基礎數學 MA-1015A

Chapter 7. Concepts of Analysis

§7.1 Convergent Sequences

Recall that a sequence is a function with domain $\mathbb{N}$. For $n \in \mathbb{N}$, the image of n is called the n-th term of the sequence and is written as x_n. In the following discussion, we only consider real sequences.

Definition

Let $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ be a sequence. $\{x_n\}_{n=1}^{\infty}$ is said to be ***convergent*** if there exists $L \in \mathbb{R}$ such that for every $\varepsilon > 0$,

$$\#\{n \in \mathbb{N} \,|\, x_n \notin (L-\varepsilon, L+\varepsilon)\} < \infty\,.$$

Such an L is called a ***limit*** of the sequence. In notation,

$$\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{R} \text{ is convergent}$$
$$\Leftrightarrow \quad (\exists\, L \in \mathbb{R})(\forall\, \varepsilon > 0)\big(\#\{n \in \mathbb{N} \,|\, x_n \notin (L-\varepsilon, L+\varepsilon)\} < \infty\big).$$

If L is a limit of $\{x_n\}_{n=1}^{\infty}$, we say $\{x_n\}_{n=1}^{\infty}$ converges to L and write $x_n \to L$ as $n \to \infty$. If $\{x_n\}_{n=1}^{\infty}$ is not convergent, we say that $\{x_n\}_{n=1}^{\infty}$ diverges or is divergent.

§7.1 Convergent Sequences

Example

Let $x_n = \dfrac{(-1)^n}{n+1}$. We show that $\{x_n\}_{n=1}^{\infty}$ converges to 0. By definition, we need to show for every $\varepsilon > 0$ the set

$$A_\varepsilon = \left\{ n \in \mathbb{N} \,\middle|\, x_n \notin (-\varepsilon, \varepsilon) \right\}$$

is finite. Note that $A_\varepsilon = \left\{ n \in \mathbb{N} \,\middle|\, |x_n| \geqslant \varepsilon \right\}$; thus

$$A_\varepsilon = \left\{ n \in \mathbb{N} \,\middle|\, \frac{1}{n+1} \geqslant \varepsilon \right\} = \left\{ n \in \mathbb{N} \,\middle|\, n \leqslant \frac{1}{\varepsilon} - 1 \right\}.$$

Therefore, $\#A_\epsilon = \left[\frac{1}{\varepsilon}\right] - 1 < \infty$ which implies that $\{x_n\}_{n=1}^{\infty}$ converges to 0.

§7.1 Convergent Sequences

Example

The sequence $\{y_n\}_{n=1}^{\infty}$ given by $y_n = \dfrac{3+(-1)^n}{2}$ diverges. To see this, we have to show that any real number L cannot be the limit of $\{y_n\}_{n=1}^{\infty}$.

Let $L \in \mathbb{R}$ be given and $\varepsilon = \dfrac{1}{2}$. Then $(L-\varepsilon, L+\varepsilon)$ at most contains one integer. Since y_n only takes value 1 or 2 and $\#\{n \in \mathbb{N} \,|\, y_n = 1\} = \#\{n \in \mathbb{N} \,|\, y_n = 2\} = \infty$, we find that

$$\#\Big\{n \in \mathbb{N} \,\Big|\, y_n \notin (L-\varepsilon, L+\varepsilon)\Big\} = \infty$$

which implies $\{y_n\}_{n=1}^{\infty}$ cannot converges to L.

§7.1 Convergent Sequences

Example

Recall that a permutation of a non-empty set A is a one-to-one correspondence from A onto A. Let $\pi : \mathbb{N} \to \mathbb{N}$ be a permutation of $\mathbb{N}$, and $\{x_n\}_{n=1}^{\infty}$ be a convergent sequence. Then $\{x_{\pi(n)}\}_{n=1}^{\infty}$ is also convergent since if L is the limit of $\{x_n\}_{n=1}^{\infty}$ and $\varepsilon > 0$,

$$\begin{aligned}&\#\{n \in \mathbb{N} \,|\, x_{\pi(n)} \notin (x-\varepsilon, x+\varepsilon)\}\\ &\quad = \#\{n \in \mathbb{N} \,|\, x_n \notin (x-\varepsilon, x+\varepsilon)\} < \infty\,.\end{aligned}$$

Theorem

Let $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ be a sequence and L be a real number. Then $\{x_n\}_{n=1}^{\infty}$ converges to L if and only if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_n - L| < \varepsilon$ whenever $n \geqslant N$. In notation,

$$(\forall\, \varepsilon > 0)(\exists\, N \in \mathbb{N})(n \geqslant N \Rightarrow |x_n - L| < \varepsilon)\,.$$

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Proof.

"⇒" Let $\varepsilon > 0$ be given, and $A_\epsilon = \{n \in \mathbb{N} \,|\, x_n \notin (L-\varepsilon, L+\varepsilon)\}$. Since $\{x_n\}_{n=1}^\infty$ converges to L, $k \equiv \#A_\epsilon < \infty$. Suppose that $n_1 < n_2 < \cdots < n_k$ belongs to A_ϵ. Let $N = n_k + 1$. Then $N \in \mathbb{N}$ and if $n \geqslant N$, $n \notin A_\epsilon$ which implies that if $n \geqslant N$, $x_n \in (L-\varepsilon, L+\varepsilon)$ or equivalently,

$$|x_n - L| < \varepsilon \quad \text{whenever} \quad n \geqslant N.$$

"⇐" Let $\varepsilon > 0$ be given. Then for some $N \in \mathbb{N}$, if $n \geqslant N$, we have $|x_n - L| < \varepsilon$ or equivalently, if $n \geqslant N$, $x_n \in (L-\varepsilon, L+\varepsilon)$. This implies that

$$\#\{n \in \mathbb{N} \,|\, x_n \notin (L-\varepsilon, L+\varepsilon)\} < N < \infty.$$

□

§7.1 Convergent Sequences

Theorem

If $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ is a sequence such that $x_n \to x$ and $x_n \to y$ as $n \to \infty$, then $x = y$. (The uniqueness of the limit).

Proof.

Assume the contrary that $x \neq y$. W.L.O.G. we may assume that $x < y$, and let $\varepsilon = \frac{y-x}{2} > 0$. Then

$$\#\{n \in \mathbb{N} \,|\, x_n \notin (x-\varepsilon, x+\varepsilon)\} < \infty\,, \tag{$\star$}$$

and

$$\#\{n \in \mathbb{N} \,|\, x_n \notin (y-\varepsilon, y+\varepsilon)\} < \infty\,.$$

Note that the latter implies that $\#\{n \in \mathbb{N} \,|\, x_n \in (y-\varepsilon, y+\varepsilon)\} = \infty$ which contradicts to ($\star$) since

$$(x-\varepsilon, x+\varepsilon) \cap (y-\varepsilon, y+\varepsilon) = \varnothing\,.$$

□

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Alternative proof using ε-N definition.

