^2 \delta \le \frac{1}{4}$$

and then choose $T > 0$ small enough so that

$$C(M) N(u_0, F)^2 C(\delta) T \le \frac{1}{4}.$$

Then for a.a. $0 < t \le T$,

$$\Big[\|v(t)\|^2_{H^2(\Omega)} + \|v_t(t)\|^2_{L^2(\Omega)} + \|h(t)\|^2_{H^4(\Gamma)} + \|h_t(t)\|^2_{H^2(\Gamma)} \Big]$$
$$+ \int_0^t \Big[\|v\|^2_{H^3(\Omega)} + \|v_t\|^2_{H^1(\Omega)} + \|h_t\|^2_{H^{2.5}(\Gamma)} + \|h_{tt}\|^2_{H^{0.5}(\Gamma)} \Big] ds$$
$$\le C_2 N(u_0, F) + \frac{1}{2},$$

and therefore

$$\sup_{0 \le t \le T} \Big[\|v(t)\|^2_{H^2(\Omega)} + \|v_t(t)\|^2_{L^2(\Omega)} + \|h(t)\|^2_{H^4(\Gamma)} + \|h_t(t)\|^2_{H^2(\Gamma)} \Big]$$
$$+ \|v\|^2_{\mathcal{V}^3(T)} + \|h\|^2_{\mathcal{H}(T)} \le 2C_2 N(u_0, F) + 1, \tag{10.14}$$

or in other words,

$$\|(v, h)\|^2_{Y(T)} \le 2C_2 N(u_0, F) + 1.$$

REMARK 19. *Equation* (10.14) *implies that for* $(\tilde{v}, \tilde{h}) \in C_T(M)$ *(with* M *and* T *chosen as above), the corresponding solution to the linear problem* (7.1) $(v, h) = \Theta_T(\tilde{v}, \tilde{h})$ *is also in* $C_T(M)$.

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10.4. Weak continuity of the mapping Θ_T.

LEMMA 10.4. *The mapping Θ_T is weakly sequentially continuous from $C_T(M)$ into $C_T(M)$ (endowed with the norm of X_T).*

Proof. Let $(v_p, h_p)_{p\in\mathbb{N}}$ be a given sequence of elements of $C_T(M)$ weakly convergent (in Y_T) toward a given element $(v, h) \in C_T(M)$ (where $C_T(M)$ is sequentially weakly closed as a closed convex set) and let $(v_{\sigma(p)}, h_{\sigma(p)})_{p\in\mathbb{N}}$ be any subsequence of this sequence.

Since $\mathcal{V}^3(T)$ is compactly embedded into $L^2(0, T; H^2(\Omega))$, we deduce the following strong convergence results in $L^2(0, T; L^2(\Omega))$ as $p \to \infty$:

$$(a_\ell^j)_p(a_\ell^k)_p \to a_\ell^j a_\ell^k \quad \text{and} \quad (a_\ell^j)_p(a_k^\ell)_p \to a_\ell^j a_k^\ell, \tag{10.15a}$$

$$[(a_\ell^j)_p(a_\ell^k)_p]_{,j} \to (a_\ell^j a_\ell^k)_{,j} \quad \text{and} \quad [(a_\ell^j)_p(a_k^\ell)_p]_{,j} \to (a_\ell^j a_k^\ell)_{,j}, \tag{10.15b}$$

$$(a_i^k)_p \to a_i^k. \tag{10.15c}$$

Now let $(w_p, g_p) = \Theta_T(v_p, h_p)$ and let q_p be the associated pressure so that $(q_p)_{p\in\mathbb{N}}$ is in a bounded set of $\mathcal{V}^2(T)$. Since X_T is a reflexive Hilbert space, let $(w_{\sigma(p)}, g_{\sigma(p)}, q_{\sigma(p)})_{p\in\mathbb{N}}$ be a subsequence weakly converging in $X_T \times \mathcal{V}^2(T)$ toward an element $(w, g, q) \in X_T \times \mathcal{V}^2(T)$. Since $C_T(M)$ is weakly closed in X_T, we also have $(w, g) \in C_T(M)$.

For each $\phi \in L^2(0, T; H^1(\Omega))$, we deduce from (7.3) (and Remark 6) that

$$\int_0^T \left[(w_t, \phi)_{L^2(\Omega)} + \frac{\mu}{2}\int_\Omega D_\eta w : D_\eta \phi dx + \sigma \int_\Gamma L_h(g)(g_{,\alpha}\phi_\alpha - \phi_z) dS \right.$$
$$\left. + \int_\Omega q a_i^j \phi^i_{,j} dx\right] dt = \int_0^T \langle F, \phi\rangle dt,$$

which with the fact that, from (10.15), for all $t \in [0, T]$, $w \in \mathcal{V}_v$, provides that (w, g) is a solution of (2.16) in $C_T(M)$, i.e., $(w, g) = \Theta_T(v, h)$.

Therefore, we deduce that the whole sequence $(\Theta_T(v_n, h_n))_{n\in\mathbb{N}}$ weakly converges in $C_T(M)$ toward $\Theta_T(v, h)$, which concludes the lemma. ∎

10.5. Uniqueness. For the uniqueness result, we assume that u_0, F, and Γ are smooth enough (e.g., $u_0 \in H^{5.5}(\Omega)$, $F \in \mathcal{V}^4(T)$, Γ is a $H^{8.5}$ surface) so that u_0 and the associated u_1, q_0 satisfy compatibility condition (4.4). Therefore, the solution (v, h, q) is such that $v \in \mathcal{V}^6(T)$, $q \in L^2(0, T; H^5(\Omega))$ and $h \in L^\infty(0, T; H^7(\Gamma)) \cap L^2(0, T; H^{8.5}(\Gamma))$, $h_t \in L^\infty(0, T; H^5(\Gamma)) \cap L^2(0, T; H^{5.5}(\Gamma))$, $h_{tt} \in L^\infty(0, T; H^2(\Gamma)) \cap L^2(0, T; H^{3.5}(\Gamma))$. This implies $a \in L^\infty(0, T; H^5(\Omega))$, and hence by studying the elliptic equation

$$(a_i^\ell a_i^k q_{t,k})_{,\ell} = \Big[\nu a_i^\ell(a_p^k a_p^j v^i_{,j})_{,k\ell} + a_{it}^\ell v^i_{,\ell} + a_i^\ell F_{,\ell}\Big]_t - [(a_i^\ell a_i^k)_t q_{,k}]_{,\ell} \quad \text{in} \quad \Omega,$$
$$q_t = J_h^{-2}\Big[\Big(\sigma L_h(h) N_i - \nu D_\eta(v)_i^\ell a_i^j N_j\Big)_t - (a_i^j N_j)_t q\Big] a_i^\ell N_\ell \quad \text{on} \quad \Gamma,$$

we find that $q_t \in L^2(0, T; H^2(\Omega))$, and this implies $v_{tt} \in L^2(0, T; H^1(\Omega))$. By the interpolation theorem, we also conclude that $v_t \in \mathcal{C}^0([0, T]; H^{2.5}(\Omega))$.

Suppose (v, h, q) and $(\tilde{v}, \tilde{h}, \tilde{q})$ are two sets of solutions of (1.1). Then

$$(v - \tilde{v})_t - \nu[a_\ell^k D_\eta(v - \tilde{v})^i_\ell]_{,k} = -a_i^k(q - \tilde{q})_{,k} + \delta F, \tag{10.16a}$$

$$a_i^j (v - \tilde{v})^i_{,j} = \delta a, \tag{10.16b}$$

$$\Big[\nu[D_\eta(v - \tilde{v})]_i^\ell - (q - \tilde{q})\delta_i^\ell\Big] a_\ell^j N_j = \sigma\Theta\Big[L_h(h - \tilde{h})(-\nabla_0 h, 1)\Big] \circ \eta^\tau$$
$$+ \delta L_1 + \delta L_2 + \delta L_3, \tag{10.16c}$$

