# Vector Analysis MA2014-* Midterm Exam 2

National Central University, Dec. 1 2015

學號：______________________ 姓名：______________________ 系級班別：_______________

| 1 | 2 | 3(1) | 3(2) | 4 | 5 | 6 | Total |
|---|---|---|---|---|---|---|---|
| | | | | | | | |

**Problem 1.** (20%) Evaluate the integral $\int_0^1 \left[\int_0^{1-x} \left(\int_y^1 \frac{\sin(\pi z)}{z(2-z)}\, dz\right) dy\right] dx$. (Hint: Change of order of integration to $dxdydz$)

*Solution*: Let $R$ denote the region of integration:

$$R = \left\{(x, y, z) \in \mathbb{R}^3 \,\middle|\, 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1-x, y \leqslant z \leqslant 1\right\}.$$

Then $R$ can also be expressed as

$$R = \left\{(x, y, z) \in \mathbb{R}^3 \,\middle|\, 0 \leqslant z \leqslant 1, 0 \leqslant y \leqslant z, 0 \leqslant x \leqslant 1-y\right\}.$$

Therefore,

$$\begin{aligned}\int_0^1 \int_0^{1-x} \int_y^1 \frac{\sin(\pi z)}{z(2-z)} dzdydx &= \int_0^1 \int_0^z \int_0^{1-y} \frac{\sin(\pi z)}{z(2-z)} dxdydz = \int_0^1 \int_0^z \frac{(1-y)\sin(\pi z)}{z(2-z)} dydz \\ &= \int_0^1 \frac{(z - \frac{z^2}{2})\sin(\pi z)}{z(2-z)} dz = \frac{1}{2}\int_0^1 \sin(\pi z) dz = -\frac{1}{2\pi}\cos(\pi z)\Big|_{z=0}^{z=1} = \frac{1}{\pi}.\end{aligned}$$

**Problem 2.** (15%) Evaluate the integral $\int_R e^{x+y}\, d(x,y)$, where $R$ is the region $\{(x,y) \,|\, |x|+|y| \leqslant a\}$.

**Solution 1**: Since the region $R$ is the disjoint union of the two regions $R_1$ and $R_2$ given by

$$R_1 = \{(x,y) \in \mathbb{R}^2 \,|\, 0 \leqslant x \leqslant a, x-a \leqslant y \leqslant a-x\},$$
$$R_2 = \{(x,y) \in \mathbb{R}^2 \,|\, -a \leqslant x \leqslant 0, -x-a \leqslant y \leqslant x+a\},$$

the desired integral can be computed by

$$\begin{aligned}
\int_R e^{x+y}\, d(x,y) &= \int_{R_1} e^{x+y}\, d(x,y) + \int_{R_2} e^{x+y}\, d(x,y) = \int_0^a \int_{x-a}^{a-x} e^{x+y}\, dydx + \int_{-a}^0 \int_{-x-a}^{x+a} e^{x+y}\, dydx \\
&= \int_0^a e^{x+y}\Big|_{y=x-a}^{y=a-x} dx + \int_{-a}^0 e^{x+y}\Big|_{y=-x-a}^{y=x+a} dx = \int_0^a \left(e^a - e^{2x-a}\right) dx + \int_{-a}^0 \left(e^{2x+a} - e^{-a}\right) dx \\
&= \left(xe^a - \frac{1}{2}e^{2x-a}\right)\Big|_{x=0}^{x=a} + \left(\frac{1}{2}e^{2x+a} - xe^{-a}\right)\Big|_{x=-a}^{x=0} \\
&= ae^a - \frac{1}{2}e^a + \frac{1}{2}e^{-a} + \frac{1}{2}e^a - \frac{1}{2}e^{-a} - ae^{-a} = a(e^a - e^{-a}).
\end{aligned}$$

**Solution 2**: Let $u = x+y$ and $v = x-y$. Then $R = \{(u,v) \in \mathbb{R}^2 \,|\, |u| \leqslant a, |v| \leqslant a\}$. Since $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{\partial(u,v)}{\partial(x,y)}^{-1} = -\dfrac{1}{2}$, we find that

$$\int_R e^{x+y}\, d(x,y) = \int_{-a}^a \int_{-a}^a e^u \frac{1}{2} d(u,v) = a(e^a - e^{-a}).$$

**Problem 3.** Let $T$ be the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$. Evaluate the integral $\int_T e^{(y-x)/(y+x)} d(x,y)$

1. (15%) by transforming to polar coordinates, and
2. (10%) by using the transformation $u = y - x$ and $v = y + x$.

