# (Matlab) Assignment 2

Due Apr. 15. 2022

**Problem 1.** For matrices $A = [a_{k\ell}]$ and $B = [b_{k\ell}]$ of the same size $m \times n$, define the Hadamard product of $A$ and $B$, denoted by $A \odot B$, as an $m \times n$ matrix whose $(k, \ell)$-entry is give by $a_{k\ell} b_{k\ell}$; that is,

$$C = A \odot B\,, \quad C = [c_{k\ell}]\,, \quad c_{k\ell} = a_{k\ell} b_{k\ell}\,. \tag{0.1}$$

In matlab®, the Hadamard product of $A$ and $B$ can be computed by $A \odot B = A .* B$. **In the following, we will always use $.*$ to denote the Hadamard product.**

Let $\mathrm{H}_n$ be the **unnormalized** Hadamard matrix whose $(k, \ell)$-entry is given by $(-1)^{(k-1)\bullet(\ell-1)}$, and $\boldsymbol{r}_j$ be the $(j+1)$-th row of $\mathrm{H}_n$. Define $\varphi : \{0, 1\}^n \to \{\boldsymbol{r}_0, \boldsymbol{r}_1, \cdots, \boldsymbol{r}_{2^n-1}\}$ by

$$\varphi(j_1, j_2, \cdots, j_n) = \boldsymbol{r}_j \quad \text{if} \;\; j = (j_1 j_2 \cdots j_n)_2\,.$$

For example, for the case $n = 2$ the map $\varphi$ is given by

$$\varphi : \left\{ \begin{array}{ccc} (0,0) & \mapsto & \boldsymbol{r}_0 = \\ (0,1) & \mapsto & \boldsymbol{r}_1 = \\ (1,0) & \mapsto & \boldsymbol{r}_2 = \\ (1,1) & \mapsto & \boldsymbol{r}_3 = \end{array} \right. \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \equiv \mathrm{H}_2\ . \tag{$\star$}$$

Show that $\varphi : (\{0, 1\}^n, \oplus) \to \big(\{\boldsymbol{r}_0, \boldsymbol{r}_1, \cdots, \boldsymbol{r}_{2^n-1}\}, .*\big)$ is a group isomorphism, where $\oplus$ is the element-wise addition in $\mathbb{Z}_2$; that is,

$$(x_1, x_2, \cdots, x_n) \oplus (y_1, y_2, \cdots, y_n) = (x_1 \oplus y_1, x_2 \oplus y_2, \cdots, x_n \oplus y_n)\,.$$

In other words, show that $\varphi : \{0, 1\}^n \to \{\boldsymbol{r}_0, \boldsymbol{r}_1, \cdots, \boldsymbol{r}_{2^n-1}\}$ defined above is a bijection and

$$\varphi\big((k_1, \cdots, k_n) \oplus (\ell_1, \cdots, \ell_n)\big) = \boldsymbol{r}_k .* \boldsymbol{r}_\ell \qquad \forall\, k = (k_1 k_2 \cdots k_n)_2 \text{ and } \ell = (\ell_1 \ell_2 \cdots \ell_n)_2\,. \tag{$\diamond$}$$

For example, in the example above $(\star)$ implies that

$$\varphi\big((0,1) \oplus (1,1)\big) = \varphi(1,0) = \boldsymbol{r}_2$$

while

$$\varphi(0,1).*\varphi(1,1) = \boldsymbol{r}_1.*\boldsymbol{r}_3 = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}.*\begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix} = \boldsymbol{r}_2$$

so that $\varphi\big((0,1) \oplus (1,1)\big) = \varphi(0,1).*\varphi(1,1)$.

在此次作業中，證明可以選擇直接（手寫）證明（數學系學生尤其鼓勵這樣做），或是選擇使用 matlab® 程式執行證明。選擇使用 matlab® 程式證明的學生，在程式中要呈現「給定一自然數 $n$ 則可以驗證在 $\mathrm{H}_n$ 上有上述性質（讓電腦跑完所有可能性看看 $(\diamond)$ 是否恆成立）」。