# 最佳化方法與應用 MA5037
## Homework Assignment 2

Due Nov. 01. 2023

**Problem 1.** Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable. Consider the iteration $x_{k+1} = x_k + \alpha_k p_k$, where $p_k$ is a descent direction and $\alpha_k$ satisfies the Wolfe conditions

$$\begin{aligned} f(x_k + \alpha_k p_k) &\leqslant f(x_k) + c_1 \alpha_k \nabla f_k^{\mathrm{T}} p_k \,, \\ (\nabla f)(x_k + \alpha_k p_k)^{\mathrm{T}} p_k &\geqslant c_2 \nabla f_k^{\mathrm{T}} p_k \,, \end{aligned}$$

with $c_1 < 1/2$. If the sequence $\{x_k\}_{k=1}^{\infty}$ converges to a point $x_*$ such that $(\nabla f)(x_*) = 0$ and $(\nabla^2 f)(x_*)$ is positive definite. Show that if the search direction $p_k$ satisfies

$$\lim_{k \to \infty} \frac{\|\nabla f_k + \nabla^2 f_k p_k\|}{\|p_k\|} = 0 \,,$$

then the step length $\alpha_k = 1$ is admissible for all $k \gg 1$.
**Hint**: Use Taylor's theorem to expand the function value up to $(\nabla^2 f)(x_k)$ term and apply the condition $\|\nabla f_k + \nabla^2 f_k p_k\| = o(\|p_k\|)$.

**Problem 2.** Let $A$ be an $n \times n$ symmetric matrix with spectral decomposition $A = Q\Lambda Q^{\mathrm{T}}$, where $Q$ is orthogonal and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$ is diagonal.

1. Show that for a given constant $\delta > 0$, the minimizer $\Delta A_*$ of the minimization problem

$$\min_{\Delta A} \|\Delta A\|_F \quad \text{subject to} \quad \lambda_{\min}(A + \Delta A) \geqslant \delta, \Delta A \text{ is symmetric}$$

   is given by $\Delta A_* = Q\mathrm{diag}(\tau_1, \cdots, \tau_n)$ with $\tau_i$'s satisfying

$$\tau_i = \begin{cases} 0 & \text{if } \lambda_i \geqslant \delta \,, \\ \delta - \lambda_i & \text{if } \lambda_i < \delta \,. \end{cases}$$

2. Show that for a given constant $\delta > 0$, the minimizer $\Delta A_*$ of the minimization problem

$$\min_{\Delta A} \|\Delta A\|_2 \quad \text{subject to} \quad \lambda_{\min}(A + \Delta A) \geqslant \delta, \Delta A \text{ is symmetric}$$

   is given by

$$\Delta A_* = \tau \mathrm{I} \quad \text{with} \quad \tau = \max\big\{0, \delta - \lambda_{\min}(A)\big\} \,.$$

Here $\|\cdot\|_F$ denotes the Frobenius norm of matrices defined by $\|A\|_F^2 = \mathrm{tr}(A^{\mathrm{T}}A) = \mathrm{tr}(AA^{\mathrm{T}})$, and $\|\cdot\|_2$ denotes the Euclidean norm of matrices defined by $\|A\|_2^2 =$ maximum eigenvalue of $A^{\mathrm{T}}A$, and $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue of matrices.