微分方程 II 上課講義

Lecture Note of Differential Equations II

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Contents

Chapter 8

System of Linear First-Order Differential Equations

8.1 Preliminary Theory - Linear Systems

Definition 8.1. A system of n first-order equations

$$\begin{aligned}\frac{dx_1}{dt} &= g_1(t, x_1, x_2, \cdots, x_n)\,,\\ \frac{dx_2}{dt} &= g_2(t, x_1, x_2, \cdots, x_n)\,,\\ &\vdots \qquad\qquad \vdots\\ \frac{dx_n}{dt} &= g_n(t, x_1, x_2, \cdots, x_n)\,,\end{aligned} \tag{8.1}$$

is called a first-order system. If each of the functions g_1, g_2, $\cdots$, g_n in (8.1) in is linear in the dependent variables x_1, x_2, $\cdots$, x_n, equation (8.1) is called the normal form of a first-order linear equations (or simply called a linear system). In other words, a linear system is of the form

$$\begin{aligned}\frac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t)\,,\\ \frac{dx_2}{dt} &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t)\,,\\ &\vdots \qquad\qquad\qquad\qquad \vdots\\ \frac{dx_n}{dt} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t)\,.\end{aligned} \tag{8.2}$$

The linear system (8.2) is said to be homogeneous if $f_i(t) = 0$ for all $1 \leqslant i \leqslant n$; otherwise it is non-homogeneous.

There are several reasons that we should consider system of first order ODEs, and here we provide two of them.

1. In real life, a lot of phenomena can be modelled by system of first order ODE. For example, the Lotka–Volterra equation or the predator-prey equation

$$\begin{aligned} p' &= -\gamma p + \alpha pq = (-\gamma + \alpha q)p\,, \\ q' &= \beta q - \delta pq = (\beta - \delta p)q\,, \end{aligned}$$

can be used to described a predator-prey system. Let $\boldsymbol{x} \equiv (x_1, x_2) = (p, q)^{\mathrm{T}}$ and $\boldsymbol{F}(t, \boldsymbol{x}) = (\gamma x_1 - \alpha x_1 x_2, \beta x_2 + \delta x_1 x_2)^{\mathrm{T}}$. Then the Lotka-Volterra equation can also be written as

$$\boldsymbol{x}'(t) = \boldsymbol{F}\big(t, \boldsymbol{x}(t)\big)\,. \tag{8.3}$$

2. Suppose that we are considering a scalar n-th order ODE

$$y^{(n)}(t) = f\big(t, y(t), y'(t), \cdots, y^{(n-1)}(t)\big)\,.$$

Let $x_1(t) = y(t)$, $x_2(t) = y'(t)$, $\cdots$, $x_n(t) = y^{(n-1)}(t)$. Then $(x_1, \cdots, x_n)$ satisfies

$$x_1'(t) = x_2(t)\,, \tag{8.4a}$$
$$x_2'(t) = x_3(t)\,, \tag{8.4b}$$
$$\vdots \quad = \quad \vdots \tag{8.4c}$$
$$x_n'(t) = f\big(t, x_1(t), x_2(t), \cdots, x_n(t)\big)\,. \tag{8.4d}$$

Let $\boldsymbol{x} = (x_1, \cdots, x_n)^{\mathrm{T}}$ be a vector in $\mathbb{R}^n$, and $\boldsymbol{F}(t, \boldsymbol{x}) = \big(x_2, \cdots, x_n, f(t, x_1, \cdots, x_n)\big)^{\mathrm{T}}$ be a vector-valued function. Then (8.4) can also be written as (8.3).

• **Matrix Form of a Linear System** If $\boldsymbol{X}$, $\boldsymbol{A}(t)$, $\boldsymbol{F}(t)$ denote the respective matrices

$$\boldsymbol{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \boldsymbol{A}(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}, \quad \boldsymbol{F}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix},$$

then the linear system (8.2) can be written in matrix form as

$$\boldsymbol{X}' = \boldsymbol{A}(t)\boldsymbol{X} + \boldsymbol{F}(t)\,. \tag{8.5}$$

and a homogeneous linear system in matrix form as

$$\boldsymbol{X}' = \boldsymbol{A}(t)\boldsymbol{X}\,. \tag{8.6}$$

Definition 8.2. A solution vector (or simply solution) on an interval I to the linear system (8.5) is a vector-valued function

$$\boldsymbol{X}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

whose entries are differentiable functions satisfying the system (8.5) on the interval.

Example 8.3. The functions

$$\boldsymbol{X}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} = \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix} \quad \text{and} \quad \boldsymbol{X}_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix} e^{6t} = \begin{bmatrix} 3e^{6t} \\ 5e^{6t} \end{bmatrix}$$

are solution vectors of

$$\boldsymbol{X}' = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \boldsymbol{X}\,.$$

• Initial-Value Problem: Let t_0 denote a number in an interval I and

$$\boldsymbol{X}_0 = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{bmatrix},$$

where γ_i's are constants for all $1 \leqslant i \leqslant n$. Then the problem

$$\begin{aligned} &\textbf{Solve}: && \boldsymbol{X}' = \boldsymbol{A}(t)\boldsymbol{X} + \boldsymbol{F}(t) \\ &\textbf{Subject to}: && \boldsymbol{X}(t_0) = \boldsymbol{X}_0 \end{aligned} \tag{8.7}$$

is called an initial-value problem on the interval.

Theorem 8.4. *Let the entries of the matrices $\boldsymbol{A}(t)$ and $\boldsymbol{F}(t)$ be functions continuous on a common interval I that contains the point t_0. Then there exists a unique solution of the initial-value problem* (8.7) *on the interval.*

Theorem 8.5 (Superposition Principle)**.** *Let $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, $\cdots$, $\boldsymbol{X}_k$ be a set of solution vectors of the homogeneous system* (8.6) *on an interval I. Then the linear combination*

$$\boldsymbol{X} = c_1\boldsymbol{X}_1 + c_2\boldsymbol{X}_2 + \cdots + c_k\boldsymbol{X}_k\,,$$

where c_i's are constants for all $1 \leqslant i \leqslant k$, is also a solution of the system on the interval.

Example 8.6. The functions

$$\boldsymbol{X}_1 = \begin{bmatrix} \cos t \\ -\dfrac{1}{2}\cos t + \dfrac{1}{2}\sin t \\ -\cos t - \sin t \end{bmatrix} \quad \text{and} \quad \boldsymbol{X}_2 = \begin{bmatrix} 0 \\ e^t \\ 0 \end{bmatrix}$$

are solutions of the system

$$\boldsymbol{X}' = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \boldsymbol{X} .$$

By the superposition principle the linear combination

$$\boldsymbol{X} = c_1\boldsymbol{X}_1 + c_2\boldsymbol{X}_2 = c_1 \begin{bmatrix} \cos t \\ -\dfrac{1}{2}\cos t + \dfrac{1}{2}\sin t \\ -\cos t - \sin t \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^t \\ 0 \end{bmatrix}$$

is also a solution of the system.

Definition 8.7 (Fundamental Set of Solutions and Fundamental Matrix)**.** Any set $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, $\cdots$, $\boldsymbol{X}_n$ of n linearly independent solution vectors of the homogeneous system (8.6) on an interval I called a fundamental set of solutions of (8.6) on the interval, and the matrix

$$\boldsymbol{\Phi}(t) = \left[\boldsymbol{X}_1(t) \vdots \boldsymbol{X}_2(t) \vdots \cdots \vdots \boldsymbol{X}_n(t)\right]$$

is called a fundamental matrix of (8.6) on the interval.

Remark 8.8. A fundamental matrix $\boldsymbol{\Phi}$ of the homogeneous system (8.6) satisfies that $\boldsymbol{\Phi}' = \boldsymbol{A}(t)\boldsymbol{\Phi}$.

Theorem 8.9. *Let the entries of the matrices $\boldsymbol{A}(t)$ be continuous on an interval I. Then there exists a fundamental set of solutions of the homogeneous system* (8.6) *on the interval.*

Theorem 8.10. *Let the entries of the matrices $\boldsymbol{A}(t)$ be continuous on an interval I, and $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, $\cdots$, $\boldsymbol{X}_n$ be a fundamental set of solutions of the homogeneous system* (8.6) *on the interval. Then the general solution of the system on the interval can be expressed as a linear combination of $\boldsymbol{X}_1, \cdots, \boldsymbol{X}_n$ in exact one way; that is, there exists a unique constant vector $(c_1, \cdots, c_n)$ such that*

$$\boldsymbol{X}(t) = c_1\boldsymbol{X}_1(t) + \cdots + c_n\boldsymbol{X}_n(t) . \tag{8.8}$$

Proof. Let $t_0 \in I$. By Theorem 8.4, for each $\boldsymbol{e}_i = (\underbrace{0, \cdots, 0}_{(i-1) \text{ slots}}, 1, 0, \cdots, 0)$, there exists a unique solution $\boldsymbol{X} = \boldsymbol{\varphi}_i(t)$ to (8.6) satisfying the initial data $\boldsymbol{X}(t_0) = \boldsymbol{e}_i$. The set $\{\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \cdots, \boldsymbol{\varphi}_n\}$ are linearly independent for otherwise there exists a non-zero vector $(c_1, \cdots, c_n)$ such that

$$c_1\boldsymbol{\varphi}_1(t) + c_2\boldsymbol{\varphi}_2(t) + \cdots + c_n\boldsymbol{\varphi}_n(t) = \mathbf{0}$$

which, by setting $t = t_0$, would imply that $(c_1, c_2, \cdots, c_n) = \mathbf{0}$, a contradiction.