*Solution*:

1. Let $x = r\cos\theta$ and $y = r\sin\theta$. Since the line $x + y = 1$ in polar coordinate is $r = \frac{1}{\sin\theta + \cos\theta}$, we find that

$$T = \left\{(r,\theta) \,\middle|\, 0 \leqslant r \leqslant \frac{1}{\sin\theta + \cos\theta}, 0 \leqslant \theta \leqslant \frac{\pi}{2}\right\},.$$

Therefore,

$$\int_T e^{(y-x)/(y+x)}\, d(x,y) = \int_0^{\frac{\pi}{2}} \int_0^{\frac{1}{\sin\theta+\cos\theta}} e^{\frac{\sin\theta-\cos\theta}{\cos\theta+\sin\theta}} r\,dr\,d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} e^{\frac{\sin\theta-\cos\theta}{\cos\theta+\sin\theta}} \frac{1}{(\cos\theta+\sin\theta)^2} d\theta\,.$$

Letting $u = \frac{\sin\theta - \cos\theta}{\cos\theta + \sin\theta}$, we find that

$$du = \frac{(\cos\theta + \sin\theta)^2 + (\cos\theta - \sin\theta)^2}{(\cos\theta + \sin\theta)^2} d\theta = \frac{2}{(\cos\theta + \sin\theta)^2} d\theta\,;$$

thus

$$\int_T e^{(y-x)/(y+x)}\, d(x,y) = \frac{1}{2}\int_{-1}^{1} e^u \frac{1}{2} du = \frac{1}{4}(e - e^{-1})\,.$$

2. Let $u = y - x$ and $v = y + x$. Then $\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial(u,v)}{\partial(x,y)}^{-1} = -\frac{1}{2}$ and

$$T = \left\{(u,v) \,\middle|\, 0 \leqslant v \leqslant 1, -v \leqslant u \leqslant v\right\}.$$

Therefore,

$$\int_T e^{(y-x)/(y+x)}\, d(x,y) = \int_0^1 \int_{-v}^{v} e^{\frac{u}{v}} \frac{1}{2} du\,dv = \frac{1}{2}\int_0^1 v e^{\frac{u}{v}} \Big|_{u=-v}^{u=v} dv = \frac{1}{4}(e - e^{-1})\,.$$

**Problem 4.** (20%) Find $\int_R x\,d(x,y,z)$ and $\int_R z\,d(x,y,z)$, where $R$ is the region given by

$$R = \left\{(x,y,z) \in \mathbb{R}^3 \,\middle|\, x \geqslant 0, y \geqslant 0, 0 \leqslant z \leqslant h \times \Big(1 - \frac{\sqrt{x^2+y^2}}{a}\Big)\right\}.$$

*Solution*: Consider the change of variable: $x = r\cos\theta$, $y = r\sin\theta$ and $z = hz'$. Then

$$R = \left\{(r,\theta,z') \,\middle|\, 0 \leqslant r \leqslant a, 0 \leqslant \theta \leqslant \frac{\pi}{2} \text{ and } 0 \leqslant z' \leqslant 1 - \frac{r}{a}\right\}.$$

Moreover, $\dfrac{\partial(x,y,z)}{\partial(r,\theta,z')} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & h \end{vmatrix} = hr$. Therefore,

$$\begin{aligned}\int_R x\,d(x,y,z) &= \int_0^{\frac{\pi}{2}}\int_0^a\int_0^{1-\frac{r}{a}} r\cos\theta hr dz'drd\theta = h\int_0^{\frac{\pi}{2}}\int_0^a\int_0^{1-\frac{r}{a}} r^2\cos\theta dz'drd\theta \\ &= h\int_0^a (r^2 - \frac{r^3}{a})dr = h(\frac{r^3}{3} - \frac{r^4}{4a})\Big|_{r=0}^{r=a} = \frac{ha^3}{12}\end{aligned}$$

and

$$\begin{aligned}\int_R z\,d(x,y,z) &= \int_0^{\frac{\pi}{2}}\int_0^a\int_0^{1-\frac{r}{a}} hz'hr dz'drd\theta = \frac{h^2\pi}{2}\int_0^a\int_0^{1-\frac{r}{a}} z'rdz'dr \\ &= \frac{h^2\pi}{4}\int_0^a \big(1 - \frac{r}{a}\big)^2 rdr = \frac{h^2\pi}{4}(\frac{r^2}{2} - \frac{2r^3}{3a} + \frac{r^4}{4a^2})\Big|_{r=0}^{r=a} = \frac{h^2a^2\pi}{48}.\end{aligned}$$

**Problem 5.** (15%) Find the volume of the region under the surface $z = x^2 \sin(y^4)$ and above the triangle in the $xy$-plane with vertices $(0,0)$, $(0,\pi^{1/4})$ and $(\pi^{1/4},\pi^{1/4})$.