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$$(h - \tilde{h})_t \circ \eta^\tau = [h_{,\alpha} \circ \eta^\tau](v_\alpha - \tilde{v}_\alpha) - (v_z - \tilde{v}_z) + \delta h_1 + \delta h_2 + \delta h_3, \tag{10.16d}$$

$$(v - \tilde{v})(0) = 0, \tag{10.16e}$$

$$(h - \tilde{h})(0) = 0, \tag{10.16f}$$

where

$$\delta F = f \circ \eta - f \circ \tilde{\eta} + \nu[(a^k_\ell a^j_\ell - \tilde{a}^k_\ell \tilde{a}^j_\ell)\tilde{v}^i_{,j}]_{,k} + \nu[(a^k_\ell a^j_i - \tilde{a}^k_\ell \tilde{a}^j_i)\tilde{v}^\ell_{,j}]_{,k} - (a^k_i - \tilde{a}^k_i)\tilde{q}_{,k}, \tag{10.17a}$$

$$\delta a = (a^j_i - \tilde{a}^j_i)\tilde{v}^i_{,j}, \tag{10.17b}$$

$$\delta L_1 = \sigma\Theta\Big[L_h(\tilde{h})(\nabla_0 h - \nabla_0 \tilde{h}, 0)\Big] \circ \eta^\tau - \nu(a^k_i a^j_\ell - \tilde{a}^k_i \tilde{a}^j_\ell)\tilde{v}^\ell_{,k} N_j - \nu(a^k_\ell a^j_\ell - \tilde{a}^k_\ell \tilde{a}^j_\ell)\tilde{v}^i_{,k} N_j + (a^j_i - \tilde{a}^j_i)\tilde{q} N_j, \tag{10.17c}$$

$$\delta L_2 = \tilde{\Theta}[L_{\tilde{h}}(\tilde{h}) \circ \eta^\tau](\nabla_0 \tilde{h} \circ \eta^\tau - \nabla_0 \tilde{h} \circ \tilde{\eta}^\tau, 0) + \Big[\Theta L_h(\tilde{h}) \circ \eta^\tau - \tilde{\Theta} L_h(\tilde{h}) \circ \tilde{\eta}^\tau\Big](\nabla_0 \tilde{h} \circ \tilde{\eta}^\tau, -1), \tag{10.17d}$$

$$\delta L_3 = \Big[[L_h(\tilde{h}) - L_{\tilde{h}}(\tilde{h})](\nabla_0 \tilde{h}, -1)\Big] \circ \tilde{\eta}^\tau, \tag{10.17e}$$

$$\delta h_1 = (h_{,\alpha} \circ \eta^\tau - h_{,\alpha} \circ \tilde{\eta}^\tau)\tilde{v}_\alpha, \tag{10.17f}$$

$$\delta h_2 = \Big[(h_{,\alpha} - \tilde{h}_{,\alpha}) \circ \tilde{\eta}^\tau\Big]\tilde{v}_\alpha, \tag{10.17g}$$

$$\delta h_3 = -(\tilde{h}_t \circ \eta^\tau - \tilde{h}_t \circ \tilde{\eta}^\tau). \tag{10.17h}$$

We will also use δL and δh to denote $\sum_{k=1}^3 L_k$ and $\sum_{k=1}^3 \delta h_k$, respectively.

Similar to (11.3) in [8], we also have the following estimates.

LEMMA 10.5. *For $f \in H^2(\Omega)$ and $g \in H^{1.5}(\Gamma)$,*

$$\|f \circ \eta - f \circ \tilde{\eta}\|_{L^2(\Omega)} \le C\sqrt{t}\|f\|_{H^2(\Omega)}\left[\int_0^t \|v - \tilde{v}\|^2_{H^1(\Omega)} ds\right]^{1/2}, \tag{10.18}$$

$$\|g \circ \eta^\tau - g \circ \tilde{\eta}^\tau\|_{L^2(\Gamma)} \le C\sqrt{t}\|g\|_{H^{1.5}(\Gamma)}\left[\int_0^t \|v - \tilde{v}\|^2_{H^1(\Omega)} ds\right]^{1/2} \tag{10.19}$$

for some constant C.

REMARK 20. *Assuming the regularity of h, h_t, and h_{tt} given in the beginning of this section, we have*

$$\|\delta L_2\|_{H^2(\Gamma)} + \|\delta h_1 + \delta h_3\|_{H^{2.5}(\Gamma)} \le C\sqrt{t}\left[\int_0^t \|v - \tilde{v}\|^2_{H^3(\Omega)} ds\right]^{1/2} \tag{10.20}$$

and

$$\|(\delta L_2)_t\|_{L^2(\Gamma)} + \|(\delta h_1 + \delta h_3)_t\|_{H^1(\Gamma)} \le C\left[\|v - \tilde{v}\|_{H^1(\Omega)} + \sqrt{t}\left(\int_0^t \|v - \tilde{v}\|^2_{H^2(\Omega)} ds\right)^{1/2}\right] \tag{10.21}$$

and

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$$\|\nabla_0^2(\delta h_3)_t\|_{L^2(\Gamma)} \le C\Bigg[\|v-\tilde v\|_{H^1(\Omega)} + \|v-\tilde v\|_{H^3(\Omega)} \\ + \sqrt{t}\|\tilde h_{tt}\|_{H^{3.5}(\Gamma)}\left(\int_0^t \|v-\tilde v\|^2_{H^3(\Omega)} ds\right)^{1/2}\Bigg]. \tag{10.22}$$

By using (10.18) to estimate $\|\delta F\|_{L^2(\Omega)}$, we find that

$$\|\nabla(v-\tilde v)(t)\|^2_{L^2(\Omega)} + \int_0^t \|(v-\tilde v)_t\|^2_{L^2(\Omega)} ds \\ \le C(\delta)\int_0^t \left[\|v-\tilde v\|^2_{H^1(\Omega)} + \|h-\tilde h\|^2_{H^4(\Gamma)}\right] ds + (C(\delta)t^2+\delta)\int_0^t \|v-\tilde v\|^2_{H^2(\Omega)} ds \\ + \delta \int_0^t \left[\|(v-\tilde v)_t\|^2_{H^1(\Omega)} + \|q-\tilde q\|^2_{H^1(\Omega)}\right] ds. \tag{10.23}$$

For the $L^2_t H^3_x$-estimate for $v-\tilde v$ and the $L^2_t H^1_x$-estimate for $(v-\tilde v)_t$, we have

$$\frac{1}{2}\frac{d}{dt}\left[\|\zeta_1 \nabla_0^2(v-\tilde v)\|^2_{L^2(\Omega)} + 2\sigma E_h(\nabla_0^2(h-\tilde h))\right] + \frac{\nu}{4}\|\zeta_1 D_{\bar\eta}\nabla_0^2(v-\tilde v)\|^2_{L^2(\Omega)} \\ \le C\Big[\|\delta F\|^2_{H^1(\Omega)} + \|(v-\tilde v)_t\|^2_{L^2(\Omega)} + \|\nabla(v-\tilde v)\|^2_{L^2(\Omega)} + \|\nabla\nabla_0(v-\tilde v)\|^2_{L^2(\Omega_1')} \\ + \|\nabla_0^4(h-\tilde h)\|^2_{L^2(\Gamma)}\Big] + \delta\|v-\tilde v\|^2_{H^3(\Omega)} + D_1 + D_2 + D_3$$

and

$$\frac{1}{2}\frac{d}{dt}\left[\|(v-\tilde v)_t\|^2_{L^2(\Omega)} + 2\sigma E_h((h-\tilde h)_t)\right] + \frac{\nu}{4}\|\nabla(v-\tilde v)_t\|^2_{L^2(\Omega)} \\ \le C\Big[(\|\nabla_0^4(h-\tilde h)\|^2_{L^2(\Gamma)} + \|\nabla_0^2(h-\tilde h)_t\|^2_{L^2(\Gamma)}) + \|\delta F_t\|^2_{H^1(\Omega)'}\Big] + \delta\|v-\tilde v\|^2_{H^3(\Omega)} \\ + E_1 + E_2 + E_3,$$

where

$$D_1 := \int_\Omega \zeta_1^2 \nabla_0^2(q-\tilde q)\nabla_0^2 \delta a\, dx, \qquad D_2 := \int_\Gamma \Theta\Big[[L_h(h-\tilde h)]\circ\eta^\tau\Big](\nabla_0^4 \delta h) dS, \\ D_3 := \int_\Gamma \delta L\cdot \nabla_0^4(v-\tilde v) dS$$

and

$$E_1 := \int_\Omega (q-\tilde q)_t(\delta a)_t dx, \qquad E_2 := \int_\Gamma \Big[\Theta[L_h(h-\tilde h)]\circ\eta^\tau\Big]_t (\delta h)_t dS, \\ E_3 := \int_\Gamma (\delta L)_t\cdot(v-\tilde v)_t dS.$$

By using (10.20) to estimate D_i and (10.21), (10.22) to estimate E_i, we obtain

$$\left[\|\nabla_0^2(v-\tilde v)(t)\|^2_{L^2(\Omega_1)} + \|\nabla_0^4(h-\tilde h)(t)\|^2_{L^2(\Gamma)}\right] + \int_0^t \|\nabla\nabla_0^2(v-\tilde v)\|^2_{L^2(\Omega_1)} ds \\ \le C(\delta)\int_0^t \left[\|(v-\tilde v)_t\|^2_{L^2(\Omega)} + \|\nabla_0(v-\tilde v)\|^2_{L^2(\Omega)} + \|\nabla_0^4(h-\tilde h)\|^2_{L^4(\Gamma)}\right] ds \\ + (C(\delta)t^2+\delta)\int_0^t \|v-\tilde v\|^2_{H^3(\Omega)} ds + \delta\int_0^t \|q-\tilde q\|^2_{H^2(\Omega)} ds \tag{10.24}$$