We note that every solution $\boldsymbol{X}(t) = \big[x_1(t), \cdots, x_n(t)\big]^{\mathrm{T}}$ to (8.6) can be uniquely expressed by

$$\boldsymbol{X}(t) = x_1(0)\boldsymbol{\varphi}_1(t) + x_2(0)\boldsymbol{\varphi}_2(t) + \cdots + x_n(0)\boldsymbol{\varphi}_n(t)\,. \tag{8.9}$$

In fact, $x_1(0)\boldsymbol{\varphi}_1(t) + \cdots + x_n(0)\boldsymbol{\varphi}_n(t)$ is a solution to (8.6) satisfying the initial data

$$\boldsymbol{X}(0) = \big[x_1(0), \cdots, x_n(0)\big]^{\mathrm{T}};$$

thus (8.9) is concluded from the uniqueness of the solution.

Now, since $\boldsymbol{X}_1, \cdots, \boldsymbol{X}_n$ are solution to (8.6), we find that

$$\operatorname{span}(\boldsymbol{X}_1, \cdots, \boldsymbol{X}_n) \subseteq \operatorname{span}(\boldsymbol{\varphi}_1, \cdots, \boldsymbol{\varphi}_n)\,.$$

Since $\{\boldsymbol{X}_1, \cdots, \boldsymbol{X}_n\}$ are linearly independent, $\dim\big(\operatorname{span}(\boldsymbol{X}_1, \cdots, \boldsymbol{X}_n)\big) = n$; thus by the fact that $\dim\big(\operatorname{span}(\boldsymbol{\varphi}_1, \cdots, \boldsymbol{\varphi}_n)\big) = n$, we must have

$$\operatorname{span}(\boldsymbol{X}_1, \cdots, \boldsymbol{X}_n) = \operatorname{span}(\boldsymbol{\varphi}_1, \cdots, \boldsymbol{\varphi}_n)\,.$$

Therefore, every solution $\boldsymbol{X} = \boldsymbol{\varphi}(t)$ of (8.6) can be (uniquely) expressed by (8.8). □

Theorem 8.11. *Let $\boldsymbol{X}_p$ be a given solution of the non-homogeneous system* (8.5) *on an interval I and let*

$$\boldsymbol{X}_c = c_1\boldsymbol{X}_1 + c_2\boldsymbol{X}_2 + \cdots + c_n\boldsymbol{X}_n\,,$$

denote the general solution on the same interval of the associated homogeneous system (8.6)*. Then the general solution of the non-homogeneous system on the interval is*

$$\boldsymbol{X} = \boldsymbol{X}_c + \boldsymbol{X}_p\,.$$

Remark 8.12. The general solution $\boldsymbol{X}_c$ of the associated homogeneous system (8.6) is called the complementary function of the non-homogeneous system (8.5).

Example 8.13. The function $\boldsymbol{X}_p = \begin{bmatrix} 3t-4 \\ -5t+6 \end{bmatrix}$ is a particular solution of the non-homogeneous system

$$\boldsymbol{X}' = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \boldsymbol{X} + \begin{bmatrix} 12t-11 \\ -3 \end{bmatrix}. \tag{8.10}$$

From Example 8.3, we find that the general solution to (8.10) is

$$\boldsymbol{X} = c_1 \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} 3e^{6t} \\ 5e^{6t} \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \boldsymbol{X} + \begin{bmatrix} 12t-11 \\ -3 \end{bmatrix}.$$

Next we provide a tool to determine if n solution vectors to the homogeneous system (8.6) are linearly independent. We recall that in linear algebra n vectors $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n$ in $\mathbb{R}^n$ are linearly independent if and only if the determinant $\det([\boldsymbol{v}_1 \vdots \boldsymbol{v}_2 \vdots \cdots \vdots \boldsymbol{v}_n]) \neq 0$. This motivates the following

Definition 8.14. Let

$$\boldsymbol{X}_1 = \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{bmatrix}, \quad \boldsymbol{X}_2 = \begin{bmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{bmatrix}, \quad \cdots \quad \boldsymbol{X}_k = \begin{bmatrix} x_{1k} \\ x_{2k} \\ \vdots \\ x_{nk} \end{bmatrix}, \quad \cdots \quad \boldsymbol{X}_n = \begin{bmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{bmatrix}$$

be n vector-valued functions defined on an interval I. The Wronskian (or Wronskian determinant) of $\boldsymbol{X}_1, \cdots, \boldsymbol{X}_n$ at $t \in I$, denoted by $W[\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_n](t)$, is the number

$$W[\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_n](t) \equiv \det\big([\boldsymbol{X}_1(t) \vdots \boldsymbol{X}_2(t) \vdots \cdots \vdots \boldsymbol{X}_n(t)]\big) = \begin{vmatrix} x_{11}(t) & x_{12}(t) & \cdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \cdots & x_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1}(t) & x_{n2}(t) & \cdots & x_{nn}(t) \end{vmatrix}.$$

Suppose that $\boldsymbol{X}_1, \cdots, \boldsymbol{X}_n$ are functions (but not necessary solution vectors of the homogeneous system (8.6)). The following two statements are true:

1. if $W[\boldsymbol{X}_1, \cdots, \boldsymbol{X}_n](t) \neq 0$ for some $t \in I$, then $\boldsymbol{X}_1, \cdots, \boldsymbol{X}_n$ are linearly independent;
2. if $\boldsymbol{X}_1, \cdots, \boldsymbol{X}_n$ are linearly dependent, then $W[\boldsymbol{X}_1, \cdots, \boldsymbol{X}_n] = 0$.

However, the converse statement for the two statements above is not true. For example, the two functions

$$\boldsymbol{X}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{X}_2 = \begin{bmatrix} t \\ 0 \end{bmatrix}$$

are linearly independent, but $W[\boldsymbol{X}_1, \boldsymbol{X}_2](t) = 0$ for all $t \in \mathbb{R}$. In other words, the Wronskian is not a very reliable tool to determine the linear independence/dependence of functions. Nevertheless, we have the following

Theorem 8.15. *Let*

$$\boldsymbol{X}_1 = \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{bmatrix}, \quad \boldsymbol{X}_2 = \begin{bmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{bmatrix}, \quad \cdots \quad \boldsymbol{X}_k = \begin{bmatrix} x_{1k} \\ x_{2k} \\ \vdots \\ x_{nk} \end{bmatrix}, \quad \cdots \quad \boldsymbol{X}_n = \begin{bmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{bmatrix}$$

be n solution vectors of the homogeneous system (8.6) *on an interval I. Then the Wronskian $W[\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_n]$ vanishes on the interval I if and only if it vanishes at some $t_0 \in I$.*

Proof. Let $\boldsymbol{A}(t) = [a_{ij}(t)]_{n\times n}$, and rewrite x_{ij} as $x_j^{(i)}$. Since $\boldsymbol{X}_j' = \boldsymbol{A}\boldsymbol{X}_j$, we must have

$$x_j^{(i)\prime} = \text{the } i\text{-th component of } \boldsymbol{A}\boldsymbol{X}_j = \sum_{k=1}^n a_{ik} x_j^{(k)};$$

thus using the properties of the determinants we find that

$$\begin{vmatrix} x_1^{(1)} & x_2^{(1)} & \cdots & \cdots & x_n^{(1)} \\ \vdots & \vdots & & & \vdots \\ x_1^{(i-1)} & x_2^{(i-1)} & \cdots & \cdots & x_n^{(i-1)} \\ x_1^{(i)\prime} & x_2^{(i)\prime} & \cdots & \cdots & x_n^{(i)\prime} \\ x_1^{(i+1)} & x_2^{(i+1)} & \cdots & \cdots & x_n^{(i+1)} \\ \vdots & \vdots & & & \vdots \\ x_1^{(n)} & x_2^{(n)} & \cdots & \cdots & x_n^{(n)} \end{vmatrix} = \begin{vmatrix} x_1^{(1)} & x_2^{(1)} & \cdots & \cdots & x_n^{(1)} \\ \vdots & \vdots & & & \vdots \\ x_1^{(i-1)} & x_2^{(i-1)} & \cdots & \cdots & x_n^{(i-1)} \\ \sum\limits_{k=1}^n a_{ik} x_1^{(k)} & \sum\limits_{k=1}^n a_{ik} x_2^{(k)} & \cdots & \cdots & \sum\limits_{k=1}^n a_{ik} x_n^{(k)} \\ x_1^{(i+1)} & x_2^{(i+1)} & \cdots & \cdots & x_n^{(i+1)} \\ \vdots & \vdots & & & \vdots \\ x_1^{(n)} & x_2^{(n)} & \cdots & \cdots & x_n^{(n)} \end{vmatrix}$$

$$\overset{\text{“row operations”}}{=} \begin{vmatrix} x_1^{(1)} & x_2^{(1)} & \cdots & \cdots & x_n^{(1)} \\ \vdots & \vdots & & & \vdots \\ x_1^{(i-1)} & x_2^{(i-1)} & \cdots & \cdots & x_n^{(i-1)} \\ a_{ii}x_1^{(i)} & a_{ii}x_2^{(i)} & \cdots & \cdots & a_{ii}x_n^{(i)} \\ x_1^{(i+1)} & x_2^{(i+1)} & \cdots & \cdots & x_n^{(i+1)} \\ \vdots & \vdots & & & \vdots \\ x_1^{(n)} & x_2^{(n)} & \cdots & \cdots & x_n^{(n)} \end{vmatrix} = a_{ii}\mathrm{W}.$$

