# 最佳化方法與應用二
# MA5038-*

# Chapter 12. Theory of Constrained Optimization

§12.1 Examples

§12.2 Tangent Cone and Constraint Qualifications

§12.3 First-Order Optimality Conditions

§12.4 First-Order Optimality Conditions: Proof

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# Introduction

The second part of the textbook is about minimizing functions **subject to constraints** on the variables:

$$\min_{x\in\mathbb{R}^n} f(x) \quad \text{subject to} \quad \begin{cases} c_i(x) = 0, i \in \mathcal{E}, \\ c_i(x) \geqslant 0, i \in \mathcal{I}, \end{cases} \tag{1}$$

where $f$ and the functions $c_i$ are all smooth, real-valued functions on a subset of $\mathbb{R}^n$, and $\mathcal{I}$ and $\mathcal{E}$ are two finite sets of indices. As before, we call $f$ the objective function, while $c_i$, $i \in \mathcal{E}$, are the equality constraints and $c_i$, $i \in \mathcal{I}$, are the inequality constraints. We define the **feasible set** $\Omega$ by

$$\Omega = \left\{x \,\middle|\, (\forall\, i \in \mathcal{E})(c_i(x) = 0) \text{ and } (\forall\, i \in \mathcal{I})(c_i(x) \geqslant 0)\right\},$$

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In this chapter we derive mathematical characterizations of the solutions of (2). Two types of optimality conditions are discussed:

1. Necessary conditions are conditions that must be satisfied by any solution point (under certain assumptions).
2. Sufficient conditions are those that, if satisfied at a certain point $x_*$, guarantee that $x_*$ is in fact a solution.

Optimality conditions for **unconstrained** optimization problems are:

1. Necessary conditions: Local unconstrained minimizers $x_*$ satisfies that $(\nabla f)(x_*) = 0$ and $(\nabla^2 f)(x_*)$ positive semi-definite.
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# Introduction

- **Local and global solutions**

We have seen already that global solutions are difficult to find even when there are no constraints. The situation may be improved when we add constraints, since the feasible set might exclude many of the local minima and it may be comparatively easy to pick the global minimum from those that remain. However, constraints can also make things more difficult. As an example, consider the problem

$$\min (x_2 + 100)^2 + 0.01 x_1^2 \quad \text{subject to} \quad x_2 - \cos x_1 \geqslant 0 \,,$$

illustrated in Figure 1. Without the constraint, the problem has the unique solution $(0, -100)^{\mathrm{T}}$. With the constraint, there are local solutions near the points

$$x^{(k)} = (k\pi, -1)^{\mathrm{T}} \quad \text{for } k = \pm 1, \pm 3, \pm 5, \cdots .$$

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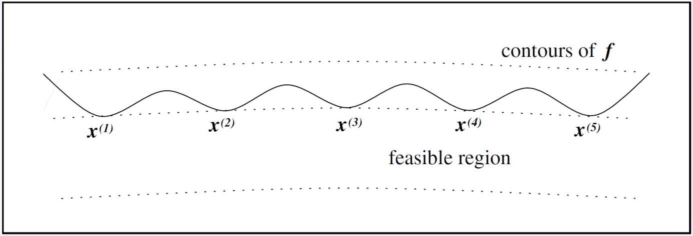


Figure 1: Constrained problem with many isolated local solutions.

# Introduction

Definitions of the different types of local solutions are simple extensions of the corresponding definitions for the unconstrained case.

## Definition

1. A vector $x_*$ is a local solution of the problem (2) if $x_* \in \Omega$ and there is a neighborhood $\mathcal{N}$ of $x_*$ such that $f(x) \geqslant f(x_*)$ for $x \in \mathcal{N} \cap \Omega$.
2. A vector $x_*$ is a strict local solution (also called a strong local solution) if $x_* \in \Omega$ and there is a neighborhood $\mathcal{N}$ of $x_*$ such that $f(x) > f(x_*)$ for all $x \in \mathcal{N} \cap \Omega$ with $x \neq x_*$.
3. A point $x_*$ is an isolated local solution if $x_* \in \Omega$ and there is a neighborhood $\mathcal{N}$ of $x_*$ such that $x_*$ is the only local solution in $\mathcal{N} \cap \Omega$.

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# Introduction

- **Smoothness**

Smoothness of objective functions and constraints is an important issue in characterizing solutions, just as in the unconstrained case. It ensures that the objective function and the constraints all behave in a reasonably predictable way and therefore allows algorithms to make good choices for search directions.

We saw in Chapter 2 that graphs of non-smooth functions contain "kinks" or "jumps" where the smoothness breaks down. If we plot the feasible region for any given constrained optimization problem, we usually observe many kinks and sharp edges. Does this mean that the constraint functions that describe these regions are non-smooth? The answer is often no, because the non-smooth boundaries can often be described by a collection of smooth constraint functions.

# Introduction

For example, Figure 2 shows a diamond-shaped feasible region in $\mathbb{R}^2$ that could be described by the single non-smooth constraint

$$\|x\|_1 \equiv |x_1| + |x_2| \leqslant 1 .$$

It can also be described by the following set of smooth (in fact, linear) constraints:

$$x_1 + x_2 \leqslant 1 , \quad x_1 - x_2 \leqslant 1 , \quad -x_1 + x_2 \leqslant 1 , \quad -x_1 - x_2 \leqslant 1 . \qquad (3)$$

Each of the four constraints represents one edge of the feasible polytope. In general, the constraint functions are chosen so that each one represents a smooth piece of the boundary of $\Omega$.

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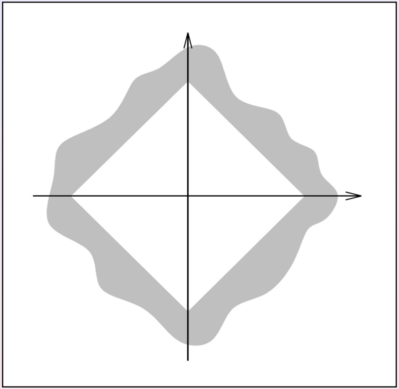

Figure 2: A feasible region with a non-smooth boundary can be described by smooth constraints.

# Introduction

Non-smooth, unconstrained optimization problems can sometimes be reformulated as smooth constrained problems. An example is the unconstrained minimization of a function

$$f(x) = \max\{x^2, x\},$$

which has kinks at $x = 0$ and $x = 1$, and the solution at $x_* = 0$. We obtain a smooth, constrained formulation of this problem by adding an artificial variable t and writing

$$\min t \quad \text{s.t.} \quad t \geqslant x, t \geqslant x^2. \tag{4}$$

Reformulation techniques such as (3) and (4) are used often in cases where $f$ is a maximum of a collection of functions or when $f$ is a 1-norm or $\infty$-norm of a vector function.

# Introduction

In the examples above we expressed inequality constraints in a slightly different way from the form $c_i(x) \geqslant 0$ that appears in the definition (1). However, any collection of inequality constraints with $\geqslant$ and $\leqslant$ and nonzero right-hand sides can be expressed in the form $c_i(x) \geqslant 0$ by simple rearrangement of the inequality.

# §12.1 Examples

To introduce the basic principles behind the characterization of solutions of constrained optimization problems, we work through three simple examples.

We begin with the definition of one important terminology.

**Definition**

The **active set** $\mathcal{A}(x)$ at any feasible $x$ consists of the equality constraint indices from $\mathcal{E}$ together with the indices of the inequality constraints $i$ for which $c_i(x) = 0$; that is,

$$\mathcal{A}(x) = \mathcal{E} \cup \left\{ i \in \mathcal{I} \,\middle|\, c_i(x) = 0 \right\}.$$

At a feasible point $x$, the inequality constraint $i \in \mathcal{I}$ is said to be active if $c_i(x) = 0$ and inactive if the strict inequality $c_i(x) > 0$ is satisfied.

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- **A Single equality constraint**

**Example**

Our first example is a two-variable problem with a single equality constraint:

$$\min(x_1 + x_2) \quad \text{subject to} \quad x_1^2 + x_2^2 - 2 = 0 \tag{5}$$

(see Figure 12.3). In the language of (1), we have $f(x) = x_1 + x_2$, $\mathcal{I} = \emptyset$, $\mathcal{E} = \{1\}$, and $c_1(x) = x_1^2 + x_2^2 - 2$. We can see by inspection that the feasible set for this problem is the circle of radius $\sqrt{2}$ centered at the origin – just the boundary of this circle, not its interior. The solution $x_*$ is obviously $(-1,-1)^{\mathrm{T}}$. From any other point on the circle, it is easy to find a way to move that stays feasible (that is, remains on the circle) while decreasing $f$. For instance, from the point $x = (\sqrt{2}, 0)^{\mathrm{T}}$ any move in the clockwise direction around the circle has the desired effect.

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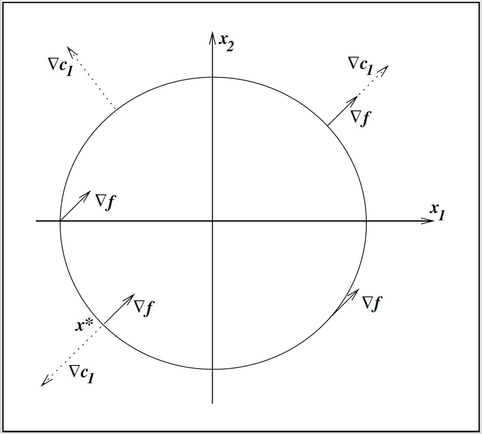


Figure 3: Problem (5), showing constraint and function gradients at various feasible points.

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**Example (cont'd)**

We also see from Figure 3 that at the solution $x_*$, the constraint normal $\nabla c_1(x_*)$ is parallel to $(\nabla f)(x_*)$. That is, there is a scalar $\lambda_1^*$ (in this case $\lambda_1^* = -1/2$) such that

$$(\nabla f)(x_*) = \lambda_1^* \nabla c_1(x_*) \,. \tag{6}$$

We can derive (6) by examining first-order Taylor series approximations to the objective and constraint functions. To retain feasibility with respect to the function $c_1(x) = 0$, we require any small (but nonzero) step $s$ to satisfy that $c_1(x+s) = 0$; that is,

$$0 = c_1(x+s) \approx c_1(x) + \nabla c_1(x)^{\mathrm{T}} s = \nabla c_1(x)^{\mathrm{T}} s \,.$$

Hence, the step $s$ retains feasibility with respect to $c_1$, to first order, when it satisfies

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Similarly, if we want $s$ to produce a decrease in $f$, we would have

$$0 > f(x+s) - f(x) \approx \nabla f(x)^{\mathrm{T}} s\,,$$

or, to first order,

$$\nabla f(x)^{\mathrm{T}} s < 0\,. \tag{8}$$

Existence of a small step $s$ that satisfies both (7) and (8) strongly suggests existence of a direction $d$ (where the size of $d$ is not small; we could have $d \approx s/\|s\|$ to ensure that the norm of $d$ is close to 1) with the same properties, namely

$$\nabla c_1(x)^{\mathrm{T}} d = 0 \quad \text{and} \quad \nabla f(x)^{\mathrm{T}} d < 0\,. \tag{9}$$

If, on the other hand, there is no direction $d$ with the properties (9), then is it likely that we cannot find a small step $s$ with the properties (7) and (8). In this case, $x_*$ would appear to be a local minimizer.

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By drawing a picture, the reader can check that the only way that a $d$ satisfying (9) does **not** exist is if $\nabla f(x)$ and $\nabla c_1(x)$ are parallel; that is, if the condition $\nabla f(x) = \lambda_1 \nabla c_1(x)$ holds at $x$, for some scalar $\lambda_1$. If in fact $\nabla f(x)$ and $\nabla c_1(x)$ are not parallel, we can set

$$\bar{d} = -\Big(\mathrm{I} - \frac{\nabla c_1(x) \nabla c_1(x)^{\mathrm{T}}}{\|\nabla c_1(x)\|^2}\Big) \nabla f(x); \quad d = \frac{\bar{d}}{\|\bar{d}\|}.$$

It is easy to verify that this d satisfies (9).

# §12.1 Examples

Introduce the Lagrangian function

$$\mathcal{L}(x, \lambda_1) = f(x) - \lambda_1 c_1(x) . \tag{10}$$

Since $\nabla_x \mathcal{L}(x, \lambda_1) = \nabla f(x) - \lambda_1 \nabla c_1(x)$, we can state the condition

$$(\nabla f)(x_*) = \lambda_1^* \nabla c_1(x_*) \tag{6}$$

equivalently as follows: At the solution $x_*$, there is a scalar $\lambda_1^*$ such that

$$\nabla_x \mathcal{L}(x_*, \lambda_1^*) = 0 . \tag{11}$$

This observation suggests that we can search for solutions of the equality-constrained problem (5) by seeking stationary points of the Lagrangian function. The scalar quantity $\lambda_1$ in (10) is called a **Lagrange multiplier** for the constraint $c_1(x) = 0$.

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- **A single inequality constraint**

**Example**

This is a slight modification of the first example, in which the equality constraint is replaced by an inequality. Consider

$$\min(x_1 + x_2) \quad \text{s.t.} \quad 2 - x_1^2 - x_2^2 \geqslant 0\,, \tag{12}$$

for which the feasible region consists of the circle of problem (5) and its interior (see Figure 4). Note that the constraint normal $\nabla c_1$ points toward the interior of the feasible region at each point on the boundary of the circle. By inspection, we see that the solution is still $(-1,-1)^{\mathrm{T}}$ and that the condition (6) holds for the value $\lambda_1^* = 1/2$. However, this inequality-constrained problem differs from the equality-constrained problem (5) of the first example in that the sign of the Lagrange multiplier plays a significant role.

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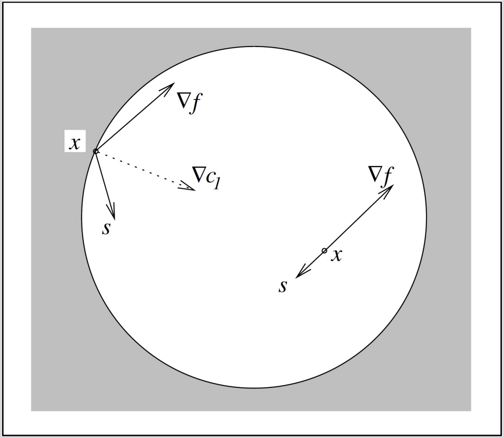


Figure 4: Improvement directions $s$ from two feasible points $x$ for the problem $c$ at which the constraint is active and inactive, respectively.

# §12.1 Examples

As before, we conjecture that a given feasible point $x$ is not optimal if we can find a small step $s$ that both retains feasibility and decreases the objective function $f$ to first order. The main difference between problems (5) and (12) comes in the handling of the feasibility condition. As in (8), the step $s$ improves the objective function, to first order, if $\nabla f(x)^{\mathrm{T}} s < 0$. Meanwhile, $s$ retains feasibility if

$$0 \leqslant c_1(x+s) \approx c_1(x) + \nabla c_1(x)^{\mathrm{T}} s,$$

so, to first order, feasibility is retained if

$$c_1(x) + \nabla c_1(x)^{\mathrm{T}} s \geqslant 0. \tag{13}$$

In determining whether a step $s$ exists that satisfies both (8) and (13), we consider two cases, which are illustrated in Figure 4.

# §12.1 Examples

**Case I**: Consider first the case in which x lies strictly inside the circle, so that the strict inequality $c_1(x) > 0$ holds. In this case, any step vector $s$ satisfies the condition

$$c_1(x) + \nabla c_1(x)^{\mathrm{T}} s \geqslant 0\,, \tag{13}$$

provided only that its length is sufficiently small. In fact, whenever $\nabla f(x) \neq 0$, we can obtain a step $s$ that satisfies both

$$\nabla f(x)^{\mathrm{T}} s < 0 \tag{8}$$

and (13) by setting

$$s = -\alpha \nabla f(x)$$

for any positive scalar $\alpha$ sufficiently small. However, this definition does **not** give a step $s$ with the required properties when

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**Case II**: Consider now the case in which $x$ lies on the boundary of the circle, so that $c_1(x) = 0$. The conditions (8) and (13) therefore become

$$\nabla f(x)^{\mathrm{T}} s < 0\,, \quad \nabla c_1(x)^{\mathrm{T}} s \geqslant 0\,.$$

The first of these conditions defines an open half-space, while the second defines a closed half-space, as illustrated in Figure 5 in the next slide. It is clear from this figure that the intersection of these two regions is empty only when $\nabla f(x)$ and $\nabla c_1(x)$ point in the same direction; that is, when

$$\nabla f(x) = \lambda_1 \nabla c_1(x) \quad \text{for some } \lambda_1 \geqslant 0. \tag{15}$$

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# §12.1 Examples

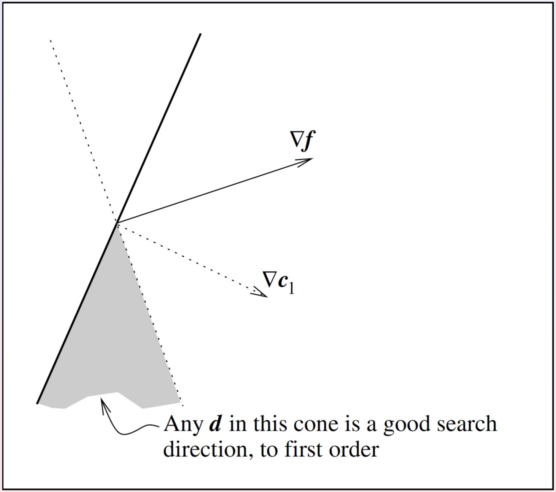


Figure 5: A direction $d$ that satisfies both (8) and (13) lies in the intersection of a closed half-plane and an open half-plane.

# §12.1 Examples

Note that the sign of the multiplier is significant here. If (6) were satisfied with a negative value of $\lambda_1$, then $\nabla f(x)$ and $\nabla c_1(x)$ would point in opposite directions, and we see from Figure 5 that the set of directions that satisfy both (8) and (13) would make up an entire open half-plane.

The optimality conditions for both cases I and II can again be summarized neatly using the Lagrangian function $\mathcal{L}$ defined in

$$\mathcal{L}(x, \lambda_1) = f(x) - \lambda_1 c_1(x) \,. \tag{10}$$

When no first-order feasible descent direction exists at some point $x_*$, we have that

$$\nabla_x \mathcal{L}(x_*, \lambda_1^*) = 0 \text{ for some } \lambda_1^* \geqslant 0 \,, \tag{16}$$

where we also require that

$$\lambda_1^* c_1(x_*) = 0 \,. \tag{17}$$

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# §12.1 Examples

Condition

$$\lambda_1^* c_1(x_*) = 0 \tag{17}$$

is known as a **complementarity condition** (互補條件); it implies that the Lagrange multiplier $\lambda_1$ can be strictly positive only when the corresponding constraint $c_1$ is active. Conditions of this type play a central role in constrained optimization, as we see in the sections that follow. In case I, we have that $c_1(x_*) > 0$, so (17) requires that $\lambda_1^* = 0$. Hence,

$$(\nabla f)(x_*) = \lambda_1^* \nabla c_1(x_*) \tag{6}$$

reduces to $(\nabla f)(x_*) = 0$, as required by (14). In case II, (17) allows $\lambda_1^*$ to take on a non-negative value, so (16) becomes equivalent to

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# §12.1 Examples

- **Two inequality constraints**

**Example**

Suppose we add an extra constraint to the problem (12) to obtain

$$\min(x_1 + x_2) \quad \text{s.t.} \quad 2 - x_1^2 - x_2^2 \geqslant 0\,, x_2 \geqslant 0\,, \tag{18}$$

for which the feasible region is the half-disk illustrated in Figure 6. It is easy to see that the solution lies at $(-\sqrt{2}, 0)^{\mathrm{T}}$, a point at which both constraints are active. By repeating the arguments for the previous examples, we would expect a direction $d$ of first-order feasible descent to satisfy

$$\nabla c_i(x)^{\mathrm{T}} d \geqslant 0 \quad \text{for} \quad i \in \mathcal{I} = \{1, 2\} \tag{19a}$$

and

$$\nabla f(x)^{\mathrm{T}} d < 0\,. \tag{19b}$$

# §12.1 Examples

## Example (cont'd)

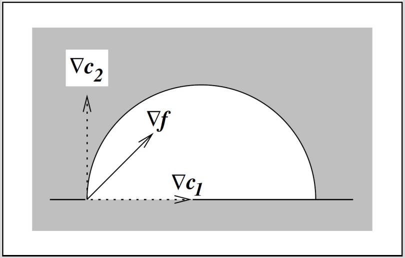


Figure 6: Problem (18), illustrating the gradients of the active constraints and objective at the solution.