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and

$$\begin{aligned}
&\Big[\|(v-\tilde v)_t(t)\|^2_{L^2(\Omega)} + \|\nabla_0^2(h-\tilde h)_t\|^2_{L^2(\Gamma)}\Big] + \int_0^t \|\nabla(v-\tilde v)_t\|^2_{L^2(\Omega)}ds \\
&\le C(\delta)\int_0^t \Big[\|v-\tilde v\|^2_{H^1(\Omega)} + \|\nabla_0^4(h-\tilde h)\|^2_{L^2(\Gamma)} + (1+\|\tilde h_{tt}\|^2_{H^{4.5}(\Gamma)}) \\
&\qquad \times \|\nabla_0^2(h-\tilde h)_t\|^2_{L^2(\Gamma)}\Big]ds \\
&\quad + (C(\delta)(t+t^2)+\delta)\int_0^t \|v-\tilde v\|^2_{H^3(\Omega)}ds + \delta\|q-\tilde q\|^2_{L^2(\Omega)} \\
&\quad + \delta\int_0^t \Big[\|(v-\tilde v)_t\|^2_{H^1(\Omega)} + \|q-\tilde q\|^2_{H^2(\Omega)}\Big]ds.
\end{aligned} \tag{10.25}$$

Summing (10.23), (10.24), and (10.25), we find that

$$Y(t) + \int_0^t Z(s)ds \le C(\delta)\int_0^t k(s)Y(s)ds + (C(\delta)(t^2+t)+\delta)\int_0^t Z(s)ds, \tag{10.26}$$

where

$$\begin{aligned}
k(t) &= 1 + \|\tilde h_{tt}(t)\|^2_{H^{3.5}(\Gamma)}, \\
Y(t) &= \Big[\|v-\tilde v(t)\|^2_{H^1(\Omega)} + \|\nabla_0^2(v-\tilde v)(t)\|^2_{L^2(\Omega_1)} + \|(v-\tilde v)_t(t)\|^2_{L^2(\Omega)} \\
&\qquad + \|h-\tilde h\|^2_{H^4(\Gamma)} + \|(h-\tilde h)_t\|^2_{H^2(\Gamma)}\Big], \\
Z(t) &= \|(v-\tilde v)_t(t)\|^2_{H^1(\Omega)} + \|\nabla\nabla_0^2(v-\tilde v)(t)\|^2_{L^2(\Omega_1)}.
\end{aligned}$$

By letting $\delta = 1/4$ and choosing $T_u \le T$ so that $C(\delta)(T_u^2 + T_u) \le 1/4$,

$$Y(t) + \int_0^t Z(s)ds \le C\int_0^t k(s)Y(s)ds \tag{10.27}$$

for all $0 < t \le T_u$. Since $Y(0) = 0$, the uniqueness of the solution follows from that $Y(t) = 0$ for all $0 < t \le T_u$.

11. The analysis of the membrane traction.

The analysis of the membrane traction consists of four parts: (1) the modified linearized (and regularized) problem; (2) the κ-independent estimates; (3) the fixed-point argument; and (4) the uniqueness of the solution.

11.1. The modified linearized and regularized problem.

Recall that the membrane traction is

$$\mathrm{t}_{\text{mem}} = \Big[\mathcal{J}\mathcal{P}''(\mathcal{J}) + 2\mathcal{P}'(\mathcal{J})\Big]\mathcal{J}_{,\beta}g^{\alpha\beta}\eta_{,\alpha} + \Big[\mathcal{J}\mathcal{P}'(\mathcal{J}) + \mathcal{P}(\mathcal{J})\Big]Hn.$$

For given $\bar v = \rho_{\epsilon_1} * \tilde v$ (and hence $\bar\eta$, $\bar g$, etc.), we define (for fixed but small $\epsilon > 0$)

$$L^\epsilon_{\bar m} = \frac{1}{2}\bar{\mathcal{J}}^{-1}\left[(\partial_\beta\rho_\epsilon) * \left(\frac{\bar g}{g_0}\right)\right]\Big[\bar{\mathcal{J}}\mathcal{P}''(\bar{\mathcal{J}}) + 2\mathcal{P}'(\bar{\mathcal{J}})\Big]\bar g^{\alpha\beta}\bar\eta_{,\alpha} + \Big[\bar{\mathcal{J}}\mathcal{P}'(\bar{\mathcal{J}}) + \mathcal{P}(\bar{\mathcal{J}})\Big]\bar H\bar n.$$

For the linearized problem, we change the boundary condition (7.1c) to

$$\begin{aligned}
[\nu D_{\tilde\eta}(v)^j_i - q\delta^j_i]\tilde a^\ell_j N_\ell = (L^\epsilon_{\bar m})^i + \sigma\tilde\Theta\Big[\mathcal{L}_{\tilde h}(h)(-\nabla_0\tilde h, 1)\Big]\circ\tilde\eta^\tau \quad \text{on} \quad (0,T)\times\Gamma \\
+ \sigma\tilde\Theta\Big[[\mathcal{M}(\tilde h)(-\nabla_0\tilde h, 1)]\circ\tilde\eta^\tau\Big],
\end{aligned} \tag{11.1}$$

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where we recall that $\bar{\Theta} = \det(\nabla_0 \bar{\eta}^\tau)\sqrt{\det(G_{\bar{h}}) \circ \bar{\eta}^\tau}$. Note that here we treat the membrane traction as a given forcing on the boundary. The regularized problem consists of adding the artificial viscosity, as introduced in (7.2c), in (11.1). Note that here we also mollify $\bar{\mathcal{J}}_{,\beta}$ and use the equality $(\rho_\epsilon * f)_{,\beta} = \rho_{\epsilon,\beta} * f$.

Since $L_{\bar{m}}$ is given as a forcing, all the estimates are essentially the same as those in the previous sections. Therefore, we have a unique solution (v_κ, h_κ) to the regularized problem (with ϵ_1-, ϵ-, and κ-dependent estimates).

11.2. The κ-independent estimates. The introduction of the artificial viscosity is to provide enough regularity for the solution to the linearized problem. As in Appendix A, the κ-independent estimates are obtained by studying the normal component of (A.1). Note that with the help of the mollification operation in (11.1), the corresponding f in (A.1) is also a function in $L^2(0, T; H^{1.5}(\Gamma))$. Therefore, (A.7) is still valid. This κ-independent estimate will enable us to take the limit as $\kappa \to 0$ and obtain the solution $(v_{\epsilon_1}, h_{\epsilon_1})$. Essentially the same proof as in section 9.4 shows that (9.12) still holds, and hence taking the limit as $\epsilon_1 \to 0$, the weak limit (v_ϵ, h_ϵ) solves the linearized problem (7.1), and all the estimates in the previous sections hold with $C(M)$ replaced by $C(M, \epsilon)$.

REMARK 21. *The estimate for (v_ϵ, h_ϵ) still depends on ϵ, where the extra ϵ-regularization is used in the $L_t^2 H_x^3$-estimates, which requires estimating the following boundary integral:*

$$\int_\Gamma \frac{1}{2} \bar{\mathcal{J}}^{-1} \left[(\partial_\beta \rho_\epsilon) * \left(\frac{\bar{g}}{g_0} \right) \right] \left[\bar{\mathcal{J}} \mathcal{P}''(\bar{\mathcal{J}}) + 2\mathcal{P}'(\bar{\mathcal{J}}) \right] \bar{g}^{\alpha\beta} \bar{\eta}_{,\alpha} \nabla_0^4 v dS.$$

Moreover, even though the estimate for h_{ϵ_1} depends only on the normal component of $L_{\bar{m}}$, in the linearized problem, there are still contributions to the normal direction made by $\bar{g}^{\alpha\beta}\bar{\eta}_{,\alpha}$.