Therefore,

$$\begin{aligned}\frac{d}{dt}\mathrm{W} &= \begin{vmatrix} x_1^{(1)\prime} & x_2^{(1)\prime} & \cdots & x_n^{(1)\prime} \\ x_1^{(2)} & x_2^{(2)} & \cdots & x_n^{(2)} \\ \vdots & & \ddots & \vdots \\ x_1^{(n)} & x_2^{(n)} & \cdots & x_n^{(n)} \end{vmatrix} + \cdots + \begin{vmatrix} x_1^{(1)} & x_2^{(1)} & \cdots & x_n^{(1)} \\ \vdots & & \ddots & \vdots \\ x_1^{(n-1)} & x_2^{(n-1)} & \cdots & x_n^{(n-1)} \\ x_1^{(n)\prime} & x_2^{(n)\prime} & \cdots & x_n^{(n)\prime} \end{vmatrix} \\ &= (a_{11} + \cdots + a_{nn})\mathrm{W} = \mathrm{tr}(\boldsymbol{A})\mathrm{W}\,;\end{aligned}$$

thus by the method of integrating factors,

$$\mathrm{W}(t) = \exp\Big(\int_{t_0}^{t} \mathrm{tr}(\boldsymbol{A})(s)\,ds\Big)\mathrm{W}(t_0)$$

which implies that the Wronskian is identically zero (if $\mathrm{W}(t_0)$ is zero) or else never vanishes (if $\mathrm{W}(t_0) \neq 0$). □

Remark 8.16. The theorem above implies that if $\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_n$ are n solution vectors of the homogeneous system (8.6) on an interval I, then exactly one of the following cases holds:

1. $W[\boldsymbol{X}_1, \cdots, \boldsymbol{X}_n](t) = 0$ for all $t \in I$, in which case $\boldsymbol{X}_1, \cdots, \boldsymbol{X}_n$ are linearly dependent;
2. $W[\boldsymbol{X}_1, \cdots, \boldsymbol{X}_n](t) \neq 0$ for NO values of $t \in I$, in which case $\boldsymbol{X}_1, \cdots, \boldsymbol{X}_n$ are linearly independent.

8.2 Homogeneous Linear Systems

In this section, we consider the equation

$$\boldsymbol{X}' = \boldsymbol{A}\boldsymbol{X}\,, \tag{8.11}$$

where $\boldsymbol{A}$ is a **constant** $n \times n$ matrix.

Example 8.17. For $a, b, c, d \in \mathbb{R}$ being given constants, we consider the linear system

$$\boldsymbol{X}' = \boldsymbol{A}\boldsymbol{X} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \boldsymbol{X} \tag{8.12}$$

or equivalently,

$$x_1' = ax_1 + bx_2\,, \tag{8.13a}$$

$$x_2' = cx_1 + dx_2\,, \tag{8.13b}$$

where $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\boldsymbol{X} = [x_1, x_2]^{\mathrm{T}}$. Using (8.13a), we have $bx_2 = x_1' - ax_2$; thus (8.13b) implies that x_1 satisfies

$$x_1'' - (a+d)x_1' + (ad - bc)x_1 = 0\,.$$

Similarly, x_2 satisfies

$$x_2'' - (a+d)x_2' + (ad - bc)x_1 = 0\,.$$

We note that the characteristic equation for the two ODEs above are exactly the characteristic equation of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

1. Suppose that $\lambda_1 \neq \lambda_2$ are distinct zeros of the characteristic equation, then

$$x_1(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} \qquad \text{and} \qquad x_2(t) = k_3 e^{\lambda_1 t} + k_4 e^{\lambda_2 t}$$

for some constants k_1, k_2, k_3, k_4 satisfying

$$\begin{aligned} \lambda_1 k_1 e^{\lambda_1 t} + \lambda_2 k_2 e^{\lambda_2 t} &= (ak_1 + bk_3)e^{\lambda_1 t} + (ak_2 + bk_4)e^{\lambda_2 t}\,, \\ \lambda_1 k_3 e^{\lambda_1 t} + \lambda_2 k_2 e^{\lambda_2 t} &= (ck_1 + dk_3)e^{\lambda_1 t} + (ck_2 + dk_4)e^{\lambda_2 t}\,. \end{aligned}$$

Since $\{e^{\lambda_1 t}, e^{\lambda_2 t}\}$ are linearly independent, we must have that k_1, k_2, k_3, k_4 satisfy

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} k_1 \\ k_3 \end{bmatrix} = \lambda_1 \begin{bmatrix} k_1 \\ k_3 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} k_2 \\ k_4 \end{bmatrix} = \lambda_2 \begin{bmatrix} k_2 \\ k_4 \end{bmatrix}\,.$$

Observing that

$$\boldsymbol{X}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} \\ k_3 e^{\lambda_1 t} + k_4 e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} k_1 \\ k_3 \end{bmatrix} e^{\lambda_1 t} + \begin{bmatrix} k_2 \\ k_4 \end{bmatrix} e^{\lambda_2 t}\,,$$

by letting $\boldsymbol{K}_1 = [k_1, k_3]^{\mathrm{T}}$ and $\boldsymbol{K}_2 = [k_2, k_4]^{\mathrm{T}}$, we conclude that the solution of the system (8.12) can be expressed as

$$\boldsymbol{X}(t) = \boldsymbol{K}_1 e^{\lambda_1 t} + \boldsymbol{K}_2 e^{\lambda_2 t}\,,$$

where $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ satisfies that $(\boldsymbol{A} - \lambda_1 \mathbf{I})\boldsymbol{K}_1 = (\boldsymbol{A} - \lambda_2 \mathbf{I})\boldsymbol{K}_2 = \mathbf{0}$. In particular, if $\boldsymbol{K}_1, \boldsymbol{K}_2$ are respective eigenvectors associated with λ_1 and λ_2, then $\{\boldsymbol{K}_1 e^{\lambda_1 t}, \boldsymbol{K}_2 e^{\lambda_2 t}\}$ is a fundamental set of solutions of (8.12).

2. Suppose that λ is a repeated eigenvalue of the characteristic equation, then
$$x_1(t) = k_1 te^{\lambda t} + k_2 e^{\lambda t} \quad \text{and} \quad x_2(t) = k_3 te^{\lambda t} + k_4 e^{\lambda t}$$
for some k_1, k_2, k_3, k_4 satisfying
$$\begin{aligned}(k_1 + k_2\lambda)e^{\lambda t} + k_1\lambda te^{\lambda t} &= (ak_1 + bk_3)te^{\lambda t} + (ak_2 + bk_4)e^{\lambda t}\,,\\ (k_3 + k_4\lambda)e^{\lambda t} + k_3\lambda te^{\lambda t} &= (ck_3 + dk_4)te^{\lambda t} + (ck_3 + dk_4)e^{\lambda t}\,.\end{aligned}$$
Since $\{e^{\lambda t}, te^{\lambda t}\}$ are linearly independent,
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} k_1 \\ k_3 \end{bmatrix} = \lambda \begin{bmatrix} k_1 \\ k_3 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} k_2 \\ k_4 \end{bmatrix} = \lambda \begin{bmatrix} k_2 \\ k_4 \end{bmatrix} + \begin{bmatrix} k_1 \\ k_3 \end{bmatrix}.$$
Let $\boldsymbol{K}_1 = [k_1, k_3]^{\mathrm{T}}$ and $\boldsymbol{K}_2 = [k_2, k_4]^{\mathrm{T}}$. Then the two identities above show that
$$(\boldsymbol{A} - \lambda\mathbf{I})\boldsymbol{K}_1 = \mathbf{0}\,, \tag{8.14a}$$
$$(\boldsymbol{A} - \lambda\mathbf{I})\boldsymbol{K}_2 = \boldsymbol{K}_1\,. \tag{8.14b}$$
As a consequence,
   (i) If $\boldsymbol{A}$ has two linearly independent eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, then there exists no solution to (8.14b) if $\boldsymbol{K}_1$ is a non-zero vector satisfying (8.14a). In this case, $\boldsymbol{K}_1$ can only be zero vector so that $\boldsymbol{K}_2$ is an eigenvector of $\boldsymbol{A}$ associated with eigenvalue λ. Therefore, $\{\boldsymbol{v}_1 e^{\lambda t}, \boldsymbol{v}_2 e^{\lambda t}\}$ is a fundamental set of solutions of (8.12).
   (ii) If $\boldsymbol{A}$ has only one linearly independent eigenvector $\boldsymbol{K}_1$, then there exists a non-zero $\boldsymbol{K}_2$ satisfying $(A - \lambda\mathbf{I})\boldsymbol{K}_2 = \boldsymbol{K}_1$. In this case, $\{\boldsymbol{K}_1 e^{\lambda t}, \boldsymbol{K}_1 te^{\lambda t} + \boldsymbol{K}_2 e^{\lambda t}\}$ is a fundamental set of solutions of (8.12). We also note that in this case $\boldsymbol{K}_2$ satisfies that $(\boldsymbol{A} - \lambda\mathbf{I})^2\boldsymbol{K}_2 = \mathbf{0}$ but $(\boldsymbol{A} - \lambda\mathbf{I})\boldsymbol{K}_2 \neq \mathbf{0}$ so that $\boldsymbol{K}_2$ is a generalized eigenvector of $\boldsymbol{A}$ (associated with λ).