However, it is clear from Figure 6 that no such direction can exist when $x = (-\sqrt{2}, 0)^{\mathrm{T}}$. The conditions $\nabla c_i(x)^{\mathrm{T}} d \geqslant 0$ are satisfied for $i = 1, 2$ only if $d = \alpha_1 \nabla c_1(x) + \alpha_2 \nabla c_2(x)$ for some $c_1, c_2 \geqslant 0$, but it is clear by inspection that all such vectors $d$ satisfy $\nabla f(x)^{\mathrm{T}} d \geqslant 0$.

# §12.1 Examples

## Example (cont'd)

Let us see how the Lagrangian and its derivatives behave for the problem (18) and the solution point $(-\sqrt{2}, 0)^{\mathrm{T}}$. First, we include an additional term $\lambda_i c_i(x)$ in the Lagrangian for each additional constraint, so the definition of $\mathcal{L}$ becomes

$$\mathcal{L}(x, \lambda) = f(x) - \lambda_1 c_1(x) - \lambda_2 c_2(x),$$

where $\lambda = (\lambda_1, \lambda_2)^{\mathrm{T}}$ is the vector of Lagrange multipliers. The extension of condition (16) to this case is

$$\nabla_x \mathcal{L}(x_*, \lambda_*) = 0 \quad \text{for some } \lambda_* \geqslant 0, \tag{20}$$

where the inequality $\lambda_* \geqslant 0$ means that all components of $\lambda_*$ are required to be non-negative. The non-negativity of the Lagrange multipliers is an important feature in the inequality constrained problem, and (20) will be shown in the next slide.

# §12.1 Examples

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# §12.1 Examples

## Example (cont'd)

By applying the complementarity condition (17) to both inequality constraints, we obtain

$$\lambda_1^* c_1(x_*) = 0\,, \quad \lambda_2^* c_2(x_*) = 0\,. \tag{21}$$

When $x_* = (-\sqrt{2}, 0)^{\mathrm{T}}$, we have

$$(\nabla f)(x_*) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \nabla c_1(x_*) = \begin{bmatrix} 2\sqrt{2} \\ 0 \end{bmatrix}, \quad \nabla c_2(x_*) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

so that it is easy to verify that $\nabla_x \mathcal{L}(x_*, \lambda_*) = 0$ when we select $\lambda_*$ as follows:

$$\lambda_* = \begin{bmatrix} 1/(2\sqrt{2}) \\ 1 \end{bmatrix}.$$

Note that both components of $\lambda_*$ are positive, so that (20) is satisfied.

# §12.1 Examples

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# §12.1 Examples

## Example (cont'd)

We consider now some other feasible points that are not solutions of (18), and examine the properties of the Lagrangian and its gradient at these points. For the point $x = (\sqrt{2}, 0)^{\mathrm{T}}$, we again have that both constraints are active (see Figure 7 in the next slide). However, it is easy to identify vectors $d$ that satisfies

$$\nabla c_i(x)^{\mathrm{T}} d \geqslant 0 \quad \text{for} \quad i \in \mathcal{I} = \{1, 2\}\,, \tag{19a}$$

$$\nabla f(x)^{\mathrm{T}} d < 0\,. \tag{19b}$$

In fact, $d = (-1, 0)^{\mathrm{T}}$ is one such vector (there are many others). For this value of $x$ it is easy to verify that the condition $\nabla_x \mathcal{L}(x, \lambda) = 0$ is satisfied only when $\lambda = (-1/(2\sqrt{2}), 1)^{\mathrm{T}}$. Note that the first component $\lambda_1$ is negative, so that the conditions (20) are not satisfied at this point.

# §12.1 Examples

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# §12.1 Examples

## Example (cont'd)

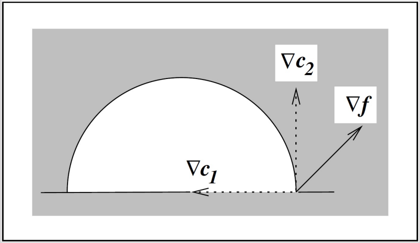


Figure 7: Problem (18), illustrating the gradients of the active constraints and objective at a non-optimal point.

# §12.1 Examples

## Example (cont'd)

Finally, we consider the point $x = (1,0)^{\mathrm{T}}$, at which only the second constraint $c_2$ is active. Since any small step $s$ away from this point will continue to satisfy $c_1(x+s) > 0$, we need to consider only the behavior of $c_2$ and $f$ in determining whether $s$ is indeed a feasible descent step. Using the same reasoning as in the earlier examples, we find that the direction of feasible descent $d$ must satisfy

$$\nabla c_2(x)^{\mathrm{T}} d \geqslant 0\,, \quad \nabla f(x)^{\mathrm{T}} d < 0\,. \tag{22}$$

By noting that

$$\nabla f(x) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \nabla c_2(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

it is easy to verify that the vector $d = (-1/2, 1/4)^{\mathrm{T}}$ satisfies (22) and is therefore a descent direction.

# §12.1 Examples

## Example (cont'd)

To show that optimality conditions

$$\nabla_x \mathcal{L}(x_*, \lambda_*) = 0 \quad \text{for some } \lambda_* \geqslant 0 \tag{20}$$

and

$$\lambda_1^* c_1(x_*) = 0\,, \quad \lambda_2^* c_2(x_*) = 0\,, \tag{21}$$

fail, we note first from (21) that since $c_1(x) > 0$, we must have $\lambda_1 = 0$. Therefore, in trying to satisfy $\nabla_x \mathcal{L}(x, \lambda) = 0$, we are left to search for a value $\lambda_2$ such that $\nabla f(x) - \lambda_2 \nabla c_2(x) = 0$. No such $\lambda_2$ exists, and thus this point fails to satisfy the optimality conditions.

# §12.2 Tangent Cone and Constraint Qualifications

In the previous section, we determined whether or not it was possible to take a feasible descent step from a given feasible point $x$ by examining the first derivatives of the objective function $f$ and the constraint functions $c_i$. We used the first-order Taylor series of these functions about $x$ to form an approximate problem in which both objective and constraints are linear. This approach makes sense only if the linearized approximation captures the essential geometric features of the feasible set near the point $x$ in question. If the linearization is fundamentally different from the feasible set, then we cannot expect the linear approximation to yield useful information about the original problem. Hence, we need to make assumptions about the nature of the constraints $c_i$ that are active at $x$ to ensure that the linear approximation is similar to the feasible set, near $x$.

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# §12.2 Tangent Cone and Constraint Qualifications

In this section we define the tangent cone $T_\Omega(x)$ to the closed convex set $\Omega$ at a point $x \in \Omega$, and also the set $\mathcal{F}(x)$ of first-order feasible directions at $x$. We also discuss **constraint qualifications**, assumptions that ensure similarity of the constraint set $\Omega$ and its linearized approximation, in a neighborhood of $x$.

# §12.2 Tangent Cone and Constraint Qualifications

Recall that the feasible set is denoted by $\Omega$.

**Definition**

1. Given a feasible point $x$, we call $\{z_k\}$ a feasible sequence approaching $x$ if $z_k \in \Omega$ (for all $k$ sufficiently large) and $z_k \to x$.
2. A tangent is a limiting direction of a feasible sequence. To be more precise, a vector $d$ is said to be a tangent (or tangent vector) to $\Omega$ at a point $x$ if there are a feasible sequence $\{z_k\}$ approaching $x$ and a sequence of positive scalars $\{t_k\}$ with $t_k \to 0$ such that
$$\lim_{k\to\infty} \frac{z_k - x}{t_k} = d\,. \tag{23}$$
The collection of all tangents to $\Omega$ at $x$ is called the **tangent cone** to the set $\Omega$ at $x$ and is denoted by $T_\Omega(x)$.

# §12.2 Tangent Cone and Constraint Qualifications

It is easy to see that the tangent cone $T_\Omega(x)$ is indeed a cone:

1. $0 \in T_\Omega(x)$: $z_k \equiv x$ for all $k$ is a feasible sequence.
2. $d \in T_\Omega(x)$ and $\alpha > 0 \Rightarrow \alpha d \in T_\Omega(x)$: If $\{z_k\}$ and $\{t_k\}$ satisfy (23), then by replacing $t_k$ by $\alpha^{-1} t_k$, we find that $\alpha d \in T_\Omega(x)$.

Later, we characterize a local solution of

$$\min_{x\in\mathbb{R}^n} f(x) \quad \text{subject to} \quad \begin{cases} c_i(x) = 0, i \in \mathcal{E}, \\ c_i(x) \geqslant 0, i \in \mathcal{I}, \end{cases} \tag{1}$$

as a point $x$ at which all feasible sequences approaching $x$ have the property that $f(z_k) \geqslant f(x)$ for all $k$ sufficiently large, and we will derive practical, verifiable conditions under which this property holds. We lay the groundwork in this section by characterizing the directions in which we can step away from $x$ while remaining feasible.

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# §12.2 Tangent Cone and Constraint Qualifications

We turn now to the linearized feasible direction set. Recall that for a feasible point $x$, the active set $\mathcal{A}(x)$ is $\mathcal{E} \cup \{i \in \mathcal{I} \mid c_i(x) = 0\}$.

**Definition**

Given a feasible point $x$ and the active constraint set $\mathcal{A}(x)$, the set of linearized feasible directions $\mathcal{F}(x)$ is

$$\mathcal{F}(x) = \left\{ d \;\middle|\; \begin{array}{l} d^{\mathrm{T}}\nabla c_i(x) = 0 \text{ for all } i \in \mathcal{E}, \\ d^{\mathrm{T}}\nabla c_i(x) \geqslant 0 \text{ for all } i \in \mathcal{A}(x) \cap \mathcal{I} \end{array} \right\}.$$

As with the tangent cone, it is easy to verify that $\mathcal{F}(x)$ is a cone.

It is important to note that the definition of tangent cone does not rely on the algebraic specification of the set $\Omega$, only on its geometry. The linearized feasible direction set does, however, depend on the definition of the constraint functions $c_i$, $i \in \mathcal{E} \cup \mathcal{I}$.

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# §12.2 Tangent Cone and Constraint Qualifications

We illustrate the tangent cone and the linearized feasible direction set by revisiting two examples in Section 12.1.

**Example (Revisit of the 1st example)**

Recall the equality-constrained problem

$$\min(x_1 + x_2) \quad \text{s.t.} \quad x_1^2 + x_2^2 - 2 = 0\,. \tag{5}$$

Near the non-optimal point $x = (-\sqrt{2}, 0)^{\mathrm{T}}$, Figure 8 shows a feasible sequence approaching $x$ given by

$$z_k = \begin{bmatrix} -\sqrt{2 - 1/k^2} \\ -1/k \end{bmatrix}. \tag{24}$$

By choosing $t_k = \|z_k - x\|$, we find that $d = (0, -1)^{\mathrm{T}}$ is a tangent. Note that the objective function $f(x) = x_1 + x_2$ increases strictly as we move along the sequence (24); that is, $f(z_{k+1}) > f(z_k)$ for all $k = 2, 3, \cdots$. So $x$ cannot be a solution of (5).

# §12.2 Tangent Cone and Constraint Qualifications

## Example (cont'd)

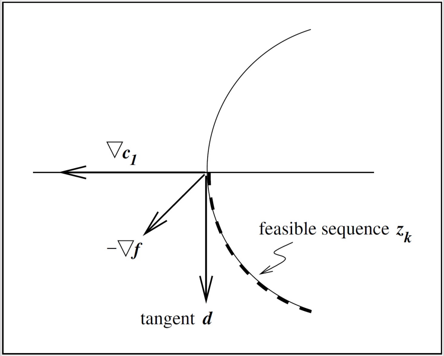


Figure 8: Constraint normal, objective gradient, and feasible sequence for problem (5).

# §12.2 Tangent Cone and Constraint Qualifications

## Example (cont'd)

Another feasible sequence is one that approaches $x = (-\sqrt{2}, 0)^{\mathrm{T}}$ from the opposite direction given by

$$z_k = \begin{bmatrix} -\sqrt{2 - 1/k^2} \\ 1/k \end{bmatrix}.$$

It is easy to show that $f$ decreases along this sequence and that the tangents corresponding to this sequence are $d = (0, \alpha)^{\mathrm{T}}$. In summary, the tangent cone at $x = (-\sqrt{2}, 0)^{\mathrm{T}}$ is $\{(0, d_2)^{\mathrm{T}} \mid d_2 \in \mathbb{R}\}$.

By the definition of the linearized feasible direction, we have

$$d = (d_1, d_2)^{\mathrm{T}} \in \mathcal{F}(x) \quad \Leftrightarrow \quad 0 = \nabla c_1(x)^{\mathrm{T}} d = -2\sqrt{2}\, d_1 .$$

Therefore, we obtain $\mathcal{F}(x) = \{(0, d_2)^{\mathrm{T}} \mid d_2 \in \mathbb{R}\}$. In this case, we have $T_\Omega(x) = \mathcal{F}(x)$.

# §12.2 Tangent Cone and Constraint Qualifications

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# §12.2 Tangent Cone and Constraint Qualifications

## Example (cont'd)

Suppose that the feasible set is defined instead by the formula $\Omega = \{x \,|\, c_1(x) = 0\}$, where

$$c_1(x) = (x_1^2 + x_2^2 - 2)^2 = 0\,. \tag{25}$$

Note that $\Omega$ is the same, but its algebraic specification has changed. The vector $d$ belongs to the linearized feasible set if

$$0 = \nabla c_1(x)^{\mathrm{T}} d = \begin{bmatrix} 4(x_1^2 + x_2^2 - 2)x_1 & 4(x_1^2 + x_2^2 - 2)x_2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix},$$

which is true for all $(d_1, d_2)^{\mathrm{T}}$. Hence, we have $\mathcal{F}(x) = \mathbb{R}^2$, so for this algebraic specification of $\Omega$, the tangent cone and linearized feasible sets differ.

# §12.2 Tangent Cone and Constraint Qualifications

## Example (Revisit of the 2nd example)

We now reconsider the problem

$$\min(x_1 + x_2) \quad \text{s.t.} \quad 2 - x_1^2 - x_2^2 \geqslant 0. \tag{12}$$

The solution $x = (-1, -1)^{\mathrm{T}}$ is the same as the previous case, but there is a much more extensive collection of feasible sequences that converge to any given feasible point (see Figure 9).

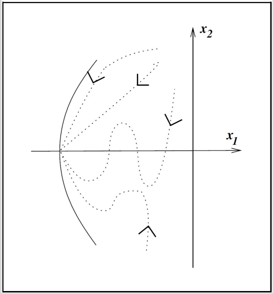


Figure 9: Feasible sequences converging to a particular feasible point for the region defined by $x_1^2 + x_2^2 \leqslant 2$.

# §12.2 Tangent Cone and Constraint Qualifications

## Example (cont'd)

From the point $x = (-\sqrt{2}, 0)^{\mathrm{T}}$, the various feasible sequences defined above for the equality-constrained problem are still feasible for (12). There are also infinitely many feasible sequences that converge to $x = (-\sqrt{2}, 0)^{\mathrm{T}}$ along a straight line from the interior of the circle. These sequences have the form

$$z_k = (-\sqrt{2}, 0)^{\mathrm{T}} + (1/k)w,$$

where $w$ is any vector whose first component is positive ($w_1 > 0$). The point $z_k$ is feasible provided that $\|z_k\| \leqslant \sqrt{2}$; that is,

$$(-\sqrt{2} + w_1/k)^2 + (w_2/k)^2 \leqslant 2,$$

which is true when $k \geqslant (w_1^2 + w_2^2)/(2\sqrt{2}w_1)$.

# §12.2 Tangent Cone and Constraint Qualifications

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# §12.2 Tangent Cone and Constraint Qualifications

## Example (cont'd)

In addition to these straight-line feasible sequences, we can also define an infinite variety of sequences that approach $(-\sqrt{2}, 0)^{\mathrm{T}}$ along a curve from the interior of the circle. To summarize, the tangent cone to this set at $(-\sqrt{2}, 0)^{\mathrm{T}}$ is $\{(w_1, w_2)^{\mathrm{T}} \mid w_1 \geqslant 0\}$.

For the definition (12) of this feasible set, we have

$$d \in \mathcal{F}(x) \quad \Leftrightarrow \quad 0 \leqslant \nabla c_1(x)^{\mathrm{T}} d = \begin{bmatrix} -2x_1 & -2x_2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = 2\sqrt{2}\, d_1 \, .$$

Hence, we obtain $\mathcal{F}(x) = T_{\Omega}(x)$ for this particular algebraic specification of the feasible set.

# §12.2 Tangent Cone and Constraint Qualifications

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Hence, we obtain $\mathcal{F}(x) = T_\Omega(x)$ for this particular algebraic specification of the feasible set.

# §12.2 Tangent Cone and Constraint Qualifications

In general, we have the following

**Lemma**

*Let $x$ be a feasible point. Then $T_\Omega(x) \subseteq \mathcal{F}(x)$.*

**Proof.**

Let $d \in T_\Omega(x)$. Then there exist a feasible sequence $\{z_k\}$ and a sequence of positive scalars $\{t_k\}$ satisfying $\lim\limits_{k\to\infty} t_k = 0$ and

$$\lim_{k\to\infty} \frac{z_k - x}{t_k} = d\,.$$

From the limit above, we have

$$z_k = x + t_k d + o(t_k)\,;$$

thus Taylor's Theorem implies that

$$\begin{aligned} c_i(z_k) &= c_i(x) + \nabla c_i(x)^{\mathrm{T}}(z_k - x) + o(\|z_k - x\|) \\ &= c_i(x) + t_k \nabla c_i(x)^{\mathrm{T}} d + o(t_k)\,. \end{aligned}$$

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# §12.2 Tangent Cone and Constraint Qualifications

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while for $i \in \mathcal{A}(x) \cap \mathcal{I}$ we have

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Passing to the limit as $k \to \infty$, we obtain

1. $\nabla c_i(x)^{\mathrm{T}} d = 0$ if $i \in \mathcal{E}$.
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# §12.2 Tangent Cone and Constraint Qualifications

Constraint qualifications are conditions under which the linearized feasible set $\mathcal{F}(x)$ is similar to the tangent cone $T_\Omega(x)$. In fact, most constraint qualifications ensure that these two sets are identical. As mentioned earlier, these conditions ensure that the $\mathcal{F}(x)$, which is constructed by linearizing the algebraic description of the set $\Omega$ at $x$, captures the essential geometric features of the set $\Omega$ in the vicinity of $x$, as represented by $T_\Omega(x)$.

Both $T_\Omega(x)$ and $\mathcal{F}(x)$ in the first example consist of the vertical axis, which is qualitatively similar to the set $\Omega\backslash\{x\}$ in the neighborhood of $x$. As a further example, consider the constraints

$$c_1(x) = 1 - x_1^2 - (x_2 - 1)^2 \geqslant 0\,, \quad c_2(x) = -x_2 \geqslant 0\,, \tag{26}$$

for which the feasible set is the single point $\Omega = \{(0,0)^{\mathrm{T}}\}$ (see Figure 10).

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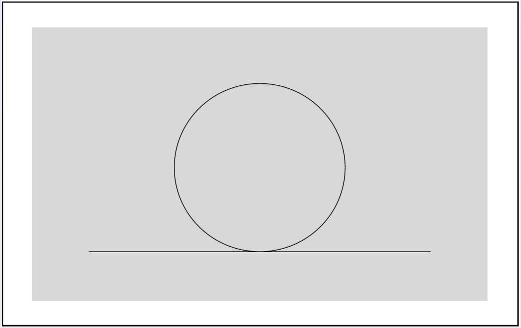

Figure 10: Problem (26), for which the feasible set is the single point of intersection between circle and line.

# §12.2 Tangent Cone and Constraint Qualifications

For this point $x = (0,0)^{\mathrm{T}}$, it is obvious that tangent cone is $T_{\Omega}(x) = \{(0,0)^{\mathrm{T}}\}$, since all feasible sequences approaching $x$ must have $z_k = x = (0,0)^{\mathrm{T}}$ for all $k$ sufficiently large. Moreover, it is easy to show that linearized approximation to the feasible set $\mathcal{F}(x)$ is

$$\mathcal{F}(x) = \left\{(d_1, 0)^{\mathrm{T}} \,\middle|\, d_1 \in \mathbb{R}\right\},$$

that is, the entire horizontal axis. In this case, the linearized feasible direction set does not capture the geometry of the feasible set, so constraint qualifications are not satisfied. The constraint qualification most often used in the design of algorithms is the subject of the next definition.

