量子計算的數學基礎 MA5501

Homework Assignment 1

Due Mar. 20. 2023

在此次作業中所提到的 2-qubit 量子電路圖，上面一條線所對應的 qubit 視為該二位元數的最高位元，而下面一條線所對應的 qubit 視為該二位元數的最低位元，亦即若上面那條線在某一時間所對應的 qubit 之量子態為 $\alpha_0|0\rangle+\alpha_1|1\rangle$ 且下面那條線所對應的 qubit 之量子態為 $\beta_0|0\rangle+\beta_1|1\rangle$，則在該時間二者合一來看其所表示的量子態為

$$\big(\alpha_0|0\rangle+\alpha_1|1\rangle\big)\otimes\big(\beta_0|0\rangle+\beta_1|1\rangle\big)=\alpha_0\beta_0|0\rangle+\alpha 0\beta_1|1\rangle+\alpha_1\beta_0|2\rangle+\alpha_1\beta_1|3\rangle\,.$$

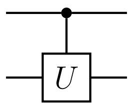
另外，為了以矩陣來表示量子邏輯閘，對於單一 qubit 量子態我們使用有序基底 $\{|0\rangle,|1\rangle\}$ 而對於 2-qubit 量子態我們使用有序基底 $\{|0\rangle,|1\rangle,|2\rangle,|3\rangle\}$（即 $\{|0\rangle\otimes|0\rangle,|0\rangle\otimes|1\rangle,|1\rangle\otimes|0\rangle,|1\rangle\otimes|1\rangle\}$），亦即量子態 $\alpha_0|0\rangle+\alpha_1|1\rangle$ 以及 $\beta_0|0\rangle+\beta_1|1\rangle+\beta_2|2\rangle+\beta_3|3\rangle$ 以矩陣表示可分別寫成

$$\begin{bmatrix}\alpha_0\\ \alpha_1\end{bmatrix} \qquad \text{與} \qquad \begin{bmatrix}\beta_0\\ \beta_1\\ \beta_2\\ \beta_3\end{bmatrix}.$$

Problem 1. 給定一個作用在 1-qubit 量子態上的量子邏輯閘 U，其定義為

$$U\big(\alpha_0|0\rangle+\alpha_1|1\rangle\big)=\big(\alpha_0u_{11}+\alpha_1u_{12}\big)|0\rangle+\big(\alpha_0u_{21}+\alpha_1u_{22}\big)|1\rangle$$

1. 於課堂上我們提到了 Controlled-U 的量子邏輯閘，其量子電路符號為



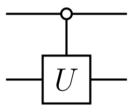
它是一個作用在 2-qubits 上的邏輯閘，其作用是

$$|0b\rangle\mapsto|0b\rangle \quad \text{且} \quad |1b\rangle\mapsto|1\rangle\otimes U|b\rangle \qquad \forall\, b\in\{0,1\}\,.$$

證明上述的 Controlled-U 閘的矩陣表示為

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&u_{11}&u_{12}\\0&0&u_{21}&u_{22}\end{bmatrix}.$$

2. 考慮另一種 2-qubit 量子邏輯閘，其量子電路表示法為



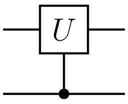
且其作用是

$$|0b\rangle\mapsto|0\rangle\otimes U|b\rangle \quad \text{且} \quad |1b\rangle\mapsto|1b\rangle \qquad \forall\, b\in\{0,1\}\,.$$

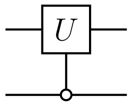
試證明此邏輯閘的矩陣表示是

$$\begin{bmatrix}u_{11}&u_{12}&0&0\\u_{21}&u_{22}&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}.$$

3. 考慮上述的兩個 2-qubit 量子邏輯閘中的 control quit 與 target quib 對調的情況。證明量子電路分別為



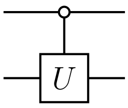
與



的矩陣表示分別為

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & u_{11} & 0 & u_{12} \\ 0 & 0 & 1 & 0 \\ 0 & u_{21} & 0 & u_{22} \end{bmatrix} \quad \text{及} \quad \begin{bmatrix} u_{11} & 0 & u_{12} & 0 \\ 0 & 1 & 0 & 0 \\ u_{21} & 0 & u_{22} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

4. 以所給定的方式證明以下兩量子電路作用相同：



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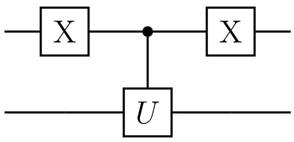