Assume the contrary that $x \neq y$. W.L.O.G. we may assume that $x < y$, and let $\varepsilon = \dfrac{y-x}{2} > 0$ $(x+\varepsilon = y-\varepsilon)$. Since $x_n \to x$ and $x_n \to y$ as $n \to \infty$,

$$(\exists\, N_1 \in \mathbb{N})(n \geqslant N_1 \Rightarrow |x_n - x| < \varepsilon)\,,$$

and

$$(\exists\, N_2 \in \mathbb{N})(n \geqslant N_2 \Rightarrow |x_n - y| < \varepsilon)\,.$$

Define $N \equiv \max\{N_1, N_2\}$. Then $N \in \mathbb{N}$. Moreover, if $n \geqslant N$, we have both $|x_n - x| < \varepsilon$ and $|x_n - y| < \varepsilon$ for all $n \geqslant N$. As a consequence, $x_n < x+\varepsilon$ and $x_n > y-\varepsilon$ for all $n \geqslant N$, a contradiction. So $x = y$. □

§7.1 Convergent Sequences

Notation: Since the limit of a convergent sequence $\{x_n\}_{n=1}^{\infty}$ is unique, we use $\lim\limits_{n\to\infty} x_n$ to denote the limit of $\{x_n\}_{n=1}^{\infty}$ when $\{x_n\}_{n=1}^{\infty}$ is convergent.

Remark: A sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ diverges if (and only if)

$$(\forall\, L \in \mathbb{R})(\exists\, \varepsilon > 0)\big(\#\{n \in \mathbb{N} \,|\, x_n \notin (L-\varepsilon, L+\varepsilon)\} = \infty\big)$$

which is equivalent to that

$$(\forall\, L \in \mathbb{R})(\exists\, \varepsilon > 0)(\forall\, N \in \mathbb{N})(\exists\, n \geqslant N)(|x_n - L| \geqslant \varepsilon)\,.$$

§7.1 Convergent Sequences

Example

Let $x_n = \dfrac{(-1)^n}{n+1}$. We show that $\{x_n\}_{n=1}^{\infty}$ converges to 0 using ε-N argument.

Let $\varepsilon > 0$ be given. Define $N = \left[\frac{1}{\varepsilon}\right] + 1$. Then $N \in \mathbb{N}$. Since $\left[\frac{1}{\varepsilon}\right] > \frac{1}{\varepsilon} - 1$, if $n \geqslant N$ we must have $n > \frac{1}{\varepsilon}$; thus if $n \geqslant N$, $\dfrac{1}{n+1} < \dfrac{1}{n} < \varepsilon$. Therefore,

$$|x_n - 0| < \varepsilon \quad \text{whenever} \quad n \geqslant N$$

which implies that $\{x_n\}_{n=1}^{\infty}$ converges to 0.

§7.1 Convergent Sequences

Example

In this example we use ε-N argument to show that the sequence $\{y_n\}_{n=1}^{\infty}$ given by $y_n = \dfrac{3+(-1)^n}{2}$ diverges. We need to show that

$$(\forall\, L \in \mathbb{R})(\exists\, \varepsilon > 0)(\forall\, N \in \mathbb{N})(\exists\, n \geqslant N)(|y_n - L| \geqslant \varepsilon)\,.$$

Let $L \in \mathbb{R}$ be given. Choose $\varepsilon = \dfrac{1}{2}$. For $N \in \mathbb{N}$, define

$$n = \begin{cases} N+1 & \text{if } |y_N - L| < \varepsilon\,, \\ N+2 & \text{if } |y_N - L| \geqslant \varepsilon\,. \end{cases}$$

Then $n \geqslant N$. Moreover, if $|y_N - L| < \varepsilon$, then $|y_n - L| \geqslant |y_n - y_N| - |y_N - L| > 1 - \varepsilon = \varepsilon$, while if $|y_N - L| \geqslant \varepsilon$ then clearly $|y_n - L| \geqslant \varepsilon$. Therefore,

$$(\forall\, L \in \mathbb{R})(\exists\, \varepsilon > 0)(\forall\, N \in \mathbb{N})(\exists\, n \geqslant N)(|y_n - L| \geqslant \varepsilon)\,.$$

§7.1 Convergent Sequences

Example

Let $\pi : \mathbb{N} \to \mathbb{N}$ be a permutation of $\mathbb{N}$, and $\{x_n\}_{n=1}^{\infty}$ be a convergent sequence. We show that $\{x_{\pi(n)}\}_{n=1}^{\infty}$ converges using the ε-N argument.

Suppose that $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence with limit L, and $\varepsilon > 0$ be given. Then by the convergence of $\{x_n\}_{n=1}^{\infty}$ to L, there exists $N_1 \in \mathbb{N}$ such that if $n \geqslant N_1$, we have $|x_n - L| < \varepsilon$. Define $N = \max\{\pi^{-1}(1), \pi^{-1}(2), \cdots, \pi^{-1}(N_1)\}$. Then if $n \geqslant N$, $\pi(n) \geqslant N_1$ which implies that

$$\left|x_{\pi(n)} - L\right| < \varepsilon \quad \text{whenever} \quad n \geqslant N .$$

Therefore, $\{x_{\pi(n)}\}_{n=1}^{\infty}$ converges to L.

§7.1 Convergent Sequences

Theorem (Squeeze/Sandwich)

Suppose that $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ *and* $\{c_n\}_{n=1}^{\infty}$ *are sequences of real numbers such that* $a_n \leqslant b_n \leqslant c_n$ *for all* $n \in \mathbb{N}$. *If* $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, *then* $\lim\limits_{n\to\infty} b_n = L$.

Proof.

Let $\varepsilon > 0$ be given. Since $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = L$, by definition

$$(\exists\, N_1 \in \mathbb{N})(n \geqslant N_1 \Rightarrow L - \varepsilon < a_n < L + \varepsilon)\,,$$

and

$$(\exists\, N_2 \in \mathbb{N})(n \geqslant N_2 \Rightarrow L - \varepsilon < b_n < L + \varepsilon)\,.$$

Let $N = \max\{N_1, N_2\}$. Then $N \in \mathbb{N}$ and if $n \geqslant N$, $L - \varepsilon < a_n \leqslant c_n \leqslant b_n < L + \varepsilon$; thus $\lim\limits_{n\to\infty} c_n = L$. □

§7.1 Convergent Sequences

Example

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence given by $x_n = \dfrac{\sin n}{n}$. Then $\lim\limits_{n\to\infty} \dfrac{\sin n}{n} = 0$.

Definition

Let $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ be a sequence.

1. $\{x_n\}_{n=1}^{\infty}$ is said to be ***bounded***（有界的）if there exists $M > 0$ such that $|x_n| \leqslant M$ for all $n \in \mathbb{N}$.
2. $\{x_n\}_{n=1}^{\infty}$ is said to be ***bounded from above***（有上界）if there exists $M \in \mathbb{R}$, called an ***upper bound*** of the sequence, such that $x_n \leqslant M$ for all $n \in \mathbb{N}$.
3. $\{x_n\}_{n=1}^{\infty}$ is said to be ***bounded from below***（有下界）if there exists $m \in \mathbb{R}$, called a ***lower bound*** of the sequence, such that $m \leqslant x_n$ for all $n \in \mathbb{N}$.

§7.1 Convergent Sequences

Theorem

A convergent sequence is bounded（數列收斂必有界）.

Proof.

Let $\{x_n\}_{n=1}^{\infty}$ be a convergent sequence with limit x. Then there exists $N > 0$ such that

$$|x_n - x| < 1 \quad \text{whenever} \quad n \geqslant N$$

or equivalently,

$$x_n \in (x-1, x+1) \quad \text{whenever} \quad n \geqslant N.$$

Let $M = \max\{|x_1|, |x_2|, \cdots, |x_{N-1}|, |x|+1\}$. Then $|x_n| \leqslant M$ for all $n \in \mathbb{N}$. □

§7.1 Convergent Sequences

Theorem

Suppose that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. Then

1. $x_n \pm y_n \to x \pm y$ *as* $n \to \infty$.
2. $x_n \cdot y_n \to x \cdot y$ *as* $n \to \infty$.
3. *If* $y_n, y \neq 0$, *then* $\dfrac{x_n}{y_n} \to \dfrac{x}{y}$ *as* $n \to \infty$.