# §12.2 Tangent Cone and Constraint Qualifications

## Definition (LICQ)

For a given feasible point $x$ (with corresponding active set $\mathcal{A}(x)$), we say that the linear independence constraint qualification (LICQ) holds at $x$ if the set of active constraint gradients $\{\nabla c_i(x) \mid i \in \mathcal{A}(x)\}$ is linearly independent.

Note that this condition is not satisfied for the examples (25) and (26). In general, if LICQ holds, none of the active constraint gradients can be zero. We will mention other constraint qualifications in Section 12.6.

# §12.2 Tangent Cone and Constraint Qualifications

- **The relation between $T_\Omega(x)$ and $\mathcal{F}(x)$ given LICQ**

In the following, we use $A(x)$ to represent the matrix whose **rows** are the active constraint gradients at the optimal point; that is,

$$A(x)^{\mathrm{T}} = [\nabla c_i(x)]_{i\in\mathcal{A}(x)} . \tag{27}$$

We first establish the following

**Lemma**

*Let $x$ be a feasible point at which the LICQ condition holds. Then for every $d \in \mathcal{F}(x)$ and sequence $\{t_k\}$ of positive scalars satisfying $\lim\limits_{k\to\infty} t_k = 0$, there exists a feasible sequence $\{z_k\}$ such that*

$$\lim_{k\to\infty} \frac{z_k - x}{t_k} = d \tag{23}$$

*and*

$$c_i(z_k) = t_k \nabla c_i(x)^{\mathrm{T}} d \qquad \forall\, i \in \mathcal{A}(x) \text{ and } k \gg 1 . \tag{28}$$

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# §12.2 Tangent Cone and Constraint Qualifications

**Proof.**

W.L.O.G. we can assume that all the constraints $c_i$, $i = 1, 2, \cdots, m$ are active at $x$. Let $d \in \mathcal{F}(x)$ be given, and suppose that $\{t_k\}_{k=0}^{\infty}$ is any sequence of positive scalars such $\lim_{k \to \infty} t_k = 0$. We first note that the $m \times n$ matrix $A(x)$ of active constraint gradients has full row rank $m$ since the LICQ holds at $x$. By the fact that

$$\text{the null space of } A(x) \oplus \text{the range of } A(x)^{\mathrm{T}} = \mathbb{R}^n,$$

there exists an $n \times (n-m)$ matrix $Z$ whose columns are a basis for the null space of $A(x)$; that is,

$$Z \in \mathbb{R}^{n \times (n-m)}, \quad Z \text{ has full column rank}, \quad A(x)Z = 0. \tag{29}$$

With $c \equiv [c_i]_{i \in \mathcal{A}(x)}$, define $R : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ by

$$R(z,t) = \begin{bmatrix} c(z) - tA(x)d \\ Z^{\mathrm{T}}(z - x - td) \end{bmatrix}.$$

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# §12.2 Tangent Cone and Constraint Qualifications

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# §12.2 Tangent Cone and Constraint Qualifications

## Proof (cont'd).

Note that $R(x,0)=0$. Moreover, the Jacobian of $R(\cdot,\cdot)$ with respect to $z$ at point $(z,t)=(x,0)$ is

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$$R(z,t) = \begin{bmatrix} c(z) - tA(x)d \\ Z^{\mathrm{T}}(z-x-td) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (30)$$

has a unique solution $z_k$ ($\approx x$) for all $t_k > 0$ sufficiently small. The Implicit Function Theorem also shows that $\lim\limits_{k\to\infty} z_k = x$.

We claim that $\{z_k\}$ is a feasible sequence and satisfies desired properties (23) and (28). □

# §12.2 Tangent Cone and Constraint Qualifications

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has a unique solution $z_k$ $(\approx x)$ for all $t_k>0$ sufficiently small. The Implicit Function Theorem also shows that $\lim\limits_{k\to\infty} z_k=x$.

We claim that $\{z_k\}$ is a feasible sequence and satisfies desired properties (23) and (28). □

# §12.2 Tangent Cone and Constraint Qualifications

## Proof (cont'd).

First we show that

$$\lim_{k\to\infty} \frac{z_k - x}{t_k} = d \qquad (23)$$

holds for this choice of $\{z_k\}$. Using the facts that

1. $R(z_k, t_k) = 0$ for sufficiently large $k$ and
2. $c(x) = [c_i(x)]_{i\in\mathcal{A}(x)} = 0$,

Taylor's Theorem implies that for $k$ sufficiently large,

$$\begin{aligned} 0 = R(z_k, t_k) &= \begin{bmatrix} c(z_k) - t_k A(x) d \\ Z^{\mathrm{T}}(z_k - x - t_k d) \end{bmatrix} \\ &= \begin{bmatrix} A(x)(z_k - x) + o(\|z_k - x\|) - t_k A(x) d \\ Z^{\mathrm{T}}(z_k - x - t_k d) \end{bmatrix} \\ &= \begin{bmatrix} A(x) \\ Z^{\mathrm{T}} \end{bmatrix} (z_k - x - t_k d) + o(\|z_k - x\|). \end{aligned}$$

□

# §12.2 Tangent Cone and Constraint Qualifications

## Proof (cont'd).

By dividing this expression by $t_k$ and using non-singularity of the coefficient matrix in the first term, we obtain

$$\frac{z_k - x}{t_k} = d + o\Big(\frac{\|z_k - x\|}{t_k}\Big),$$

from which it follows that (23) is satisfied (details required). Moreover, since

$$R(z_k, t_k) = \begin{bmatrix} c(z_k) - t_k A(x) d \\ Z^{\mathrm{T}}(z_k - x - t_k d) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

for sufficiently large $k$, we find that

$$c_i(z_k) = t_k \nabla c_i(x)^{\mathrm{T}} d \qquad \forall\, i \in \mathcal{A}(x) \text{ and } k \gg 1\,. \tag{28}$$

To conclude the lemma, we show that $\{z_k\}$ is a feasible sequence; that is, $c_i(z_k) = 0$ if $i \in \mathcal{E}$ and $c_i(z_k) \geqslant 0$ if $i \in \mathcal{I}$ for all sufficiently large $k$. □

# §12.2 Tangent Cone and Constraint Qualifications

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# §12.2 Tangent Cone and Constraint Qualifications

## Proof (cont'd).

Since $d \in \mathcal{F}(x)$, the definition of the set of linearized feasible directions implies that for sufficiently large $k$,

$$\begin{aligned} i \in \mathcal{E} \quad &\Rightarrow \quad c_i(z_k) = t_k \nabla c_i(x)^{\mathrm{T}} d = 0\,, \\ i \in \mathcal{A}(x) \cap \mathcal{I} \quad &\Rightarrow \quad c_i(z_k) = t_k \nabla c_i(x)^{\mathrm{T}} d \geqslant 0\,. \end{aligned}$$

Moreover, if $i \in \mathcal{I} \cap \mathcal{A}(x)^{\complement}$, we must have $c_i(x) > 0$; thus by the fact that $\lim\limits_{k\to\infty} z_k = x$ we have

$$c_i(z_k) > 0 \quad \forall\, k \gg 1\,.$$

Therefore, the continuity of $c_i$ shows that for sufficiently large $k$,

$$i \in \mathcal{I} \cap \mathcal{A}(x)^{\complement} \quad \Rightarrow \quad c_i(z_k) > 0\,.$$

Combining all the cases discussed above, we conclude that $\{z_k\}$ is indeed feasible. □

# §12.2 Tangent Cone and Constraint Qualifications

**Proof (cont'd).**

Since $d \in \mathcal{F}(x)$, the definition of the set of linearized feasible directions implies that for sufficiently large $k$,

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Combining all the cases discussed above, we conclude that $\{z_k\}$ is indeed feasible. □

# §12.2 Tangent Cone and Constraint Qualifications

Note that for a feasible point $x$, by the definition of the tangent cone $T_\Omega(x)$, the lemma above shows that $\mathcal{F}(x) \subseteq T_\Omega(x)$ provided that the LICQ condition holds at $x$. Combining with the lemma about $T_\Omega(x) \subseteq \mathcal{F}(x)$, gives the following

**Corollary**

*Let $x$ be a feasible point at which the LICQ condition holds. Then $T_\Omega(x) = \mathcal{F}(x)$.*

# §12.3 First-Order Optimality Conditions

In this section, we state first-order **necessary** conditions for $x_*$ to be a local minimizer and show how these conditions are satisfied on a small example. The proof of the result is presented in subsequent sections.

As a preliminary to stating the necessary conditions, we define the Lagrangian function for the general problem (1):

$$\mathcal{L}(x, \lambda) = f(x) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i c_i(x) . \tag{31}$$

The necessary conditions defined in the following theorem are called first-order conditions because they are concerned with properties of the gradients (first-derivative vectors) of the objective and constraint functions. These conditions are the foundation for many of the algorithms described in the remaining chapters of the book.

# §12.3 First-Order Optimality Conditions

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# §12.3 First-Order Optimality Conditions

## Theorem (First-Order Necessary Conditions)

*Suppose that* $x_*$ *is a local solution of problem*

$$\min_{x\in\mathbb{R}^n} f(x) \quad \textit{subject to} \quad \begin{cases} c_i(x) = 0, i \in \mathcal{E}\,, \\ c_i(x) \geqslant 0, i \in \mathcal{I}\,, \end{cases} \tag{1}$$

*that the functions* $f$ *and* $c_i$ *in* (1) *are continuously differentiable, and that the LICQ holds at* $x_*$. *Then there is a Lagrange multiplier vector* $\lambda_*$, *with components* $\lambda_i^*$, $i \in \mathcal{E} \cup \mathcal{I}$, *such that the following conditions are satisfied.*

$$\nabla_x \mathcal{L}(x_*, \lambda_*) = 0\,, \tag{32a}$$

$$c_i(x_*) = 0 \quad \textit{for all } i \in \mathcal{E}, \tag{32b}$$

$$c_i(x_*) \geqslant 0 \quad \textit{for all } i \in \mathcal{I}, \tag{32c}$$

$$\lambda_i^* \geqslant 0 \quad \textit{for all } i \in \mathcal{I}, \tag{32d}$$

$$\lambda_i^* c_i(x_*) = 0 \quad \textit{for all } i \in \mathcal{E} \cup \mathcal{I}. \tag{32e}$$

# §12.3 First-Order Optimality Conditions

1. The conditions (32) are often known as the Karush-Kuhn-Tucker conditions, or **KKT conditions** for short.
2. The conditions

$$\lambda_i^* c_i(x_*) = 0 \quad \text{for all } i \in \mathcal{E} \cup \mathcal{I} \tag{32e}$$

are **complementarity conditions**; they imply that either constraint $i$ is active or $\lambda_i^* = 0$, or possibly both. In particular, the Lagrange multipliers corresponding to inactive inequality constraints are zero, we can omit the terms for indices $i \notin \mathcal{A}(x_*)$ from (32a) and rewrite this condition as

$$0 = \nabla_x \mathcal{L}(x_*, \lambda_*) = \nabla f(x_*) - \sum_{i \in \mathcal{A}(x_*)} \lambda_i^* \nabla c_i(x_*) .$$

The proof of the 1st order necessary condition is quite complex, but it is important to our understanding of constrained optimization, so we present it in the next section.

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# §12.3 First-Order Optimality Conditions

A special case of complementarity is important and deserves its own definition.

**Definition (Strict Complementarity)**

Given a local solution $x_*$ of problem (1) and a vector $\lambda_*$ satisfying (32), we say that the **strict** complementarity condition holds if exactly one of $\lambda_i^*$ and $c_i(x_*)$ is zero for each index $i \in \mathcal{I}$; that is, $\lambda_i^* > 0$ for each $i \in \mathcal{I} \cap \mathcal{A}(x_*)$.

Satisfaction of the strict complementarity property usually makes it easier for algorithms to determine the active set $\mathcal{A}(x_*)$ and converge rapidly to the solution $x_*$. For a given problem (1) and solution point $x_*$, there may be many vectors $\lambda_*$ for which the conditions (32) are satisfied. When the LICQ holds at $x_*$, however, the optimal $\lambda_*$ is unique.

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# §12.3 First-Order Optimality Conditions

We illustrate the KKT conditions with another example before finishing this section.

## Example

Consider the minimization problem

$$\min_{x} \left(x_1 - \frac{3}{2}\right)^2 + \left(x_2 - \frac{1}{2}\right)^4 \quad \text{s.t.} \quad \begin{bmatrix} 1 - x_1 - x_2 \\ 1 - x_1 + x_2 \\ 1 + x_1 - x_2 \\ 1 + x_1 + x_2 \end{bmatrix} \geqslant 0 \,. \quad (33)$$

From Figure 11 we see that the solution is $x_* = (1,0)^{\mathrm{T}}$ at which the first and second constraints in (33) are active. Denoting them by $c_1$ and $c_2$ (and the inactive constraints by $c_3$ and $c_4$), we have

$$\nabla f(x_*) = \begin{bmatrix} -1 \\ -\frac{1}{2} \end{bmatrix}, \quad \nabla c_1(x_*) = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad \nabla c_2(x_*) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

# §12.3 First-Order Optimality Conditions

## Example (cont'd)

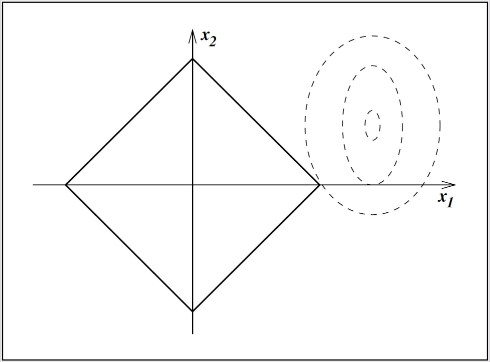


Figure 11: Inequality-constrained problem (33) with solution at $(1, 0)^{\mathrm{T}}$.

Therefore, the KKT conditions (32a)-(32e) are satisfied when we set $\lambda_* = (3/4, 1/4, 0, 0)^{\mathrm{T}}$.

# §12.4 First-Order Optimality Conditions: Proof

- **A fundamental necessary condition**

As mentioned above, a local solution of

$$\min_{x\in\mathbb{R}^n} f(x) \quad \text{subject to} \quad \begin{cases} c_i(x) = 0, i \in \mathcal{E}, \\ c_i(x) \geqslant 0, i \in \mathcal{I}, \end{cases} \tag{1}$$

is a point $x$ at which all feasible sequences have the property that $f(z_k) \geqslant f(x)$ for all $k$ sufficiently large. The following result shows that if such a sequence exists, then its limiting directions must make a non-negative inner product with the objective function gradient.

**Theorem**

*If $x_*$ is a local solution of (1), then we have*

$$(\nabla f)(x_*)^{\mathrm{T}} d \geqslant 0 \quad \textit{for all } d \in T_\Omega(x_*) .$$

# §12.4 First-Order Optimality Conditions: Proof

**Proof.**

Suppose the contrary that there exists $d \in T_\Omega(x_*)$ for which $(\nabla f)(x_*)^{\mathrm{T}} d < 0$. Let $\{z_k\} \subseteq \Omega$ and $\{t_k\} \subseteq \mathbb{R}^+$ be sequences satisfying

$$\lim_{k\to\infty} t_k = 0 \quad \text{and} \quad \lim_{k\to\infty} \frac{z_k - x_*}{t_k} = d\,.$$

By Taylor's Theorem,

$$\begin{aligned} f(z_k) &= f(x_*) + (z_k - x_*)^{\mathrm{T}}(\nabla f)(x_*) + o(\|z_k - x_*\|) \\ &= f(x_*) + t_k d^{\mathrm{T}}(\nabla f)(x_*) + o(t_k)\,. \end{aligned}$$

Since $d^{\mathrm{T}}(\nabla f)(x_*) < 0$, the remainder term is eventually dominated by the first-order term; thus

$$f(z_k) < f(x_*) + \frac{1}{2} t_k d^{\mathrm{T}}(\nabla f)(x_*) \qquad \forall\, k \gg 1\,.$$

Since $d^{\mathrm{T}}(\nabla f)(x_*) < 0$, $x_*$ cannot be a local solution. □

# §12.4 First-Order Optimality Conditions: Proof

The converse of this result is not necessarily true. That is, we may have $(\nabla f)(x_*)^{\mathrm{T}} d \geqslant 0$ for all $d \in T_\Omega(x_*)$, yet $x_*$ is not a local minimizer. An example is the following problem in two unknowns, illustrated in Figure 12.

$$\min x_2 \quad \text{subject to} \quad x_2 \geqslant -x_1^2\,. \tag{34}$$

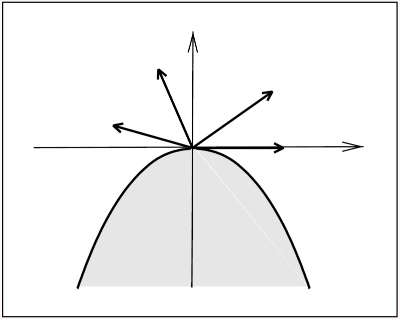

Figure 12: Problem (34), showing various limiting directions of feasible sequences at the point $(0,0)^{\mathrm{T}}$.

# §12.4 First-Order Optimality Conditions: Proof

This problem is actually unbounded, but let us examine its behavior at $x_* = (0,0)^{\mathrm{T}}$. It is not difficult to show that all limiting directions $d$ of feasible sequences must have $d_2 \geqslant 0$, so that

$$(\nabla f)(x_*)^{\mathrm{T}} d = d_2 \geqslant 0 \,.$$

However, $x_*$ is clearly not a local minimizer; the point $(\alpha, -\alpha^2)^{\mathrm{T}}$ for $\alpha > 0$ has a smaller function value than $x_*$, and can be brought arbitrarily close to $x_*$ by setting $\alpha$ sufficiently small.

# §12.4 First-Order Optimality Conditions: Proof

- **Farkas' lemma**

The most important step in proving the 1st-order necessary condition is a classical theorem of the alternative known as Farkas' Lemma. This lemma considers a cone $K$ defined as follows:

$$K = \{By + Cw \,|\, y \geqslant 0\}, \tag{35}$$

where $B$ and $C$ are **given** matrices of dimension $n \times m$ and $n \times p$, respectively, and $y$ and $w$ are **arbitrary** vectors of appropriate dimensions. Given a vector $g \in \mathbb{R}^n$, Farkas' Lemma states that one (and only one) of two alternatives is true: either $g \in K$, or else there is a vector $d \in \mathbb{R}^n$ such that

$$g^{\mathrm{T}} d < 0, \quad B^{\mathrm{T}} d \geqslant 0, \quad C^{\mathrm{T}} d = 0. \tag{36}$$

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# §12.4 First-Order Optimality Conditions: Proof

The two cases are illustrated in Figure 13 for the case of $B$ with three columns, $C$ null, and $n = 2$. Note that in the second case, the vector $d$ defines a separating hyperplane, which is a plane in $\mathbb{R}^n$ that separates the vector $g$ from the cone $K$.

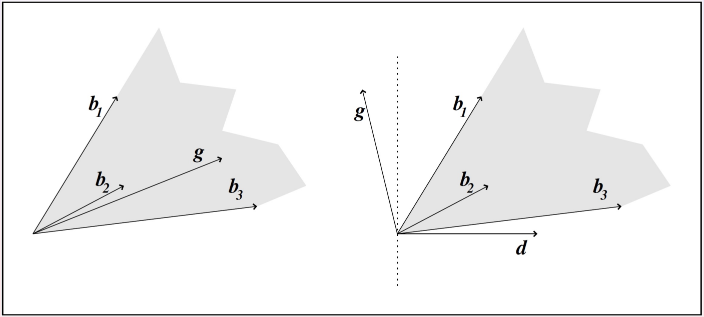


Figure 13: Farkas' Lemma: either $g \in K$ (left) or there is a separating hyperplane (right).