(a) 假設上面那條線的輸入是 $\alpha_0|0\rangle + \alpha_1|1\rangle$ 而下面那條線的輸入是 $\beta_0|0\rangle + \beta_1|1\rangle$，驗證兩個量子電路輸出一致的量子態。

(b) 求得 $\mathrm{X} \otimes \mathrm{I}$ 之矩陣表示，然後計算右方電路三個串聯的量子邏輯閘其對應的矩陣表示與左方電路中的量子邏輯閘之矩陣表示相同。

Solution. 1. Let CU denote the controlled-U gate given in the problem. Then

$$\begin{aligned}
&\mathrm{CU}\big(\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle\big) \\
&\quad = \mathrm{CU}|0\rangle \otimes \big(\alpha_{00}|0\rangle + \alpha_{01}|1\rangle\big) + \mathrm{CU}|1\rangle \otimes \big(\alpha_{10}|0\rangle + \alpha_{11}|1\rangle\big) \\
&\quad = |0\rangle \otimes \big(\alpha_{00}|0\rangle + \alpha_{01}|1\rangle\big) + |1\rangle \otimes U\big(\alpha_{10}|0\rangle + \alpha_{11}|1\rangle\big) \\
&\quad = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + |1\rangle \otimes \big[\big(\alpha_{10}u_{11} + \alpha_{11}u_{12}\big)|0\rangle + \big(\alpha_{10}u_{21} + \alpha_{11}u_{22}\big)|1\rangle\big)\big] \\
&\quad = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \big(\alpha_{10}u_{11} + \alpha_{11}u_{12}\big)|10\rangle + \big(\alpha_{10}u_{21} + \alpha_{11}u_{22}\big)|11\rangle .
\end{aligned}$$

If [CU] is the matrix representation of CU w.r.t. to the computational basis $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$, then

$$[\mathrm{CU}] \begin{bmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{bmatrix} = \begin{bmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10}u_{11} + \alpha_{11}u_{12} \\ \alpha_{10}u_{21} + \alpha_{11}u_{22} \end{bmatrix}$$

for all $\alpha_{00}, \alpha_{01}, \alpha_{10}, \alpha_{11} \in \mathbb{C}$. Therefore,

$$[\mathrm{CU}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u_{11} & u_{12} \\ 0 & 0 & u_{21} & u_{22} \end{bmatrix}.$$

2. Let OCU denote the 2-qubit gate given in the problem. Then

$$\begin{aligned}
&\mathrm{OCU}\big(\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle\big) \\
&\quad = \mathrm{OCU}|0\rangle \otimes \big(\alpha_{00}|0\rangle + \alpha_{01}|1\rangle\big) + \mathrm{OCU}|1\rangle \otimes \big(\alpha_{10}|0\rangle + \alpha_{11}|1\rangle\big) \\
&\quad = |0\rangle \otimes U\big(\alpha_{00}|0\rangle + \alpha_{01}|1\rangle\big) + |1\rangle \otimes \big(\alpha_{10}|0\rangle + \alpha_{11}|1\rangle\big) \\
&\quad = |0\rangle \otimes \big[\big(\alpha_{00}u_{11} + \alpha_{01}u_{12}\big)|0\rangle + \big(\alpha_{00}u_{21} + \alpha_{01}u_{22}\big)|1\rangle\big] + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle\big) \\
&\quad = \big(\alpha_{00}u_{11} + \alpha_{01}u_{12}\big)|00\rangle + \big(\alpha_{00}u_{21} + \alpha_{01}u_{22}\big)|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle\big)\,.
\end{aligned}$$

If [OCU] is the matrix representation of CU w.r.t. to the computational basis $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$, then

$$[\mathrm{OCU}]\begin{bmatrix}\alpha_{00}\\ \alpha_{01}\\ \alpha_{10}\\ \alpha_{11}\end{bmatrix} = \begin{bmatrix}\alpha_{00}u_{11} + \alpha_{01}u_{12}\\ \alpha_{00}u_{21} + \alpha_{01}u_{22}\\ \alpha_{10}\\ \alpha_{11}\end{bmatrix}$$

for all $\alpha_{00}, \alpha_{01}, \alpha_{10}, \alpha_{11} \in \mathbb{C}$. Therefore,

$$[\mathrm{CU}] = \begin{bmatrix} u_{11} & u_{12} & 0 & 0\\ u_{21} & u_{22} & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