11.3. The fixed-point argument. Similar fixed-point arguments as in section 10 guarantee the existence of a fixed point (which is still denoted by (v_ϵ, h_ϵ)) in the space X_{T_ϵ}; that is, there is a fixed point $(v_\epsilon, h_\epsilon) \in \mathcal{V}^3(T_\epsilon) \times \mathcal{H}(T_\epsilon)$. This fixed point satisfies the boundary condition

$$\begin{aligned}(11.2) \qquad [\nu D_{\eta_\epsilon}(v_\epsilon)_i^j - q_\epsilon \delta_i^j](a_\epsilon)_j^\ell N_\ell = (L_m^\epsilon)^i + \sigma\Theta_\epsilon \left[\mathcal{L}_{h_\epsilon}(h_\epsilon)(-\nabla_0 h_\epsilon, 1) \right] \circ {\eta_\epsilon}^\tau \\ + \sigma\Theta_\epsilon \left[[\mathcal{M}(h_\epsilon)(-\nabla_0 h_\epsilon, 1)] \circ {\eta_\epsilon}^\tau \right]\end{aligned}$$

on $(0, T) \times \Gamma$, where

$$L_m^\epsilon = \frac{1}{2} \mathcal{J}_\epsilon^{-1} \left[\rho_\epsilon * \left(\frac{g_\epsilon}{g_0} \right) \right]_{,\beta} \left[\mathcal{J}_\epsilon \mathcal{P}''(\mathcal{J}_\epsilon) + 2\mathcal{P}'(\mathcal{J}_\epsilon) \right] g_\epsilon^{\alpha\beta} \eta_{\epsilon,\alpha} + \left[\mathcal{J}_\epsilon \mathcal{P}'(\mathcal{J}_\epsilon) + \mathcal{P}(\mathcal{J}_\epsilon) \right] H_\epsilon n_\epsilon.$$

By studying the tangential component of (11.2), we find that for $\gamma = 1, 2$,

$$(11.3) \quad \mathcal{J}_\epsilon^{-1} \left[\rho_\epsilon * \left(\frac{g_\epsilon}{g_0} \right) \right]_{,\gamma} \left[\mathcal{J}_\epsilon \mathcal{P}''(\mathcal{J}_\epsilon) + 2\mathcal{P}'(\mathcal{J}_\epsilon) \right] = 2[\nu D_{\eta_\epsilon}(v_\epsilon)_i^j - q_\epsilon \delta_i^j](a_\epsilon)_j^\ell N_\ell {\eta_\epsilon}_{,\gamma}^i.$$

Take T_ϵ even smaller so that

$$\begin{aligned}&\frac{1}{2} \le \|\Theta_\epsilon\|_{H^{1.5}(\Gamma)} \le \frac{3}{2}, && \frac{1}{2} \le \|a_\epsilon\|_{H^2(\Omega)} \le \frac{3}{2}, \\ &\|v_\epsilon\|_{L^2(0,T_\epsilon;H^3(\Omega))} \le \|u_0\|_{H^3(\Omega)}^2 + 1, && \|\eta_\epsilon\|_{H^3(\Omega)} \le |\Omega| + 1.\end{aligned}$$

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With these bounds, (11.3) together with the assumptions that $\mathcal{P}$ is strictly convex and $\mathcal{P}$ attains its minimum at $\mathcal{J} = 1$ (that assure that the second bracket of the left-hand side of (11.3) is bounded away from zero) implies that

$$\left\| \nabla_0 \left[\rho_\epsilon * \left(\frac{g_\epsilon}{g_0} \right) \right] \right\|_{H^{1.5}(\Gamma)} \leq C(u_0, \Omega). \tag{11.4}$$

Since (11.4) is independent of the ϵ, we find that

$$\|g_\epsilon\|_{H^{2.5}(\Gamma)} \leq C(u_0, g_0, \Omega). \tag{11.5}$$

Having (11.5), we no longer need ϵ-regularization to estimate the boundary integral in Remark 21 and the study of (A.1), and hence all the estimates in the previous sections are still valid with $C(M)$ replaced by $C(u_0, g_0, \Omega)$. These ϵ-independent estimates allow us to construct a solution (v_ϵ, h_ϵ) in $X(T)$ (where T is independent of ϵ) with the same estimates. The solution of the original problem (1.1) is then the limit of (v_ϵ, h_ϵ) as $\epsilon \to 0$.

11.4. The uniqueness of the solution. The uniqueness of the solution follows from the elliptic estimate

$$\|g - \tilde{g}\|^2_{H^{2.5}(\Gamma)} \leq C \Big[\|v - \tilde{v}\|^2_{H^3(\Omega)} + \|v_t - \tilde{v}_t\|^2_{H^1(\Omega)} \Big],$$

which follows from the equation

$$\left(\frac{g - \tilde{g}}{g_0} \right)_{,\gamma} \mathcal{Q}(\eta) + \left(\frac{\tilde{g}}{g_0} \right)_{,\gamma} \Big[\mathcal{Q}(\eta) - \mathcal{Q}(\tilde{\eta}) \Big] = F(v, q)^\gamma - F(\tilde{v}, \tilde{q})^\gamma,$$

where

$$\mathcal{Q}(\eta) = \mathcal{J}^{-1} \Big[\mathcal{J}\mathcal{P}''(\mathcal{J}) + 2\mathcal{P}'(\mathcal{J}) \Big] \quad \text{and} \quad F(v, q)^\gamma = 2[\nu D_\eta(v)^j_i - q\delta^j_i] a^\ell_j N_\ell \eta^i_{,\gamma}.$$

Appendix A. Elliptic regularity. We establish a κ-independent elliptic estimate for solutions of

$$\frac{\bar{\Theta}}{\sqrt{\det(g_0)}} \Big[\Big(\sqrt{\det(g_0)} \bar{A}^{\alpha\beta\gamma\delta} h_{\kappa,\alpha\beta} \Big)_{,\gamma\delta} (-\nabla_0 \bar{h}, 1) \Big] \circ \bar{\eta}^\tau + \kappa \Delta_0^2 v_\kappa = f, \tag{A.1}$$

where h_κ and v_κ satisfy (7.4) with $h_\kappa \in H^4(\Gamma)$, $v_\kappa \in H^4(\Gamma)$, and $f \in H^{1.5}(\Gamma)$. Letting $w = v_\kappa \circ \bar{\eta}^{-\tau}$, (A.1) is equivalent to

$$\frac{\bar{\Theta}}{\sqrt{\det(g_0)}} \Big[\sqrt{\det(g_0)} \bar{A}^{\alpha\beta\gamma\delta} h_{\kappa,\alpha\beta} \Big]_{,\gamma\delta} (-\nabla_0 \bar{h}, 1) + \kappa \Delta_0^2 w = f \circ \bar{\eta}^\tau, \tag{A.2}$$

which implies that

$$\begin{aligned} &\frac{\bar{\Theta}}{\sqrt{\det(g_0)}} \Big[\sqrt{\det(g_0)} \bar{A}^{\alpha\beta\gamma\delta} h_{\kappa,\alpha\beta} \Big]_{,\gamma\delta} + \kappa J_{\bar{h}}^{-2} \Delta_0^2 w \cdot (-\nabla_0 \bar{h}, 1) \\ &= J_{\bar{h}}^{-2} f \circ \bar{\eta}^\tau \cdot (-\nabla_0 \bar{h}, 1). \end{aligned} \tag{A.3}$$

Recall that $w \cdot (-\nabla_0 \bar{h}, 1) = h_{\kappa t}$.

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Let D_h denote the difference quotients (with respect to the surface coordinate system). Taking the inner product of (A.3) with $D_{-h}D_h\nabla_0^4 h_\kappa$, by Corollary 7.1 we find that

$$\nu_1\int_0^t \|D_h\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma)}ds \le C(\epsilon_1)\int_0^t \Big[\|h_\kappa\|^2_{H^2(\Gamma)}+\|f\|^2_{H^1(\Gamma)}+\kappa\|w\|^2_{H^4(\Gamma)}\Big]ds.$$

Since the right-hand side is independent of difference parameter h, it follows that $h_\kappa \in H^5(\Gamma)$ (as it is already a H^4-function) with the estimate

$$\int_0^t \|\nabla_0^5 h_\kappa\|^2_{L^2(\Gamma)}ds \le C(\epsilon_1)\int_0^t \Big[\|h_\kappa\|^2_{H^2(\Gamma)}+\|f\|^2_{H^1(\Gamma)}+\kappa\|w\|^2_{H^4(\Gamma)}\Big]ds. \tag{A.4}$$