# §12.4 First-Order Optimality Conditions: Proof

## Lemma (Farkas)

*Let $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{n\times p}$ be given, and $K$ be a set defined by*

$$K = \left\{ By + Cw \,\middle|\, y \geqslant 0\,, w \in \mathbb{R}^p \right\}. \tag{35}$$

*For a given vector $g \in \mathbb{R}^n$, we have either that $g \in K$ or that there exists $d \in \mathbb{R}^n$ satisfying*

$$g^{\mathrm{T}} d < 0, \quad B^{\mathrm{T}} d \geqslant 0, \quad C^{\mathrm{T}} d = 0\,, \tag{36}$$

*but **not** both.*

### Proof.

We show first that the two alternatives cannot hold simultaneously. If $g \in K$, there exist vectors $y \geqslant 0$ and $w$ such that $g = By + Cw$. If there also exists a $d$ with the property (36), we have

$$0 > d^{\mathrm{T}} g = d^{\mathrm{T}} By + d^{\mathrm{T}} Cw = (B^{\mathrm{T}} d)^{\mathrm{T}} y + (C^{\mathrm{T}} d)^{\mathrm{T}} w \geqslant 0\,.$$

Hence, we cannot have both alternatives holding at once. □

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# §12.4 First-Order Optimality Conditions: Proof

## Proof (cont'd).

We now show that one of the alternatives holds. To be precise, we show how to construct $d$ with the properties (36) in the case that $g \notin K$. For this part of the proof, we need to use the property that $K$ is a closed set – a fact that is intuitively obvious but not trivial to prove (see Lemma 12.15 in the textbook). Let $\{s_k\} \subseteq K$ be a minimizing sequence satisfying

$$\inf_{s \in K} \|s - g\| \leqslant \|s_k - g\| < \inf_{s \in K} \|s - g\| + \frac{1}{k}\,.$$

Then the fact that $\{s_k\} \subseteq K \cap B\Big[g, \inf_{s \in K} \|s - g\| + 1\Big]$ implies that there exists a convergent subsequence $\{s_{k_j}\}$ with limit $\widehat{s} \in K$. Such $\widehat{s}$ is the vector in $K$ that is closet to $g$ in the sense of the Euclidean norm. Since $\widehat{s} \in K$, we have from the fact that $K$ is a cone that $\alpha \widehat{s} \in K$ for all scalars $\alpha \geqslant 0$. □

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# §12.4 First-Order Optimality Conditions: Proof

## Proof (cont'd).

Since $\|\alpha \widehat{s} - g\|^2$ is minimized by $\alpha = 1$, we have

$$\frac{d}{d\alpha}\Big|_{\alpha=1} \|\alpha \widehat{s} - g\|^2 = 0 \quad \Rightarrow \quad \widehat{s}^{\mathrm{T}}(\widehat{s} - g) = 0 \,. \tag{37}$$

Now, let s be any other vector in $K$. Since $K$ is convex, we have by the minimizing property of $\widehat{s}$ that

$$\|\widehat{s} + \theta(s - \widehat{s}) - g\|^2 \geqslant \|\widehat{s} - g\|^2 \quad \text{for all } \theta \in [0,1] \,,$$

and hence

$$2\theta(s - \widehat{s})^{\mathrm{T}}(\widehat{s} - g) + \theta^2 \|s - \widehat{s}\|^2 \geqslant 0 \quad \text{for all } \theta \in [0,1] \,.$$

By dividing this expression by $\theta$ and taking the limit as $\theta \searrow 0$, we have $(s - \widehat{s})^{\mathrm{T}}(\widehat{s} - g) \geqslant 0$. Therefore, because of (37),

$$s^{\mathrm{T}}(\widehat{s} - g) \geqslant 0 \quad \text{for all } s \in K \,. \tag{38}$$

# §12.4 First-Order Optimality Conditions: Proof

## Proof (cont'd).

Since $\|\alpha\widehat{s}-g\|^2$ is minimized by $\alpha=1$, we have

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$$\|\widehat{s}+\theta(s-\widehat{s})-g\|^2 \geqslant \|\widehat{s}-g\|^2 \quad \text{for all } \theta\in[0,1]\,,$$

and hence

$$2\theta(s-\widehat{s})^{\mathrm{T}}(\widehat{s}-g)+\theta^2\|s-\widehat{s}\|^2 \geqslant 0 \quad \text{for all } \theta\in[0,1]\,.$$

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# §12.4 First-Order Optimality Conditions: Proof

**Proof (cont'd).**

We claim now that the vector $d = \hat{s} - g$ satisfies the conditions

$$g^{\mathrm{T}}d < 0, \quad B^{\mathrm{T}}d \geqslant 0, \quad C^{\mathrm{T}}d = 0\,. \tag{36}$$

Note that $d \neq 0$ because $g \notin K$. We have from (37) that

$$d^{\mathrm{T}}g = d^{\mathrm{T}}(\hat{s} - d) = (\hat{s} - g)^{\mathrm{T}}\hat{s} - d^{\mathrm{T}}d = -\|d\|^2 < 0\,,$$

so that $d$ satisfies the first property in (36).

From (38), we have that $d^{\mathrm{T}}s \geqslant 0$ for all $s \in K$, so that

$$d^{\mathrm{T}}(By + Cw) \geqslant 0 \quad \text{for all } y \geqslant 0 \text{ and all } w.$$

By fixing $y = 0$ we have that $(C^{\mathrm{T}}d)^{\mathrm{T}}w \geqslant 0$ for all $w$, which is true only if $C^{\mathrm{T}}d = 0$. By fixing $w = 0$, we have that $(B^{\mathrm{T}}d)^{\mathrm{T}}y \geqslant 0$ for all $y \geqslant 0$, which is true only if $B^{\mathrm{T}}d \geqslant 0$. Hence, $d$ also satisfies the second and third properties in (36), and our proof is complete. □

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# §12.4 First-Order Optimality Conditions: Proof

By applying Farkas' Lemma to the cone $N$ defined by

$$N = \left\{ \sum_{i \in \mathcal{A}(x_*)} \lambda_i \nabla c_i(x_*) \,\middle|\, \{\lambda_i\}_{i \in \mathcal{A}(x_*)} \subseteq \mathbb{R}, \lambda_i \geqslant 0 \text{ if } i \in \mathcal{A}(x_*) \cap \mathcal{I} \right\},$$

(with $B = \big[\nabla c_i(x_*)\big]_{i \in \mathcal{A}(x_*) \cap \mathcal{I}}$ and $C = \big[\nabla c_i(x_*)\big]_{i \in \mathcal{A}(x_*) \setminus \mathcal{I}}$ in the definition of the cone $K$ in Farkas' Lemma) and setting $g = (\nabla f)(x_*)$, we have that either

$$(\nabla f)(x_*) = \sum_{i \in \mathcal{A}(x_*)} \lambda_i \nabla c_i(x_*)$$

for some collection of multipliers $\{\lambda_i\}_{i \in \mathcal{A}(x_*)} \subseteq \mathbb{R}$ with $\lambda_i \geqslant 0$ if $i \in \mathcal{A}(x_*) \cap \mathcal{I}$, or else there is a direction $d$ such that

$$d^{\mathrm{T}}(\nabla f)(x_*) < 0, \big[\nabla c_i(x_*)\big]^{\mathrm{T}}_{i \in \mathcal{A}(x_*) \cap \mathcal{I}} d \geqslant 0, \big[\nabla c_i(x_*)\big]^{\mathrm{T}}_{i \in \mathcal{A}(x_*) \setminus \mathcal{I}} d = 0 .$$

Note that by the definition of the linearized feasible direction, the latter two imply that $d \in \mathcal{F}(x_*)$ (and vice versa).

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$$d^{\mathrm{T}} (\nabla f)(x_*) < 0\,.$$

Therefore, if $\nabla f(x_*)^{\mathrm{T}} d \geqslant 0$ for all $d \in \mathcal{F}(x_*)$, then $(\nabla f)(x_*)$ belongs to $N$.

# §12.4 First-Order Optimality Conditions: Proof

By applying Farkas' Lemma to the cone $N$ defined by

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# §12.4 First-Order Optimality Conditions: Proof

## Theorem (First-Order Necessary Conditions)

*Suppose that $x_*$ is a local solution of problem*

$$\min_{x\in\mathbb{R}^n} f(x) \quad \textit{subject to} \quad \begin{cases} c_i(x) = 0, i \in \mathcal{E}, \\ c_i(x) \geqslant 0, i \in \mathcal{I}, \end{cases} \tag{1}$$

*that the functions $f$ and $c_i$ in (1) are continuously differentiable, and that the LICQ holds at $x_*$. Then there is a Lagrange multiplier vector $\lambda_*$, with components $\lambda_i^*$, $i \in \mathcal{E} \cup \mathcal{I}$, such that the following conditions are satisfied.*

$$\nabla_x \mathcal{L}(x_*, \lambda_*) = 0, \tag{32a}$$

$$c_i(x_*) = 0 \quad \textit{for all } i \in \mathcal{E}, \tag{32b}$$

$$c_i(x_*) \geqslant 0 \quad \textit{for all } i \in \mathcal{I}, \tag{32c}$$

$$\lambda_i^* \geqslant 0 \quad \textit{for all } i \in \mathcal{I}, \tag{32d}$$

$$\lambda_i^* c_i(x_*) = 0 \quad \textit{for all } i \in \mathcal{E} \cup \mathcal{I}. \tag{32e}$$

# §12.4 First-Order Optimality Conditions: Proof

**Proof.**

Suppose that $x_* \in \mathbb{R}^n$ is a local solution of (1) at which the LICQ holds. Then the established lemmas and theorem show that

$$T_\Omega(x_*) = \mathcal{F}(x_*) \quad \text{and} \quad d^{\mathrm{T}}(\nabla f)(x_*) \geqslant 0 \ \ \forall\, d \in T_\Omega(x_*)\,.$$

Therefore, $d^{\mathrm{T}}(\nabla f)(x_*) \geqslant 0$ for all $d \in \mathcal{F}(x_*)$; thus Farkas' Lemma implies that there are multipliers $\{\lambda_i\}_{i\in\mathcal{A}(x_*)} \subseteq \mathbb{R}$ such that

$$(\nabla f)(x_*) = \sum_{i\in\mathcal{A}(x_*)} \lambda_i \nabla c_i(x_*), \quad \lambda_i \geqslant 0 \text{ if } i \in \mathcal{A}(x_*) \cap \mathcal{I}\,. \tag{39}$$

Define the vector $\lambda_*$ by

$$\lambda_i^* = \begin{cases} \lambda_i & \text{if } i \in \mathcal{A}(x_*), \\ 0 & \text{if } i \in \mathcal{I} \setminus \mathcal{A}(x_*), \end{cases} \tag{40}$$

We claim that this choice of $\lambda_*$, together with our local solution $x_*$, satisfies the conditions (32). □

# §12.4 First-Order Optimality Conditions: Proof

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Suppose that $x_* \in \mathbb{R}^n$ is a local solution of (1) at which the LICQ holds. Then the established lemmas and theorem show that

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# §12.4 First-Order Optimality Conditions: Proof

**Proof (cont'd).**

We check these conditions in turn.

1. The condition (32a) follows immediately from (39) and the definitions (31) of the Lagrangian function and (40) of $\lambda_*$.
2. Since $x_*$ is feasible, the conditions (32b) and (32c) are satisfied.
3. We have from (39) that $\lambda_i^* \geqslant 0$ for $i \in \mathcal{A}(x_*) \cap \mathcal{I}$, while from (40), $\lambda_i^* = 0$ for $i \in \mathcal{I} \setminus \mathcal{A}(x_*)$. Hence, $\lambda_i^* \geqslant 0$ for $i \in \mathcal{I}$, so that (32d) holds.
4. We have for $i \in \mathcal{A}(x_*) \cap \mathcal{I}$ that $c_i(x_*) = 0$, while for $i \in \mathcal{I} \setminus \mathcal{A}(x_*)$, we have $\lambda_i^* = 0$. Hence $\lambda_i^* c_i(x_*) = 0$ for $i \in \mathcal{I}$, so that (32e) is satisfied as well.

The proof is complete. □

# §12.4 First-Order Optimality Conditions: Proof

From the proof of the theorem, the requirement of LICQ at a local solution $x_*$ is to motivate the condition

$$T_\Omega(x_*) = \mathcal{F}(x_*)\,. \tag{41}$$

Therefore, we indeed have the following

**Theorem (First-Order Necessary Conditions)**

*Suppose that $x_*$ is a local solution of (1) in which the functions $f$ and $c_i$ are continuously differentiable. If (41) holds, then there is a Lagrange multiplier vector $\lambda_*$ with components $\lambda_i^*$ such that*

$$\nabla_x \mathcal{L}(x_*, \lambda_*) = 0\,, \tag{32a}$$

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$$\lambda_i^* c_i(x_*) = 0 \quad \text{for all } i \in \mathcal{E} \cup \mathcal{I}. \tag{32e}$$

# §12.5 Second-Order Conditions

So far, we have described first-order conditions – the KKT conditions – which tell us how the first derivatives of $f$ and the active constraints $c_i$ are related to each other at a solution $x_*$. When these conditions are satisfied, a move along any vector $w$ from $\mathcal{F}(x_*)$ either increases the first-order approximation to the objective function (that is, $w^{\mathrm{T}}(\nabla f)(x_*) > 0$), or else keeps this value the same (that is, $w^{\mathrm{T}}(\nabla f)(x_*) = 0$).

# §12.5 Second-Order Conditions

What role do the second derivatives of $f$ and the constraints $c_i$ play in optimality conditions? We see in this section that second derivatives play a "tiebreaking" role. For the directions $w \in \mathcal{F}(x_*)$ for which $w^{\mathrm{T}}(\nabla f)(x_*) = 0$, we cannot determine from first derivative information alone whether a move along this direction will increase or decrease the objective function $f$. Second-order conditions examine the second derivative terms in the Taylor series expansions of $f$ and $c_i$, to see whether this extra information resolves the issue of increase or decrease in $f$. Essentially, the second-order conditions concern the curvature of the Lagrangian function in the "undecided" directions – the directions $w \in \mathcal{F}(x_*)$ for which $w^{\mathrm{T}}(\nabla f)(x_*) = 0$.

# §12.5 Second-Order Conditions

Given $\mathcal{F}(x_*)$ and some Lagrange multiplier vector $\lambda_*$ satisfying the KKT conditions (32), we define the critical cone $\mathcal{C}(x_*, \lambda_*)$ as follows:

$$\mathcal{C}(x_*, \lambda_*) = \left\{ w \in \mathcal{F}(x_*) \,\middle|\, \nabla c_i(x_*)^{\mathrm{T}} w = 0 \text{ if } i \in \mathcal{A}(x_*) \cap \mathcal{I} \;\&\; \lambda_i^* > 0 \right\}$$

or equivalently,

$$w \in \mathcal{C}(x_*, \lambda_*) \Leftrightarrow \begin{cases} \nabla c_i(x_*)^{\mathrm{T}} w = 0 & \text{if } i \in \mathcal{E}, \\ \nabla c_i(x_*)^{\mathrm{T}} w = 0 & \text{if } i \in \mathcal{A}(x_*) \cap \mathcal{I} \text{ and } \lambda_i^* > 0, \\ \nabla c_i(x_*)^{\mathrm{T}} w \geqslant 0 & \text{if } i \in \mathcal{A}(x_*) \cap \mathcal{I} \text{ and } \lambda_i^* = 0. \end{cases} \tag{42}$$

The critical cone contains those directions $w$ that would tend to "adhere" to the active inequality constraints even when we were to make small changes to the objective (those indices $i \in \mathcal{I}$ for which the Lagrange multiplier component $\lambda_i^*$ is positive), as well as to the equality constraints.

# §12.5 Second-Order Conditions

Given $\mathcal{F}(x_*)$ and some Lagrange multiplier vector $\lambda_*$ satisfying the KKT conditions (32), we define the critical cone $\mathcal{C}(x_*, \lambda_*)$ as follows:

$$\mathcal{C}(x_*, \lambda_*) = \left\{ w \in \mathcal{F}(x_*) \,\middle|\, \nabla c_i(x_*)^{\mathrm{T}} w = 0 \text{ if } i \in \mathcal{A}(x_*) \cap \mathcal{I} \;\&\; \lambda_i^* > 0 \right\}$$

or equivalently,

$$w \in \mathcal{C}(x_*, \lambda_*) \Leftrightarrow \begin{cases} \nabla c_i(x_*)^{\mathrm{T}} w = 0 & \text{if } i \in \mathcal{E}, \\ \nabla c_i(x_*)^{\mathrm{T}} w = 0 & \text{if } i \in \mathcal{A}(x_*) \cap \mathcal{I} \text{ and } \lambda_i^* > 0, \\ \nabla c_i(x_*)^{\mathrm{T}} w \geqslant 0 & \text{if } i \in \mathcal{A}(x_*) \cap \mathcal{I} \text{ and } \lambda_i^* = 0. \end{cases} \tag{42}$$

The critical cone contains those directions $w$ that would tend to "adhere" to the active inequality constraints even when we were to make small changes to the objective (those indices $i \in \mathcal{I}$ for which the Lagrange multiplier component $\lambda_i^*$ is positive), as well as to the equality constraints.

# §12.5 Second-Order Conditions

From the definition (42) and the fact that $\lambda_i^* = 0$ for all inactive components $i \in \mathcal{I} \setminus \mathcal{A}(x_*)$, it follows immediately that

$$w \in \mathcal{C}(x_*, \lambda_*) \quad \Rightarrow \quad \lambda_i^* \nabla c_i(x_*)^{\mathrm{T}} w = 0 \text{ if } i \in \mathcal{E} \cup \mathcal{I}. \tag{43}$$

Hence, from the first KKT condition (32a) and the definition (31) of the Lagrangian function, we have that

$$w \in \mathcal{C}(x_*, \lambda_*) \;\Rightarrow\; w^{\mathrm{T}} (\nabla f)(x_*) = \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i^* w^{\mathrm{T}} \nabla c_i(x_*) = 0 \,.$$

Hence the critical cone $\mathcal{C}(x_*, \lambda_*)$ contains directions from $\mathcal{F}(x_*)$ for which it is not clear from first derivative information alone whether $f$ will increase or decrease.

# §12.5 Second-Order Conditions

## Example

Consider the problem

$$\min x_1 \quad \text{subject to} \quad x_2 \geqslant 0\,, 1-(x_1-1)^2-x_2^2 \geqslant 0\,, \qquad (44)$$

illustrated in Figure 14. It is not difficult to see that the solution is $x_* = (0,0)^{\mathrm{T}}$, with active set $\mathcal{A}(x_*) = \{1,2\}$ and a unique optimal Lagrange multiplier $\lambda_* = (0, 0.5)^{\mathrm{T}}$. Since the gradients of the active constraints at $x_*$ are $(0,1)^{\mathrm{T}}$ and $(2,0)^{\mathrm{T}}$, respectively, the LICQ holds, so the optimal multiplier is unique. The linearized feasible set is then

$$\mathcal{F}(x_*) = \left\{ d \,\middle|\, d^{\mathrm{T}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \geqslant 0, d^{\mathrm{T}} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \geqslant 0 \right\} = \{ d \,|\, d \geqslant 0 \}\,,$$

while the critical cone is

$$\mathcal{C}(x_*, \lambda_*) = \left\{ w \in \mathcal{F}(x_*) \,\middle|\, w^{\mathrm{T}} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 0 \right\} = \{ (0, w_2)^{\mathrm{T}} \,|\, w_2 \geqslant 0 \}\,.$$

# §12.5 Second-Order Conditions

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# §12.5 Second-Order Conditions

## Example (cont'd)

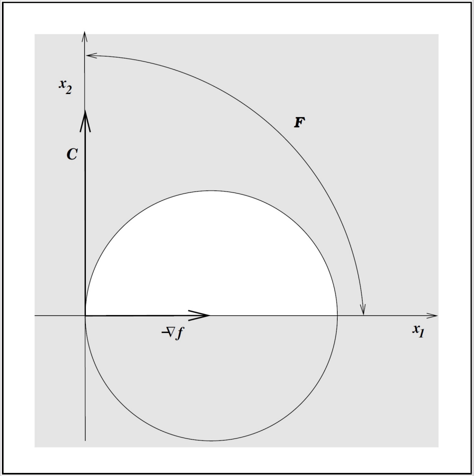


Figure 14: Problem (44), showing $\mathcal{F}(x_*)$ and $\mathcal{C}(x_*, \lambda_*)$.

# §12.5 Second-Order Conditions

The theorem in the next slide defines a necessary condition involving the second derivatives: If $x_*$ is a local solution, then the Hessian of the Lagrangian has nonnegative curvature along critical directions (that is, the directions in $\mathcal{C}(x_*, \lambda_*)$).