3. Let UC and UOC denote the first and the second 2-qubit gate given in the problem, respectively. Then

$$\begin{aligned}
&\mathrm{UC}\big(\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle\big) \\
&\quad = \mathrm{UC}\big(\alpha_{00}|0\rangle + \alpha_{10}|1\rangle\big) \otimes |0\rangle + \mathrm{UC}\big(\alpha_{01}|0\rangle + \alpha_{11}|1\rangle\big) \otimes |1\rangle \\
&\quad = \big(\alpha_{00}|0\rangle + \alpha_{10}|1\rangle\big) \otimes |0\rangle + \big[\big(\alpha_{01}u_{11} + \alpha_{11}u_{22}\big)|0\rangle + \big(\alpha_{01}u_{21} + \alpha_{11}u_{12}\big)|1\rangle\big] \otimes |1\rangle \\
&\quad = \alpha_{00}|00\rangle + \alpha_{10}|10\rangle + \big(\alpha_{01}u_{11} + \alpha_{11}u_{22}\big)|01\rangle + \big(\alpha_{01}u_{21} + \alpha_{11}u_{12}\big)|11\rangle
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{UOC}\big(\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle\big) \\
&\quad = \mathrm{UOC}\big(\alpha_{00}|0\rangle + \alpha_{10}|1\rangle\big) \otimes |0\rangle + \mathrm{UOC}\big(\alpha_{01}|0\rangle + \alpha_{11}|1\rangle\big) \otimes |1\rangle \\
&\quad = U\big(\alpha_{00}|0\rangle + \alpha_{10}|1\rangle\big) \otimes |0\rangle + \big(\alpha_{01}|0\rangle + \alpha_{11}|1\rangle\big) \otimes |1\rangle \\
&\quad = \big[\big(\alpha_{00}u_{11} + \alpha_{10}u_{12}\big)|0\rangle + \big(\alpha_{00}u_{21} + \alpha_{10}u_{22}\big)|1\rangle\big] \otimes |0\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle\big) \\
&\quad = \big(\alpha_{00}u_{11} + \alpha_{10}u_{12}\big)|00\rangle + \big(\alpha_{00}u_{21} + \alpha_{10}u_{22}\big)|10\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle\big)\,.
\end{aligned}$$

If [UC] and [UOC] denote the matrix representation of UC and UOC w.r.t. the computational basis $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$, respectively, then

$$[\mathrm{UC}]\begin{bmatrix}\alpha_{00}\\ \alpha_{01}\\ \alpha_{10}\\ \alpha_{11}\end{bmatrix} = \begin{bmatrix}\alpha_{00}\\ \alpha_{10}\\ \alpha_{01}u_{11} + \alpha_{11}u_{12}\\ \alpha_{01}u_{21} + \alpha_{11}u_{22}\end{bmatrix}$$

and

$$[\mathrm{UOC}]\begin{bmatrix}\alpha_{00}\\ \alpha_{01}\\ \alpha_{10}\\ \alpha_{11}\end{bmatrix} = \begin{bmatrix}\alpha_{00}u_{11} + \alpha_{10}u_{12}\\ \alpha_{00}u_{21} + \alpha_{10}u_{22}\\ \alpha_{10}\\ \alpha_{11}\end{bmatrix}.$$

Therefore,

$$[\mathrm{UC}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & u_{11} & 0 & u_{12} \\ 0 & 0 & 1 & 0 \\ 0 & u_{21} & 0 & u_{22} \end{bmatrix} \qquad \text{and} \qquad [\mathrm{UOC}] = \begin{bmatrix} u_{11} & 0 & u_{12} & 0 \\ 0 & 1 & 0 & 0 \\ u_{21} & 0 & u_{22} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

4. (a) Given the input state $|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$, by the computation of OCU above we find that

$$\begin{aligned} \mathrm{OCU}|\psi\rangle &= \mathrm{OCU}\big(\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle\big) \\ &= \big(\alpha_{00}u_{11} + \alpha_{01}u_{12}\big)|00\rangle + \big(\alpha_{00}u_{21} + \alpha_{01}u_{22}\big)|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle\big) . \end{aligned}$$