Motivated by Example 8.17, we are prompted to ask whether we can always find a solution of the form
$$\boldsymbol{X}(t) = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} e^{\lambda t} = \boldsymbol{K}e^{\lambda t}\,,$$
where $\boldsymbol{K}$ is a non-zero constant vector and λ is a constant, for the general homogeneous linear first-order system
$$\boldsymbol{X}' = \boldsymbol{A}\boldsymbol{X}\,, \tag{8.15}$$

where $\boldsymbol{A}$ is an $n \times n$ matrix with constant entries. Suppose that $\boldsymbol{X}(t) = \boldsymbol{K}e^{\lambda t}$ is indeed a solution to (8.15), then by the fact that $\dfrac{d}{dx}\boldsymbol{K}e^{\lambda t} = \lambda \boldsymbol{K}e^{\lambda t}$, we find that λ and $\boldsymbol{K}$ satisfy that

$$(\boldsymbol{A} - \lambda \mathbf{I})\boldsymbol{K} = \mathbf{0}\,.$$

In other words, for $\boldsymbol{X}(t) = \boldsymbol{K}e^{\lambda t}$ being a non-trivial solution to (8.15), λ must be an eigenvalue of $\boldsymbol{A}$ and $\boldsymbol{K}$ is a corresponding eigenvector.

8.2.1 The case that $\boldsymbol{A}$ has n linearly independent eigenvectors

Theorem 8.18. *Let $\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n$ be n eigenvalues of the coefficient matrix $\boldsymbol{A}$ of the homogeneous system* (8.15) *and let $\boldsymbol{K}_1,\ \boldsymbol{K}_2,\ \cdots,\ \boldsymbol{K}_n$ be the corresponding eigenvectors such that $\boldsymbol{K}_1, \cdots, \boldsymbol{K}_n$ are linearly independent. Then the general solution of* (8.15) *on $\mathbb{R}$ is given by*

$$\boldsymbol{X}(t) = c_1\boldsymbol{K}_1 e^{\lambda_1 t} + c_2\boldsymbol{K}_2 e^{\lambda_2 t} + \cdots + c_n\boldsymbol{K}_n e^{\lambda_n t}\,.$$

Proof. Note that the computation above shows that $\boldsymbol{X}_j \equiv \boldsymbol{K}_j e^{\lambda_j t}$ of the homogeneous system (8.15). By the fact that $\boldsymbol{K}_1, \cdots, \boldsymbol{K}_n$ are linearly independent, we find that

$$W[\boldsymbol{X}_1, \cdots, \boldsymbol{X}_n](t) = e^{(\lambda_1+\lambda_2+\cdots+\lambda_n)t}\det\left(\left[\boldsymbol{K}_1 \vdots \boldsymbol{K}_2 \vdots \cdots \vdots \boldsymbol{K}_n\right]\right) \neq 0\,.$$

The desired result is then concluded from Theorem 8.10 and 8.15. □

Example 8.19. Consider the linear system

$$\boldsymbol{X}' = \begin{bmatrix} -4 & 1 & 1 \\ 1 & 5 & -1 \\ 0 & 1 & -3 \end{bmatrix}\boldsymbol{X}\,. \tag{8.16}$$

Let $\boldsymbol{A}$ be the 3×3 matrix in (8.16). If λ is an eigenvalue of $\boldsymbol{A}$, then

$$0 = \det(\boldsymbol{A} - \lambda \mathbf{I}) = \begin{vmatrix} -4-\lambda & 1 & 1 \\ 1 & 5-\lambda & -1 \\ 0 & 1 & -3-\lambda \end{vmatrix} = (4+\lambda)(5-\lambda)(3+\lambda)\,;$$

thus $\lambda = -3$, -4 or 5. If $\boldsymbol{K}_1$, $\boldsymbol{K}_2$ and $\boldsymbol{K}_3$ are respective eigenvectors associated with eigenvalues $\lambda_1 = -3$, $\lambda_2 = -4$ and $\lambda_3 = 5$, then

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & 8 & -1 \\ 0 & 1 & 0 \end{bmatrix}\boldsymbol{K}_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 9 & -1 \\ 0 & 1 & 1 \end{bmatrix}\boldsymbol{K}_2 = \begin{bmatrix} -9 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -8 \end{bmatrix}\boldsymbol{K}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and we choose

$$\boldsymbol{K}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{K}_2 = \begin{bmatrix} 10 \\ -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{K}_3 = \begin{bmatrix} 1 \\ 8 \\ 1 \end{bmatrix}.$$

Therefore, the general solution of (8.16) is

$$\boldsymbol{X}(t) = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 10 \\ -1 \\ 1 \end{bmatrix} e^{-4t} + c_3 \begin{bmatrix} 1 \\ 8 \\ 1 \end{bmatrix} e^{5t}.$$

Example 8.20. Consider the linear system

$$\boldsymbol{X}' = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \boldsymbol{X}.$$

The characteristic equation of the matrix given in the linear system is $-(\lambda+1)^2(\lambda-5)=0$ so that the eigenvalues of the matrix are -1 and t. There are two linearly independent eigenvectors $\boldsymbol{K}_1 = [1,1,0]^{\mathrm{T}}$ and $\boldsymbol{K}_2 = [0,1,1]^{\mathrm{T}}$ associated with $\lambda = -1$, while $\boldsymbol{K}_3 = [1,-1,1]^{\mathrm{T}}$ is an eigenvector associated with $\lambda = 5$. Therefore, Theorem 8.18 implies that the general solution to the linear system above is

$$\boldsymbol{X}(t) = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{-5t}.$$

Example 8.21. Consider the linear system

$$\boldsymbol{X}' = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix} \boldsymbol{X}. \tag{8.17}$$

Let $\boldsymbol{A}$ be the 2×2 matrix in (8.17). If λ is an eigenvalue of $\boldsymbol{A}$, then

$$0 = \det(\boldsymbol{A} - \lambda\mathbf{I}) = \begin{vmatrix} 6-\lambda & -1 \\ 5 & 4-\lambda \end{vmatrix} = (6-\lambda)(4-\lambda)+5 = \lambda^2 - 10\lambda + 29;$$

thus $\lambda = 5 \pm 2i$. If $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ are eigenvectors of $\boldsymbol{A}$ associated with the eigenvalues $\lambda_1 = 5+2i$ and $\lambda_2 = 5-2i$, then

$$\begin{bmatrix} 1+2i & -1 \\ 5 & -1+2i \end{bmatrix} \boldsymbol{K}_1 = \begin{bmatrix} 1-2i & -1 \\ 5 & -1-2i \end{bmatrix} \boldsymbol{K}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and we choose

$$\boldsymbol{K}_1 = \begin{bmatrix} 1 \\ 1-2i \end{bmatrix} \quad \text{and} \quad \boldsymbol{K}_2 = \begin{bmatrix} 1 \\ 1+2i \end{bmatrix}.$$

Therefore, the general solution of (8.17) is

$$\boldsymbol{X}(t) = c_1 \begin{bmatrix} 1 \\ 1-2i \end{bmatrix} e^{(5+2i)t} + c_2 \begin{bmatrix} 1 \\ 1+2i \end{bmatrix} e^{(5-2i)t}. \tag{8.18}$$

Theorem 8.22. *Let $\boldsymbol{A}$ be the coefficient matix having real entries of the homogeneous system* (8.11)*, and let $\boldsymbol{K}$ be an eigenvector corresponding to the complex eigenvalue λ. Then*

$$\boldsymbol{X}_1(t) = \boldsymbol{K}e^{\lambda t} \qquad \textit{and} \qquad \boldsymbol{X}_2(t) = \bar{\boldsymbol{K}}e^{\bar{\lambda}t}$$

are solutions to (8.11)*, where $^-$ denotes the complex conjugate.*

Proof. The theorem is concluded from the fact that if λ is a complex eigenvalue of a real matrix $\boldsymbol{A}$ with a corresponding eigenvector $\boldsymbol{K}$, then $\bar{\lambda}$ is also an eigenvalue of $\boldsymbol{A}$ with a corresponding eigenvector $\bar{\boldsymbol{K}}$. □