# §12.5 Second-Order Conditions

## Theorem (Second-Order Necessary Conditions)

*Suppose that* $x_*$ *is a local solution of*

$$\min_{x\in\mathbb{R}^n} f(x) \quad \textit{subject to} \quad \begin{cases} c_i(x) = 0, i \in \mathcal{E}, \\ c_i(x) \geqslant 0, i \in \mathcal{I}, \end{cases} \tag{1}$$

*and that the LICQ condition is satisfied at* $x_*$. *Let* $\lambda_*$ *be the Lagrange multiplier vector for which the KKT conditions*

$$\nabla_x \mathcal{L}(x_*, \lambda_*) = 0, \tag{32a}$$

$$c_i(x_*) = 0 \quad \textit{for all } i \in \mathcal{E}, \tag{32b}$$

$$c_i(x_*) \geqslant 0 \quad \textit{for all } i \in \mathcal{I}, \tag{32c}$$

$$\lambda_i^* \geqslant 0 \quad \textit{for all } i \in \mathcal{I}, \tag{32d}$$

$$\lambda_i^* c_i(x_*) = 0 \quad \textit{for all } i \in \mathcal{E} \cup \mathcal{I}. \tag{32e}$$

*are satisfied. Then*

$$w^{\mathrm{T}} \nabla_{xx}^2 \mathcal{L}(x_*, \lambda_*) w \geqslant 0 \quad \textit{for all } w \in \mathcal{C}(x_*, \lambda_*). \tag{45}$$

# §12.5 Second-Order Conditions

## Proof.

Let $w \in \mathcal{C}(x_*, \lambda_*)$ be given. Since the LICQ condition holds at $x_*$ and $\mathcal{C}(x_*, \lambda_*) \subseteq \mathcal{F}(x_*)$, there exist a feasible sequence $\{z_k\}$ approaching $x_*$ and a sequence $\{t_k\}$ of positive scalars approaching $0$ such that

$$\lim_{k\to\infty} \frac{z_k - x_*}{t_k} = w \tag{23}$$

and

$$c_i(z_k) = t_k \nabla c_i(x_*)^{\mathrm{T}} w \qquad \forall\, i \in \mathcal{A}(x_*) \text{ and } k \gg 1\,. \tag{28}$$

The fact that the multiplier corresponding to inactive constraint is zero implies that for $k$ sufficiently large,

$$\mathcal{L}(z_k, \lambda_*) = f(z_k) - \sum_{i\in\mathcal{E}\cup\mathcal{I}} \lambda_i^* c_i(z_k) = f(z_k) - \sum_{i\in\mathcal{A}(x_*)} \lambda_i^* c_i(z_k)$$

$$= f(z_k) - t_k \sum_{i\in\mathcal{A}(x_*)} \lambda_i^* \nabla c_i(x_*)^{\mathrm{T}} w\,.$$

□

# §12.5 Second-Order Conditions

**Proof.**

Let $w \in \mathcal{C}(x_*, \lambda_*)$ be given. Since the LICQ condition holds at $x_*$ and $\mathcal{C}(x_*, \lambda_*) \subseteq \mathcal{F}(x_*)$, there exist a feasible sequence $\{z_k\}$ approaching $x_*$ and a sequence $\{t_k\}$ of positive scalars approaching $0$ such that

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$$\begin{aligned}\mathcal{L}(z_k, \lambda_*) &= f(z_k) - \sum_{i\in\mathcal{E}\cup\mathcal{I}} \lambda_i^* c_i(z_k) = f(z_k) - \sum_{i\in\mathcal{A}(x_*)} \lambda_i^* c_i(z_k) \\ &= f(z_k) - t_k \sum_{i\in\mathcal{A}(x_*)} \lambda_i^* \nabla c_i(x_*)^{\mathrm{T}} w.\end{aligned}$$

□

# §12.5 Second-Order Conditions

## Proof (cont'd).

Since $w \in \mathcal{C}(x_*, \lambda_*)$, using (43) (which shows that $\lambda_i^* \nabla c_i(x_*)^{\mathrm{T}} w = 0$ for all $i \in \mathcal{E} \cup \mathcal{I}$) we obtain that

$$\mathcal{L}(z_k, \lambda_*) = f(z_k).$$

On the other hand, using Taylor's Theorem expression and continuity of the Hessians $\nabla^2 f$ and $\nabla^2 c_i$, $i \in \mathcal{E} \cup \mathcal{I}$, we obtain

$$\begin{aligned}\mathcal{L}(z_k, \lambda_*) &= \mathcal{L}(x_*, \lambda_*) + (z_k - x_*)^{\mathrm{T}} \nabla_x \mathcal{L}(x_*, \lambda_*) \\ &\quad + \frac{1}{2}(z_k - x_*)^{\mathrm{T}} \nabla^2_{xx} \mathcal{L}(x_*, \lambda_*)(z_k - x_*) + o(\|z_k - x_*\|^2). \end{aligned} \tag{46}$$

By the complementarity conditions (32e), $\mathcal{L}(x_*, \lambda_*) = f(x_*)$. From (32a), $\nabla_x \mathcal{L}(x_*, \lambda_*) = 0$ so the second term on the right-hand side is zero. Also note that the limit (23) can be rewritten as

$$z_k - x_* = t_k w + o(t_k).$$

□

# §12.5 Second-Order Conditions

## Proof (cont'd).

Since $w \in \mathcal{C}(x_*, \lambda_*)$, using (43) (which shows that $\lambda_i^* \nabla c_i(x_*)^{\mathrm{T}} w = 0$ for all $i \in \mathcal{E} \cup \mathcal{I}$) we obtain that

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$$\mathcal{L}(z_k, \lambda_*) = \mathcal{L}(x_*, \lambda_*) + (z_k - x_*)^{\mathrm{T}} \nabla_x \mathcal{L}(x_*, \lambda_*) + \frac{1}{2}(z_k - x_*)^{\mathrm{T}} \nabla^2_{xx} \mathcal{L}(x_*, \lambda_*)(z_k - x_*) + o(\|z_k - x_*\|^2)\,. \tag{46}$$

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# §12.5 Second-Order Conditions

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□

# §12.5 Second-Order Conditions

## Proof (cont'd).

Therefore,

$$f(z_k) = f(x_*) + \frac{1}{2} t_k^2 w^{\mathrm{T}} \nabla_{xx}^2 \mathcal{L}(x_*, \lambda_*) w + o(t_k^2).$$

If $w^{\mathrm{T}} \nabla_{xx}^2 \mathcal{L}(x_*, \lambda_*) w < 0$, then $f(z_k) < f(x_*)$ for $k \gg 1$, contradicting the fact that $x_*$ is a local solution. Hence, the condition

$$w^{\mathrm{T}} \nabla_{xx}^2 \mathcal{L}(x_*, \lambda_*) w \geqslant 0 \quad \text{for all } w \in \mathcal{C}(x_*, \lambda_*) \tag{45}$$

must hold, as claimed. □

The second-order sufficient condition stated in the next theorem looks very much like the necessary condition just discussed, but it differs in that

1. **the constraint qualification is not required**, and
2. the inequality in (45) is replaced by a **strict** inequality

$$w^{\mathrm{T}} \nabla_{xx}^2 \mathcal{L}(x_*, \lambda_*) w > 0 \quad \text{for all } w \in \mathcal{C}(x_*, \lambda_*) \backslash \{0\}. \tag{48}$$

# §12.5 Second-Order Conditions

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# §12.5 Second-Order Conditions

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# §12.5 Second-Order Conditions

## Theorem (Second-Order Sufficient Conditions)

*Suppose that for some feasible point* $x_* \in \mathbb{R}^n$ *there is a Lagrange multiplier vector* $\lambda_*$ *such that the KKT conditions*

$$\nabla_x \mathcal{L}(x_*, \lambda_*) = 0\,, \tag{32a}$$

$$c_i(x_*) = 0 \quad \text{for all } i \in \mathcal{E}, \tag{32b}$$

$$c_i(x_*) \geqslant 0 \quad \text{for all } i \in \mathcal{I}, \tag{32c}$$

$$\lambda_i^* \geqslant 0 \quad \text{for all } i \in \mathcal{I}, \tag{32d}$$

$$\lambda_i^* c_i(x_*) = 0 \quad \text{for all } i \in \mathcal{E} \cup \mathcal{I}. \tag{32e}$$

*are satisfied. Suppose also that*

$$w^{\mathrm{T}} \nabla^2_{xx} \mathcal{L}(x_*, \lambda_*) w > 0 \quad \text{for all } w \in \mathcal{C}(x_*, \lambda_*) \backslash \{0\}\,. \tag{48}$$

*Then* $x_*$ *is a strict local solution for*

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{subject to} \quad \begin{cases} c_i(x) = 0, i \in \mathcal{E}\,, \\ c_i(x) \geqslant 0, i \in \mathcal{I}\,. \end{cases} \tag{1}$$

# §12.5 Second-Order Conditions

## Proof.

Before proceeding, note that the set $\bar{\mathcal{C}} = \left\{d \in \mathcal{C}(x_*, \lambda_*) \,\middle|\, \|d\| = 1\right\}$ is a compact subset of $\mathcal{C}(x_*, \lambda_*)$, so by (48),

$$\min_{d\in\bar{\mathcal{C}}} d^{\mathrm{T}}\nabla^2_{xx}\mathcal{L}(x_*, \lambda_*)d \equiv \sigma > 0\,.$$

Since $\mathcal{C}(x_*, \lambda_*)$ is a cone, we have that $w/\|w\| \in \bar{\mathcal{C}}$ if and only if $w \in \mathcal{C}(x_*, \lambda_*)\backslash\{0\}$. Therefore, condition (48) implies that

$$w^{\mathrm{T}}\nabla^2_{xx}\mathcal{L}(x_*, \lambda_*)w \geqslant \sigma\|w\|^2 \quad \text{for all } w \in \mathcal{C}(x_*, \lambda_*)\,, \tag{49}$$

where $\sigma > 0$ is defined above. Moreover, by Taylor's Theorem the KKT condition (32a) shows that

$$\begin{aligned}\mathcal{L}(x, \lambda_*) = f(x_*) + \frac{1}{2}(x - x_*)^{\mathrm{T}}\nabla^2_{xx}\mathcal{L}(x_*, \lambda_*)(x - x_*)\\ + o(\|x - x_*\|^2)\,.\end{aligned} \tag{50}$$

# §12.5 Second-Order Conditions

**Proof.**

Before proceeding, note that the set $\bar{\mathcal{C}} = \big\{ d \in \mathcal{C}(x_*, \lambda_*) \,\big|\, \|d\| = 1 \big\}$ is a compact subset of $\mathcal{C}(x_*, \lambda_*)$, so by (48),

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□

# §12.5 Second-Order Conditions

## Proof (cont'd).

Now we prove the result by showing that every feasible sequence $\{z_k\}$ approaching $x_*$ satisfies

$$f(z_k) \geqslant f(x_*) + \frac{\sigma}{4}\|z_k - x_*\|^2 \quad \forall\, k \gg 1\,.$$

Suppose the contrary that there is a feasible sequence $\{z_k\}$ approaching $x_*$ with

$$f(z_k) < f(x_*) + \frac{\sigma}{4}\|z_k - x_*\|^2 \quad \forall\, k \gg 1\,. \tag{51}$$

By taking a subsequence if necessary, we can identify a limiting direction $d$ such that

$$\lim_{k\to\infty} \frac{z_k - x_*}{\|z_k - x_*\|} = d\,.$$

We then have $d \in T_\Omega(x_*)$, and the fact that $T_\Omega(x_*) \subseteq \mathcal{F}(x_*)$ shows that $d \in \mathcal{F}(x_*)$. Next we show that $d \in \mathcal{C}(x_*, \lambda_*)$. □

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By taking a subsequence if necessary, we can identify a limiting direction $d$ such that

$$z_k - x_* = t_k d + o(t_k), \quad t_k = \|z_k - x_*\|\,.$$

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# §12.5 Second-Order Conditions

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Suppose the contrary that there is a feasible sequence $\{z_k\}$ approaching $x_*$ with

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By taking a subsequence if necessary, we can identify a limiting direction $d$ such that

$$z_k - x_* = t_k d + o(t_k), \quad t_k = \|z_k - x_*\|\,.$$

We then have $d \in T_\Omega(x_*)$, and the fact that $T_\Omega(x_*) \subseteq \mathcal{F}(x_*)$ shows that $d \in \mathcal{F}(x_*)$. Next we show that $d \in \mathcal{C}(x_*, \lambda_*)$. □

# §12.5 Second-Order Conditions

## Proof (cont'd).

If $d$ were not in $\mathcal{C}(x_*, \lambda_*)$, we could identify some index $j \in \mathcal{A}(x_*) \cap \mathcal{I}$ such that the strict positivity condition

$$\lambda_j^* \nabla c_j(x_*)^{\mathrm{T}} d > 0 \tag{52}$$

is satisfied, while for the remaining indices $i \in \mathcal{A}(x_*)$, we have

$$\lambda_i^* \nabla c_i(x_*)^{\mathrm{T}} d \geqslant 0 \,.$$

From Taylor's Theorem, for this particular value of $j$ we have that

$$\begin{aligned} \lambda_j^* c_j(z_k) &= \lambda_j^* c_j(x_*) + \lambda_j^* \nabla c_j(x_*)^{\mathrm{T}}(z_k - x_*) + o(\|z_k - x_*\|) \\ &= t_k \lambda_j^* \nabla c_j(x_*)^{\mathrm{T}} d + o(t_k). \end{aligned}$$

Recall the Lagrange function

$$\mathcal{L}(x, \lambda) = f(x) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i c_i(x) \,.$$

□

# §12.5 Second-Order Conditions

Proof (cont'd).

If $d$ were not in $\mathcal{C}(x_*, \lambda_*)$, we could identify some index $j \in \mathcal{A}(x_*) \cap \mathcal{I}$ such that the strict positivity condition

$$\lambda_j^* \nabla c_j(x_*)^{\mathrm{T}} d > 0 \tag{52}$$

is satisfied, while for the remaining indices $i \in \mathcal{A}(x_*)$, we have

$$\lambda_i^* \nabla c_i(x_*)^{\mathrm{T}} d \geqslant 0\,.$$

From Taylor's Theorem, for this particular value of $j$ we have that

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# §12.5 Second-Order Conditions

## Proof (cont'd).

Since $\{z_k\}$ is feasible, the KKT condition (32d) implies that

$$\begin{aligned}\mathcal{L}(z_k, \lambda_*) &= f(z_k) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i^* c_i(z_k) \leqslant f(z_k) - \lambda_j^* c_j(z_k) \\ &\leqslant f(z_k) - t_k \lambda_j^* \nabla c_j(x_*)^{\mathrm{T}} d + o(t_k).\end{aligned} \tag{53}$$

On the other hand, (50) shows that

$$\mathcal{L}(z_k, \lambda_*) = f(x_*) + \frac{1}{2} t_k^2 d^{\mathrm{T}} \nabla_{xx}^2 \mathcal{L}(x_*, \lambda_*) d + o(t_k^2)\,;$$

thus, combining the equality above and (53), we conclude that

$$f(z_k) \geqslant f(x_*) + t_k \lambda_j^* \nabla c_j(x_*)^{\mathrm{T}} d + o(t_k),$$

which, because of (52), is a contradiction to

$$f(z_k) < f(x_*) + \frac{\sigma}{4} \|z_k - x_*\|^2 \quad \forall\, k \gg 1\,. \tag{51}$$

□

# §12.5 Second-Order Conditions

## Proof (cont'd).

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# §12.5 Second-Order Conditions

## Proof (cont'd).

Therefore, $d \in \mathcal{C}(x_*, \lambda_*)$, and hence (49) shows that

$$d^{\mathrm{T}} \nabla^2_{xx} \mathcal{L}(x_*, \lambda_*) d \geqslant \sigma \|d\|^2.$$

By the Taylor series estimate (50), we obtain that

$$\begin{aligned} f(z_k) &\geqslant f(z_k) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i^* c_i(z_k) = \mathcal{L}(z_k, \lambda_*) \\ &= f(x_*) + \frac{1}{2} t_k^2 d^{\mathrm{T}} \nabla^2_{xx} \mathcal{L}(x_*, \lambda_*) d + o(t_k^2) \\ &\geqslant f(x_*) + \frac{\sigma}{2} \|z_k - x_*\|^2 + o(\|z_k - x_*\|^2). \end{aligned}$$

This inequality again yields the contradiction to (51). Therefore, every feasible sequence $\{z_k\}$ approaching $x_*$ must satisfy

$$f(z_k) \geqslant f(x_*) + \frac{\sigma}{4} \|z_k - x_*\|^2 \quad \forall\, k \gg 1,$$

so $x_*$ is a strict local solution. □

# §12.5 Second-Order Conditions

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so $x_*$ is a strict local solution. □

# §12.5 Second-Order Conditions

## Example

We now return to the 2nd example in Section 12.1 to check the second-order conditions for problem

$$\min(x_1 + x_2) \quad \text{s.t.} \quad 2 - x_1^2 - x_2^2 \geqslant 0\,. \tag{12}$$

In this problem we have the Lagrange function

$$\mathcal{L}(x, \lambda) = (x_1 + x_2) - \lambda_1(2 - x_1^2 - x_2^2),$$

and $\mathcal{E} = \varnothing$, $\mathcal{I} = \{1\}$. The KKT conditions (32) are satisfied by $x_* = (-1, -1)^{\mathrm{T}}$, with $\lambda_1^* = 1/2$. The Lagrangian Hessian at $x_*$ is

$$\nabla^2_{xx}\mathcal{L}(x_*, \lambda_*) = \begin{bmatrix} 2\lambda_1^* & 0 \\ 0 & 2\lambda_1^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which is positive definite, so it certainly satisfies the conditions of the theorem above. We conclude that $x_* = (-1, -1)^{\mathrm{T}}$ is a strict local solution for (12).

# §12.5 Second-Order Conditions

## Example

For a more complex example, consider the problem

$$\min -0.1(x_1-4)^2 + x_2^2 \quad \text{s.t.} \quad x_1^2 + x_2^2 - 1 \geqslant 0\,, \tag{54}$$

in which we seek to minimize a non-convex function over the **exterior** of the unit circle. Obviously, the objective function is not bounded below on the feasible region, since we can take the feasible sequence

$$\begin{bmatrix} 10 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 20 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 30 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 40 \\ 0 \end{bmatrix}, \quad \cdots\cdots$$

and note that $f(x)$ approaches $-\infty$ along this sequence. Therefore, no global solution exists, but it may still be possible to identify a strict local solution on the boundary of the constraint. We search for such a solution by using the KKT conditions (32) and the second-order conditions of in the previous theorem.

# §12.5 Second-Order Conditions

## Example (cont'd)

By defining the Lagrangian for (54) in the usual way, it is easy to verify that

$$\nabla_x \mathcal{L}(x,\lambda) = \begin{bmatrix} -0.2(x_1-4) - 2\lambda_1 x_1 \\ 2x_2 - 2\lambda_1 x_2 \end{bmatrix}, \tag{55a}$$

$$\nabla^2_{xx} \mathcal{L}(x,\lambda) = \begin{bmatrix} -0.2 - 2\lambda_1 & 0 \\ 0 & 2 - 2\lambda_1 \end{bmatrix}. \tag{55b}$$

The point $x_* = (1,0)^{\mathrm{T}}$ satisfies the KKT conditions with $\lambda_1^* = 0.3$ and the active set $\mathcal{A}(x_*) = \{1\}$. To check that the second-order sufficient conditions are satisfied at this point, we note that

$$\nabla c_1(x_*) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

so that the critical cone is simply

$$\mathcal{C}(x_*,\lambda_*) = \left\{ (0,w_2)^{\mathrm{T}} \,\middle|\, w_2 \in \mathbb{R} \right\}.$$

# §12.5 Second-Order Conditions

## Example (cont'd)

Now, by substituting $x_*$ and $\lambda_*$ into (55b), we have for any $w \in \mathcal{C}(x_*, \lambda_*)$ with $w \neq 0$ that $w_2 \neq 0$ and thus

$$\begin{aligned} w^{\mathrm{T}} \nabla^2_{xx} \mathcal{L}(x_*, \lambda_*) w &= \begin{bmatrix} 0 & w_2 \end{bmatrix} \begin{bmatrix} -0.4 & 0 \\ 0 & 1.4 \end{bmatrix} \begin{bmatrix} 0 \\ w_2 \end{bmatrix} \\ &= 1.4 w_2^2 > 0. \end{aligned}$$

Hence, the second-order sufficient conditions are satisfied, and we conclude from the previous theorem that $(1, 0)^{\mathrm{T}}$ is a strict local solution for (54).

