Vector Analysis MA2014-* Final Exam

National Central University, Jan. 14 2016

Problem 1. Let $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field given by $\boldsymbol{F}(x, y, z) = \big(M(x, y), N(x, y), 0\big)$, where $M, N : \mathbb{R}^2 \to \mathbb{R}$ be $\mathscr{C}^1$-functions, and C be a simple closed plane curve $r(t) = \big(x(t), y(t), 0\big)$ for $t \in [a, b]$ and $\boldsymbol{r}(t)$ moves counter-clockwise as t increases. Suppose that C is the boundary of a region $\mathrm{R} \subseteq \mathbb{R}^2$.

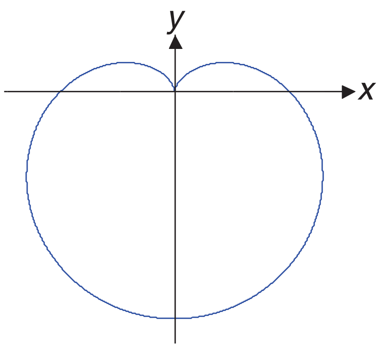
1. (10%) Show the Green theorem

$$\oint_C (M, N, 0) \cdot d\boldsymbol{r} = \iint_{\mathrm{R}} (N_x - M_y)\, d\mathbb{A}\,.$$

2. (10%) Use $M(x, y) = -y$ and $N(x, y) = x$ to show that the area of R is given by

$$\mathcal{A}(\mathrm{R}) = \frac{1}{2}\int_a^b \big(x(t)y'(\mathrm{t}) - y(t)x'(t)\big)\, dt\,.$$

3. (15%) Compute the area enclosed by the Cardioid which has a polar representation $r = (1 - \sin\theta)$ with $\theta \in [0, 2\pi]$.



Proof. 1. Since $\mathrm{curl}\boldsymbol{F} = (0, 0, N_x - M_y)$, the Stokes theorem implies that

$$\oint_C (M, N, 0) \cdot d\boldsymbol{r} = \oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_R \mathrm{curl}\boldsymbol{F} \cdot \mathbf{N}\, d\mathbb{A} = \iint_R (0, 0, N_x - M_y) \cdot (0, 0, 1)\, d\mathbb{A}$$
$$= \iint_R (N_x - M_y)\, d\mathbb{A}\,.$$

2. Letting $M(x, y) = -y$ and $N(x, y) = x$, we have $N_x(x, y) - M_y(x, y) = 2$; thus 1 implies that

$$\frac{1}{2}\int_a^b \big(x(t)y'(t) - y(t)x'(t)\big)\, dt = \frac{1}{2}\oint_C (M, N, 0) \cdot dr = \frac{1}{2}\iint_{\mathrm{R}} 2d\mathbb{A} = \mathcal{A}(\mathrm{R})\,.$$

3. A parametrization of the Cardioid is

$$\boldsymbol{r}(t) = \big(x(t), y(t)\big) = \big((1 - \sin t)\cos t, (1 - \sin t)\sin t\big) \qquad t \in [0, 2\pi]\,.$$

Then 2 implies that the area enclosed by the Cardioid is

$$\begin{aligned}
&\frac{1}{2}\int_0^{2\pi} \Big[(1-\sin t)\cos t\big(-\cos t\sin t + (1-\sin t)\cos t\big) \\
&\qquad - (1-\sin t)\sin t\big(-\cos^2 t - (1-\sin t)\sin t\big)\Big]dt \\
&\qquad = \frac{1}{2}\int_0^{2\pi}(1-\sin t)\big[\cos^2 t - 2\sin t\cos^2 t + \sin t\cos^2 t + \sin^2 t - \sin^3 t\big]dt \\
&\qquad = \frac{1}{2}\int_0^{2\pi}(1-\sin t)(1-\sin t\cos^2 t - \sin^3 t)dt = \frac{1}{2}\int_0^{2\pi}(1-\sin t)^2 dt = \frac{3\pi}{2}\,.
\end{aligned}$$

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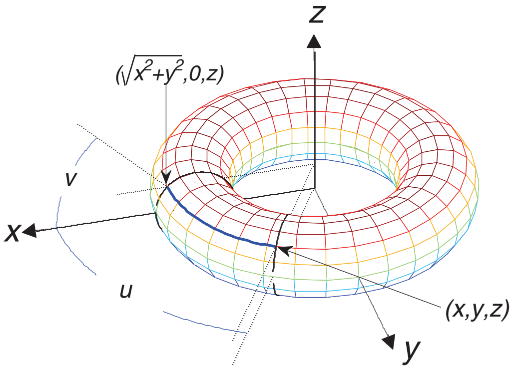
Problem 2. Let D be the solid region enclosed by the torus $\mathbb{T}^2 \equiv \psi([0,2\pi]\times[0,2\pi])$, where ψ is given by