In Example 8.21, it is desirable to rewrite a solution such as (8.18) in terms of real-valued functions. Using the Euler formula

$$e^{\alpha+i\beta} = e^{\alpha}(\cos\beta + i\sin\beta) \qquad \forall\, \alpha, \beta \in \mathbb{R},$$

letting $C_1 = c_1 + c_2$ and $C_2 = (c_1 - c_2)i$, (8.18) becomes

$$\begin{aligned}\boldsymbol{X}(t) &= \frac{C_1 - C_2 i}{2}\begin{bmatrix} 1 \\ 1-2i \end{bmatrix} e^{5t}(\cos 2t + i\sin 2t) + \frac{C_1 + C_2 i}{2}\begin{bmatrix} 1 \\ 1+2i \end{bmatrix} e^{5t}(\cos 2t - i\sin 2t) \\ &= C_1\Big(\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t}\cos 2t - \begin{bmatrix} 0 \\ -2 \end{bmatrix} e^{5t}\sin 2t\Big) + C_2\Big(\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t}\sin 2t + \begin{bmatrix} 0 \\ -2 \end{bmatrix} e^{5t}\cos 2t\Big).\end{aligned}$$

In general, if $\boldsymbol{K}$ is an eigenvector of $\boldsymbol{A}$ (with real entries) associated with the complex eigenvalue $\lambda = \alpha + i\beta$, where $\alpha, \beta \in \mathbb{R}$, then

$$\boldsymbol{K}e^{\lambda t} = \boldsymbol{K}e^{\alpha t}(\cos\beta t + i\sin\beta t) \qquad \text{and} \qquad \bar{\boldsymbol{K}}e^{\bar{\lambda}t} = \bar{\boldsymbol{K}}e^{\alpha t}(\cos\beta t - i\sin\beta t).$$

By the superposition principle,

$$\boldsymbol{X}_1(t) = \frac{1}{2}\big(\boldsymbol{K}e^{\lambda t} + \bar{\boldsymbol{K}}e^{\bar{\lambda}t}\big) = \frac{1}{2}(\boldsymbol{K} + \bar{\boldsymbol{K}})e^{\alpha t}\cos\beta t + \frac{i}{2}(\boldsymbol{K} - \bar{\boldsymbol{K}})e^{\alpha t}\sin\beta t$$

and

$$\boldsymbol{X}_2(t) = \frac{i}{2}\big(-\boldsymbol{K}e^{\lambda t} + \bar{\boldsymbol{K}}e^{\bar{\lambda}t}\big) = -\frac{i}{2}(\boldsymbol{K} - \bar{\boldsymbol{K}})e^{\alpha t}\cos\beta t + \frac{1}{2}(\boldsymbol{K} + \bar{\boldsymbol{K}})e^{\alpha t}\sin\beta t$$

are also solutions to the homogeneous system (8.11). By defining

$$\boldsymbol{B}_1 = \frac{1}{2}(\boldsymbol{K} + \bar{\boldsymbol{K}}) \qquad \text{and} \qquad \boldsymbol{B}_2 = -\frac{i}{2}(\boldsymbol{K} - \bar{\boldsymbol{K}})\,, \tag{8.19}$$

we conclude the following

Theorem 8.23. *Let $\lambda = \alpha + i\beta$ be a complex eigenvalue of the coefficient matrix $\boldsymbol{A}$ with real entries in the homogeneous system* (8.11)*, and let $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ denote the column vectors defined in* (8.19)*. Then*

$$\boldsymbol{X}_1(t) = (\boldsymbol{B}_1 \cos\beta t - \boldsymbol{B}_2 \sin\beta t)e^{\alpha t} \qquad \textit{and} \qquad \boldsymbol{X}_2(t) = (\boldsymbol{B}_2 \cos\beta t + \boldsymbol{B}_1 \sin\beta t)e^{\alpha t}$$

are linearly independent solutions of (8.11) *on $\mathbb{R}$.*

Remark 8.24. We note that $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ are the real part and the imaginary part of $\boldsymbol{K}$, respectively.

8.2.2 The case that $\boldsymbol{A}$ does not have n linearly independent eigenvectors

In this sub-section we consider, in contrast to the previous sub-section, the case that the matrix $\boldsymbol{A}$ does not have n linearly independent eigenvectors. Then there exists at least one eigenvalue λ of A whose corresponding eigenspace has dimension less than the algebraic multiplicity of λ. To be more precise, there exist λ and $m \in \mathbb{N}$ such that $(x-\lambda)^m$ is a factor of the characteristic equation $\det(\boldsymbol{A} - x\mathbf{I}) = 0$ while $(x-\lambda)^{m+1}$ is not a factor, but there are only p linearly independent eigenvectors $\boldsymbol{K}_1, \cdots, \boldsymbol{K}_p$ of $\boldsymbol{A}$ associated with λ for some $p < m$.

Suppose the simplest case that λ is the only eigenvalue of $\boldsymbol{A}$. Then Theorem 8.9 and 8.10 show that there are $(n-p)$ solutions whose union with the set $\left\{\boldsymbol{K}_j e^{\lambda t} \,\middle|\, 1 \leqslant j \leqslant p\right\}$ forms a basis of the solution space. Motivated by Example 8.17, in this case we look for non-zero constant vectors $\boldsymbol{K}$ satisfying that

$$(\boldsymbol{A} - \lambda\mathbf{I})^2\boldsymbol{K} = \mathbf{0} \qquad \text{and} \qquad (\boldsymbol{A} - \lambda\mathbf{I})\boldsymbol{K} \neq \mathbf{0}\,,$$

and expect that $(\boldsymbol{A} - \lambda\mathbf{I})\boldsymbol{K}te^{\lambda t} + \boldsymbol{K}e^{\lambda t}$ is a non-trivial solution of the homogeneous system (8.11). We note that if $(\boldsymbol{A} - \lambda\mathbf{I})^2\boldsymbol{K} = \mathbf{0}$, then $\boldsymbol{X}(t) = (\boldsymbol{A} - \lambda\mathbf{I})\boldsymbol{K}te^{\lambda t} + \boldsymbol{K}e^{\lambda t}$ is indeed a solution to (8.11) since

$$\begin{aligned}\boldsymbol{X}' - \boldsymbol{A}\boldsymbol{X} &= \frac{d}{dt}\Big[(\boldsymbol{A}-\lambda\mathbf{I})\boldsymbol{K}te^{\lambda t} + \boldsymbol{K}e^{\lambda t}\Big] - \boldsymbol{A}\big[(\boldsymbol{A}-\lambda\mathbf{I})\boldsymbol{K}te^{\lambda t} + \boldsymbol{K}e^{\lambda t}\big]\\ &= (\boldsymbol{A}-\lambda\mathbf{I})\boldsymbol{K}e^{\lambda t} + \lambda(\boldsymbol{A}-\lambda\mathbf{I})\boldsymbol{K}te^{\lambda t} + \lambda\boldsymbol{K}e^{\lambda t} - \boldsymbol{A}(\boldsymbol{A}-\lambda\mathbf{I})\boldsymbol{K}te^{\lambda t} - \boldsymbol{A}\boldsymbol{K}e^{\lambda t}\\ &= -(\boldsymbol{A}-\lambda\mathbf{I})^2\boldsymbol{K}te^{\lambda t} = \mathbf{0}\,.\end{aligned}$$

Suppose that the dimension of the null space of the matrix $(\boldsymbol{A}-\lambda\mathbf{I})^2$ is q, and $\boldsymbol{K}_{p+1}$, $\boldsymbol{K}_{p+2}$, $\cdots$, $\boldsymbol{K}_q$ are linearly independent vectors satisfying

$$(\boldsymbol{A}-\lambda\mathbf{I})^2\boldsymbol{K}_j = \mathbf{0} \quad \text{and} \quad (\boldsymbol{A}-\lambda\mathbf{I})\boldsymbol{K}_j \neq \mathbf{0} \qquad \forall\, p+1 \leqslant j \leqslant q\,.$$

By the fact that the set

$$\big\{\boldsymbol{K}_j e^{\lambda t}\,\big|\, 1 \leqslant j \leqslant p\big\} \cup \big\{(\boldsymbol{A}-\lambda\mathbf{I})\boldsymbol{K}_j te^{\lambda t} + \boldsymbol{K}_j e^{\lambda t}\,\big|\, p+1 \leqslant j \leqslant q\big\} \tag{8.20}$$

is a linearly independent set, we find that if $q = n$, then the set given in (8.20) is a fundamental set of solutions of (8.11).

On the other hand, if $q < n$ then there exist $(n-q)$ linearly independent solutions whose union with the set given in (8.20) forms a fundamental set of solutions of (8.11). Having the experience from Example 8.17 and the solutions of second-order linear ODEs with constant coefficients in Chapter 4, we expect that there is a solution of the form

$$\boldsymbol{L}\,t^2e^{\lambda t} + \boldsymbol{J}te^{\lambda t} + \boldsymbol{K}e^{\lambda t}$$

for some vectors $\boldsymbol{K}$, $\boldsymbol{J}$ and $\boldsymbol{L}$. This is indeed the case, and we have the following

Theorem 8.25. *Let $\boldsymbol{A}$ be the coefficient matrix in the homogeneous system* (8.11)*, and λ be an eigenvalue of $\boldsymbol{A}$ with algebraic multiplicity m. Then m linearly independent solution to* (8.11) *are given by*

$$\begin{aligned}\boldsymbol{X}_1(t) &= \boldsymbol{K}_{11}e^{\lambda t}\,,\\ \boldsymbol{X}_2(t) &= \boldsymbol{K}_{21}te^{\lambda t} + \boldsymbol{K}_{22}e^{\lambda t}\,,\\ &\ \ \vdots\\ \boldsymbol{X}_m(t) &= \boldsymbol{K}_{m1}\frac{t^{m-1}}{(m-1)!}e^{\lambda t} + \boldsymbol{K}_{m2}\frac{t^{m-2}}{(m-2)!}e^{\lambda t} + \cdots + \boldsymbol{K}_{mm}e^{\lambda t}\,,\end{aligned}$$

where $\boldsymbol{K}_{ij}$ are vectors for all $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant i$.