# §12.5 Second-Order Conditions

- **Second-order conditions and projected Hessians**

The second-order conditions are sometimes stated in a form that is slightly weaker but **easier to verify** than

$$w^{\mathrm{T}}\nabla^2_{xx}\mathcal{L}(x_*,\lambda_*)w \geqslant 0 \quad \text{for all } w \in \mathcal{C}(x_*,\lambda_*) \tag{45}$$

and

$$w^{\mathrm{T}}\nabla^2_{xx}\mathcal{L}(x_*,\lambda_*)w > 0 \quad \text{for all } w \in \mathcal{C}(x_*,\lambda_*)\backslash\{0\}\,. \tag{48}$$

This form uses a two-sided projection of the Lagrangian Hessian $\nabla^2_{xx}\mathcal{L}(x_*,\lambda_*)$ onto subspaces that are related to $\mathcal{C}(x_*,\lambda_*)$.

# §12.5 Second-Order Conditions

The simplest case is obtained when the multiplier $\lambda_*$ that satisfies the KKT conditions (32) is unique (as happens, for example, when the LICQ condition holds) and strict complementarity ($\lambda_i^* > 0$ for each $i \in \mathcal{I} \cap \mathcal{A}(x_*)$) holds. In this case, the definition of $\mathcal{C}(x_*, \lambda_*)$ reduces to

$$\mathcal{C}(x_*, \lambda_*) = \text{Null}\Big(\big[\nabla c_i(x_*)^{\mathrm{T}}\big]_{i \in \mathcal{A}(x_*)}\Big) = \text{Null}(A(x_*)),$$

where $A(x_*) \equiv \big[\nabla c_i(x_*)^{\mathrm{T}}\big]_{i \in \mathcal{A}(x_*)}$ is defined as in (27). In other words, in such a case $\mathcal{C}(x_*, \lambda_*)$ is the null space of the matrix whose rows are the active constraint gradients at $x_*$. As in (29), we can define the matrix $Z$ with full column rank whose columns span the space $\mathcal{C}(x_*, \lambda_*)$; that is,

$$\mathcal{C}(x_*, \lambda_*) = \Big\{ Zu \,\Big|\, u \in \mathbb{R}^{n-|\mathcal{A}(x_*)|} \Big\}.$$

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$$\mathcal{C}(x_*, \lambda_*) = \Big\{ Zu \,\Big|\, u \in \mathbb{R}^{n - |\mathcal{A}(x_*)|} \Big\}.$$

# §12.5 Second-Order Conditions

Hence, the condition

$$w^{\mathrm{T}} \nabla^2_{xx} \mathcal{L}(x_*, \lambda_*) w \geqslant 0 \quad \text{for all } w \in \mathcal{C}(x_*, \lambda_*) \tag{45}$$

can be restated as

$$u^{\mathrm{T}} Z^{\mathrm{T}} \nabla^2_{xx} \mathcal{L}(x_*, \lambda_*) Z u \geqslant 0 \quad \forall\, u \in \mathbb{R}^{|\mathcal{A}(x_*)|},$$

or, more succinctly,

$$Z^{\mathrm{T}} \nabla^2_{xx} \mathcal{L}(x_*, \lambda_*) Z \text{ is positive semi-definite.}$$

Similarly, the condition

$$w^{\mathrm{T}} \nabla^2_{xx} \mathcal{L}(x_*, \lambda_*) w > 0 \quad \text{for all } w \in \mathcal{C}(x_*, \lambda_*) \backslash \{0\} \tag{48}$$

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As we show next, $Z$ can be computed numerically, so that the positive (semi-)definiteness conditions can actually be checked.

# §12.5 Second-Order Conditions

Hence, the condition

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# §12.5 Second-Order Conditions

Hence, the condition

$$w^{\mathrm{T}}\nabla_{xx}^{2}\mathcal{L}(x_*,\lambda_*)w \geqslant 0 \quad \text{for all } w \in \mathcal{C}(x_*,\lambda_*) \tag{45}$$

can be restated as

$$u^{\mathrm{T}}Z^{\mathrm{T}}\nabla_{xx}^{2}\mathcal{L}(x_*,\lambda_*)Zu \geqslant 0 \quad \forall\, u \in \mathbb{R}^{|\mathcal{A}(x_*)|},$$

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As we show next, $Z$ can be computed numerically, so that the positive (semi-)definiteness conditions can actually be checked.

# §12.5 Second-Order Conditions

One way to compute the matrix $Z$ is to apply a QR factorization to the matrix of active constraint gradients $A(x_*)$ whose null space we seek. In the simplest case above (in which the multiplier $\lambda_*$ is unique and strictly complementary holds), we define $A(x_*)$ as in (27) and write the QR factorization of its transpose as

$$A(x_*)^{\mathrm{T}} = Q\begin{bmatrix} R \\ 0 \end{bmatrix} = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}\begin{bmatrix} R \\ 0 \end{bmatrix} = Q_1 R,$$

where $R$ is a square upper triangular matrix and $Q$ is $n \times n$ orthogonal. If $R$ is non-singular, we can set $Z = Q_2$. If $R$ is singular (indicating that the active constraint gradients are linearly dependent), a slight enhancement of this procedure that makes use of column pivoting during the QR procedure can be used to identify $Z$.

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# §12.6 Other Constraint Qualifications

We now reconsider constraint qualifications, the conditions discussed in Sections 12.2 and 12.4 that ensure that the linearized approximation to the feasible set $\Omega$ captures the essential shape of $\Omega$ in a neighborhood of $x_*$.

One situation in which the linearized feasible direction set $\mathcal{F}(x_*)$ is obviously an adequate representation of the actual feasible set occurs when all the active constraints are already linear; that is,

$$c_i(x) = a_i^{\mathrm{T}} x + b_i, \tag{56}$$

for some $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$. It is not difficult to prove a version of the following lemma for this situation.

# §12.6 Other Constraint Qualifications

**Lemma**

*Suppose that at some $x_* \in \Omega$, all active constraints $c_i(\cdot)$, $i \in \mathcal{A}(x_*)$, are linear functions. Then $\mathcal{F}(x_*) = T_\Omega(x_*)$.*

*Proof.*

From previous lemma it suffices to show that $\mathcal{F}(x_*) \subseteq T_\Omega(x_*)$.

Let $w \in \mathcal{F}(x_*)$. By the definition of feasible direction set and the form (56) of the constraints, we have

$$\mathcal{F}(x_*) = \left\{ d \;\middle|\; \begin{array}{l} a_i^{\mathrm{T}} d = 0 \text{ for all } i \in \mathcal{E}, \\ a_i^{\mathrm{T}} d \geqslant 0 \text{ for all } i \in \mathcal{A}(x_*) \cap \mathcal{I} \end{array} \right\}.$$

First, note that there is a positive scalar $\bar{t}$ such that the inactive constraint remain inactive at $x_* + tw$, for all $t \in [0, \bar{t}\,]$; that is,

$$c_i(x_* + tw) > 0 \quad \text{for all } i \in \mathcal{I} \setminus \mathcal{A}(x_*) \text{ and all } t \in [0, \bar{t}\,].$$

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# §12.6 Other Constraint Qualifications

## Proof (cont'd).

Now define the sequence $z_k$ by

$$z_k = x_* + (\bar{t}/k)w, \quad k = 1, 2, \cdots.$$

By the choice of $\bar{t}$, we find that $z_k$ is feasible with respect to the inactive inequality constraints $i \in \mathcal{I} \setminus \mathcal{A}(x_*)$. Moreover, since $a_i^{\mathrm{T}} w \geqslant 0$ for all $i \in \mathcal{I} \cap \mathcal{A}(x_*)$, we find that for all $i \in \mathcal{I} \cap \mathcal{A}(x_*)$,

$$c_i(z_k) = c_i(z_k) - c_i(x_*) = a_i^{\mathrm{T}}(z_k - x_*) = \frac{\bar{t}}{k} a_i^{\mathrm{T}} w \geqslant 0,$$

so that $z_k$ is also feasible with respect to the active inequality constraints $c_i$, $i \in \mathcal{I} \cap \mathcal{A}(x_*)$. Finally, for $i \in \mathcal{E}$, by the fact that $x_*$ is feasible and $w \in \mathcal{F}(x_*)$, we have $a_i^{\mathrm{T}} w = 0$ so that

$$a_i^{\mathrm{T}} z_k + b_i = a_i^{\mathrm{T}}(x_* + (\bar{t}/k)w) + b_i = a_i^{\mathrm{T}} x_* + b_i = 0;$$

thus $z_k$ is feasible with respect to the equality constraints $c_i$, $i \in \mathcal{E}$. □

# §12.6 Other Constraint Qualifications

## Proof (cont'd).

Now define the sequence $z_k$ by

$$z_k = x_* + (\bar{t}/k)w, \quad k = 1, 2, \cdots .$$

By the choice of $\bar{t}$, we find that $z_k$ is feasible with respect to the inactive inequality constraints $i \in \mathcal{I} \setminus \mathcal{A}(x_*)$. Moreover, since $a_i^{\mathrm{T}} w \geqslant 0$ for all $i \in \mathcal{I} \cap \mathcal{A}(x_*)$, we find that for all $i \in \mathcal{I} \cap \mathcal{A}(x_*)$,

$$c_i(z_k) = c_i(z_k) - c_i(x_*) = a_i^{\mathrm{T}}(z_k - x_*) = \frac{\bar{t}}{k} a_i^{\mathrm{T}} w \geqslant 0,$$

so that $z_k$ is also feasible with respect to the active inequality constraints $c_i$, $i \in \mathcal{I} \cap \mathcal{A}(x_*)$. Finally, for $i \in \mathcal{E}$, by the fact that $x_*$ is feasible and $w \in \mathcal{F}(x_*)$, we have $a_i^{\mathrm{T}} w = 0$ so that

$$a_i^{\mathrm{T}} z_k + b_i = a_i^{\mathrm{T}}(x_* + (\bar{t}/k)w) + b_i = a_i^{\mathrm{T}} x_* + b_i = 0\,;$$

thus $z_k$ is feasible with respect to the equality constraints $c_i$, $i \in \mathcal{E}$. □

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# §12.6 Other Constraint Qualifications

**Proof (cont'd).**

Hence, $z_k$ is feasible for each $k = 1, 2, \cdots$. In addition, we have that

$$\frac{z_k - x_*}{\bar{t}/k} = \frac{(\bar{t}/k)w}{\bar{t}/k} = w,$$

so that $w$ is the limiting direction of $\{z_k\}$. Hence, $w \in T_\Omega(x_*)$, and the proof is complete. □

We conclude from this result that the condition that all active constraints be linear is another possible constraint qualification. It is neither weaker nor stronger than the LICQ condition; that is, there are situations in which one condition is satisfied but not the other.

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# §12.6 Other Constraint Qualifications

Another useful generalization of the LICQ is the Mangasarian–Fromovitz constraint qualification (MFCQ).

**Definition (MFCQ)**

We say that the Mangasarian-Fromovitz constraint qualification (MFCQ) holds at $x_*$ if there exists a vector $w \in \mathbb{R}^n$ such that

$$\begin{aligned} \nabla c_i(x_*)^{\mathrm{T}} w > 0 \quad & \text{for all } i \in \mathcal{A}(x_*) \cap \mathcal{I}, \\ \nabla c_i(x_*)^{\mathrm{T}} w = 0 \quad & \text{for all } i \in \mathcal{E}, \end{aligned}$$

and the set of **equality** constraint gradients $\{\nabla c_i(x_*) \,|\, i \in \mathcal{E}\}$ is linearly independent.

Note the "strict" inequality involving the active inequality constraints.

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# §12.6 Other Constraint Qualifications

**Remark**: (Applying the duality theory in §12.9) we can show that

> There exists $w \in \mathbb{R}^n$ satisfying
> $$\nabla c_i(x_*)^{\mathrm{T}} w > 0 \text{ for all } i \in \mathcal{A}(x_*) \cap \mathcal{I},$$
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is equivalent to that

> The extreme value of the constrained optimization problem
> $$\max_{\lambda \in \mathbb{R}^{|\mathcal{A}(x_*)|}} \sum_{i \in \mathcal{A}(x_*) \cap \mathcal{I}} \lambda_i \text{ subject to } \begin{cases} \displaystyle\sum_{i \in \mathcal{A}(x_*)} \lambda_i \nabla c_i(x_*) = 0, \\ \lambda_i \geqslant 0, \ i \in \mathcal{A}(x_*) \cap \mathcal{I}, \end{cases}$$
> is zero.

Note that the extreme value is zero means that

$$\left\{ \left[\nabla c_i(x_*)\right]_{i \in \mathcal{A}(x_*) \cap \mathcal{I}} y \,\middle|\, y \geqslant 0 \right\} \cap \left\{ \left[\nabla c_i(x_*)\right]_{i \in \mathcal{E}} w \,\middle|\, w \in \mathbb{R}^{|\mathcal{E}|} \right\} = \{0\}.$$

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# §12.6 Other Constraint Qualifications

Adding the condition that the set $\left\{\nabla c_i(x_*) \,\middle|\, i \in \mathcal{E}\right\}$ is linearly independent, we obtain that

**Theorem**

*Let $x \in \Omega$. Then MFCQ holds at $x$ if and only if the system (for $\lambda$)*

$$\begin{aligned} \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i \nabla c_i(x) &= 0\,, \\ \lambda_i c_i(x) &= 0\,, \quad i \in \mathcal{I}\,, \\ \lambda_i &\geqslant 0\,, \quad i \in \mathcal{I}\,, \end{aligned}$$

*only has zero solution.*

因此，MFCQ 條件與 LICQ 條件不同之處在於驗證向量間的“線性獨立性”時，在 active constraint gradients 的“線性組合”中不等式限制所對應的係數必須非負。然後該注意到的是我們無法由上述定理下結論說若 MFCQ 在 local solution $x_*$ 滿足，則其對應滿足 KKT 條件的 $\lambda_*$（若存在的話）的唯一性。

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# §12.6 Other Constraint Qualifications

The MFCQ is a weaker condition than LICQ. If LICQ is satisfied, then the system of equalities defined by

$$\begin{aligned} \nabla c_i(x_*)^{\mathrm{T}} w = 1 & \quad \text{for all } i \in \mathcal{A}(x_*) \cap \mathcal{I}, \\ \nabla c_i(x_*)^{\mathrm{T}} w = 0 & \quad \text{for all } i \in \mathcal{E}, \end{aligned}$$

has a solution $w$, by full rank of the active constraint gradients. Hence, we can choose this $w$ be precisely the vector in the definition of MFCQ. On the other hand, it is easy to construct examples in which the MFCQ is satisfied but the LICQ is not.

# §12.6 Other Constraint Qualifications

## Example

Let

$$c_1(x_1, x_2) = 2 - (x_1 - 1)^2 - (x_2 - 1)^2,$$
$$c_2(x_1, x_2) = 2 - (x_1 - 1)^2 - (x_2 + 1)^2,$$
$$c_3(x_1, x_2) = x_1$$

be the constraint functions for inequality constraints. Then MFCQ holds at $x = (0, 0)^{\mathrm{T}}$ but LICQ does not hold at $x$.

# §12.6 Other Constraint Qualifications

It is possible to prove a version of the first-order necessary condition result in which MFCQ replaces LICQ in the assumptions. MFCQ gives rise to the nice property that it is equivalent to boundedness of the set of Lagrange multiplier vectors $\lambda_*$ for which the KKT conditions (32) are satisfied. (In the case of LICQ, this set consists of a unique vector $\lambda_*$, and so is trivially bounded.)

Note that constraint qualifications are sufficient conditions for the linear approximation to be adequate, not necessary conditions. For instance, consider the set defined by $x_2 \geqslant -x_1^2$ and $x_2 \leqslant x_1^2$ and the feasible point $x_* = (0,0)^{\mathrm{T}}$. None of the constraint qualifications we have discussed are satisfied, but the linear approximation

$$\mathcal{F}(x_*) = \left\{(w_1, 0)^{\mathrm{T}} \,\middle|\, w_1 \in \mathbb{R}\right\}$$

accurately reflects the geometry of the feasible set near $x_*$.

# §12.7 A Geometric Viewpoint

Finally, we mention an alternative first-order optimality condition that depends only on the geometry of the feasible set $\Omega$ and not on any of its algebraic description in terms of the constraint functions $c_i$, $i \in \mathcal{E} \cup \mathcal{I}$. In geometric terms, our problem (1) can be stated as

$$\min f(x) \quad \text{subject to} \quad x \in \Omega, \tag{57}$$

where $\Omega$ is the feasible set.

We first define the normal cone to the set $\Omega$ at a feasible point $x$.

**Definition**

The normal cone to the set $\Omega$ at a point $x \in \Omega$ is defined as

$$N_\Omega(x) = \left\{ v \,\middle|\, v^{\mathrm{T}} w \leqslant 0 \text{ for all } w \in T_\Omega(x) \right\},$$

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The first-order necessary condition for (57) is delightfully simple.

**Theorem**

*Suppose that $x_*$ is a local minimizer of $f$ in $\Omega$. Then*

$$-\nabla f(x_*) \in N_\Omega(x_*) .$$

**Proof.**

Let $d \in T_\Omega(x_*)$ be given, there exist a feasible sequence $\{z_k\}$ and a sequence of positive scalars $\{t_k\}$ such that

$$z_k = x_* + t_k d + o(t_k) \quad \forall\, k \in \mathbb{N} .$$

Since $x_*$ is a local solution and $f$ is continuously differentiable, by Taylor's Theorem we have

$$0 \leqslant f(z_k) - f(x_*) = t_k \nabla f(x_*)^{\mathrm{T}} d + o(t_k) .$$

By dividing by $t_k$ and passing to the limit as $k \to \infty$, we find that $\nabla f(x_*)^{\mathrm{T}} d \geqslant 0$. □

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# §12.7 A Geometric Viewpoint

**Proof (cont'd).**

Therefore,

$$-\nabla f(x_*)^{\mathrm{T}} d \leqslant 0 \quad \forall\, d \in T_\Omega(x_*)\,.$$

We then conclude from the definition of the normal cone that $-\nabla f(x_*) \in N_\Omega(x_*)$. □

This result suggests a close relationship between $N_\Omega(x_*)$ and the conic combination of active constraint gradients given by

$$N = \left\{ \sum_{i\in\mathcal{A}(x_*)} \lambda_i \nabla c_i(x_*) \,\middle|\, \{\lambda_i\}_{i\in\mathcal{A}(x_*)} \subseteq \mathbb{R}, \lambda_i \geqslant 0 \text{ if } i \in \mathcal{A}(x_*) \cap \mathcal{I} \right\}.$$

When the linear independence constraint qualification holds, identical (to within a change of sign).

# §12.7 A Geometric Viewpoint

**Lemma**

*Suppose that the LICQ holds at $x_*$. Then the normal cone $N_\Omega(x_*)$ is simply $-N$, where $N$ is the set defined by*

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*Proof.*

By Farkas' Lemma, we have that

$$g \in N \quad \Leftrightarrow \quad g^{\mathrm{T}} d \geqslant 0 \ \text{ for all } d \in \mathcal{F}(x_*).$$

Since LICQ holds at $x_*$, $\mathcal{F}(x_*) = T_\Omega(x_*)$; thus it follows by switching the sign of this expression that

$$g \in -N \quad \Leftrightarrow \quad g^{\mathrm{T}} d \leqslant 0 \ \text{ for all } d \in T_\Omega(x_*).$$

We then conclude from the definition of the normal cone that $N_\Omega(x_*) = -N$, as claimed. □

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# §12.8 Lagrange Multipliers and Sensitivity

What are the intuitive significance of Lagrange multipliers in optimality theory? We will "show" in this section that each Lagrange multiplier $\lambda_i^*$ tells us something about the sensitivity of the optimal objective value $f(x_*)$ to the presence of the constraint $c_i$. To put it another way, $\lambda_i^*$ indicates how hard $f$ is "pushing" or "pulling" the solution $x_*$ against the particular constraint $c_i$.