Proof.

1. Let $\varepsilon > 0$ be given. Since $x_n \to x$ and $y_n \to y$ as $n \to \infty$, there exist $N_1, N_2 \in \mathbb{N}$ such that $|x_n - x| < \dfrac{\varepsilon}{2}$ for all $n \geqslant N_1$ and $|y_n - x| < \dfrac{\varepsilon}{2}$ for all $n \geqslant N_2$. Define $N = \max\{N_1, N_2\}$. Then $N \in \mathbb{N}$ and if $n \geqslant N$,
$$|(x_n \pm y_n) - (x \pm y)| \leqslant |x_n - x| + |y_n - y| < \varepsilon\,;$$
thus $x_n \pm y_n \to x \pm y$ as $n \to \infty$. □

§7.1 Convergent Sequences

Proof (Cont'd).

2. Since $x_n \to x$ and $y_n \to y$ as $n \to \infty$, by the boundedness of convergent sequences, there exists $M > 0$ such that $|x_n| \leqslant M$ and $|y_n| \leqslant M$. Let $\varepsilon > 0$ be given. Then

$$(\exists\, N_1 \in \mathbb{N})\Big(n \geqslant N_1 \Rightarrow |x_n - x| < \frac{\varepsilon}{2M}\Big)\,,$$

and

$$(\exists\, N_2 \in \mathbb{N})\Big(n \geqslant N_2 \Rightarrow |y_n - y| < \frac{\varepsilon}{2M}\Big)\,.$$

Define $N = \max\{N_1, N_2\}$. Then $N \in \mathbb{N}$ and if $n \geqslant N$,

$$\begin{aligned}
|x_n \cdot y_n - x \cdot y| &= |x_n \cdot y_n - x_n \cdot y + x_n \cdot y - x \cdot y| \\
&\leqslant |x_n \cdot (y_n - y)| + |y \cdot (x_n - x)| \\
&\leqslant M \cdot |y_n - y| + M \cdot |x_n - x| \\
&< M \cdot \frac{\varepsilon}{2M} + M \cdot \frac{\varepsilon}{2M} = \varepsilon.
\end{aligned}$$

□

§7.1 Convergent Sequences

Proof (Cont'd).

3. It suffices to show that $\lim\limits_{n\to\infty} \frac{1}{y_n} = \frac{1}{y}$ if $y_n, y \neq 0$ (because of ②). Since $\lim\limits_{n\to\infty} y_n = y$, there exists $N_1 \in \mathbb{N}$ such that $|y_n - y| < \frac{|y|}{2}$ whenever $n \geqslant N_1$. Therefore, $|y| - |y_n| < \frac{|y|}{2}$ for all $n \geqslant N_1$ which further implies that $|y_n| > \frac{|y|}{2}$ for all $n \geqslant N_1$.

   Let $\varepsilon > 0$ be given. Since $\lim\limits_{n\to\infty} y_n = y$, there exists $N_2 \in \mathbb{N}$ such that $|y_n - y| < \frac{|y|^2}{2}\varepsilon$ whenever $n \geqslant N_2$. Define $N = \max\{N_1, N_2\}$. Then $N \in \mathbb{N}$ and if $n \geqslant N$,

$$\left|\frac{1}{y_n} - \frac{1}{y}\right| = \frac{|y_n - y|}{|y_n||y|} < \frac{|y|^2}{2}\varepsilon \cdot \frac{1}{|y|}\frac{2}{|y|} = \varepsilon\,.$$

□

§7.1 Convergent Sequences

Definition

A sequence $\{y_j\}_{j=1}^{\infty}$ is called a ***subsequence*** of a sequence $\{x_n\}_{n=1}^{\infty}$ if there exists an increasing function $f: \mathbb{N} \to \mathbb{N}$ such that $y_j = x_{f(j)}$. In this case, we often write $f(j) = n_j$ and $y_j = x_{n_j}$.

In other words, a subsequence of a sequence is derived by deleting some elements without changing the order of remaining elements.

Example

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence. Then $\{x_{2n}\}_{n=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$. It is obtained by deleting all the odd terms of $\{x_n\}_{n=1}^{\infty}$. On the other hand, the sequence $\{x_{2n-1}\}_{n=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$ and is obtained by deleting all the even terms of $\{x_n\}_{n=1}^{\infty}$.

§7.1 Convergent Sequences

Theorem

A sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ converges if and only if every subsequence of $\{x_n\}_{n=1}^{\infty}$ converges (to the same limit).

Proof.

Since $\{x_n\}_{n=1}^{\infty}$ itself is a subsequence of $\{x_n\}_{n=1}^{\infty}$, it suffices to show the implication from LHS to RHS.

Suppose that $\lim\limits_{n\to\infty} x_n = L$. We claim that every subsequence of $\{x_n\}_{n=1}^{\infty}$ also converges to L.

Let $\varepsilon > 0$ be given. Since $\lim\limits_{n\to\infty} x_n = L$, there exists $N \in \mathbb{N}$ such that $|x_n - L| < \varepsilon$ whenever $n \geqslant N$. Choose $J > 0$ such that $n_J \geqslant N$ (this is possible since $n_j \to \infty$ as $j \to \infty$). Then if $j \geqslant J$, $n_j \geqslant n_J \geqslant N$, we must have $\left|x_{n_j} - L\right| < \varepsilon$. □

§7.2 Limits and Continuity of Real-Valued Functions

Definition

Let $I \subseteq \mathbb{R}$ be an interval, $a \in I$, and f be a real-valued function defined on $I - \{a\}$. We say that ***the limit of*** f ***as*** x ***approaches*** a exists if for every sequence $\{a_n\}_{n=1}^{\infty} \subseteq I$ satisfying

1. $a_n \neq a$ for all $n \in \mathbb{N}$,
2. $\lim\limits_{n\to\infty} a_n = a$,

the sequence $\{b_n\}_{n=1}^{\infty}$ given by $b_n = f(a_n)$ converges.

（一函數在 a 的極限存在如果「所有在 I 中取值不是 a 但收斂到 a 的數列其函數值所形成的數列都收斂」）

Using the logic notation, the limit of f at a exists if

$$\left(\forall \{a_n\}_{n=1}^{\infty} \subseteq I - \{a\}\right)\left(\lim_{n\to\infty} a_n = a \Rightarrow \lim_{n\to\infty} f(a_n) \text{ exists}\right).$$

§7.2 Limits and Continuity of Real-Valued Functions

Theorem

Let $I \subseteq \mathbb{R}$ be an interval, $a \in I$, and f be a real-valued function defined on $I - \{a\}$. If the limit of f as x approaches a exists, then the limit is unique; that is, there exists a unique $L \in \mathbb{R}$ such that $\lim\limits_{n\to\infty} f(a_n) = L$ for every sequence $\{a_n\}_{n=1}^{\infty} \subseteq I - \{a\}$ which converges to a.

Proof.

Suppose that contrary that there exist two sequences $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty} \subseteq I - \{a\}$ and two numbers L_1, L_2 such that $a_n \to a$, $b_n \to a$ as $n \to \infty$ and

$$\lim_{n\to\infty} f(a_n) = L_1 \quad \text{and} \quad \lim_{n\to\infty} f(b_n) = L_2 .$$

□

§7.2 Limits and Continuity of Real-Valued Functions

Proof (Cont'd).