Remark 8.26. Assume the conditions in Theorem above, and let

$$\boldsymbol{X}(t) = \boldsymbol{K}_\ell \frac{t^{\ell-1}}{(\ell-1)!}e^{\lambda t} + \boldsymbol{K}_{\ell-1}\frac{t^{\ell-2}}{(\ell-2)!}e^{\lambda t} + \cdots + \boldsymbol{K}_2 t e^{\lambda t} + \boldsymbol{K}_1 e^{\lambda t}. \tag{8.21}$$

Then

$$\begin{aligned}\boldsymbol{X}' - \boldsymbol{A}\boldsymbol{X} &= \boldsymbol{K}_\ell \frac{t^{\ell-2}}{(\ell-2)!}e^{\lambda t} + \boldsymbol{K}_{\ell-1}\frac{t^{\ell-3}}{(\ell-3)!}e^{\lambda t} + \cdots + \boldsymbol{K}_2 e^{\lambda t} - (\boldsymbol{A} - \lambda\mathbf{I})\boldsymbol{X}(t)\\ &= -(\boldsymbol{A} - \lambda\mathbf{I})\boldsymbol{K}_\ell \frac{t^{\ell-1}}{(\ell-1)!}e^{\lambda t} + \big[\boldsymbol{K}_\ell - (\boldsymbol{A} - \lambda\mathbf{I})\boldsymbol{K}_{\ell-1}\big]\frac{t^{\ell-2}}{(\ell-2)!}e^{\lambda t}\\ &\quad + \big[\boldsymbol{K}_{\ell-1} - (\boldsymbol{A} - \lambda\mathbf{I})\boldsymbol{K}_{\ell-2}\big]\frac{t^{\ell-3}}{(\ell-3)!}e^{\lambda t} + \cdots + \big[\boldsymbol{K}_2 - (\boldsymbol{A} - \lambda\mathbf{I})\boldsymbol{K}_1\big]e^{\lambda t}\end{aligned}$$

so that the fact that $\big\{e^{\lambda t}, te^{\lambda t}, t^2 e^{\lambda t}, \cdots, t^{\ell-1}e^{\lambda t}\big\}$ is a linearly independent set implies that

$$\begin{aligned}\boldsymbol{X} \text{ is a solution to (8.11)} &\Leftrightarrow (\boldsymbol{A} - \lambda\mathbf{I})\boldsymbol{K}_\ell = \mathbf{0} \text{ and } \boldsymbol{K}_j = (\boldsymbol{A} - \lambda\mathbf{I})\boldsymbol{K}_{j-1} \text{ for all } 2 \leqslant j \leqslant \ell\\ &\Leftrightarrow (\boldsymbol{A} - \lambda\mathbf{I})\boldsymbol{K}_\ell = \mathbf{0} \text{ and } \boldsymbol{K}_j = (\boldsymbol{A} - \lambda\mathbf{I})^{j-1}\boldsymbol{K}_1 \text{ for all } 2 \leqslant j \leqslant \ell.\end{aligned}$$

We then immediately conclude that $\boldsymbol{X}$ given by (8.21) is a solution to (8.11) with leading coefficient vector $\boldsymbol{K}_\ell \neq \mathbf{0}$ if and only if

$$(\boldsymbol{A} - \lambda\mathbf{I})^\ell \boldsymbol{K}_1 = \mathbf{0}, \quad (\boldsymbol{A} - \lambda\mathbf{I})^{\ell-1}\boldsymbol{K}_1 \neq \mathbf{0} \quad \text{and} \quad \boldsymbol{K}_j = (\boldsymbol{A} - \lambda\mathbf{I})^{j-1}\boldsymbol{K}_1 \text{ for all } 2 \leqslant j \leqslant \ell.$$

The statement above provides a way to find solutions to (8.11) of the form given by (8.11).

Example 8.27. Consider the linear system

$$\boldsymbol{X}' = \boldsymbol{A}\boldsymbol{X} = \begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix}\boldsymbol{X}.$$

The matrix $\boldsymbol{A}$ has a repeated eigenvalue $\lambda = -3$ with algebraic multiplicity 2, but the corresponding eigenspace is 1-dimensional and is spanned by the eigenvector $\boldsymbol{K}_1 = [3, 1]^{\mathrm{T}}$. Therefore, we immediately obtain one solution $\boldsymbol{X}_1(t) = \boldsymbol{K}_1 e^{-3t}$ and there exists a solution of the form

$$\boldsymbol{X}_2(t) = (\boldsymbol{A} + 3\mathbf{I})\boldsymbol{K}te^{-3t} + \boldsymbol{K}e^{-3t}$$

for some $\boldsymbol{K}$ satisfying $(\boldsymbol{A} + 3\mathbf{I})^2\boldsymbol{K} = \mathbf{0}$ and $(\boldsymbol{A} + 3\mathbf{I})\boldsymbol{K} \neq \mathbf{0}$. Since

$$(\boldsymbol{A} + 3\mathbf{I})^2 = \begin{bmatrix} 6 & -18 \\ 2 & -6 \end{bmatrix}\begin{bmatrix} 6 & -18 \\ 2 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

we can choose $\boldsymbol{K} = [1; 0]^{\mathrm{T}}$ (or any $\boldsymbol{K}$ which is not an eigenvector of $\boldsymbol{A}$) and obtain another solution

$$\boldsymbol{X}_2(t) = \begin{bmatrix} 6 \\ 2 \end{bmatrix} te^{-3t} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} te^{-3t}.$$

Example 8.28. Consider the linear system

$$\boldsymbol{X}' = \boldsymbol{A}\boldsymbol{X} = \begin{bmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix} \boldsymbol{X}\,.$$

The matrix $\boldsymbol{A}$ has a repeated eigenvalue $\lambda = 2$ with algebraic multiplicity 3, and the corresponding eigenspace is 1-dimensional and is spanned by the eigenvector $\boldsymbol{K}_1 = [1, 0, 0]^{\mathrm{T}}$. Therefore, we immediately obtain one solution $\boldsymbol{X}_1(t) = \boldsymbol{K}_1 e^{2t}$. To obtain another two linearly independent solution, we first look for solutions of the form

$$(\boldsymbol{A} - 2\mathbf{I})\boldsymbol{K} t e^{2t} + \boldsymbol{K} e^{2t}\,,$$

where $\boldsymbol{K}$ satisfies $(\boldsymbol{A} - 2\mathbf{I})^2 \boldsymbol{K} = 0$ but $(\boldsymbol{A} - 2\mathbf{I})\boldsymbol{K} \neq \mathbf{0}$. Since

$$(\boldsymbol{A} - 2\mathbf{I})^2 = \begin{bmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

we choose $\boldsymbol{K} = [0, 1, 0]^{\mathrm{T}}$ and obtain another solution

$$\boldsymbol{X}_2(t) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t e^{2t} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{t}\,.$$

We note that there will be no other linearly independent solution of the form $(\boldsymbol{A} - 2\mathbf{I})\boldsymbol{K} t e^{2t} + \boldsymbol{K} e^{2t}$ since the null space of $(\boldsymbol{A} - 2\mathbf{I})^2$ is 2-dimensional. Therefore, there must be a solution of the form

$$(\boldsymbol{A} - 2\mathbf{I})^2 \boldsymbol{K} \frac{t^2}{2} e^{2t} + (\boldsymbol{A} - 2\mathbf{I})\boldsymbol{K} t e^{2t} + \boldsymbol{K} e^{2t}\,,$$

where $\boldsymbol{K}$ satisfies $(\boldsymbol{A} - 2\mathbf{I})^3 \boldsymbol{K} = 0$ but $(\boldsymbol{A} - 2\mathbf{I})^2 \boldsymbol{K} \neq \mathbf{0}$. We then choose $\boldsymbol{K} = [0, 0, 1]^{\mathrm{T}}$ and obtain

$$\boldsymbol{X}_3(t) = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} \frac{t^2}{2} e^{t} + \begin{bmatrix} 6 \\ 5 \\ 0 \end{bmatrix} t e^{t} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{t}$$

so that $\{\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3\}$ is a fundamental set of solutions to the system given above.

8.3 Non-homogeneous Linear Systems

In this section we focus on solving the linear system

$$\boldsymbol{X}' = \boldsymbol{A}(t)\boldsymbol{X} + \boldsymbol{F}(t)\,, \tag{8.22}$$

where $\boldsymbol{A}$ is an $n \times n$ matrix. By Theorem 8.11, it suffices to find a particular solution so that general solutions to (8.22) can be expressed as the sum of this particular solution and linear combinations of functions in a fundamental set of solution of (8.6).