Next, we obtain a κ-independent estimate of $\kappa\|w\|^2_{H^4(\Gamma)}$. By taking the inner product of (A.2) with $\nabla_0^2 w$ and $\nabla_0^4 w$, we find that

$$\begin{aligned}&\|\nabla_0^3 h_\kappa(t)\|^2_{L^2(\Gamma)}+\kappa\int_0^t\|w\|^2_{H^3(\Gamma)}ds\\ &\quad\le C(\epsilon_1)\int_0^t\Big[\|\nabla_0^3 h_\kappa\|^2_{L^2(\Gamma)}+\|f\|^2_{L^2(\Gamma)}+\|w\|^2_{H^{2.5}(\Omega)}\Big]ds\end{aligned} \tag{A.5}$$

and

$$\begin{aligned}&\|\nabla_0^4 h_\kappa(t)\|^2_{L^2(\Gamma)}+\kappa\int_0^t\|w\|^2_{H^4(\Gamma)}ds\\ &\le C(\epsilon_1,\delta_1)\int_0^t\Big[\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma)}+\|f\|^2_{H^{1.5}(\Gamma)}+\|w\|^2_{H^3(\Omega)}\Big]ds+\delta_1\int_0^t\|\nabla_0^5 h_\kappa\|^2_{L^2(\Gamma)}dS,\end{aligned} \tag{A.6}$$

where we use (A.5) to estimate $\kappa\int_0^t\|w\|_{H^3(\Gamma)}ds$. Equation (A.6) provides a κ-independent estimate for $\kappa\|w\|^2_{H^4(\Gamma)}$; hence by choosing $\delta_1>0$ small enough, (A.4) implies that for all $t\in[0,T]$,

$$\int_0^t\|\nabla_0^2 h_\kappa\|^2_{H^3(\Gamma)}ds \le C'\int_0^t\Big[\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma)}+\|f\|^2_{H^{1.5}(\Gamma)}+\|w\|^2_{H^3(\Omega)}\Big]ds \tag{A.7}$$

for some constant C' depending on ϵ_1.

Appendix B. Inequalities in the estimates for $\nabla_0^2 v$ near the boundary.

B.1. κ-independent estimates. Since $\zeta_1\equiv 1$ on Γ and

$$\begin{aligned}(-\nabla_0\bar h\circ\bar\eta^\tau,1)\cdot\nabla_0^4 v_\kappa &= \nabla_0^4((-\nabla_0\bar h\circ\bar\eta^\tau,1)\cdot v_\kappa)-\nabla_0^4(-\nabla_0\bar h\circ\bar\eta^\tau,1)\cdot v_\kappa\\ &\quad-4\nabla_0^3(-\nabla_0\bar h\circ\bar\eta^\tau,1)\cdot\nabla_0 v_\kappa-6\nabla_0^2(-\nabla_0\bar h\circ\bar\eta^\tau,1)\cdot\nabla_0^2 v_\kappa\\ &\quad-4\nabla_0(-\nabla_0\bar h\circ\bar\eta^\tau,1)\cdot\nabla_0^3 v_\kappa,\end{aligned}$$

we find that

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$$
\begin{aligned}
&\int_\Gamma \bar\Theta\Big[L_{\bar h}(h_\kappa)\circ\bar\eta^\tau\Big]((-\nabla_0\bar h\circ\bar\eta^\tau,1)\cdot\nabla_0^2(\zeta_1^2\nabla_0^2 v_\kappa))dS\\
=&-\int_\Gamma \bar\Theta\Big[L_{\bar h}(h_\kappa)\circ\bar\eta^\tau\Big]\Big[\nabla_0^4(-\nabla_0\bar h\circ\bar\eta^\tau,1)\cdot v_\kappa+4\nabla_0^3(-\nabla_0\bar h\circ\bar\eta^\tau,1)\cdot\nabla_0 v_\kappa\\
&\qquad\qquad+6\nabla_0^2(-\nabla_0\bar h\circ\bar\eta^\tau,1)\cdot\nabla_0^2 v_\kappa\Big]dS \qquad (\equiv I_1)\\
&-4\int_\Gamma \bar\Theta\Big[L_{\bar h}(h_\kappa)\circ\bar\eta^\tau\Big](\nabla_0(-\nabla_0\bar h\circ\bar\eta^\tau,1)\cdot\nabla_0^3 v_\kappa)dS \qquad (\equiv I_2)\\
&+\int_\Gamma \frac{\bar\Theta}{\sqrt{\det(g_0)}}\nabla_0^2\Big[\sqrt{\det(g_0)}\Big(L_1^{\alpha\beta\gamma}\tilde h_{,\alpha\beta\gamma}+L_2\Big)\circ\bar\eta^\tau\Big]\nabla_0^2(h_{\kappa t}\circ\bar\eta^\tau)dS \qquad (\equiv I_3)\\
&+\int_\Gamma \frac{2\nabla_0\bar\Theta}{\sqrt{\det(g_0)}}\nabla_0\Big[\sqrt{\det(g_0)}\Big(L_1^{\alpha\beta\gamma}\tilde h_{,\alpha\beta\gamma}+L_2\Big)\circ\bar\eta^\tau\Big]\nabla_0^2(h_{\kappa t}\circ\bar\eta^\tau)dS \qquad (\equiv I_4)\\
&+\int_\Gamma (\nabla_0^2\bar\Theta)\Big[\Big(L_1^{\alpha\beta\gamma}\tilde h_{,\alpha\beta\gamma}+L_2\Big)\circ\bar\eta^\tau\Big]\nabla_0^2(h_{\kappa t}\circ\bar\eta^\tau)dS \qquad (\equiv I_5)\\
&+\int_\Gamma \frac{\bar\Theta}{\sqrt{\det(g_0)}}\Big[(\sqrt{\det(g_0)}\bar A^{\alpha\beta\gamma\delta}h_{\kappa,\alpha\beta})_{,\gamma\delta}\circ\bar\eta^\tau\Big]\nabla_0^4(h_{\kappa t}\circ\bar\eta^\tau)dS.
\end{aligned}
$$

The last term of the identity above, by a change of coordinates, can be written as

$$
\begin{aligned}
&\int_\Gamma \frac{\bar\Theta}{\sqrt{\det(g_0)}}\Big[(\sqrt{\det(g_0)}\bar A^{\alpha\beta\gamma\delta}h_{\kappa,\alpha\beta})_{,\gamma\delta}\circ\bar\eta^\tau\Big]\nabla_0^4(h_{\kappa t}\circ\bar\eta^\tau)dS\\
=&\int_\Gamma \frac{B}{\sqrt{\det(g_0)}}\nabla_0^2(\sqrt{\det(g_0)}\bar A^{\alpha\beta\gamma\delta}h_{\kappa,\alpha\beta})_{,\gamma\delta}\nabla_0^2 h_{\kappa t}dS+R_1\\
&+2\int_\Gamma \frac{\nabla_0\bar\Theta}{\sqrt{\det(g_0)}}\nabla_0\Big[(\sqrt{\det(g_0)}\bar A^{\alpha\beta\gamma\delta}h_{\kappa,\alpha\beta})_{,\gamma\delta}\circ\bar\eta^\tau\Big]\nabla_0^2(h_{\kappa t}\circ\bar\eta^\tau)dS \qquad (\equiv J_1)\\
&+\int_\Gamma \frac{\nabla_0^2\bar\Theta}{\sqrt{\det(g_0)}}\Big[(\sqrt{\det(g_0)}\bar A^{\alpha\beta\gamma\delta}h_{\kappa,\alpha\beta})_{,\gamma\delta}\circ\bar\eta^\tau\Big]\nabla_0^2(h_{\kappa t}\circ\bar\eta^\tau)dS \qquad (\equiv J_2)\\
=&\frac12\frac{d}{dt}\int_\Gamma B\bar A^{\alpha\beta\gamma\delta}\nabla_0^2 h_{\kappa,\alpha\beta}\nabla_0^2 h_{\kappa,\gamma\delta}dS+R_1',
\end{aligned}
$$

where $B=b^t\otimes b^t\otimes b^t\otimes b^t$ with $b=\nabla_0\bar\eta^\tau$, and

$$
\begin{aligned}
R_1(t)=&\int_\Gamma b^t\otimes b^t\otimes(\nabla_0 b^t)\otimes(\nabla_0 b^t)\nabla_0(\sqrt{\det(g_0)}\bar A^{\alpha\beta\gamma\delta}h_{\kappa,\alpha\beta})_{,\gamma\delta}\nabla_0 h_{\kappa t}dS \quad (\equiv J_3)\\
&+\int_\Gamma b^t\otimes b^t\otimes b^t\otimes(\nabla_0 b^t)\nabla_0(\sqrt{\det(g_0)}\bar A^{\alpha\beta\gamma\delta}h_{\kappa,\alpha\beta})_{,\gamma\delta}\nabla_0^2 h_{\kappa t}dS \quad (\equiv J_4)\\
&+\int_\Gamma b^t\otimes b^t\otimes b^t\otimes(\nabla_0 b^t)\nabla_0^2(\sqrt{\det(g_0)}\bar A^{\alpha\beta\gamma\delta}h_{\kappa,\alpha\beta})_{,\gamma\delta}\nabla_0 h_{\kappa t}dS \quad (\equiv J_5)
\end{aligned}
$$

and

$$
\begin{aligned}
R_1'(t)=&R_1(t)+J_1(t)+J_2(t)-\frac12\int_\Gamma(B\bar A^{\alpha\beta\gamma\delta})_t\nabla_0^2 h_{\kappa,\alpha\beta}\nabla_0^2 h_{\kappa,\gamma\delta}dS \quad (\equiv J_6)\\
&+2\int_\Gamma\frac{B}{\sqrt{\det(g_0)}}\nabla_0(\sqrt{\det(g_0)}\bar A^{\alpha\beta\gamma\delta})\nabla_0 h_{\kappa,\alpha\beta}\nabla_0^2 h_{\kappa t,\gamma\delta}dS \quad (\equiv J_7)\\
&+\int_\Gamma\frac{B}{\sqrt{\det(g_0)}}\nabla_0^2(\sqrt{\det(g_0)}\bar A^{\alpha\beta\gamma\delta})h_{\kappa,\alpha\beta}\nabla_0^2 h_{\kappa t,\gamma\delta}dS \quad (\equiv J_8)
\end{aligned}
$$