$$\psi(u,v) = \big((2+\cos v)\cos u, (2+\cos v)\sin u, \sin v\big)\,.$$



1. (10%) Compute $\psi_u \times \psi_v$ as well as $\|\psi_u \times \psi_v\|_{\mathbb{R}^3}$.
2. (10%) Determine which one of the two vectors $\dfrac{\psi_u\times\psi_v}{\|\psi_u\times\psi_v\|_{\mathbb{R}^3}}\circ\psi^{-1}$ and $\dfrac{\psi_v\times\psi_u}{\|\psi_u\times\psi_v\|_{\mathbb{R}^3}}\circ\psi^{-1}$ is the outward-pointing unit normal $\mathbf{N}$ on $\mathbb{T}^2$.
3. (15%) Let $\boldsymbol{F}:\mathbb{R}^3\to\mathbb{R}^3$ be given by $\boldsymbol{F}(x,y,z) = (y+z^2, xz, e^x\sin y)$. Use the divergence theorem to compute the flux integral $\displaystyle\iint_{\mathbb{T}^2}\boldsymbol{F}\cdot\mathbf{N}\,dS$.

Solution:

1. Since

$$\begin{aligned}
\psi_u(u,v) &= \big(-(2+\cos v)\sin u, (2+\cos v)\cos u, 0\big)\,,\\
\psi_v(u,v) &= \big(-\sin v\cos u, -\sin v\sin u, \cos v\big)\,,
\end{aligned}$$

we have

$$\begin{aligned}
&(\psi_u\times\psi_v)(u,v)\\
&\quad = \big((2+\cos v)\cos u\cos v, (2+\cos v)\sin u\cos v, (2+\cos v)(\sin^2 u\sin v + \sin v\cos^2 u)\big)\\
&\quad = (2+\cos v)\big(\cos u\cos v, \sin u\cos v, \sin v\big)\,.
\end{aligned}$$

Therefore, $\|\psi_u\times\psi_v)(u,v)\|_{\mathbb{R}^3} = (2+\cos v)$.

2. We note that the outward-pointing unit normal at $\psi(0,0) = (3,0,0)$ is $(1,0,0)$. Since

$$\frac{\psi_u \times \psi_v}{\|\psi_u \times \psi_v\|_{\mathbb{R}^3}}(0,0) = (1,0,0)\,,$$

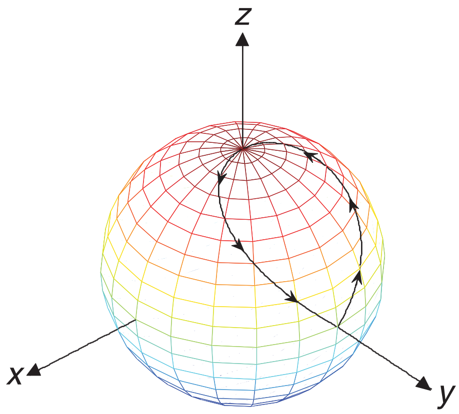
we find that $\dfrac{\psi_u \times \psi_v}{\|\psi_u \times \psi_v\|_{\mathbb{R}^3}} \circ \psi^{-1}$ is the outward-pointing unit normal $\mathbf{N}$ on $\mathbb{T}^2$.

3. Since $\operatorname{div}\boldsymbol{F} = 0$, by the divergence theorem,

$$\iint\limits_{\mathbb{T}^2} \boldsymbol{F} \cdot \mathbf{N}\, dS = \int_{\mathrm{D}} \operatorname{div}\boldsymbol{F} dx = 0\,.$$

Problem 3. Let C be a smooth curve parametrized by

$$\boldsymbol{r}(t) = (\cos t \sin t, \sin t \sin t, \cos t)\,, \qquad t \in \Big[-\frac{\pi}{2}, \frac{\pi}{2}\Big]\,.$$



1. (5%) Show that C is a curve on the sphere $\mathbb{S}^2$ centered at the origin with radius one.

2. (10%) Let $\psi : \mathrm{R} \equiv (0, 2\pi) \times (0, \pi) \to \mathbb{R}^3$ given by $\psi(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$ be a local parametrization of $\mathbb{S}^2$. Find a curve on R such that the image of this curve under ψ is C (with the north pole of the sphere being excluded).

3. (15%) The curve C divides $\mathbb{S}^2$ into two parts, and let Σ be the part with smaller area. Find the area of Σ.

4. (15%) Let $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field given by $\boldsymbol{F}(x, y, z) = (y, -x, 0)$. Compute the line integral $\displaystyle\oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$ using the definition of line integral.

5. (15%) Use the Stokes theorem to find the line integral $\displaystyle\oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$.