8.3.1 The method of undetermined coefficients

When $\boldsymbol{A}$ is a **constant** matrix, the method of undetermined coefficients sometimes provides a quick way of finding a particular solution when the entries $\boldsymbol{F}$ are constants, polynomials, exponential functions, sines and cosines or finite sums and products of these functions. We will ignore this method here since there are certain restrictions, depending on the form for applying this method.

8.3.2 The method of variation of parameters

Let $\{\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_n\}$ be a fundamental set of solutions of the homogeneous system (8.6) on an interval I (here we do not exclude the possibility that $\boldsymbol{A} = \boldsymbol{A}(t)$), and $\boldsymbol{\Phi}(t) = [\boldsymbol{X}_1(t) \vdots \cdots \vdots \boldsymbol{X}_n(t)]$ be a fundamental matrix of the homogeneous system (8.11) on the interval. Then

1. $\boldsymbol{\Phi}' = \boldsymbol{A}(t)\boldsymbol{\Phi}$;
2. Theorem 8.15 implies that $\boldsymbol{\Phi}(t)$ is non-singular for all $t \in I$;
3. Theorem 8.10 implies that every solution $\boldsymbol{X}$ to (8.11) on the interval can be expressed as $\boldsymbol{X}(t) = \boldsymbol{\Phi}(t)\,\boldsymbol{C}$ for some constant vector $\boldsymbol{C} = [c_1, c_2, \cdots, c_n]^{\mathrm{T}}$.

The method of variation of parameters of finding a particular solution to (8.22) is to find a solution of the form

$$\boldsymbol{X}_p(t) = \boldsymbol{\Phi}(t)\,\boldsymbol{U}(t)$$

for some vector-valued function $\boldsymbol{U}$. By the product rule,

$$\frac{d}{dt}\big[\boldsymbol{\Phi}(t)\,\boldsymbol{U}(t)\big] = \boldsymbol{\Phi}'(t)\,\boldsymbol{U}(t) + \boldsymbol{\Phi}(t)\,\boldsymbol{U}'(t) = \boldsymbol{A}(t)\boldsymbol{\Phi}(t)\,\boldsymbol{U}(t) + \boldsymbol{\Phi}(t)\,\boldsymbol{U}'(t)\,;$$

thus $\boldsymbol{X}_p = \boldsymbol{\Phi}\,\boldsymbol{U}$ is a particular solution to (8.22) if and only if

$$\boldsymbol{\Phi}(t)\,\boldsymbol{U}'(t) = \boldsymbol{F}(t)\,.$$

Since $\mathbf{\Phi}$ is non-singular, we find that

$$\boldsymbol{X}_p = \mathbf{\Phi}\,\boldsymbol{U} \text{ is a particular solution to (8.22) if and only if } \boldsymbol{U}'(t) = \mathbf{\Phi}^{-1}(t)\boldsymbol{F}(t)\,.$$

Therefore, a particular solution can be expressed as

$$\boldsymbol{X}_p(t) = \mathbf{\Phi}(t)\int \mathbf{\Phi}^{-1}(t)\boldsymbol{F}(t)\,dt \tag{8.23}$$

and the general solution of (8.22) can be expressed as

$$\boldsymbol{X}(t) = \mathbf{\Phi}(t)\,\boldsymbol{C} + \mathbf{\Phi}(t)\int \mathbf{\Phi}^{-1}(t)\boldsymbol{F}(t)\,dt$$

for some constant vector $\boldsymbol{C} = [c_1, c_2, \cdots, c_n]^{\mathrm{T}}$.

Example 8.29. Consider the linear system

$$\boldsymbol{X}' = \boldsymbol{A}\boldsymbol{X} + \boldsymbol{F} = \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix}\boldsymbol{X} + \begin{bmatrix} 3t \\ e^{-t} \end{bmatrix}$$

on $\mathbb{R}$. We first find a fundamental set of the associated homogeneous system. Note that there are two distinct eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -5$ with corresponding eigenvectors $\boldsymbol{K}_1 = [1,1]^{\mathrm{T}}$ and $\boldsymbol{K}_2 = [1,-2]^{\mathrm{T}}$. Therefore, $\boldsymbol{X}_1(t) \equiv \boldsymbol{K}_1 e^{-2t}$ and $\boldsymbol{X}_2(t) \equiv \boldsymbol{K}_2 e^{-5t}$ forms a fundamental set of the associated homogeneous system. Therefore,

$$\mathbf{\Phi}(t) = \begin{bmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{bmatrix}$$

is a fundamental matrix. Since $\mathbf{\Phi}^{-1}(t) = \begin{bmatrix} \frac{2}{3}e^{2t} & \frac{1}{3}e^{2t} \\ \frac{1}{3}e^{5t} & -\frac{1}{3}e^{5t} \end{bmatrix}$, using (8.23) we find that

$$\begin{aligned}
\boldsymbol{X}_p(t) &= \begin{bmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{bmatrix}\int \begin{bmatrix} \frac{2}{3}e^{2t} & \frac{1}{3}e^{2t} \\ \frac{1}{3}e^{5t} & -\frac{1}{3}e^{5t} \end{bmatrix}\begin{bmatrix} 3t \\ e^{-t} \end{bmatrix} dt \\
&= \begin{bmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{bmatrix}\int \begin{bmatrix} 2te^{t} + \frac{1}{3}e^{t} \\ te^{5t} - \frac{1}{3}e^{4t} \end{bmatrix} dt = \begin{bmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{bmatrix}\begin{bmatrix} te^{2t} - \frac{1}{2}e^{2t} + \frac{1}{3}e^{t} \\ \frac{1}{5}te^{5t} - \frac{1}{25}e^{5t} - \frac{1}{12}e^{4t} \end{bmatrix} \\
&= \begin{bmatrix} \frac{6}{5}t - \frac{27}{50} + \frac{1}{4}e^{-t} \\ \frac{3}{5}t - \frac{21}{50} + \frac{1}{2}e^{-t} \end{bmatrix}
\end{aligned}$$

is a particular solution of the linear system above. Therefore, the general solution to the linear system is

$$\boldsymbol{X}(t) = \begin{bmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} \frac{6}{5}t - \frac{27}{50} + \frac{1}{4}e^{-t} \\ \frac{3}{5}t - \frac{21}{50} + \frac{1}{2}e^{-t} \end{bmatrix}.$$

Example 8.30. Consider the linear system

$$\boldsymbol{X}' = \boldsymbol{A}(t)\boldsymbol{X} + \boldsymbol{F}(t) = \begin{bmatrix} 0 & 1 \\ 0 & \frac{2}{t} \end{bmatrix} \boldsymbol{X} + \begin{bmatrix} e^t \\ t^2 \end{bmatrix}.$$

Note that the associated homogeneous system $\boldsymbol{X}' = \boldsymbol{A}(t)\boldsymbol{X}$ has two linearly independent solution

$$\boldsymbol{X}_1(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{and} \quad \boldsymbol{X}_2(t) = \begin{bmatrix} t^3 \\ 3t^2 \end{bmatrix}.$$

Let $\boldsymbol{\Phi}(t) = \begin{bmatrix} 1 & t^3 \\ 0 & 3t^2 \end{bmatrix}$. Using (8.23) we obtain a particular solution

$$\begin{aligned} \boldsymbol{X}_p(t) &= \begin{bmatrix} 1 & t^3 \\ 0 & 3t^2 \end{bmatrix} \int \begin{bmatrix} 1 & t^3 \\ 0 & 3t^2 \end{bmatrix}^{-1} \begin{bmatrix} e^t \\ t^2 \end{bmatrix} dt = \begin{bmatrix} 1 & t^3 \\ 0 & 3t^2 \end{bmatrix} \int \begin{bmatrix} 1 & -\frac{t}{3} \\ 0 & \frac{1}{3t^2} \end{bmatrix} \begin{bmatrix} e^t \\ t^2 \end{bmatrix} dt \\ &= \begin{bmatrix} 1 & t^3 \\ 0 & 3t^2 \end{bmatrix} \int \begin{bmatrix} e^t - \frac{t^3}{3} \\ \frac{1}{3} \end{bmatrix} dt = \begin{bmatrix} 1 & t^3 \\ 0 & 3t^2 \end{bmatrix} \begin{bmatrix} e^t - \frac{t^4}{12} \\ \frac{t}{3} \end{bmatrix} = \begin{bmatrix} e^t + \frac{t^4}{4} \\ t^3 \end{bmatrix}. \end{aligned}$$

Therefore, the general solution to the non-homogeneous linear system is

$$\boldsymbol{X}(t) = \begin{bmatrix} 1 & t^3 \\ 0 & 3t^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} e^t + \frac{t^4}{4} \\ t^3 \end{bmatrix}.$$