Define a sequence $\{c_n\}_{n=1}^{\infty}$ by $c_n = \begin{cases} a_{\frac{n+1}{2}} & \text{if } n \text{ is odd}\,, \\ b_{\frac{n}{2}} & \text{if } n \text{ is even}\,; \end{cases}$ that is, $\{c_n\}_{n=1}^{\infty} = \{a_1, b_1, a_2, b_2, a_3, b_3, \cdots\}$. Then $c_n \to a$ as $n \to \infty$; thus by the definition of the limit of functions, there exists L such that

$$\lim_{n\to\infty} f(c_n) = L\,.$$

Since $\{f(a_n)\}_{n=1}^{\infty}$ and $\{f(b_n)\}_{n=1}^{\infty}$ are subsequences of $\{f(c_n)\}_{n=1}^{\infty}$,

$$L_1 = \lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(c_n) = \lim_{n\to\infty} f(b_n) = L_2\,,$$

a contradiction. □

- **Notation**: Since the limit of a convergent sequence is unique, for a convergent sequence $\{a_n\}_{n=1}^{\infty}$, we use $\lim\limits_{x\to a} f(x)$ to denote the limit.

§7.2 Limits and Continuity of Real-Valued Functions

Example

Consider the function $f\colon [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sin \dfrac{1}{x} & \text{if } x \neq 0\,, \\ 0 & \text{if } x = 0\,. \end{cases}$$

Then f is not continuous at 0 since letting $x_n = \dfrac{1}{2n\pi}$ and $y_n = \dfrac{1}{2n\pi + \pi/2}$, we have $x_n \to 0$ and $y_n \to 0$ as $n \to \infty$ but $f(x_n) = 0$ while $f(y_n) = 1$ for all $n \in \mathbb{N}$.

§7.2 Limits and Continuity of Real-Valued Functions

Theorem

Suppose that $I \subseteq \mathbb{R}$ is an interval, $a \in I$, and f, g are two functions defined on I, except possibly at a, such that $f(x) = g(x)$ for all $x \in I - \{a\}$. If $\lim\limits_{x \to a} f(x)$ exists, then $\lim\limits_{x \to a} g(x)$ exists, and $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x)$.

Proof.

Since $\lim\limits_{x \to a} f(x)$ exists, every sequence $\{a_n\}_{n=1}^{\infty} \subseteq I - \{a\}$ converging to Let $\{a_n\}_{n=1}^{\infty} \subseteq I - \{a\}$ be a sequence converging to a. Since $\lim\limits_{x \to a} f(x)$ exists, $\lim\limits_{n \to \infty} f(a_n) = L$ for some $L \in \mathbb{R}$. By the fact that $f(x) = g(x)$ for $x \in I - \{a\}$, $\lim\limits_{n \to \infty} g(a_n) = L$. □

§7.2 Limits and Continuity of Real-Valued Functions

Theorem

Let $I \subseteq \mathbb{R}$ be an interval, $a \in I$, and f be a real-valued function defined on $I - \{a\}$. Then $\lim\limits_{x \to a} f(x) = L$ if and only if

$$(\forall\, \varepsilon > 0)(\exists\, \delta > 0)\Big[(0 < |x - a| < \delta) \wedge (x \in I) \Rightarrow |f(x) - L| < \varepsilon\Big].$$

Proof.

"$\Rightarrow$" Assume the contrary that there exists $\varepsilon > 0$ such that for all $\delta > 0$, there exists $x_\delta \in I - \{a\}$ with

$$0 < |x_\delta - a| < \delta \quad \text{and} \quad \big|f(x_\delta) - b\big| \geqslant \varepsilon\,.$$

In particular, we can find $\{x_k\}_{k=1}^{\infty} \subseteq I - \{a\}$ such that

$$0 < |x_k - a| < \frac{1}{k} \quad \text{and} \quad \big|f(x_k) - L\big| \geqslant \varepsilon\,.$$

Then $x_k \to a$ as $k \to \infty$ but $f(x_k) \nrightarrow L$ as $k \to \infty$, a contradiction. □

§7.2 Limits and Continuity of Real-Valued Functions

Goal: $\lim\limits_{x \to a} f(x) = L$ if and only if

$$(\forall\, \varepsilon > 0)(\exists\, \delta > 0)\big[(0 < |x - a| < \delta) \wedge (x \in I) \Rightarrow |f(x) - L| < \varepsilon\big]$$

Proof.

"$\Leftarrow$" Let $\{x_k\}_{k=1}^{\infty} \subseteq I - \{a\}$ be such that $x_k \to a$ as $k \to \infty$, and $\varepsilon > 0$ be given. By assumption,

$$(\exists\, \delta > 0)\big[(0 < |x - a| < \delta) \wedge (x \in I) \Rightarrow \big|f(x) - L\big| < \varepsilon\big].$$

Since $x_k \to a$ as $k \to \infty$, there exists $N > 0$ such that $|x_k - a| < \delta$ whenever $k \geqslant N$. Therefore,

$$\big|f(x_k) - L\big| < \varepsilon \quad \forall\, k \geqslant N$$

which shows that $\lim\limits_{k \to \infty} f(x_k) = L$. □

§7.2 Limits and Continuity of Real-Valued Functions

Definition

Let $I \subseteq \mathbb{R}$ be an interval, and $a \in I$. A function $f: I \to \mathbb{R}$ is said to be continuous at a if $\lim\limits_{x \to a} f(x) = f(a)$. In other words, $f: I \to \mathbb{R}$ is continuous at a if

$$(\forall\, \varepsilon > 0)(\exists\, \delta > 0)\Big[\big(|x-a| < \delta\big) \wedge (x \in I) \Rightarrow \big|f(x) - f(a)\big| < \varepsilon\Big].$$

A function $f: I \to \mathbb{R}$ is said to be continuous on I if f is continuous at every point of I.

Remark: Almost identical proof of showing the previous theorem implies that "f is continuous at a if and only if for every sequence $\{x_n\}_{n=1}^{\infty} \subseteq I$ converging to a, one has $\lim\limits_{n\to\infty} f(x_n) = f(a)$." (一函數 f 在 a 連續如果「所有在 I 中收斂到 a 的數列其函數值所形成的數列都收斂到 $f(a)$」)

§7.2 Limits and Continuity of Real-Valued Functions

Lemma

Let $I, J \subseteq \mathbb{R}$ be intervals, and $f : I \to \mathbb{R}$, $g : J \to \mathbb{R}$ be functions. If $f(I) \subseteq J$, $\lim\limits_{x \to a} f(x) = b \in J$, and g is continuous at b, then

$$\lim_{x \to a} (g \circ f)(x) = g(b)\,.$$

Proof.

Let $\{x_n\}_{n=1}^{\infty} \subseteq I - \{a\}$ such that $x_n \to a$ as $n \to \infty$. By the fact that $\lim\limits_{x \to a} f(x) = b$, we have $\lim\limits_{n \to \infty} f(x_n) = b$. Since $f(I) \subseteq J$, $\{f(x_n)\}_{n=1}^{\infty}$ is a sequence in J and converges to b; thus by the continuity of g at b and the previous remark, $\lim\limits_{n \to \infty} g(f(x_n)) = g(b)$. Therefore, for every sequence $\{x_n\}_{n=1}^{\infty} \subseteq I - \{a\}$ such that $x_n \to a$ as $n \to \infty$, one has $\lim\limits_{n \to \infty} (g \circ f)(x_n) = g(b)$. □

§7.2 Limits and Continuity of Real-Valued Functions

$f(I) \subseteq J \wedge \lim\limits_{x \to a} f(x) = b \wedge g$ is continuous at $b \Rightarrow \lim\limits_{x \to a} (g \circ f)(x) = g(b)$.

Alternative proof.