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$$+ 2\int_\Gamma \frac{B_{,\gamma}}{\sqrt{\det(g_0)}} \nabla_0^2(\sqrt{\det(g_0)}\bar{A}^{\alpha\beta\gamma\delta} h_{\kappa,\alpha\beta}) \nabla_0^2 h_{\kappa t,\delta} dS \quad (\equiv J_9)$$
$$+ \int_\Gamma \frac{B_{,\gamma\delta}}{\sqrt{\det(g_0)}} \nabla_0^2(\sqrt{\det(g_0)}\bar{A}^{\alpha\beta\gamma\delta} h_{\kappa,\alpha\beta}) \nabla_0^2 h_{\kappa t} dS \quad (\equiv J_{10}).$$

It follows that

$$|I_1| \le C(\epsilon_1)(1 + \|\nabla_0^4 h_\kappa\|_{L^2(\Gamma)})\|\nabla_0^2 v_\kappa\|_{H^1(\Omega_1')},$$
$$|I_3| + |I_4| + |I_5| \le C(M)(1 + \|\tilde{h}\|_{H^5(\Gamma)})\|\nabla_0^2 v_\kappa\|_{H^1(\Omega_1)}$$

and hence that

$$|I_1| + |I_3| + |I_4| + |I_5| \le C(\epsilon_1)\Big[\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma)} + \|\tilde{h}\|^2_{H^5(\Gamma)} + 1\Big] + \delta\|v_\kappa\|^2_{H^3(\Omega)}.$$

It follows that

$$|J_2| + |J_3| + |J_5| + |J_{10}| \le C(\epsilon_1)\|\nabla_0^4 h_\kappa\|_{L^2(\Gamma)}\|\nabla_0^2 h_{\kappa t}\|_{L^2(\Gamma)},$$
$$|J_6| \le C(M)(\|\tilde{v}\|_{H^3(\Omega)} + \|\tilde{h}_t\|_{H^{2.5}(\Gamma)})\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma)}.$$

We need only obtain κ-independent estimates for the terms I_2, J_1, J_4, J_7, J_8, and J_9. By the $H^{-0.5}(\Gamma)$-$H^{0.5}(\Gamma)$ duality pairing,

$$|I_2| \le C(M)\Big[\|\nabla_0^2 h_\kappa\|_{H^{2.5}(\Gamma)} + 1\Big]\|v_\kappa\|_{H^{2.5}(\Gamma)}.$$

Therefore, by interpolation and Young's inequality,

$$|I_2| \le C\Big[\|h_\kappa\|^2_{H^4(\Gamma)} + 1\Big] + \delta_1\|\nabla_0^2 h_\kappa\|^2_{H^3(\Gamma)} + \delta\|v_\kappa\|^2_{H^3(\Omega)} \tag{B.1}$$

for some C depending on M, δ, and δ_1.

For J_1, J_4, and J_9, we find that

$$|J_1| + |J_4| + |J_9| \le C(\epsilon_1)\|h_\kappa\|_{H^{4.5}(\Gamma)}\|v_\kappa\|_{H^{2.5}(\Gamma)}$$
$$\le C'\Big[\|\nabla_0^2 h_\kappa\|^2_{H^2(\Gamma)} + 1\Big] + \delta_1\|\nabla_0^2 h_\kappa\|^2_{H^3(\Gamma)} + \delta\|v_\kappa\|^2_{H^3(\Omega)}$$

for some constant C' depending on M, ϵ_1, δ, and δ_1.

For J_7 and J_8, by the $H^{-1.5}(\Gamma)$-$H^{1.5}(\Gamma)$ duality pairing,

$$|J_7| + |J_8| \le C(M)\|B\|_{H^{1.5}(\Gamma)}\|\bar{h}\|_{H^{3.5}(\Gamma)}\|h_\kappa\|_{H^{4.5}(\Gamma)}\|v_\kappa\|_{H^{2.5}(\Gamma)}.$$

Similarly to the estimate in (B.1), we find that

$$|J_7| + |J_8| \le C(M)\Big[\|h_\kappa\|^2_{H^4(\Gamma)} + 1\Big] + \delta_1\|\nabla_0^2 h_\kappa\|^2_{H^3(\Gamma)} + \delta\|v_\kappa\|^2_{H^3(\Omega)}.$$

Summing all the estimates and then integrating in time from 0 to t, by Corollary 7.1 and the fact that B is close to 1 in the uniform norm for T small,

$$\frac{\nu_1}{2}\|\nabla_0^4 h_\kappa(t)\|^2_{L^2(\Gamma)} \le \int_0^t\int_\Gamma \bar{\Theta}\Big[[L_{\bar{h}}(h_\kappa)(-\nabla_0\bar{h},1)]\circ\bar{\eta}^\tau\Big]\cdot\nabla_0^2(\zeta_1^2\nabla_0^2 v_\kappa)dSds$$
$$+ C'\int_0^t K(s)\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma)}ds + C'\int_0^t\Big[\|\tilde{h}\|^2_{H^5(\Gamma)} + 1\Big]ds$$
$$+ \delta\int_0^t\|v_\kappa\|^2_{H^3(\Omega)}ds + \delta_1\int_0^t\|\nabla_0^2 h_\kappa\|^2_{H^3(\Gamma)}ds$$

for some constant C' depending on M, ϵ_1, δ, and δ_1, where

$$K(s) := 1 + \|\tilde{v}\|^2_{H^3(\Omega)} + \|\tilde{h}\|^2_{H^5(\Gamma)} + \|\tilde{h}_t\|^2_{H^{2.5}(\Gamma)}.$$

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B.2. ϵ_1-independent estimates. We next obtain ϵ_1-independent estimates for the first two terms of I_1, as well as those for I_2, J_1, J_2, J_3, J_4, J_5, J_9, and J_{10} with h_κ replaced by h_{ϵ_1}. Let

$$I_1^1 = -\int_\Gamma \bar{\Theta}\Big[L_{\tilde{h}}(h_{\epsilon_1}) \circ \bar{\eta}^\tau\Big]\Big[\nabla_0^4(-\nabla_0 \bar{h} \circ \bar{\eta}^\tau, 1) \cdot v_{\epsilon_1}\Big] dS,$$
$$I_1^2 = -4\int_\Gamma \bar{\Theta}\Big[L_{\tilde{h}}(h_{\epsilon_1}) \circ \bar{\eta}^\tau\Big]\Big[\nabla_0^3(-\nabla_0 \bar{h} \circ \bar{\eta}^\tau, 1) \cdot \nabla_0 v_{\epsilon_1}\Big] dS.$$

By the $H^{-1.5}(\Gamma)$-$H^{1.5}(\Gamma)$ duality pairing,

$$|I_1^1| + |I_1^2| \le C(M)\|L_{\tilde{h}}(h_{\epsilon_1})\|_{H^{1.5}(\Gamma)}\|v_{\epsilon_1}\|_{H^{2.5}(\Gamma)}\|(\nabla_0 \tilde{h}) \circ \bar{\eta}^\tau\|_{H^{2.5}(\Gamma)}.$$

Therefore, by (6.6) and (9.12),

$$\text{(B.2)}\quad |I_1^1| + |I_1^2| \le C(M)t^{1/4}\Big[\|h_{\epsilon_1}\|^2_{H^{5.5}(\Gamma)} + 1\Big]\|v_{\epsilon_1}\|_{H^3(\Omega)}$$
$$\le Ct^{1/2}\Big[\|v_{\epsilon_1 t}\|^2_{H^1(\Omega)} + \|\nabla_0^4 h_{\epsilon_1}\|^2_{L^2(\Gamma)} + \|F\|^2_{H^1(\Omega)} + 1\Big] + (\delta + Ct^{1/2})\|v_{\epsilon_1}\|^2_{H^3(\Omega)}$$

for some constant C depending on M and δ.