Proof. 1. Let $(x, y, z) \in C$. Then $x = \cos t \sin t$, $y = \sin t \sin t, z = \cos t$ for some $t \in \big[-\frac{\pi}{2}, \frac{\pi}{2}\big]$. Therefore,

$$x^2 + y^2 + z^2 = \cos^2 t \sin^2 t + \sin^2 t \sin^2 t + \cos^2 t = \sin^2 t + \cos^2 t = 1$$

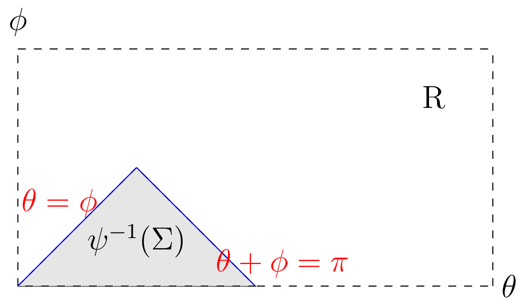
which implies that $(x, y, z) \in \mathbb{S}^2$.

2. Let $\big(\theta(t),\phi(t)\big) \in \mathrm{R}$ be such that

$$\psi\big(\theta(t),\phi(t)\big) = (\cos t \sin t, \sin t \sin t, \cos t) \qquad \forall\, t \in \big[-\frac{\pi}{2},\frac{\pi}{2}\big].$$

For $t \in \big[0,\frac{\pi}{2}\big]$, the identity $\cos t = \cos\phi(t)$ implies that $\phi(t) = t$; thus the identities $\cos t \sin t = \cos\theta(t)\cos\phi(t)$ and $\sin t \sin t = \sin\theta(t)\cos\phi(t)$ further imply that $\theta(t) = t$.

On the other hand, for $t \in \big[-\frac{\pi}{2},0\big]$, the identity $\cos t = \cos\phi(t)$, where $\phi(t) \in (0,\pi)$, implies that $\phi(t) = -t$; thus the identities $\cos t \sin t = \cos\theta(t)\sin\phi(t)$ and $\sin t \sin t = \sin\theta(t)\sin\phi(t)$ further imply that $\theta(t) = \pi + t$.



3. First, we note that the first fundamental form associated with $\{\mathrm{R},\psi\}$ is

$$\begin{aligned}
\mathrm{g}(u,v) &= \big\|(\psi_\theta \times \psi_\phi)(u,v)\big\|^2_{\mathbb{R}^3} \\
&= \big\|(-\sin\theta\sin\phi, \cos\theta\sin\phi, 0) \times (\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi)\big\|^2_{\mathbb{R}^3} \\
&= \big\|(-\cos\theta\sin^2\phi, -\sin\theta\sin^2\phi, -(\sin^2\theta+\cos^2\theta)\sin\phi\cos\phi)\big\|^2_{\mathbb{R}^3} \\
&= (\cos^2\theta+\sin^2\theta)\sin^4\phi + \sin^2\phi\cos^2\phi = \sin^2\phi\,.
\end{aligned}$$

Therefore, the area of the desired surface is

$$\begin{aligned}
\int_\Sigma dS &= \int_{\psi^{-1}(\Sigma)} \sqrt{\mathrm{g}}\, d\mathbb{A} = \int_0^{\frac{\pi}{2}} \int_\phi^{\pi-\phi} \sin\phi\, d\theta d\phi = \int_0^{\frac{\pi}{2}} (\pi - 2\phi)\sin\phi\, d\phi \\
&= \big(-\pi\cos\phi + 2\phi\cos\phi - 2\sin\phi\big)\Big|_{\phi=0}^{\phi=\frac{\pi}{2}} = \pi - 2\,.
\end{aligned}$$

4. By the definition of line integral,

$$\begin{aligned}
\oint_C \boldsymbol{F}\cdot dr &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin^2 t, -\cos t\sin t, 0)\cdot\big(\cos^2 t - \sin^2 t, 2\sin t\cos t, -\sin t\big)dt \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \big(\sin^2 t\cos^2 t - \sin^4 t - 2\sin^2 t\cos^2 t\big)dt \\
&= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 t dt = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1-\cos 2t}{2} dt = -\Big(\frac{t}{2} - \frac{\sin 2t}{4}\Big)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = -\frac{\pi}{2}\,.
\end{aligned}$$

5. Since curl$\boldsymbol{F} = (0,0,-2)$, the Stokes theorem implies that

$$\begin{aligned}\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} &= \int_\Sigma (0,0,-2)\cdot \mathbf{N}\,dS = \int_{\psi^{-1}(\Sigma)} -2\cos\phi\sin\phi\,d(\theta,\phi) = -2\int_0^{\frac{\pi}{2}}\int_\phi^{\pi-\phi} \sin\phi\cos\phi\,d\theta d\phi \\ &= -\int_0^{\frac{\pi}{2}} (\pi-2\phi)\sin 2\phi\,d\phi = \big(\frac{\pi}{2}\cos 2\phi - \phi\cos 2\phi + \frac{1}{2}\sin 2\phi\big)\Big|_{\phi=0}^{\phi=\frac{\pi}{2}} \\ &= -\frac{\pi}{2} - \frac{\pi}{2} + \frac{\pi}{2} = -\frac{\pi}{2}\,.\end{aligned}$$

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