On the other hand, by the linearity of $\mathrm{X} \otimes \mathrm{I}$,

$$\begin{aligned} (\mathrm{X}\otimes\mathrm{I})|\psi\rangle &= (\mathrm{X}\otimes\mathrm{I})\big(\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle\big) \\ &= \alpha_{00}(\mathrm{X}\otimes\mathrm{I})|00\rangle + \alpha_{01}(\mathrm{X}\otimes\mathrm{I})|01\rangle + \alpha_{10}(\mathrm{X}\otimes\mathrm{I})|10\rangle + (\mathrm{X}\otimes\mathrm{I})\alpha_{11}|11\rangle \\ &= \alpha_{00}|10\rangle + \alpha_{01}|11\rangle + \alpha_{10}|00\rangle + \alpha_{11}|01\rangle \\ &= \alpha_{10}|00\rangle + \alpha_{11}|01\rangle + \alpha_{00}|10\rangle + \alpha_{01}|11\rangle , \end{aligned} \tag{0.1}$$

so that by the computation of CU above,

$$\begin{aligned} &\mathrm{CU}(\mathrm{X}\otimes\mathrm{I})|\psi\rangle \\ &\quad = \alpha_{10}|00\rangle + \alpha_{11}|01\rangle + \big(\alpha_{00}u_{11} + \alpha_{01}u_{12}\big)|10\rangle + \big(\alpha_{00}u_{21} + \alpha_{01}u_{22}\big)|11\rangle . \end{aligned}$$

Passing the quantum state above through the gate $\mathrm{X}\otimes\mathrm{I}$, we find that

$$\begin{aligned} &(\mathrm{X}\otimes\mathrm{I})\mathrm{CU}(\mathrm{X}\otimes\mathrm{I})|\psi\rangle \\ &\quad = (\mathrm{X}\otimes\mathrm{I})\big[\alpha_{10}|00\rangle + \alpha_{11}|01\rangle + \big(\alpha_{00}u_{11} + \alpha_{01}u_{12}\big)|10\rangle + \big(\alpha_{00}u_{21} + \alpha_{01}u_{22}\big)|11\rangle\big] \\ &\quad = \alpha_{10}(\mathrm{X}\otimes\mathrm{I})|00\rangle + \alpha_{11}(\mathrm{X}\otimes\mathrm{I})|01\rangle + \big(\alpha_{00}u_{11} + \alpha_{01}u_{12}\big)(\mathrm{X}\otimes\mathrm{I})|10\rangle \\ &\qquad + \big(\alpha_{00}u_{21} + \alpha_{01}u_{22}\big)(\mathrm{X}\otimes\mathrm{I})|11\rangle \\ &\quad = \alpha_{10}|10\rangle + \alpha_{11}|11\rangle + \big(\alpha_{00}u_{11} + \alpha_{01}u_{12}\big)|00\rangle + \big(\alpha_{00}u_{21} + \alpha_{01}u_{22}\big)|01\rangle \\ &\quad = \big(\alpha_{00}u_{11} + \alpha_{01}u_{12}\big)|00\rangle + \big(\alpha_{00}u_{21} + \alpha_{01}u_{22}\big)|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle . \end{aligned}$$

Therefore, for any given 2-qubit state $|\psi\rangle$, we have

$$\mathrm{OCU}|\psi\rangle = (\mathrm{X}\otimes\mathrm{I})\mathrm{CU}(\mathrm{X}\otimes\mathrm{I})|\psi\rangle$$

so we conclude that $\mathrm{OCU} = (\mathrm{X}\otimes\mathrm{I})\mathrm{CU}(\mathrm{X}\otimes\mathrm{I})$.

(b) First we find the matrix representation of $\mathrm{X}\otimes\mathrm{I}$. If $[\mathrm{X}\otimes\mathrm{I}]$ denotes the matrix representation of $\mathrm{X}\otimes\mathrm{I}$ w.r.t. the computational basis, using (0.1) we obtain that

$$[\mathrm{X}\otimes\mathrm{I}] \begin{bmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{bmatrix} = \begin{bmatrix} \alpha_{10} \\ \alpha_{11} \\ \alpha_{00} \\ \alpha_{01} \end{bmatrix} .$$

so we have

$$[\mathrm{X} \otimes \mathrm{I}] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Having obtained the matrix representation of $\mathrm{X} \otimes \mathrm{I}$ w.r.t. the computational basis, together with part 1 we find that the matrix representation of $(\mathrm{X} \otimes \mathrm{I})\mathrm{CU}(\mathrm{X} \otimes \mathrm{I})$ w.r.t. the computational basis is given by

$$\begin{aligned} \big[(\mathrm{X} \otimes \mathrm{I})\mathrm{CU}(\mathrm{X} \otimes \mathrm{I})\big] &= [\mathrm{X} \otimes \mathrm{I}][\mathrm{CU}][\mathrm{X} \otimes \mathrm{I}] \\ &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u_{11} & u_{12} \\ 0 & 0 & u_{21} & u_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ u_{11} & u_{12} & 0 & 0 \\ u_{21} & u_{22} & 0 & 0 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & 0 & 0 \\ u_{21} & u_{22} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

which is the same as the matrix representation of OCU (from part 2). □