Let $\varepsilon > 0$ be given. Since g is continuous at b, there exists $\sigma > 0$ such that

$$\big|g(y) - g(b)\big| < \varepsilon \quad \text{whenever} \quad |y - b| < \sigma \text{ and } y \in J.$$

For such $\delta > 0$, there exists $\delta > 0$ such that

$$\big|f(x) - b\big| < \delta \quad \text{whenever} \quad 0 < |x - a| < \delta \text{ and } x \in I.$$

Therefore, if $0 < |x - a| < \delta$ and $x \in I$,

$$\big|(g \circ f)(x) - g(b)\big| = \big|g(f(x)) - g(b)\big| < \varepsilon$$

since we also have $|f(x) - b| < \sigma$ and $f(x) \in J$. $\square$

§7.2 Limits and Continuity of Real-Valued Functions

What will happen if $f(I) \subseteq J$, $\lim\limits_{x \to a} f(x) = b$ but we only have $\lim\limits_{x \to b} g(x) = c$ but not continuity of g at b? Can we still conclude that $\lim\limits_{x \to a} (g \circ f)(x) = c$ in this case?

Example

Let $f(x) = b$ be a constant function, and $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} c & \text{if } x \neq b\,, \\ c+1 & \text{if } x = b\,. \end{cases}$$

Then $\lim\limits_{x \to a} f(x) = b$ and $\lim\limits_{x \to b} g(x) = c$. By the fact that $(g \circ f)(x) = c + 1$ for all $x \in \mathbb{R}$,

$$\lim_{x \to a} (g \circ f)(x) = c + 1 \neq c\,.$$

Therefore, $\lim\limits_{x \to a} f(x) = b \wedge \lim\limits_{x \to b} g(x) = c \not\Rightarrow \lim\limits_{x \to a} (g \circ f)(x) = c$.

§7.2 Limits and Continuity of Real-Valued Functions

Theorem

Let $I, J \subseteq \mathbb{R}$ be intervals, and $f : I \to \mathbb{R}$, $g : J \to \mathbb{R}$ be functions. If $f(I) \subseteq J$, f is continuous at $a \in I$, $f(a) \in J$ and g is continuous at $f(a)$, then $g \circ f$ is continuous at a. In particular, if f is continuous on I and g is continuous on J, then $(g \circ f)$ is continuous on I.

§7.3 The Completeness Property

Definition

A set $\mathcal{F}$ is said to be a ***field*** (體) if there are two operations $+$ and $\cdot$ such that

1. $x+y \in \mathcal{F}$, $x \cdot y \in \mathcal{F}$ if $x, y \in \mathcal{F}$. (封閉性)
2. $x+y = y+x$ for all $x, y \in \mathcal{F}$. (commutativity, 加法的交換性)
3. $(x+y)+z = x+(y+z)$ for all $x, y, z \in \mathcal{F}$. (associativity, 加法的結合性)
4. There exists $0 \in \mathcal{F}$, called 加法單位元素, such that $x+0 = x$ for all $x \in \mathcal{F}$. (the existence of zero)
5. For every $x \in \mathcal{F}$, there exists $y \in \mathcal{F}$ (usually y is denoted by $-x$ and is called x 的加法反元素) such that $x+y = 0$. One writes $x - y \equiv x + (-y)$.

§7.3 The Completeness Property

Definition (Cont'd)

6. $x \cdot y = y \cdot x$ for all $x, y \in \mathcal{F}$. (乘法的交換性)
7. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in \mathcal{F}$. (乘法的結合性)
8. There exists $1 \in \mathcal{F}$, called 乘法單位元素, such that $x \cdot 1 = x$ for all $x \in \mathcal{F}$. (the existence of unity)
9. For every $x \in \mathcal{F}$, $x \neq 0$, there exists $y \in \mathcal{F}$ (usually y is denoted by x^{-1} and is called x 的乘法反元素) such that $x \cdot y = 1$. One writes $x \cdot y \equiv x \cdot x^{-1} = 1$.
10. $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in \mathcal{F}$. (distributive law, 分配律)
11. $0 \neq 1$.

§7.3 The Completeness Property

Definition

A ***partial order*** over a set P is a binary relation $\preccurlyeq$ which is reflexive, anti-symmetric and transitive, in the sense that

1. $x \preccurlyeq x$ for all $x \in P$ (reflexivity).
2. $x \preccurlyeq y$ and $y \preccurlyeq x \Rightarrow x = y$ (anti-symmetry).
3. $x \preccurlyeq y$ and $y \preccurlyeq z \Rightarrow x \preccurlyeq z$ (transitivity).

A set with a partial order is called a ***partially ordered set***.

Example

$(\mathbb{Q}, \geqslant)$ and $(2^{[0,1]}, \subseteq)$ are partially ordered sets.

Definition

Let $(P, \preccurlyeq)$ be a partially ordered set. Two elements $x, y \in P$ are said to be ***comparable*** if either $x \preccurlyeq y$ or $y \preccurlyeq x$.

§7.3 The Completeness Property

Definition

A partial order under which every pair of elements is comparable is called a ***total order*** or ***linear order***.

Example

The relation $\geqslant$ is a total order in $\mathbb{Q}$.

Definition

An ***ordered field*** is a totally ordered field $(\mathcal{F}, +, \cdot, \leqslant)$ satisfying that

1. If $x \leqslant y$, then $x + z \leqslant y + z$ for all $z \in \mathcal{F}$ (compatibility of $\leqslant$ and $+$).
2. If $0 \leqslant x$ and $0 \leqslant y$, then $0 \leqslant x \cdot y$ (compatibility of $\leqslant$ and $\cdot$).

Remark: ② in the definition above implies that $0 \leqslant 1$. In other words, we exclude that possibility that the relation $\geqslant$ is used as the total order in the ordered field $(\mathbb{Q}, +, \cdot)$ or $(\mathbb{R}, +, \cdot)$.

§7.3 The Completeness Property

Example

$(\mathbb{Q}, +, \cdot, \leqslant)$ and $(\mathbb{R}, +, \cdot, \leqslant)$ are ordered fields.

Definition

Let $(\mathcal{F}, +, \cdot, \leqslant)$ be an ordered field.

1. The relation $\geqslant$ is defined by "$x \geqslant y \Leftrightarrow y \leqslant x$".
2. The relation $<$ is defined by "$x < y \Leftrightarrow x \leqslant y \wedge x \neq y$".
3. The relation $>$ is defined by "$x > y \Leftrightarrow y < x$".

Theorem

If $a < b$ in an ordered field $(\mathcal{F}, +, \cdot, \leqslant)$, then there exists $c \in \mathcal{F}$ such that $a < c < b$.

§7.3 The Completeness Property

Definition

Let $(\mathcal{F}, +, \cdot, \leqslant)$ be an ordered field, and $\varnothing \neq A \subseteq \mathcal{F}$. A number $M \in \mathcal{F}$ is called an ***upper bound*** (上界) for A if $x \leqslant M$ for all $x \in A$, and a number $m \in \mathcal{F}$ is called a ***lower bound*** (下界) for A if $x \geqslant m$ for all $x \in A$. If there is an upper bound for A, then A is said to be ***bounded from above***, while if there is a lower bound for A, then A is said to be ***bounded from below***. A number $b \in \mathcal{F}$ is called a ***least upper bound*** (最小上界) if

1. b is an upper bound for A, and
2. if M is an upper bound for A, then $M \geqslant b$.

A number a is called a ***greatest lower bound*** (最大下界) if

1. a is a lower bound for A, and
2. if m is a lower bound for A, then $m \leqslant a$.