For J_1, we use an L^4-L^4-L^2-type of Hölder inequality and conclude that

$$|J_1| \le C(M)t^{1/2}\|h_{\epsilon_1}\|_{H^{5.5}(\Gamma)}\|v_{\epsilon_1}\|_{H^{2.5}(\Gamma)},$$

while for the other J terms, we use the $H^{0.5}(\Gamma)$-$H^{-0.5}(\Gamma)$ duality pairing to obtain

$$|J_2| + |J_3| + |J_4| + |J_5| + |J_9| + |J_{10}| \le C(M)t^{1/2}\|h_{\epsilon_1}\|_{H^{5.5}(\Gamma)}\|v_{\epsilon_1}\|_{H^{2.5}(\Gamma)},$$

and hence all the J terms are bounded by the same right-hand side of the inequality in (B.2). Therefore,

$$\frac{\nu_1}{2}\|\nabla_0^4 h_{\epsilon_1}(t)\|^2_{L^2(\Gamma)} \le \int_0^t\int_\Gamma \bar{\Theta}\Big[[L_{\bar{h}}(h_{\epsilon_1})(-\nabla_0\bar{h}, 1)] \circ \bar{\eta}^\tau\Big] \cdot \nabla_0^2(\zeta_1^2 \nabla_0^2 v_{\epsilon_1}) dS ds$$
$$+ CN_2(u_0, F) + C\int_0^t K(s)\|\nabla_0^4 h_{\epsilon_1}\|^2_{L^2(\Gamma)} ds + (\delta + Ct^{1/2})\int_0^t \|v_{\epsilon_1}\|^2_{H^3(\Omega)} ds$$
$$+ (\delta_1 + Ct^{1/2})\int_0^t \|v_{\epsilon_1 t}\|^2_{H^1(\Omega)} ds$$

for some constant C depending on M, δ, and δ_1.

Appendix C. $L_t^2 H_x^1$-estimates for v_t. By the chain rule and integrating by parts,

$$\int_\Gamma \Big[\bar{\Theta}[L_{\bar{h}}(h_\kappa)(-\nabla_0\bar{h}, 1)]\circ\bar{\eta}^\tau\Big]_t \cdot v_{\kappa t} dS = \int_\Gamma \bar{\Theta}_t\Big[L_{\bar{h}}(h_\kappa)\Big] \circ \bar{\eta}^\tau(-\nabla_0\bar{h} \circ \bar{\eta}^\tau, 1) \cdot v_{\kappa t} dS$$
$$+ \int_\Gamma \bar{\Theta}\bar{\eta}_t^\tau \cdot \Big[\nabla_0[L_{\bar{h}}(h_\kappa)](-\nabla_0\bar{h}, 1)\Big] \circ \bar{\eta}^\tau \cdot v_{\kappa t} dS \quad (\equiv K_1)$$
$$+ \int_\Gamma \bar{\Theta}\Big[[L_{\bar{h}}(h_\kappa)](\nabla_0\bar{h}, -1)\Big]_t \circ \bar{\eta}^\tau \cdot v_{\kappa t} dS \quad (\equiv K_2).$$

The first term is bounded by

$$C(M)\|\bar{v}\|_{H^3(\Omega)}\Big[\|\nabla_0^4 h_\kappa\|_{L^2(\Gamma)} + 1\Big]\|v_{\kappa t}\|_{L^2(\Gamma)}.$$

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After integrating by parts, the most difficult term to estimate in K_1 consists of the integral

$$\int_\Gamma \frac{\bar v}{\sqrt{\det(g_0)}}\Big[[\sqrt{\det(g_0)}\bar A^{\alpha\beta\gamma\delta}h_{\kappa,\alpha\beta}]_{,\gamma\delta}(\nabla_0\bar h,-1)\Big]\circ\bar\eta^\tau\nabla_0 v_{\kappa t}dS.$$

Integrating from 0 to t and integrating by parts in time, we find that

$$\begin{aligned}&\int_0^t\int_\Gamma \frac{\bar v}{\sqrt{\det(g_0)}}\Big[[\sqrt{\det(g_0)}\bar A^{\alpha\beta\gamma\delta}h_{\kappa,\alpha\beta}]_{,\gamma\delta}(\nabla_0\bar h,-1)\Big]\circ\bar\eta^\tau\nabla_0 v_{\kappa t}dSds\\ &=-\int_0^t\int_\Gamma \frac{\bar v}{\sqrt{\det(g_0)}}\Big[[\sqrt{\det(g_0)}\bar A^{\alpha\beta\gamma\delta}h_{\kappa,\alpha\beta}]_{t,\gamma\delta}(\nabla_0\bar h,-1)\Big]\circ\bar\eta^\tau\nabla_0 v_{\kappa}dSds+R_3,\end{aligned}$$

where R_3 is bounded by

$$\begin{aligned}&C\int_0^t\Big[1+\|\tilde v_t\|^2_{H^1(\Omega)}\Big]\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma)}ds+\delta_2\|\nabla_0^4h_\kappa\|^2_{L^2(\Gamma)}\\ &\quad+\delta\int_0^t\|v_\kappa\|^2_{H^3(\Omega)}ds+(\delta+Ct^{1/2})\int_0^t\|v_{\kappa t}\|^2_{H^1(\Omega)}ds\end{aligned}$$

for some constant C depending on M, δ, and δ_2. Next, using that

$$[(-\nabla_0\bar h,1)\circ\bar\eta^\tau]\cdot\nabla_0 v_\kappa=b^t(\nabla_0 h_{\kappa t})\circ\bar\eta^\tau+b^t(\nabla_0^2\bar h\circ\bar\eta^\tau,0)\cdot v_\kappa$$

and integrating by parts, we find that the integral on the right-hand side is identical to

$$\frac12\int_0^t\int_\Gamma\frac{1}{\sqrt{\det(g_0)}}\nabla_0\Big[\sqrt{\det(g_0)}\bar\Theta\bar v b^t\bar A^{\alpha\beta\gamma\delta}\Big]h_{\kappa t,\alpha\beta}h_{\kappa t,\gamma\delta}dSds+R_4,$$

where

$$|R_4|\le C(M)C(\delta)\int_0^t\|\nabla_0^4h_\kappa\|^2_{L^2(\Gamma)}ds+\delta\int_0^t\|v_\kappa\|^2_{H^3(\Omega)}ds.$$

By interpolation, the integral part is bounded by

$$C\Big[N(u_0,F)+\int_0^t\|\nabla_0^4h_\kappa\|^2_{L^2(\Gamma)}ds\Big]+\delta\int_0^t\|v_\kappa\|^2_{H^3(\Omega)}ds+Ct\int_0^t\|v_{\kappa t}\|^2_{H^1(\Omega)}ds$$

for some constant C depending on M and δ. Therefore, K_1 satisfies

(C.1)
$$\begin{aligned}\left|\int_0^t K_1ds\right|&\le C\int_0^t\Big[K(s)\Big(\|\nabla_0^4h_\kappa\|^2_{L^2(\Gamma)}+\|\nabla_0^2h_{\kappa t}\|^2_{L^2(\Gamma)}\Big)+1\Big]ds+\delta_2\|\nabla_0^4h_\kappa\|^2_{L^2(\Gamma)}\\ &\quad+(\delta+Ct^{1/2})\int_0^t\|v_\kappa\|^2_{H^3(\Omega)}ds+(\delta+Ct^{1/2})\int_0^t\|v_{\kappa t}\|^2_{H^1(\Omega)}ds\end{aligned}$$

for some constant C depending on M, δ, and δ_2.