When we choose an inactive constraint $i \notin \mathcal{A}(x_*)$ such that $c_i(x_*) > 0$, the solution $x_*$ and function value $f(x_*)$ are indifferent to whether this constraint is present or not. If we perturb $c_i$ by a tiny amount, it will still be inactive and $x_*$ will still be a local solution of the optimization problem. Since $\lambda_i^* = 0$ from the KKT condition (32e), the Lagrange multiplier indicates accurately that constraint $i$ is not significant.

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# §12.8 Lagrange Multipliers and Sensitivity

Suppose instead that constraint $i$ is active, and let us perturb the right-hand side of this constraint a little, requiring, say, that $c_i(x) \geqslant -\varepsilon\|\nabla c_i(x_*)\|$ instead of $c_i(x) \geqslant 0$. Suppose that $\varepsilon$ is sufficiently small that the perturbed solution $x_*(\varepsilon)$ still has the same set of active constraints, and that the Lagrange multipliers are not much affected by the perturbation (these conditions can be made more rigorous with the help of strict complementarity and second-order conditions). We then find that

$$\begin{aligned} -\varepsilon\|\nabla c_i(x_*)\| &= c_i(x_*(\varepsilon)) - c_i(x_*) \approx (x_*(\varepsilon) - x_*)^{\mathrm{T}} \nabla c_i(x_*), \\ 0 &= c_j(x_*(\varepsilon)) - c_j(x_*) \approx (x_*(\varepsilon) - x_*)^{\mathrm{T}} \nabla c_j(x_*), \end{aligned}$$

for all $j \in \mathcal{A}(x_*)$ with $j \neq i$.

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# §12.8 Lagrange Multipliers and Sensitivity

The value of $f(x_*(\varepsilon))$, meanwhile, can be estimated with the help of the KKT condition

$$\nabla_x \mathcal{L}(x_*, \lambda_*) = 0\,. \tag{32a}$$

By Taylor's Theorem we have

$$\begin{aligned} f(x_*(\varepsilon)) - f(x_*) &\approx (x_*(\varepsilon) - x_*)^{\mathrm{T}} \nabla f(x_*) \\ &= \sum_{j \in \mathcal{A}(x_*)} \lambda_j^* (x_*(\varepsilon) - x_*)^{\mathrm{T}} \nabla c_j(x_*) \\ &\approx -\varepsilon \|\nabla c_i(x_*)\| \lambda_i^*\,. \end{aligned}$$

By taking limits, we see that the family of solutions $x_*(\varepsilon)$ satisfies

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f(x_*(\varepsilon)) = -\lambda_i^* \|\nabla c_i(x_*)\|. \tag{58}$$

# §12.8 Lagrange Multipliers and Sensitivity

A sensitivity analysis would conclude that if $\lambda_i^* \|\nabla c_i(x_*)\|$ is large, then the optimal value is sensitive to the placement of the $i$-th constraint, while if this quantity is small, the dependence is not too strong. If $\lambda_i^*$ is exactly zero for some active constraint, small perturbations to $c_i$ in some directions will hardly affect the optimal objective value at all; the change is zero, to first order. This discussion motivates the definition below.

**Definition**

Let $x_*$ be a solution of the problem (1), and suppose that the KKT conditions (32) are satisfied. We say that an inequality constraint $c_i$ is strongly active or binding if $i \in \mathcal{A}(x_*)$ and $\lambda_i^* > 0$ for some Lagrange multiplier $\lambda_*$ satisfying (32). We say that $c_i$ is weakly active if $i \in \mathcal{A}(x_*)$ and $\lambda_i^* = 0$ for all $\lambda_*$ satisfying (32).

# §12.8 Lagrange Multipliers and Sensitivity

A sensitivity analysis would conclude that if $\lambda_i^* \|\nabla c_i(x_*)\|$ is large, then the optimal value is sensitive to the placement of the $i$-th constraint, while if this quantity is small, the dependence is not too strong. If $\lambda_i^*$ is exactly zero for some active constraint, small perturbations to $c_i$ in some directions will hardly affect the optimal objective value at all; the change is zero, to first order. This discussion motivates the definition below.

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# §12.8 Lagrange Multipliers and Sensitivity

Note that the analysis above is independent of scaling of the individual constraints. For instance, we might change the formulation of the problem by replacing some active constraint $c_i$ by $10c_i$. The new problem will actually be equivalent (that is, it has the same feasible set and same solution), but the optimal multiplier $\lambda_i^*$ corresponding to $c_i$ will be replaced by $\lambda_i^*/10$. However, since $\|\nabla c_i(x_*)\|$ is replaced by $10\|\nabla c_i(x_*)\|$, the product $\lambda_i^*\|\nabla c_i(x_*)\|$ does not change. If, on the other hand, we replace the objective function $f$ by $10f$, the multipliers $\lambda_i^*$ in (32) all will need to be replaced by $10\lambda_i^*$. Hence in (58) we see that the sensitivity of $f$ to perturbations has increased by a factor of $10$, which is exactly what we would expect.

# §12.9 Duality

在本節中，我們介紹非線性規劃的對偶理論 (duality theory) 中的一些要素。對偶理論被用來啟發和發展一些重要的演算法，包括第 17 章要提到的 Augmented Lagrangian Method。對偶理論的完整論述將超越了非線性規劃，為 convex non-smooth optimization 甚至離散最佳化領域提供了重要的洞見。其對線性規劃的特殊應用對該領域的發展至關重要；這個部份請參考第 13 章。

對偶理論告訴我們如何從原本最佳化問題的函數和數據去構建一個替代問題。這個替代原問題的「對偶」問題 (dual problem) 與原本的最佳化問題有著迷人的相關性（為了對比起見，在這種情況下有時被稱為 primal problem）。在某些情況下，對偶問題在計算上比原本的問題更容易解決。在其他情況下，對偶問題可以用來輕鬆地取得原問題中目標函數的下界。

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# §12.9 Duality

As remarked above, the dual has also been used to design algorithms for solving the primal problem. Our results in this section are mostly restricted to the special case of (1) in which

1. there are **no** equality constraints, and
2. the objective $f$ and $-c_i$ (the negatives of the inequality constraints) are all convex functions.

For simplicity we assume that there are $m$ inequality constraints. Define a vector-valued function $c(x) \equiv \big(c_1(x), c_2(x), \cdots, c_m(x)\big)^{\mathrm{T}}$, we can rewrite the problem as

$$\min_{x\in\mathbb{R}^n} f(x) \quad \text{subject to} \quad c(x) \geqslant 0 \tag{59}$$

for which the Lagrangian function (10) with Lagrange multiplier vector $\lambda \in \mathbb{R}^m$ is simply expressed as $\mathcal{L}(x,\lambda) = f(x) - \lambda^{\mathrm{T}} c(x)$.

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# §12.9 Duality

The dual objective function $q : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$q(\lambda) \equiv \inf_x \mathcal{L}(x, \lambda)\,, \tag{60}$$

where the domain of $q$ is the set of $\lambda$ for which $q$ is finite; that is,

$$\mathcal{D} \equiv \mathrm{Dom}(q) = \{\lambda \,|\, q(\lambda) > -\infty\}.$$

Note that calculation of the infimum in (60) requires finding the global minimizer of the function $\mathcal{L}(\cdot, \lambda)$ for the given $\lambda$ which, as we have noted in Chapter 2, may be extremely difficult in practice. However, when $f$ and $-c_i$ are convex functions and $\lambda \geqslant 0$ (the case in which we are most interested), the function $\mathcal{L}(\cdot, \lambda)$ is also convex. In this situation, all local minimizers are global minimizers, so computation of $q(\lambda)$ becomes a more practical proposition.

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# Convexity

Before proceeding, we develop some basic knowledge about convex functions. First we recall the definition of convex sets.

**Definition**

A subset $C$ of a vector space is said to be convex if

$$(1-t)x+ty \in C \quad \forall\, x, y \in C, t \in [0,1].$$

The definition of convex functions are given as follows.

**Definition**

Let $C$ be a convex set, and $f : C \to \mathbb{R}$ be a function.

1. $f$ is said to be convex if for all $x, y \in C$ and $t \in [0,1]$,
$$f((1-t)x+ty) \leqslant (1-t)f(x)+tf(y).$$
2. $f$ is said to be strictly convex if for all distinct $x, y \in C$ and $t \in (0,1)$,
$$f((1-t)x+ty) < (1-t)f(x)+tf(y).$$

# Convexity

## Theorem

*Let $C$ be a convex set, and $f : C \to \mathbb{R}$ be a differentiable function.*

1. *$f$ is convex if and only if*
$$f(y) \geqslant f(x) + \nabla f(x)^{\mathrm{T}}(y-x) \quad \forall\, x, y \in C.$$
2. *$f$ is strictly convex if and only if*
$$f(y) > f(x) + \nabla f(x)^{\mathrm{T}}(y-x) \quad \forall\, x, y \in C, x \neq y.$$

## Proof.

1. $(\Rightarrow)$ Let $x, y \in C$. For all $t \in [0,1]$,
$$f((1-t)x+ty) \leqslant (1-t)f(x) + tf(y);$$
thus for $t \in (0,1]$,
$$\nabla f(x)^{\mathrm{T}}(y-x) = \lim_{t\to 0^+} \frac{f((1-t)x+ty) - f(x)}{t} \leqslant f(y) - f(x).$$
Note that the limit is the directional derivative of $f$ at $x$ along direction $y-x$; thus □

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# Convexity

## Proof (cont'd).

1. ($\Leftarrow$) Let $x, y \in C$, $t \in [0,1]$, and suppose that

$$f(w) \geqslant f(z) + \nabla f(z)^{\mathrm{T}}(w-z) \quad \forall\, w, z \in C.$$

Let $z = (1-t)x + ty$ and $w = x$ or $w = y$ in the inequality above, we obtain

$$f(x) \geqslant f((1-t)x+ty) + \nabla f((1-t)x+ty)^{\mathrm{T}}(t(y-x))$$

and

$$f(y) \geqslant f((1-t)x+ty) + \nabla f((1-t)x+ty)^{\mathrm{T}}((1-t)(x-y)).$$

Therefore,

$$\begin{aligned}(1-t)f(x) + tf(y) &\geqslant (1-t)f((1-t)x+ty) + tf((1-t)x+ty)\\ &= f((1-t)x+ty);\end{aligned}$$

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# Convexity

## Proof (cont'd).

2. ($\Leftarrow$) Let $x, y \in C$, $x \neq y$, $t \in (0,1)$, and suppose that
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Let $z = (1-t)x + ty$ and $w = x$ or $w = y$ in the inequality above ($w \neq z$ since $t \in (0,1)$), we obtain
$$f(x) > f((1-t)x + ty) + t\nabla f((1-t)x + ty)^{\mathrm{T}}(y - x)$$
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## Proof (cont'd).

2. (⇒) From ① we have

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so it suffices to shows that

$$f(y) \neq f(x) + \nabla f(x)^{\mathrm{T}}(y-x) \quad \forall\, x, y \in C, x \neq y.$$

Suppose the contrary that there exist $x, y \in C$, $x \neq y$ such that

$$f(y) = f(x) + \nabla f(x)^{\mathrm{T}}(y-x).$$

Let $t \in (0,1)$, and $z = (1-t)x + ty$. Then $z - x = t(y-x)$, and the strict convexity of $f$ shows that

$$\begin{aligned} f(z) < (1-t)f(x) + tf(y) &= f(x) + t\big[f(y) - f(x)\big] \\ &= f(x) + t\nabla f(x)^{\mathrm{T}}(y-x) \\ &= f(x) + \nabla f(x)^{\mathrm{T}}(z-x) \leqslant f(z), \end{aligned}$$

a contradiction. □

# Convexity

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# §12.9 Duality

The dual problem to

$$\min_{x\in\mathbb{R}^n} f(x) \quad \text{subject to} \quad c(x) \geqslant 0 \tag{59}$$

is the constrained maximization problem

$$\max_{\lambda\in\mathbb{R}^n} q(\lambda) \quad \text{subject to} \quad \lambda \geqslant 0\,. \tag{61}$$

**Example**

Consider the problem

$$\min_{(x_1,x_2)} 0.5(x_1^2+x_2^2) \quad \text{subject to} \quad x_1 - 1 \geqslant 0\,. \tag{62}$$

The Lagrangian is

$$\mathcal{L}(x_1,x_2,\lambda) = 0.5(x_1^2+x_2^2) - \lambda(x_1-1)\,.$$

If we hold $\lambda$ fixed, $\mathcal{L}$ a convex function of $(x_1,x_2)^{\mathrm{T}}$; thus the infimum of $\mathcal{L}$ is achieved when the partial derivatives with respect to $x_1$ and $x_2$ are zero; that is, $x_1-\lambda=0$, $x_2=0$.

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# §12.9 Duality

## Example (cont'd)

By substituting these infimal values into $\mathcal{L}(x_1, x_2, \lambda)$, we obtain the dual objective (60):

$$q(\lambda) = 0.5(\lambda^2 + 0) - \lambda(\lambda - 1) = -0.5\lambda^2 + \lambda\,.$$

Hence, the dual problem of (61) is

$$\max_{\lambda \geqslant 0} -0.5\lambda^2 + \lambda \tag{63}$$

which clearly has the solution $\lambda = 1$.

In the remainder of this section, we show how the dual problem is related to

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{subject to} \quad c(x) \geqslant 0\,. \tag{59}$$

Our first result concerns concavity of $q$ and convexity of its domain $\mathcal{D}$.

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Our first result concerns concavity of $q$ and convexity of its domain $\mathcal{D}$.

# §12.9 Duality

**Theorem**

*The function $q$ defined by (60) is concave and its domain $\mathcal{D}$ is convex.*

**Proof.**

For any $\lambda_0$ and $\lambda_1$ in $\mathbb{R}^m$, any $x \in \mathbb{R}^n$, and any $\alpha \in [0,1]$, we have

$$\mathcal{L}(x,(1-\alpha)\lambda_0 + \alpha\lambda_1) = (1-\alpha)\mathcal{L}(x,\lambda_0) + \alpha\mathcal{L}(x,\lambda_1) .$$

By the fact that the infimum of a sum is greater than or equal to the sum of infimums, taking the infimum of both sides in the expression above we obtain

$$q\big((1-\alpha)\lambda_0 + \alpha\lambda_1\big) \geqslant (1-\alpha)q(\lambda_0) + \alpha q(\lambda_1) ,$$

confirming concavity of $q$. If both $\lambda_0$ and $\lambda_1$ belong to $\mathcal{D}$, this inequality implies that $q\big((1-\alpha)\lambda_0+\alpha\lambda_1\big) > -\infty$ also, and therefore $(1-\alpha)\lambda_0 + \alpha\lambda_1 \in \mathcal{D}$, verifying convexity of $\mathcal{D}$. □

# §12.9 Duality

**Theorem**

*The function $q$ defined by (60) is concave and its domain $\mathcal{D}$ is convex.*

**Proof.**

For any $\lambda_0$ and $\lambda_1$ in $\mathbb{R}^m$, any $x \in \mathbb{R}^n$, and any $\alpha \in [0,1]$, we have

$$\mathcal{L}(x, (1-\alpha)\lambda_0 + \alpha\lambda_1) = (1-\alpha)\mathcal{L}(x, \lambda_0) + \alpha\mathcal{L}(x, \lambda_1)\,.$$

By the fact that the infimum of a sum is greater than or equal to the sum of infimums, taking the infimum of both sides in the expression above we obtain

$$q\big((1-\alpha)\lambda_0 + \alpha\lambda_1\big) \geqslant (1-\alpha)q(\lambda_0) + \alpha q(\lambda_1)\,,$$

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# §12.9 Duality

In the following we prove that the optimal value of the dual problem

$$\max_{\lambda\in\mathbb{R}^n} q(\lambda) \quad \text{subject to} \quad \lambda \geqslant 0 \tag{61}$$

gives a lower bound on the optimal objective value for the primal problem

$$\min_{x\in\mathbb{R}^n} f(x) \quad \text{subject to} \quad c(x) \geqslant 0\,. \tag{59}$$

This result is a consequence of the following result.

**Theorem (Weak Duality)**

*For any $\bar{x}$ feasible for (59) and any $\bar{\lambda} \geqslant 0$, we have $q(\bar{\lambda}) \leqslant f(\bar{x})$.*

**Proof.**

By the definition of $q$,

$$q(\bar{\lambda}) = \inf_x f(x) - \bar{\lambda}^{\mathrm{T}} c(x) \leqslant f(\bar{x}) - \bar{\lambda}^{\mathrm{T}} c(\bar{x}) \leqslant f(\bar{x}),$$

where the final inequality follows from $\bar{\lambda} \geqslant 0$ and $c(\bar{x}) \geqslant 0$. □

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where the final inequality follows from $\bar{\lambda} \geqslant 0$ and $c(\bar{x}) \geqslant 0$. □

# §12.9 Duality

For the remaining results, we note that the KKT conditions (32) specialized to

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{subject to} \quad c(x) \geqslant 0 \tag{59}$$

are as follows:

$$\nabla f(\bar{x}) - \nabla c(\bar{x})\bar{\lambda} = 0, \tag{64a}$$

$$c(\bar{x}) \geqslant 0, \tag{64b}$$

$$\bar{\lambda} \geqslant 0, \tag{64c}$$

$$\bar{\lambda}_i c_i(\bar{x}) = 0, \quad i = 1, 2, \cdots, m, \tag{64d}$$

where $\nabla c(x)$ is the $n \times m$ matrix defined by

$$\nabla c(x) = [\nabla c_1(x), \nabla c_2(x), \cdots, \nabla c_m(x)] .$$

The next result shows that optimal Lagrange multipliers for (59) are solutions of the dual problem (61) under certain conditions.

# §12.9 Duality

**Theorem**

*Suppose that $f$ and $-c_i$, $i = 1, 2, \cdots, m$ are convex functions on $\mathbb{R}^n$ that are differentiable at a KKT point $\bar{x}$ to*

$$\min_{x \in \mathbb{R}^n} f(x) \quad \textit{subject to} \quad c(x) \geqslant 0, \tag{59}$$

*where $c(x) \equiv \big(c_1(x), c_2(x), \cdots, c_m(x)\big)^{\mathrm{T}}$. Then $\bar{x}$ is a solution to (59). Moreover, any $\bar{\lambda}$ for which $(\bar{x}, \bar{\lambda})$ satisfies the KKT conditions*

$$\nabla f(\bar{x}) - \nabla c(\bar{x})\bar{\lambda} = 0, \tag{64a}$$

$$c(\bar{x}) \geqslant 0, \tag{64b}$$

$$\bar{\lambda} \geqslant 0, \tag{64c}$$

$$\bar{\lambda}_i c_i(\bar{x}) = 0, \quad i = 1, 2, \cdots, m, \tag{64d}$$

*is a local solution of the dual problem*

$$\max_{\lambda \in \mathbb{R}^n} q(\lambda) \quad \textit{subject to} \quad \lambda \geqslant 0. \tag{61}$$

# §12.9 Duality

**Proof.**

Suppose that $(\bar{x}, \bar{\lambda})$ satisfies the KKT condition (64). We have from $\bar{\lambda} \geqslant 0$ that $\mathcal{L}(\cdot, \bar{\lambda})$ is a convex and differentiable function. Hence, for any $x$, we have

$$\mathcal{L}(x, \bar{\lambda}) \geqslant \mathcal{L}(\bar{x}, \bar{\lambda}) + \nabla_x \mathcal{L}(\bar{x}, \bar{\lambda})^{\mathrm{T}}(x - \bar{x}) = \mathcal{L}(\bar{x}, \bar{\lambda}),$$

where the last equality follows from (64a). Therefore, we have

$$q(\bar{\lambda}) = \inf_x \mathcal{L}(x, \bar{\lambda}) = \mathcal{L}(\bar{x}, \bar{\lambda}) = f(\bar{x}) - \bar{\lambda}^{\mathrm{T}} c(\bar{x}) = f(\bar{x}),$$

where the last equality follows from (64d).