§7.3 The Completeness Property

Definition (Cont'd)

If A is not bounded above, the least upper bound of A is set to be ∞, while if A is not bounded below, the greatest lower bound of A is set to be $-\infty$. The least upper bound of A is also called the ***supremum*** of A and is usually denoted by lub$A$ or $\sup A$, and "the" greatest lower bound of A is also called the ***infimum*** of A, and is usually denoted by glb$A$ or $\inf A$. If $A = \varnothing$, then $\sup A = -\infty$, $\inf A = \infty$.

Remark: Let $(\mathcal{F}, +, \cdot, \leqslant)$ be an ordered field.

1. If $b_1, b_2 \in \mathcal{F}$ are least upper bounds for a set $A \subseteq \mathcal{F}$, then $b_1 = b_2$. Therefore, $\sup A$ is a well-defined concept. Similarly, $\inf A$ is a well-defined concept.
2. Since the sentence "$x \in \varnothing \Rightarrow x \leqslant M$" is true for all $M \in \mathcal{F}$, we conclude that $\sup \varnothing = -\infty$. Similarly, $\inf \varnothing = \infty$.

§7.3 The Completeness Property

Example

Consider the ordered field $(\mathbb{Q}, +, \cdot, \leqslant)$ and $A = \{x \in \mathbb{Q} \,|\, x^2 < 2\}$. Then 2 is an upper bound for A; however, there is no least upper bound for A in $\mathbb{Q}$.

Reason: If $M \in \mathbb{Q}$ is an upper bound for A, then $M > \sqrt{2}$. By the property of $\mathbb{R}$ there exists a rational number $q \in (\sqrt{2}, M)$. Such q is also an upper bound for A. In other words, for any given rational upper bound for A in $\mathbb{Q}$ there exists a smaller upper bound for A in $\mathbb{Q}$; thus there is no least upper bound for A in $\mathbb{Q}$.

§7.3 The Completeness Property

Theorem

Let $(\mathcal{F}, +, \cdot, \leqslant)$ be an ordered field, and A be a subset of $\mathcal{F}$. Then $s = \sup A$ if and only if

(i) $(\forall\, \varepsilon > 0)(\forall\, x \in A)(x < s + \varepsilon)$. (ii) $(\forall\, \varepsilon > 0)(\exists\, x \in A)(x > s - \varepsilon)$.

Definition (Completeness)

Let $(\mathcal{F}, +, \cdot, \leqslant)$ be an ordered field. $\mathcal{F}$ is said to be ***complete***（完備）if every non-empty subset of $\mathcal{F}$ that has an upper bound in $\mathcal{F}$ has a supremum that is an element of $\mathcal{F}$.（非空有上界的集合必有最小上界）

Theorem

The field $(\mathbb{R}, +, \cdot, \leqslant)$ is a complete ordered field.

§7.3 The Completeness Property

Theorem (Archimedean Principle for $\mathbb{R}$)

For every real number x, there is a natural number n such that $n > x$.

Proof.

Let $x \in \mathbb{R}$. If $x < 1$, then the choice $n = 1$ validates $n > x$. Suppose $x \geqslant 1$. Define $A = \{n \in \mathbb{N} \mid n \leqslant x\}$. Then $1 \in A$ and x is an upper bound for A. By the completeness of $\mathbb{R}$, $s \equiv \sup A \in \mathbb{R}$ exists. Since s is the least upper bound for A, $s - 1$ is not an upper bound for A; thus there exists $m \in A$ such that $m > s - 1$ or $s < m + 1$. Then $m + 1 \notin A$ which implies that $m + 1 \nleqslant x$. The choice $n = m + 1$ satisfies $n > x$. □

§7.4 The Heine-Borel Theorem

Definition

Let a and δ be real numbers with $\delta > 0$. The ***δ-neighborhood*** of a is the set $\mathcal{N}(a,\delta) = \{x \in \mathbb{R} \mid |x-a| < \delta\}$.

Properties:

1. A sequence $\{x_n\}_{n=1}^{\infty}$ converges to x if for every $\varepsilon > 0$, there are only finite number of $n \in \mathbb{N}$ such that x_n lies outside the ε-neighborhood of x.
2. If $0 < \delta_1 < \delta_2$, then $\mathcal{N}(a,\delta_1) \subseteq \mathcal{N}(a,\delta_2)$.

Definition

For a set $A \subseteq \mathbb{R}$, a point x is said to be an ***interior point*** of A if there exists $\delta > 0$ such that $\mathcal{N}(a,\delta) \subseteq A$.

§7.4 The Heine-Borel Theorem

Definition

A set $A \subseteq \mathbb{R}$ is said to be ***open*** if every point of A is an interior point of A. In other words, $A \subseteq \mathbb{R}$ is open if

$$(\forall\, x \in A)(\exists\, \delta > 0)\big(\mathcal{N}(x,\delta) \subseteq A\big)\,.$$

Example

The empty set $\varnothing$ is open since the conditional statement

$$(x \in \varnothing) \Rightarrow (\exists\, \delta > 0)\big(\mathcal{N}(x,\delta) \subseteq \varnothing\big)$$

is always true.

Example

The universe $\mathbb{R}$ is open since the conditional statement

$$(x \in \mathbb{R}) \Rightarrow (\exists\, \delta > 0)\big(\mathcal{N}(x,\delta) \subseteq \mathbb{R}\big)$$

is always true.

§7.4 The Heine-Borel Theorem

Theorem

Every interval $(a, b) \subseteq \mathbb{R}$, *where* $-\infty \leqslant a < b \leqslant \infty$, *is an open set.*

Proof.

Let $x \in (a, b)$. W.L.O.G., we can assume that at least one a and b is finite. Define $\delta = \min\{x - a, b - x\}$. Then $0 < \delta < \infty$. Moreover, if $y \in \mathcal{N}(x, \delta)$, we must have $|y - x| < \delta$; thus if $y \in \mathcal{N}(x, \delta)$,

$$y - a = y - x + x - a > -\delta + x - a \geqslant 0$$

and

$$b - y = b - x + x - y > b - x - \delta \geqslant 0$$

which implies that $\mathcal{N}(x, \delta) \subseteq (a, b)$. □

§7.4 The Heine-Borel Theorem

Theorem

Let $\mathcal{F}$ be a non-empty collection of open subsets of $\mathbb{R}$. Then

1. $\bigcup_{A\in\mathcal{F}} A$ *is an open set.*
2. *If $\mathcal{F}$ has finitely many open sets, then* $\bigcap_{A\in\mathcal{F}} A$ *is an open set.*

Proof.

1. Let $x \in \bigcup_{A\in\mathcal{F}} A$. Then $x \in A$ for some $A \in \mathcal{F}$. Since A is open, x is an interior point of A; thus there exists $\delta > 0$ such that $\mathcal{N}(x,\delta) \subseteq A$. Then $\mathcal{N}(x,\delta) \subseteq \bigcup_{A\in\mathcal{F}} A$ and we establish that $\bigcup_{A\in\mathcal{F}} A$ is open. □

§7.4 The Heine-Borel Theorem

2. If $\mathcal{F}$ has finitely many open sets, then $\bigcap_{A\in\mathcal{F}} A$ is an open set.

Proof (Cont'd).