For K_2, by time differentiating the evolution equation, we find that

$$\begin{aligned}(-\nabla_0\bar h\circ\bar\eta^\tau,1)v_{\kappa t}&=h_{\kappa tt}\circ\bar\eta^\tau+\bar v^\tau\cdot(\nabla_0h_{\kappa t})\circ\bar\eta^\tau-\bar v^\tau\cdot(\nabla_0^2\bar h\circ\bar\eta^\tau,0)\cdot v_\kappa\\ &\quad-(\nabla_0\bar h_t\circ\bar\eta^\tau,0)\cdot v_\kappa,\end{aligned}$$

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and hence (after a change of coordinates)

$$
\begin{aligned}
K_2 = &\int_\Gamma [L_{\bar h}(h_\kappa)]_t h_{\kappa tt} dS + \int_\Gamma [L_{\bar h}(h_\kappa)]_t [(\bar v^\tau \circ \bar\eta^{-\tau}) \cdot (\nabla_0 h_{\kappa t})] dS \quad (\equiv K_3) \\
&- \int_\Gamma [L_{\bar h}(h_\kappa)]_t [(\nabla_0 \bar h_t, 0) \cdot (v_\kappa \circ \bar\eta^{-\tau})] dS \quad (\equiv K_4) \\
&- \int_\Gamma [L_{\bar h}(h_\kappa)]_t [(\bar v^\tau \circ \bar\eta^{-\tau}) \cdot (\nabla_0^2 \bar h, 0)(v_\kappa \circ \bar\eta^{-\tau})] dS \quad (\equiv K_5) \\
&+ \int_\Gamma [L_{\bar h}(h_\kappa)] [(\nabla_0 \bar h_t, 0) \cdot (v_{\kappa t} \circ \bar\eta^{-\tau})] dS \quad (\equiv K_6).
\end{aligned}
$$

For the first term, we have

$$
\begin{aligned}
\int_\Gamma [L_{\bar h}(h_\kappa)]_t h_{\kappa tt} dS = &\frac{1}{2}\frac{d}{dt}\int_\Gamma \bar A^{\alpha\beta\gamma\delta} h_{\kappa t,\alpha\beta} h_{\kappa t,\gamma\delta} dS \\
&+ \int_\Gamma \frac{1}{\sqrt{\det(g_0)}} \left[\sqrt{\det(g_0)} (\bar A^{\alpha\beta\gamma\delta})_t\right]_{,\gamma\delta} h_{\kappa,\alpha\beta} h_{\kappa tt} dS \quad (\equiv K_7) + R_5,
\end{aligned} \tag{C.2}
$$

where R_5 is bounded by

$$
\begin{aligned}
&C\left[1 + \|\tilde h_t\|^2_{H^{2.5}(\Gamma)}\right]\left[1 + \|\nabla_0^2 h_{\kappa t}\|^2_{L^2(\Gamma)}\right] + \delta\left[\|v_\kappa\|^2_{H^2(\Omega)} + \|\nabla_0^2 v_\kappa\|^2_{H^1(\Omega_1')}\right] \\
&+ \delta_1 \|v_{\kappa t}\|^2_{H^1(\Omega)}
\end{aligned}
$$

for some constant C depending on M, δ, and δ_1. Also, by the inequality $\|h_{\kappa tt}\|_{L^4(\Gamma)} \le C(M)\left[\|v_\kappa\|_{H^2(\Omega)} + \|v_{\kappa t}\|_{H^1(\Omega)}\right]$,

$$
\begin{aligned}
|K_7| &\le C\|[\sqrt{\det(g_0)}(\bar A^{\alpha\beta\gamma\delta})_t]_{,\gamma\delta}\|_{H^{-0.5}(\Gamma)} \left\| \frac{1}{\sqrt{\det(g_0)}} h_{\kappa,\alpha\beta} h_{\kappa tt} \right\|_{H^{0.5}(\Gamma)} \\
&\le C(M)C(\delta,\delta_1)\|\tilde h_t\|^2_{H^{2.5}(\Gamma)} \|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma)} + \delta\|v_\kappa\|^2_{H^2(\Omega)} + \delta_1\|v_{\kappa t}\|^2_{H^1(\Omega)}.
\end{aligned}
$$

REMARK 22. *The bound for K_7 can be refined even further as*

$$
|K_7| \le C(M)C(\delta)\|\tilde h_t\|^2_{H^{1.5}(\Gamma)} \|\nabla_0^2 h_\kappa\|^2_{H^{1.5}(\Gamma)} + \delta\|v_\kappa\|^2_{H^3(\Omega)} + \delta\|v_{\kappa t}\|^2_{H^1(\Omega)};
$$

it is this inequality that will be used in the proof of the fixed-point argument.

It remains to estimate K_3 to K_6. By proper use of Hölder's inequality,

$$
\begin{aligned}
|K_3| + |K_5| + |K_6| \le &C\left[1 + \|\tilde h_t\|^2_{H^{2.5}(\Gamma)}\right]\left[1 + \|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma)}\right] \\
&+ (\delta + Ct^{1/2})\|v_\kappa\|^2_{H^3(\Omega)} + \delta\|v_{\kappa t}\|^2_{H^1(\Omega)}
\end{aligned}
$$

for some constant C depending on M and δ. For K_4, most of the terms can be estimated in the same fashion, except the term

$$
\int_\Gamma \frac{1}{\sqrt{\det(g_0)}}\left[\sqrt{\det(g_0)}\bar A^{\alpha\beta\gamma\delta} h_{\kappa t,\alpha\beta}\right][(\nabla_0 \bar h_{t,\gamma\delta}, 0) \cdot (v_\kappa \circ \bar\eta^{-\tau})] dS,
$$

which is identical to

$$
\int_\Gamma \left\{ \frac{1}{\sqrt{\det(g_0)}}\left[\sqrt{\det(g_0)}\bar A^{\alpha\beta\gamma\delta} h_{\kappa t,\alpha\beta}\right][(\nabla_0 \bar h_{,\gamma\delta}, 0) \cdot (v_\kappa \circ \bar\eta^{-\tau})] \right\}_t dS \ (\equiv K_8) + R_6,
$$

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where

$$|R_6| \le C\|\tilde{h}\|^2_{H^{5.5}(\Gamma)}\Big[\|v_\kappa\|^2_{L^2(\Omega)} + \|\nabla_0^2 h_{\kappa t}\|^2_{L^2(\Gamma)}\Big] + \delta\|v_\kappa\|^2_{H^3(\Omega)} + \delta_1\|v_{\kappa t}\|^2_{H^1(\Omega)}$$

for some constant C depending on M, δ, and δ_1. Time integrating K_8 and using the interpolation inequality together with Young's inequality, we find that

$$\left|\int_0^t K_8(s)ds\right| \le C(M)\Big[\|u_0\|^2_{H^{2.5}(\Omega)} + \|\nabla_0^2 h_{\kappa t}\|_{L^2(\Omega)}\|v_\kappa\|_{L^4(\Omega)}\Big]$$

$$\le C(M)C(\delta_1,\delta_2)N_3(u_0,F) + \delta_2\|\nabla_0^2 h_{\kappa t}\|^2_{L^2(\Gamma)} + \delta_1\int_0^t \|v_{\kappa t}\|^2_{H^1(\Omega)}ds, \tag{C.3}$$

where

$$N_3(u_0,F) := \|u_0\|^2_{H^{2.5}(\Omega)} + \|u_0\|^2_{H^{4.5}(\Gamma)} + \|F\|^2_{L^2(0,T;H^1(\Omega))} + \|F_t\|^2_{L^2(0,T;H^1(\Omega)')} + \|F(0)\|^2_{H^1(\Omega)} + 1,$$

and we use $\|v_\kappa\|^2_{H^1(\Omega)} \le C\big[\int_0^t \|v_{\kappa t}\|^2_{H^1(\Omega)}ds + \|u_0\|^2_{H^1(\Omega)}\big]$ to obtain (C.3), and hence

$$\sum_{i=3}^6 |K_i| \le C\Big[1 + \|\tilde{h}\|^2_{H^{5.5}(\Gamma)} + \|\tilde{h}_t\|^2_{H^{2.5}(\Gamma)}\Big]\Big[1 + \|v_\kappa\|^2_{L^2(\Omega)} + \|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma)}\Big]$$

$$+ (\delta + Ct^{1/2})\|v_\kappa\|^2_{H^3(\Omega)} + \delta_1\|v_{\kappa t}\|^2_{H^1(\Omega)} + K_8 \tag{C.4}$$

with K_8 satisfying inequality (C.3). Finally, combining all the estimates,

(C.5)

$$\int_0^t \|\nabla_0^2 h_{\kappa t}\|^2_{L^2(\Gamma)}ds \le \int_0^t\int_\Gamma \Big[[L_{\bar{h}}(h_\kappa)(\nabla_0\bar{h},-1)]\circ\bar{\eta}^\tau\Big]_t \cdot v_{\kappa t}dS + CN_3(u_0,F)$$

$$+ C\int_0^t K(s)\Big[\|v_\kappa\|^2_{L^2(\Omega)} + \|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma)} + \|\nabla_0^2 h_{\kappa t}\|^2_{L^2(\Gamma)}\Big]ds$$

$$+ (\delta + Ct^{1/2})\int_0^t \|v_\kappa\|^2_{H^3(\Omega)}ds + (\delta_1 + Ct^{1/2})\int_0^t \|v_{\kappa t}\|^2_{H^1(\Omega)}ds + \delta_2\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma)}$$

for some constant C depending on M, δ, δ_1, and δ_2.

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