On the other hand, the weak duality implies that

$$q(\lambda) \leqslant f(\bar{x}) \quad \forall \lambda \geqslant 0;$$

thus it follows from $q(\bar{\lambda}) = f(\bar{x})$ that $\bar{x}$ is a solution to (59) and $\bar{\lambda}$ is a solution of (61). □

# §12.9 Duality

**Proof.**

Suppose that $(\bar{x}, \bar{\lambda})$ satisfies the KKT condition (64). We have from $\bar{\lambda} \geqslant 0$ that $\mathcal{L}(\cdot, \bar{\lambda})$ is a convex and differentiable function. Hence, for any $x$, we have

$$\mathcal{L}(x, \bar{\lambda}) \geqslant \mathcal{L}(\bar{x}, \bar{\lambda}) + \nabla_x \mathcal{L}(\bar{x}, \bar{\lambda})^{\mathrm{T}} (x - \bar{x}) = \mathcal{L}(\bar{x}, \bar{\lambda}),$$

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thus it follows from $q(\bar{\lambda}) = f(\bar{x})$ that $\bar{x}$ is a solution to (59) and $\bar{\lambda}$ is a solution of (61). □

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Suppose that $(\bar{x},\bar{\lambda})$ satisfies the KKT condition (64). We have from $\bar{\lambda}\geqslant 0$ that $\mathcal{L}(\cdot,\bar{\lambda})$ is a convex and differentiable function. Hence, for any $x$, we have

$$\mathcal{L}(x,\bar{\lambda})\geqslant\mathcal{L}(\bar{x},\bar{\lambda})+\nabla_x\mathcal{L}(\bar{x},\bar{\lambda})^{\mathrm{T}}(x-\bar{x})=\mathcal{L}(\bar{x},\bar{\lambda}),$$

where the last equality follows from (64a). Therefore, we have

$$q(\bar{\lambda})=\inf_x\mathcal{L}(x,\bar{\lambda})=\mathcal{L}(\bar{x},\bar{\lambda})=f(\bar{x})-\bar{\lambda}^{\mathrm{T}}c(\bar{x})=f(\bar{x}),$$

where the last equality follows from (64d).

On the other hand, the weak duality implies that

$$q(\lambda)\leqslant f(\bar{x})\quad\forall\,\lambda\geqslant 0;$$

thus it follows from $q(\bar{\lambda})=f(\bar{x})$ that $\bar{x}$ is a solution to (59) and $\bar{\lambda}$ is a solution of (61). □

# §12.9 Duality

Note that if the functions are continuously differentiable and a constraint qualification such as LICQ holds at a local solution $\bar{x}$ of (59), then an optimal Lagrange multiplier satisfying the KKT conditions is guaranteed to exist. This shows that following

**Corollary**

*Suppose that $f$ and $-c_i$, $i = 1, 2, \cdots, m$ be convex functions on $\mathbb{R}^n$ that are differentiable at a solution $\bar{x}$ to*

$$\min_{x\in\mathbb{R}^n} f(x) \quad \text{subject to} \quad c(x) \geqslant 0\,, \tag{59}$$

*If LICQ holds at $\bar{x}$; that is, $\{\nabla c_i(\bar{x}) \,|\, i \in \mathcal{A}(\bar{x})\}$ is linearly independent or equivalently, the matrix $[\nabla c_i(\bar{x})]_{i\in\mathcal{A}(\bar{x})}$ has full rank, then there is a solution of the dual problem*

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# §12.9 Duality

## Example

In previous example of solving

$$\min_{(x_1,x_2)} 0.5(x_1^2 + x_2^2) \quad \text{subject to} \quad x_1 - 1 \geqslant 0\,, \tag{62}$$

we see that $\lambda = 1$ is both an optimal Lagrange multiplier for problem (62) and a solution of its dual problem

$$\max_{\lambda \geqslant 0} -0.5\lambda^2 + \lambda\,. \tag{63}$$

Note too that the optimal objective for both problems is $0.5$.

Next we prove a partial converse of the previous theorem, which shows that solutions to the dual problem (61) can sometimes be used to derive solutions to the original problem (59).

# §12.9 Duality

## Example

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Next we prove a partial converse of the previous theorem, which shows that solutions to the dual problem (61) can sometimes be used to derive solutions to the original problem (59).

# §12.9 Duality

The essential condition is strict convexity of the function $\mathcal{L}(\cdot, \widehat{\lambda})$ for a certain value $\widehat{\lambda}$. We note that this condition holds if either $f$ is strictly convex or if $-c_i$ is strictly convex for some $i = 1, 2, \cdots, m$ with $\widehat{\lambda}_i > 0$.

**Theorem**

*Suppose that $f$ and $-c_i$, $i = 1, 2, \cdots, m$ are convex and continuously differentiable on $\mathbb{R}^n$. Suppose that*

1. *$\bar{x}$ is a solution of (59) at which LICQ holds,*
2. *$\widehat{\lambda}$ solves the dual problem (61), and the infimum $\inf_x \mathcal{L}(x, \widehat{\lambda})$ is attained at $\widehat{x}$.*

*Assume further that $\mathcal{L}(\cdot, \widehat{\lambda})$ is a strictly convex function. Then $\bar{x} = \widehat{x}$ (that is, $\widehat{x}$ is the unique solution of (59)), and $\widehat{\lambda}$ is a Lagrange multiplier for $\bar{x}$ (that is, $(\bar{x}, \widehat{\lambda})$ satisfies the KKT condition).*

# §12.9 Duality

**Theorem (Full statement of the theorem in the previous slide)**

*Suppose that* $f$ *and* $-c_i$, $i = 1, 2, \cdots, m$ *are convex and continuously differentiable on* $\mathbb{R}^n$. *Suppose that*

1. $\bar{x}$ *is a solution of*
$$\min_{x\in\mathbb{R}^n} f(x) \quad \text{subject to} \quad c(x) \geqslant 0\,, \tag{59}$$
*and LICQ holds at* $\bar{x}$ *(or* $T_\Omega(\bar{x}) = \mathcal{F}(\bar{x})$*);*
2. $\widehat{\lambda}$ *solves the dual problem*
$$\max_{\lambda\in\mathbb{R}^n} q(\lambda) \quad \text{subject to} \quad \lambda \geqslant 0\,, \tag{61}$$
*and the infimum* $\inf\limits_{x} \mathcal{L}(x, \widehat{\lambda})$ *is attained at* $\hat{x}$.

*Assume further that* $\mathcal{L}(\cdot, \widehat{\lambda})$ *is a strictly convex function. Then* $\bar{x} = \hat{x}$ *(that is,* $\hat{x}$ *is the unique solution of (59)), and* $\widehat{\lambda}$ *is a Lagrange multiplier for* $\bar{x}$ *(that is,* $(\bar{x}, \widehat{\lambda})$ *satisfies the KKT condition).*

# §12.9 Duality

**Proof.**

Suppose the contrary that $\bar{x} \neq \hat{x}$. Since $\hat{x} = \arg\min_x \mathcal{L}(x, \widehat{\lambda})$, we have $\nabla_x \mathcal{L}(\hat{x}, \widehat{\lambda}) = 0$; thus the strict convexity of $\mathcal{L}(\cdot, \widehat{\lambda})$ implies that

$$\mathcal{L}(\bar{x}, \widehat{\lambda}) - \mathcal{L}(\hat{x}, \widehat{\lambda}) > \nabla_x \mathcal{L}(\hat{x}, \widehat{\lambda})^{\mathrm{T}} (\bar{x} - \hat{x}) = 0 .$$

Since LICQ holds at $\bar{x}$, there is $\bar{\lambda}$ satisfying the KKT conditions (64). By the previous theorem $\bar{\lambda}$ solves the dual problem (61) so that

$$\mathcal{L}(\bar{x}, \bar{\lambda}) = q(\bar{\lambda}) = q(\widehat{\lambda}) = \mathcal{L}(\hat{x}, \widehat{\lambda}) .$$

Therefore,

$$\mathcal{L}(\bar{x}, \widehat{\lambda}) > \mathcal{L}(\hat{x}, \widehat{\lambda}) = \mathcal{L}(\bar{x}, \bar{\lambda}) .$$

In particular,

$$-\widehat{\lambda}^{\mathrm{T}} c(\bar{x}) > -\bar{\lambda}^{\mathrm{T}} c(\bar{x}) = 0 ,$$

where the final equality follows from the KKT condition (64d). Since $\widehat{\lambda} \geqslant 0$ and $c(\bar{x}) \geqslant 0$, we have $-\widehat{\lambda}^{\mathrm{T}} c(\bar{x}) \leqslant 0$, a contradiction. □

# §12.9 Duality

**Proof.**

Suppose the contrary that $\bar{x} \neq \widehat{x}$. Since $\widehat{x} = \arg\min_x \mathcal{L}(x, \widehat{\lambda})$, we have $\nabla_x \mathcal{L}(\widehat{x}, \widehat{\lambda}) = 0$; thus the strict convexity of $\mathcal{L}(\cdot, \widehat{\lambda})$ implies that

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Since LICQ holds at $\bar{x}$, there is $\bar{\lambda}$ satisfying the KKT conditions (64). By the previous theorem $\bar{\lambda}$ solves the dual problem (61) so that

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# §12.9 Duality

## Proof.

Suppose the contrary that $\bar{x} \neq \hat{x}$. Since $\hat{x} = \arg\min_{x} \mathcal{L}(x, \hat{\lambda})$, we have $\nabla_x \mathcal{L}(\hat{x}, \hat{\lambda}) = 0$; thus the strict convexity of $\mathcal{L}(\cdot, \hat{\lambda})$ implies that

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Therefore,

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In particular,

$$-\hat{\lambda}^{\mathrm{T}} c(\bar{x}) > -\bar{\lambda}^{\mathrm{T}} c(\bar{x}) = 0,$$

where the final equality follows from the KKT condition (64d). Since $\hat{\lambda} \geqslant 0$ and $c(\bar{x}) \geqslant 0$, we have $-\hat{\lambda}^{\mathrm{T}} c(\bar{x}) \leqslant 0$, a contradiction. □

# §12.9 Duality

## Proof (cont'd).

Therefore, $\bar{x} = \hat{x}$. Moreover, the identities (from the previous slide)

$$\mathcal{L}(\bar{x}, \bar{\lambda}) = \mathcal{L}(\hat{x}, \hat{\lambda}) \quad \text{and} \quad f(\bar{x}) = \mathcal{L}(\bar{x}, \bar{\lambda})$$

imply that $f(\bar{x}) = \mathcal{L}(\bar{x}, \hat{\lambda})$. This identity shows that $\hat{\lambda}^{\mathrm{T}} c(\bar{x}) = 0$. Since $\hat{\lambda} \geqslant 0$ and $c(\bar{x}) \geqslant 0$, we must have $\hat{\lambda}_i c_i(\bar{x}) = 0$ for all $1 \leqslant i \leqslant m$; thus the KKT condition holds at $(\bar{x}, \hat{\lambda})$. □

## Example

In previous example of solving

$$\min_{(x_1,x_2)} 0.5(x_1^2 + x_2^2) \quad \text{subject to} \quad x_1 - 1 \geqslant 0, \qquad (62)$$

at the dual solution $\lambda = 1$, the infimum of $\mathcal{L}(x_1, x_2, \lambda)$ is achieved at $(x_1, x_2) = (1, 0)^{\mathrm{T}}$, which is the solution of the original problem (62).

# §12.9 Duality

## Proof (cont'd).

Therefore, $\bar{x} = \hat{x}$. Moreover, the identities (from the previous slide)

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$$\min_{(x_1,x_2)} 0.5(x_1^2 + x_2^2) \quad \text{subject to} \quad x_1 - 1 \geqslant 0, \tag{62}$$

at the dual solution $\lambda = 1$, the infimum of $\mathcal{L}(x_1, x_2, \lambda)$ is achieved at $(x_1, x_2) = (1, 0)^{\mathrm{T}}$, which is the solution of the original problem (62).

# §12.9 Duality

An slightly different form of duality that is convenient for computations, known as the Wolfe dual, can be stated as follows:

$$\max_{x,\lambda} \mathcal{L}(x,\lambda) \quad \text{subject to} \quad \nabla_x \mathcal{L}(x,\lambda) = 0, \lambda \geqslant 0. \tag{65}$$

The following result explains the relation of the Wolfe dual to (59).

**Theorem**

*Suppose that $f$ and $-c_i$, $i = 1, 2, \cdots, m$ are convex and continuously differentiable on $\mathbb{R}^n$. Suppose that $(\bar{x}, \bar{\lambda})$ is a solution pair of (59) at which LICQ holds; that is, $\bar{x}$ is a solution of (59) and $\bar{\lambda}$ is a corresponding Lagrange multiplier vector (whose existence is guaranteed by one of previous theorem). Then $(\bar{x}, \bar{\lambda})$ solves the problem (65).*

# §12.9 Duality

**Proof.**

Since $(\bar{x}, \bar{\lambda})$ is a solution pair of (59), it holds the KKT conditions (64) so that $(\bar{x}, \bar{\lambda})$ satisfies the constraint

$$\nabla_x \mathcal{L}(x, \lambda) = 0, \quad \lambda \geqslant 0 \tag{66}$$

and that $\mathcal{L}(\bar{x}, \bar{\lambda}) = f(\bar{x})$. Therefore, for any pair $(x, \lambda)$ that satisfies (66) we have that

$$\begin{aligned} \mathcal{L}(\bar{x}, \bar{\lambda}) &= f(\bar{x}) \geqslant f(\bar{x}) - \lambda^{\mathrm{T}} c(\bar{x}) = \mathcal{L}(\bar{x}, \lambda) \\ &\geqslant \mathcal{L}(x, \lambda) + \nabla_x \mathcal{L}(x, \lambda)^{\mathrm{T}} (\bar{x} - x) = \mathcal{L}(x, \lambda), \end{aligned}$$

where the second inequality follows from the convexity of $\mathcal{L}(\cdot, \lambda)$. We have therefore shown that $(\bar{x}, \bar{\lambda})$ maximizes $\mathcal{L}$ over the constraints (66), and hence solves

$$\max_{x, \lambda} \mathcal{L}(x, \lambda) \quad \text{subject to} \quad \nabla_x \mathcal{L}(x, \lambda) = 0, \lambda \geqslant 0. \tag{65}$$

□

# §12.9 Duality

**Proof.**

Since $(\bar{x}, \bar{\lambda})$ is a solution pair of (59), it holds the KKT conditions (64) so that $(\bar{x}, \bar{\lambda})$ satisfies the constraint

$$\nabla_x \mathcal{L}(x, \lambda) = 0, \quad \lambda \geqslant 0 \tag{66}$$

and that $\mathcal{L}(\bar{x}, \bar{\lambda}) = f(\bar{x})$. Therefore, for any pair $(x, \lambda)$ that satisfies (66) we have that

$$\begin{aligned} \mathcal{L}(\bar{x}, \bar{\lambda}) = f(\bar{x}) &\geqslant f(\bar{x}) - \lambda^{\mathrm{T}} c(\bar{x}) = \mathcal{L}(\bar{x}, \lambda) \\ &\geqslant \mathcal{L}(x, \lambda) + \nabla_x \mathcal{L}(x, \lambda)^{\mathrm{T}} (\bar{x} - x) = \mathcal{L}(x, \lambda)\,, \end{aligned}$$

where the second inequality follows from the convexity of $\mathcal{L}(\cdot, \lambda)$. We have therefore shown that $(\bar{x}, \bar{\lambda})$ maximizes $\mathcal{L}$ over the constraints (66), and hence solves

$$\max_{x,\lambda} \mathcal{L}(x, \lambda) \quad \text{subject to} \quad \nabla_x \mathcal{L}(x, \lambda) = 0, \lambda \geqslant 0. \tag{65}$$

□

# §12.9 Duality

## Example (Linear Programming)

An important special case of (59) is the linear programming problem

$$\min c^{\mathrm{T}}x \quad \text{subject to} \quad Ax - b \geqslant 0\,, \tag{67}$$

for which the dual objective is

$$q(\lambda) = \inf_x c^{\mathrm{T}}x - \lambda^{\mathrm{T}}(Ax - b) = \inf_x (c - A^{\mathrm{T}}\lambda)^{\mathrm{T}}x + b^{\mathrm{T}}\lambda\,.$$

If $c - A^{\mathrm{T}}\lambda \neq 0$, the infimum is clearly $-\infty$ (we can set $x$ to be a large negative multiple of $-(c - A^{\mathrm{T}}\lambda)$ to make $q$ arbitrarily large and negative). When $c - A^{\mathrm{T}}\lambda = 0$, on the other hand, the dual objective is simply $b^{\mathrm{T}}\lambda$. In maximizing $q$, we can exclude $\lambda$ for which $c - A^{\mathrm{T}}\lambda \neq 0$ from consideration. Hence, we can write the dual problem (61) as follows:

$$\max_\lambda b^{\mathrm{T}}\lambda \quad \text{subject to} \quad A^{\mathrm{T}}\lambda = c, \lambda \geqslant 0\,. \tag{68}$$

# §12.9 Duality

## Example (Linear Programming (cont'd))

The Wolfe dual of (67) can be written as

$$\max_{\lambda} c^{\mathrm{T}}x - \lambda^{\mathrm{T}}(Ax - b) \quad \text{subject to} \quad A^{\mathrm{T}}\lambda = c, \lambda \geqslant 0\,,$$

and by substituting the constraint $A^{\mathrm{T}}\lambda - c = 0$ into the objective we obtain (68) again. For some matrices $A$, the dual problem (68) may be computationally easier to solve than the original problem (67).

## Example (Convex Quadratic Programming)

Consider

$$\min \frac{1}{2}x^{\mathrm{T}}Gx + c^{\mathrm{T}}x \quad \text{subject to} \quad Ax - b \geqslant 0\,,$$

where $G$ is a symmetric positive definite matrix. The dual objective for this problem is

$$q(\lambda) = \inf_{x} \mathcal{L}(x, \lambda) = \inf_{x} \frac{1}{2}x^{\mathrm{T}}Gx + c^{\mathrm{T}}x - \lambda^{\mathrm{T}}(Ax - b)\,.$$

# §12.9 Duality

## Example (Linear Programming (cont'd))

The Wolfe dual of (67) can be written as

$$\max_{\lambda} c^{\mathrm{T}}x - \lambda^{\mathrm{T}}(Ax - b) \quad \text{subject to} \quad A^{\mathrm{T}}\lambda = c, \lambda \geqslant 0\,,$$

and by substituting the constraint $A^{\mathrm{T}}\lambda - c = 0$ into the objective we obtain (68) again. For some matrices $A$, the dual problem (68) may be computationally easier to solve than the original problem (67).

## Example (Convex Quadratic Programming)

Consider

$$\min \frac{1}{2}x^{\mathrm{T}}Gx + c^{\mathrm{T}}x \quad \text{subject to} \quad Ax - b \geqslant 0\,,$$

where $G$ is a symmetric positive definite matrix. The dual objective for this problem is

$$q(\lambda) = \inf_{x} \mathcal{L}(x, \lambda) = \inf_{x} \frac{1}{2}x^{\mathrm{T}}Gx + c^{\mathrm{T}}x - \lambda^{\mathrm{T}}(Ax - b)\,.$$

# §12.9 Duality

## Example (Convex Quadratic Programming (cont'd))

Since $G$ is positive definite, $\mathcal{L}(\cdot,\lambda)$ is a strictly convex quadratic function; thus the infimum is achieved when $\nabla_x \mathcal{L}(x,\lambda) = 0$; that is,

$$Gx + c - A^{\mathrm{T}}\lambda = 0\,. \tag{69}$$

Hence, we can substitute for $x$ in the infimum expression and write the dual objective explicitly as follows:

$$q(\lambda) = -\frac{1}{2}(A^{\mathrm{T}}\lambda - c)^{\mathrm{T}} G^{-1} (A^{\mathrm{T}}\lambda - c)^{\mathrm{T}} + b^{\mathrm{T}}\lambda\,.$$

Alternatively, we can write the Wolfe dual form (65) by retaining $x$ as a variable and including the constraint (69) explicitly in the dual problem, to obtain

$$\max_{(\lambda,x)} \frac{1}{2}x^{\mathrm{T}}Gx + c^{\mathrm{T}}x - \lambda^{\mathrm{T}}(Ax - b) \text{ subject to } Gx + c - A^{\mathrm{T}}\lambda = 0, \lambda \geqslant 0.$$

# §12.9 Duality

## Example (Convex Quadratic Programming (cont'd))

To make it clearer that the objective is concave, we can use the constraint to substitute $(c - A^{\mathrm{T}}\lambda)^{\mathrm{T}}x = -x^{\mathrm{T}}Gx$ in the objective, and rewrite the dual formulation as follows:

$$\max_{(\lambda,x)} -\frac{1}{2}x^{\mathrm{T}}Gx + \lambda^{\mathrm{T}}b \quad \text{subject to} \quad Gx + c - A^{\mathrm{T}}\lambda = 0, \lambda \geqslant 0.$$

Note that the Wolfe dual form requires only positive semi-definiteness of $G$.