小波理論與應用 MA5110

Homework Assignment 3

Due Nov. 19. 2024

Problem 1. In class we prove the following

Theorem 6.13 (Inversion Formula)**.** *Let ψ be a wavelet and $f \in L^2(\mathbb{R})$, then f can be reconstructed by the formula*

$$f(t) = \frac{1}{C_\psi} \int_{\mathbb{R}} \int_{\mathbb{R}} W_\psi[f](a,b)\psi_{a,b}(t)\, \frac{dbda}{a^2}, \tag{6.1}$$

where the equality holds almost everywhere.

and we mention that the integral on the right-hand side of the equal sign should be understood as

$$\lim_{\varepsilon\to 0^+, A,B\to\infty} \int_{\varepsilon\leqslant |a|\leqslant A} \int_{|b|\leqslant B} W_\psi[f](a,b)\psi_{a,b}(t)\, \frac{dbda}{a^2}.$$

Prove the (6.1) holds for $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with $\widehat{f} \in L^1(\mathbb{R})$, where the integrals is in the usual Lebesgue sense, by completing the following:

1. Show that for a fixed $a \neq 0$, $W_\psi[f](a,\cdot) \in L^2(\mathbb{R})$.
2. Using the Plancherel identity to Show that
$$\int_{\mathbb{R}} W_\psi[f](a,b)\psi_{a,b}(t)\, db = |a| \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\omega) \cdot e^{it\omega} |\widehat{\psi}(a\omega)|^2\, d\omega$$
so that
$$\int_{\mathbb{R}} \int_{\mathbb{R}} W_\psi[f](a,b)\psi_{a,b}(t)\, \frac{dbda}{a^2} = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{f}(\omega) e^{it\omega} \frac{|\widehat{\psi}(a\omega)|^2}{|a|}\, d\omega da.$$
3. Show that the Fubini Theorem can be applied to obtain that
$$\int_{\mathbb{R}} \int_{\mathbb{R}} W_\psi[f](a,b)\psi_{a,b}(t)\, \frac{dbda}{a^2} = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{f}(\omega) e^{it\omega} \frac{|\widehat{\psi}(a\omega)|^2}{|a|}\, d\omega da.$$
4. Conclude (6.1).

Problem 2. Let $\{e_n\}$ is an orthonormal basis of H. Show the following statements.

(i) $\{e_1, e_1, e_2, e_2, e_3, e_3, \cdots\}$ is a tight frame with frame bounds $A = B = 2$, but it is not exact.

(ii) $\{\sqrt{2}e_1, e_2, e_3, \cdots\}$ is an exact frame but not tight since the frame bounds are easily seen as $A = 1$ and $B = 2$.

(iii) $\left\{e_1, \frac{e_2}{\sqrt{2}}, \frac{e_2}{\sqrt{2}}, \frac{e_3}{\sqrt{3}}, \frac{e_3}{\sqrt{3}}, \frac{e_3}{\sqrt{3}}, \cdots\right\}$ is a tight frame with the frame bound $A = B = 1$ but not an orthonormal basis.

(iv) $\left\{e_1, \frac{e_2}{2}, \frac{e_3}{3}, \cdots\right\}$ is a complete orthogonal sequence but is not a frame.

Problem 3. Let $H = \mathbb{R}^2$, $\langle\cdot,\cdot\rangle$ be the usual inner product in $\mathbb{R}^2$, $\{e_n\}_{n=1}^3$ be given by

$$e_1 = (0,1), \qquad e_2 = \Big(\frac{\sqrt{3}}{2}, -\frac{1}{2}\Big), \qquad e_3 = \Big(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\Big).$$

1. Show that $\{e_n\}_{n=1}^3$ is a tight frame in H.
2. Compute the frame operator Tx for $x \in H$, and find T^{-1}.
3. Show that $\sum_{n=1}^{3} \langle x, T^{-1}e_n\rangle e_n = x$ for all $x \in H$ (without using any theorem we proved in class).