2. Suppose that $\mathcal{F} = \{A_1, A_2, \cdots, A_n\}$ and A_j's are open for $1 \leqslant j \leqslant k$. Let $x \in \bigcap_{A\in\mathcal{F}} A$. Then $x \in A_j$ for all $1 \leqslant j \leqslant k$. Since each A_j is open, there exists $\delta_j > 0$ such that $\mathcal{N}(x, \delta_j) \subseteq A_j$. Define $\delta = \min\{\delta_1, \cdots, \delta_n\}$. Then $\delta > 0$ and $\mathcal{N}(x,\delta) \subseteq \mathcal{N}(x,\delta_j) \subseteq A_j$ for all $1 \leqslant j \leqslant k$. Therefore, $\mathcal{N}(x,\delta) \subseteq \bigcap_{j=1}^{k} A_j = \bigcap_{A\in\mathcal{F}} A$. □

Definition

A set A is said to be ***closed*** if its complement $A^{\complement} = \mathbb{R}\backslash A$ is open.

§7.4 The Heine-Borel Theorem

Example

The set $[a, b]$ is closed. To see this, we have to show that $[a, b]^{\complement}$ is open. Note that

$$x \in [a, b] \Leftrightarrow \left\{x \in \mathbb{R} \,\middle|\, a \leqslant x \wedge x \leqslant b\right\};$$

thus

$$x \in [a, b]^{\complement} \Leftrightarrow \left\{x \in \mathbb{R} \,\middle|\, \sim(a \leqslant x) \vee \sim(x \leqslant b)\right\}$$

or equivalently,

$$x \in [a, b]^{\complement} \Leftrightarrow \left\{x \in \mathbb{R} \,\middle|\, (a > x) \vee (x > b)\right\}.$$

Therefore, $[a, b]^{\complement} = (-\infty, a) \cup (b, \infty)$ which, by the fact that $(-\infty, a)$ and (b, ∞) are open, implies that $[a, b]^{\complement}$ is open.

§7.4 The Heine-Borel Theorem

Theorem

A subset $A \subseteq \mathbb{R}$ is closed if and only if every convergent sequence in A converges to a limit in A. In logic notation,

$$A \text{ is closed } \Leftrightarrow \left(\forall \{x_n\}_{n=1}^{\infty} \subseteq A\right)\left(\lim_{n\to\infty} x_n = x \Rightarrow x \in A\right).$$

Proof.

($\Rightarrow$) Assume the contrary that $\{x_n\}_{n=1}^{\infty} \subseteq A$, $\lim\limits_{n\to\infty} x_n = x$ but $x \notin A$. Then $x \in A^{\complement}$. By the closedness of A, there exists $\delta > 0$ such that $\mathcal{N}(x,\delta) \subseteq A^{\complement}$. Since $\{x_n\}_{n=1}^{\infty} \subseteq A$, $|x_n - x| \geqslant \delta$; thus $\lim\limits_{n\to\infty} x_n \neq x$, a contradiction.

($\Leftarrow$) Suppose the contrary that A is not closed. Then there exists $x \in A^{\complement}$ such that for all $\delta > 0$, $\mathcal{N}(x,\delta) \nsubseteq A^{\complement}$; thus for all $\delta > 0$, $\mathcal{N}(x,\delta) \cap A \neq \varnothing$. Choose $\delta = 1/n$ and $x_n \in \mathcal{N}(x, 1/n) \cap A$. Then $\left(\exists \{x_n\}_{n=1}^{\infty} \subseteq A\right)\left(\lim\limits_{n\to\infty} x_n = x \wedge \sim(x \in A)\right)$. $\square$

§7.4 The Heine-Borel Theorem

Theorem

A subset $A \subseteq \mathbb{R}$ is closed if and only if every convergent sequence in A converges to a limit in A. In logic notation,

$$A \text{ is closed} \Leftrightarrow (\forall \{x_n\}_{n=1}^{\infty} \subseteq A)\Big(\lim_{n\to\infty} x_n = x \Rightarrow x \in A\Big).$$

Proof.

($\Rightarrow$) Assume the contrary that $\{x_n\}_{n=1}^{\infty} \subseteq A$, $\lim\limits_{n\to\infty} x_n = x$ but $x \notin A$. Then $x \in A^\complement$. By the closedness of A, there exists $\delta > 0$ such that $\mathcal{N}(x,\delta) \subseteq A^\complement$. Since $\{x_n\}_{n=1}^{\infty} \subseteq A$, $|x_n - x| \geqslant \delta$; thus $\lim\limits_{n\to\infty} x_n \neq x$, a contradiction.

($\Leftarrow$) Suppose the contrary that A is not closed. Then there exists $x \in A^\complement$ such that for all $\delta > 0$, $\mathcal{N}(x,\delta) \nsubseteq A^\complement$; thus for all $\delta > 0$, $\mathcal{N}(x,\delta) \cap A \neq \varnothing$. Choose $\delta = 1/n$ and $x_n \in \mathcal{N}(x, 1/n) \cap A$. Then $\sim (\forall \{x_n\}_{n=1}^{\infty} \subseteq A)\Big(\lim\limits_{n\to\infty} x_n = x \Rightarrow x \in A\Big)$. □

§7.4 The Heine-Borel Theorem

Corollary

Let $A \subseteq \mathbb{R}$ be closed and $x \in \mathbb{R}$. If $A \cap \mathcal{N}(x, \delta) \neq \varnothing$ for all $\delta > 0$, then $x \in A$.

Theorem

If $\varnothing \neq A \subseteq \mathbb{R}$ is closed and bounded, then $\sup A \in A$ and $\inf A \in A$.

Proof.

We only prove the case that $\sup A \in A$ since the proof of the counterpart is similar.

Let $x = \sup A$. Then $x \in \mathbb{R}$, and for all $n \in \mathbb{N}$, $x - 1/n$ is no an upper bound for A which implies that there exists $x_n \in A$ such that

$$x - \frac{1}{n} < x_n \leqslant x;$$

thus we construct a sequence $\{x_n\}_{n=1}^{\infty} \subseteq A$ and $x_n \to x$ (by the squeeze theorem). The previous theorem then shows that $x \in A$. □

§7.4 The Heine-Borel Theorem

Definition

Let $A \subseteq \mathbb{R}$. A collection $\mathcal{F}$ of open subsets of $\mathbb{R}$ is called an ***open cover*** for A if $A \subseteq \bigcup_{U \in \mathcal{F}} U$. If $\mathcal{B} \subseteq \mathcal{F}$ is a sub-collection of $\mathcal{F}$ and $\mathcal{B}$ is also an open cover for A, $\mathcal{B}$ is called an ***subcover*** of $\mathcal{F}$ for A. $\mathcal{B}$ is called a ***finite subcover*** if there is only finitely many elements in $\mathcal{B}$.

Example

For $n \in \mathbb{N}$, let U_n denote the open set $\left(n - \frac{1}{n}, n + \frac{1}{n}\right)$, and $\mathcal{F}$ be the indexed family $\mathcal{F} \equiv \{U_n \mid n \in \mathbb{N}\}$. Then $\mathcal{F}$ is an open cover of $\mathbb{N}$ with no subcovers other than $\mathcal{F}$ itself.

§7.4 The Heine-Borel Theorem

Example

Since $\bigcup_{n=1}^{\infty}(-\infty, n) = \mathbb{R}$, the family $\mathcal{F} \equiv \{(-\infty, n) \,|\, n \in \mathbb{N}\}$ is an open cover for $\mathbb{R}$. There are many subcover of $\mathcal{F}$ for $\mathbb{R}$, such as

$$\{(-\infty, 2n) \,|\, n \in \mathbb{N}\} \quad \text{or} \quad \{(-\infty, 2n+1) \,|\, n \in \mathbb{N}\}.$$

However, there is no finite subcover of $\mathcal{F}$ for $\mathbb{R}$.