*Solution*: Let $T$ denote the triangle with vertices $(0,0)$, $(0,\pi^{1/4})$ and $(\pi^{1/4},\pi^{1/4})$. Then the volume under consideration is $\displaystyle\int_T x^2 \sin(y^4)\, d(x,y)$.

**Solution 1**: Since $T = \big\{(x,y) \in \mathbb{R}^2 \,\big|\, 0 \leqslant y \leqslant \pi^{1/4}, \big\}$, we have

$$\int_T x^2 \sin(y^4)\, d(x,y) = \int_0^{\pi^{1/4}} \int_0^y x^2 \sin(y^4)\, dxdy = \frac{1}{3}\int_0^{\pi^{1/4}} y^3 \sin(y^4)\, dy = -\frac{1}{12}\cos(y^4)\Big|_{y=0}^{y=\pi^{1/4}}$$
$$= -\frac{1}{12}(\cos\pi - \cos 0) = \frac{1}{6}\,.$$

**Solution 2**: Making change of variable $u = x^2$ and $v = y^4$, since

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial(u,v)}{\partial(x,y)}^{-1} = \begin{vmatrix} 2x & 0 \\ 0 & 4y^3 \end{vmatrix}^{-1} = \frac{1}{8xy^3}\Big|_{(x,y)=(\sqrt{u},\sqrt[4]{v})} = \frac{1}{8\sqrt{u}v^{\frac{3}{4}}}$$

and $T = \big\{(u,v) \,\big|\, 0 \leqslant v \leqslant \pi, 0 \leqslant u \leqslant \sqrt{v}\big\}$, we find that

$$\int_T x^2 \sin(y^4)\, d(x,y) = \frac{1}{8}\int_0^{\pi}\int_0^{\sqrt{v}} u^{\frac{1}{2}} v^{-\frac{3}{4}} \sin v\, dudv = \frac{1}{12}\int_0^{\pi} \sin v\, dv = \frac{1}{6}\,.$$

**Problem 6.** (15%) Find the volume of the region lying inside all three of the circular cylinders $x^2+y^2=a^2$, $x^2+z^2=a^2$ and $y^2+z^2=a^2$.

*Solution*: Let $R$ denote the region

$$R=\left\{(x,y)\in\mathbb{R}^2 \,\middle|\, x^2+y^2\leqslant a^2, 0\leqslant y\leqslant x\right\}.$$

By symmetry, the desired volume can be computed by $16\int_R \sqrt{a^2-x^2}\,d(x,y)$. Using the polar coordinates, we find that

$$\begin{aligned}
16\int_R \sqrt{a^2-x^2}\,d(x,y) &= 16\int_0^{\frac{\pi}{4}}\int_0^a \sqrt{a^2-r^2\cos^2\theta}\,r\,dr\,d\theta = 16\int_0^{\frac{\pi}{4}} \frac{2}{3}\frac{(a^2-r^2\cos^2\theta)^{\frac{3}{2}}}{-2\cos^2\theta}\Big|_{r=0}^{r=a} d\theta \\
&= \frac{16}{3}\int_0^{\frac{\pi}{4}} \frac{a^3(1-\sin^3\theta)}{\cos^2\theta}d\theta = \frac{16a^3}{3}\int_0^{\frac{\pi}{4}}\Big(\sec^2\theta-\frac{\sin\theta(1-\cos^2\theta)}{\cos^2\theta}\Big)d\theta \\
&= \frac{16a^3}{3}\int_0^{\frac{\pi}{4}}\big(\sec^2\theta-\sec\theta\tan\theta+\sin\theta\big)d\theta \\
&= \frac{16a^3}{3}\big(\tan\theta-\sec\theta-\cos\theta\big)\Big|_{\theta=0}^{\theta=\frac{\pi}{4}} = \frac{16a^3}{3}\Big[1-\sqrt{2}-\frac{\sqrt{2}}{2}-(0-1-1)\Big] \\
&= 8a^3(2-\sqrt{2}).
\end{aligned}$$