Definition

A subset $K \subseteq \mathbb{R}$ is said to be ***compact*** if for every open cover $\mathcal{F}$ for K, there is a finite subcover of $\mathcal{F}$ for K. In logic notation, K is compact if

$$(\forall\, \mathcal{F} \text{ open cover for } K)(\exists\, \mathcal{B} \subseteq \mathcal{F})\Big(\#\mathcal{B} < \infty \wedge K \subseteq \bigcup_{U \in \mathcal{B}} U\Big).$$

§7.4 The Heine-Borel Theorem

Example

The set $A = \{1\} \cup \left\{ \frac{n+1}{n} \,\middle|\, n \in \mathbb{N} \right\}$ is compact.

Let $\mathcal{F} = \left\{ U_\alpha \,\middle|\, \alpha \in I \right\}$ be an open cover of A. Then $1 \in U_{\alpha_0}$ for some $\alpha_0 \in I$. Since $\mathcal{U}_{\alpha_0}$ is open, there exists $\delta > 0$ such that $\mathcal{N}(1,\delta) \subseteq \mathcal{U}_{\alpha_0}$. Since $\lim\limits_{n\to\infty} \frac{n+1}{n} = 1$, there exists $N > 0$ such that $\frac{n+1}{n} \in \mathcal{N}(1,\delta)$ for all $n \geqslant N$. Therefore,

$$\{1\} \cup \left\{ \frac{n+1}{n} \,\middle|\, n \geqslant N \right\} \subseteq U_{\alpha_0}\,.$$

Let U_{α_j}, where $1 \leqslant j \leqslant N-1$, be open sets in $\mathcal{F}$ such that $\frac{j+1}{j} \in U_{\alpha_j}$. We note that such α_j exists since $\mathcal{F}$ is an open cover for A. Then

$$A \subseteq \bigcup_{j=0}^{N-1} U_{\alpha_j}\,.$$

§7.4 The Heine-Borel Theorem

Lemma

A compact set must be closed.

Proof.

Let K be a compact set. Suppose the contrary that there exists a convergent sequence $\{x_n\}_{n=1}^{\infty} \subseteq K$ with limit $x \notin K$. For each $y \in K$, the $\frac{|x-y|}{2}$-neighborhood of y is open and non-empty; thus

$$\mathcal{F} = \left\{ \mathcal{N}\left(y, \frac{|x-y|}{2}\right) \,\middle|\, y \in K \right\}$$

is an open cover of K. Since K is compact, there is a finite subcover

$$\mathcal{B} = \left\{ \mathcal{N}\left(y_j, \frac{|x-y_j|}{2}\right) \,\middle|\, 1 \leqslant j \leqslant M, y_1, \cdots, y_M \in K \right\}$$

of $\mathcal{F}$ for K. □

§7.4 The Heine-Borel Theorem

Proof (Cont'd).

Let $\delta = \min\left\{\frac{|x-y_1|}{2}, \frac{|x-y_2|}{2}, \cdots, \frac{|x-y_M|}{2}\right\}$. Then $|x-y_j| \geqslant 2\delta$ for $1 \leqslant j \leqslant M$ and $\delta > 0$. Since $x_n \to x$ as $n \to \infty$, there exists $N > 0$ such that $|x_n - x| < \delta$ whenever $n \geqslant N$. Then for $1 \leqslant j \leqslant M$ and $n \geqslant N$,

$$|y_j - x_n| \geqslant |y_j - x| - |x - x_n| > |y_j - x| - \frac{|y_j - x|}{2} = \frac{|y_j - x|}{2}.$$

Therefore, if $n \geqslant N$, $x_n \notin \mathcal{N}\left(y_j, \frac{|y_j - x|}{2}\right)$ which implies that $x_n \notin \bigcup_{U \in \mathcal{B}} U$, a contradiction (since $x_n \in K$). □

§7.4 The Heine-Borel Theorem

Lemma

A compact set must be bounded.

Proof.

Let $K \subseteq \mathbb{R}$ be a compact set. Define $\mathcal{F} \equiv \{(-n, n) \mid n \in \mathbb{N}\}$. Then clearly $\mathcal{F}$ is an open cover of K since $\mathcal{F}$ also covers $\mathbb{R}$. Since K is compact, there is a finite subcover

$$\mathcal{B} = \{(-n_k, n_k) \mid 1 \leqslant k \leqslant M, n_1, \cdots, n_M \in \mathbb{N}\}$$

of $\mathcal{F}$ for K. Let $L = \max\{n_1, \cdots, n_k\}$. Then

$$K \subseteq \bigcup_{k=1}^{M} (-n_k, n_k) \subseteq (-L, L)$$

which implies that $|x| \leqslant L$ for all $x \in K$. Therefore, K is bounded. □

§7.4 The Heine-Borel Theorem

Theorem (Heine-Borel Theorem)

A subset $K \subseteq \mathbb{R}$ is compact if and only if K is closed and bounded.

Proof.

It suffices to shows that if K is closed and bounded, then K is compact. Let $\mathcal{F} = \{U_\alpha \,|\, \alpha \in I\}$ be an open cover for K. For each $x \in R$, define $K_x = \{a \in K \,|\, a < x\}$. Define

$$D = \Big\{x \in \mathbb{R} \,\Big|\, K_x \text{ is included in a union of finitely many open sets from } \mathcal{F}\Big\}.$$

We claim that D is non-empty and D has no upper bound.

1. Since K is bounded, $\inf K \in \mathbb{R}$ exists. Let $z < \inf K$. Then K_z is empty which implies that $z \in D$. □

§7.4 The Heine-Borel Theorem

Proof (Cont'd).

2. Suppose the contrary that D is bounded from above. Then $x_0 = \sup D$ exists in $\mathbb{R}$. If there is $\delta > 0$ such that $K \cap \mathcal{N}(x_0, \delta) = \varnothing$, then $x_0 + \delta \in D$ which contradicts to that $x_0 = \sup D$. Therefore, $K \cap \mathcal{N}(x_0, \delta) \neq \varnothing$ for all $\delta > 0$. By the closedness of K, $x_0 \in K$.

   Since $\mathcal{F}$ is an open cover, $x_0 \in U_{\alpha_0}$ for some $U_{\alpha_0} \in \mathcal{F}$. Since U_{α_0} is open, there exists $\delta > 0$ such that $\mathcal{N}(x_0, \delta) \subseteq U_{\alpha_0}$. Since $x_0 = \sup D$, there exists $x_1 \in (x_0 - \delta, x_0] \cap D$. Since $x_1 \in D$ there exist $U_{\alpha_1}, U_{\alpha_2}, \cdots, U_{\alpha_n} \in \mathcal{F}$ such that $K_{x_1} \subseteq \bigcup_{j=1}^{n} U_{\alpha_j}$. Let $x_2 = x_0 + \dfrac{\delta}{2}$. Then $x_2 \in U_{\alpha_0}$; thus $K_{x_2} \subseteq \bigcup_{j=0}^{n} U_{\alpha_j}$ which implies that $x_2 \in D$ which contradicts to that $x_0 = \sup D$. □

§7.4 The Heine-Borel Theorem

Proof (Cont'd).

We have established that the set D given by

$$D = \Big\{x \in \mathbb{R} \,\Big|\, K_x \text{ is included in a union of finitely many open sets from } \mathcal{F}\Big\}$$

has no upper bound. Now, since K is bounded, $\sup K \in \mathbb{R}$. Since D has no upper bound, there exists $d \in D$ such that $d > \sup K$. Therefore, $K_d = K$ which implies that K is included in a union of finitely many open sets from $\mathcal{F}$; thus K is compact. □