Initial-Value Problem: Suppose that $\boldsymbol{\Phi}$ is a fundamental matrix of the homogeneous system $\boldsymbol{X}' = \boldsymbol{A}(t)\boldsymbol{X}$. Then the solution to the initial-value problem

$$\begin{aligned} \boldsymbol{X}' &= \boldsymbol{A}(t)\boldsymbol{X} + \boldsymbol{F}(t) \qquad \forall\, t \in I\,, \\ \boldsymbol{X}(t_0) &= \boldsymbol{X}_0\,, \end{aligned}$$

where $t_0 \in I$, is

$$\boldsymbol{X}(t) = \boldsymbol{\Phi}(t)\boldsymbol{\Phi}^{-1}(t_0)\boldsymbol{X}_0 + \boldsymbol{\Phi}(t)\int_{t_0}^t \boldsymbol{\Phi}(s)^{-1}\boldsymbol{F}(s)\,ds\,.$$

8.4 Matrix Exponential

Note that one way to see that $x(t) = ce^{\lambda t}$ is a solution to $x' = \lambda x$, where λ is a constant, is to differentiate the power series representation of the exponential function

$$\exp(\lambda t) = \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} = 1 + \lambda t + \frac{\lambda^2 t^2}{2!} + \cdots + \frac{\lambda^n t^n}{n!} + \cdots$$

and see that

$$\frac{d}{dt} e^{\lambda t} = \sum_{k=1}^{\infty} \frac{\lambda^k t^{k-1}}{(k-1)!} = \sum_{k=0}^{\infty} \frac{\lambda^{k+1} t^k}{k!} = \lambda \sum_{k=0}^{\infty} \frac{\lambda^k t^k}{k!} = \lambda e^{\lambda t}\,.$$

Motivated by the computation above, we find that the "power series" $\sum\limits_{k=0}^{\infty} \dfrac{\boldsymbol{A}^k t^k}{k!}$ is a solution to $\boldsymbol{X}' = \boldsymbol{A}\boldsymbol{X}$ since "formally" we can differentiate the series above and obtain that

$$\frac{d}{dt} \sum_{k=0}^{\infty} \frac{\boldsymbol{A}^k t^k}{k!} = \sum_{k=1}^{\infty} \frac{\boldsymbol{A}^k t^{k-1}}{(k-1)!} = \sum_{k=0}^{\infty} \frac{\boldsymbol{A}^{k+1} t^k}{k!} = \boldsymbol{A} \sum_{k=0}^{\infty} \frac{\boldsymbol{A}^k t^k}{k!}\,. \tag{8.24}$$

We note that the computation above is indeed correct since the sequence of functions $\Big\{ \sum\limits_{k=0}^{n} \dfrac{\boldsymbol{A}^k t^k}{k!} \Big\}_{n=1}^{\infty}$ converges uniformly to $\sum\limits_{k=0}^{\infty} \dfrac{\boldsymbol{A}^k t^k}{k!}$ on any bounded interval I. Motivated by the Maclaurin series of the exponential function and the computation above, we have the following

Definition 8.31. The exponential of an $n \times n$ matrix $\boldsymbol{M}$, denoted by $e^{\boldsymbol{M}}$ or $\exp(\boldsymbol{M})$, is a matrix defined by

$$e^{\boldsymbol{M}} = \exp(\boldsymbol{M}) \equiv \mathbf{I}_{n\times n} + \boldsymbol{M} + \frac{1}{2!}\boldsymbol{M}^2 + \frac{1}{3!}\boldsymbol{M}^3 + \cdots + \frac{1}{k!}\boldsymbol{M}^k + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}\boldsymbol{M}^k\,.$$

With this definition (of exponential of matrices), (8.24) can be written as $\dfrac{d}{dt} e^{\boldsymbol{A}t} = \boldsymbol{A} e^{\boldsymbol{A}t}$. In the following, we focus on the evaluation of the power series $e^{\boldsymbol{A}t} \equiv \sum\limits_{k=0}^{\infty} \dfrac{\boldsymbol{A}^k t^k}{k!}$.

By the knowledge from linear algebra,

$$\boldsymbol{A} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1} \tag{8.25}$$

for some invertible matrix $\boldsymbol{P}$, where $\boldsymbol{\Lambda}$ is of the ***Jordan canonical form***

$$\boldsymbol{\Lambda} = \begin{bmatrix} \boldsymbol{J}_1 & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{J}_2 & \ddots & \boldsymbol{O} \\ \vdots & \ddots & \ddots & \vdots \\ \boldsymbol{O} & \cdots & \boldsymbol{O} & \boldsymbol{J}_\ell \end{bmatrix} \tag{8.26}$$

in which each $\boldsymbol{O}$ is zero matrix, and each $\boldsymbol{J}_k$ is a square matrix of the form $\lambda\mathbf{I}$ or

$$\boldsymbol{J}_k = \lambda\mathbf{I} + \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 0 & 1 \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix} = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \lambda & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{bmatrix} \tag{8.27}$$

for some eigenvalue λ of $\boldsymbol{A}$. Therefore, by the fact that $\boldsymbol{A}^k = \boldsymbol{P}\boldsymbol{\Lambda}^k\boldsymbol{P}^{-1}$,

$$\begin{aligned} e^{\boldsymbol{A}t} &= \sum_{k=0}^{\infty} \frac{\boldsymbol{A}^k t^k}{k!} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(\boldsymbol{A}t)^k}{k!} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{\boldsymbol{P}(\boldsymbol{\Lambda}t)^k\boldsymbol{P}^{-1}}{k!} = \lim_{n\to\infty} \boldsymbol{P}\Big(\sum_{k=0}^{n} \frac{(\boldsymbol{\Lambda}t)^k}{k!}\Big)\boldsymbol{P}^{-1} \\ &= \boldsymbol{P}\Big(\sum_{k=0}^{\infty} \frac{(\boldsymbol{\Lambda}t)^k}{k!}\Big)\boldsymbol{P}^{-1}. \end{aligned}$$

Since $\boldsymbol{\Lambda}$ takes the form (8.26), we find that

$$(\boldsymbol{\Lambda}t)^k = \begin{bmatrix} (\boldsymbol{J}_1 t)^k & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{O} & (\boldsymbol{J}_2 t)^k & \ddots & \boldsymbol{O} \\ \vdots & \ddots & \ddots & \vdots \\ \boldsymbol{O} & \cdots & \boldsymbol{O} & (\boldsymbol{J}_\ell t)^k \end{bmatrix}.$$

If $\boldsymbol{J} = \lambda\mathbf{I}$ for some λ, then $(\boldsymbol{J}t)^k = \lambda^k t^k \mathbf{I}$ so that $e^{\boldsymbol{J}t} = e^{\lambda t}\mathbf{I}$. On the other hand, if $\boldsymbol{J}$ takes the form (8.27), then by the fact that $\lambda\mathbf{I}$ commutes with $(\boldsymbol{J} - \lambda\mathbf{I})$, we find that

$$\boldsymbol{J}^k = \big[\lambda\mathbf{I} + (\boldsymbol{J} - \lambda\mathbf{I})\big]^k = \sum_{j=0}^{k} C^k_j \lambda^{k-j} (\boldsymbol{J} - \lambda\mathbf{I})^j.$$

By the fact that

$$(\boldsymbol{J} - \lambda\mathbf{I})^j = \begin{bmatrix} \overbrace{0 \quad \cdots \quad 0}^{\text{There are } j \text{ copies of 0's here}} & 1 & 0 & \cdots & \cdots & 0 \\ 0 \quad\quad\quad \ddots & 0 & 1 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 & 1 & 0 \\ \vdots & & & \ddots & 0 & 1 \\ \vdots & & & & \ddots & 0 \\ \vdots & & & & & \vdots \\ 0 \quad \cdots \quad \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

so that with C^k_m denoting the number $\dfrac{k(k-1)\cdots(k-m+1)}{m!}$,

$$\boldsymbol{J}^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} & C^k_2\lambda^{k-2} & \cdots & \cdots & C^k_{m-1}\lambda^{k-m+1} \\ 0 & \lambda^k & k\lambda^{k-1} & \ddots & \ddots & C^k_{m-2}\lambda^{k-m+2} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & C^k_2\lambda^{k-2} \\ \vdots & \cdots & \cdots & 0 & \lambda^k & k\lambda^{k-1} \\ 0 & \cdots & \cdots & \cdots & 0 & \lambda^k \end{bmatrix}.$$

Therefore, if $\boldsymbol{J}$ is an $m \times m$ matrix taking the form (8.27),

$$e^{\boldsymbol{J}t} = \sum_{k=0}^{\infty} \frac{(\boldsymbol{J}t)^k}{k!} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!}e^{\lambda t} & \frac{t^3}{3!}e^{\lambda t} & \frac{t^4}{4!}e^{\lambda t} & \cdots & \frac{t^{m-1}}{(m-1)!}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!}e^{\lambda t} & \ddots & \ddots & \frac{t^{m-2}}{(m-2)!}e^{\lambda t} \\ \vdots & 0 & e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!}e^{\lambda t} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!}e^{\lambda t} \\ \vdots & \vdots & \ddots & \ddots & 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & e^{\lambda t} \end{bmatrix}.$$

Using (8.25), letting $\boldsymbol{Y} = \boldsymbol{P}^{-1}\boldsymbol{X}$, by the fact that $\boldsymbol{P}$ is a constant matrix we find that

$$\boldsymbol{Y}' = \boldsymbol{P}^{-1}\boldsymbol{X}' = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{X} = \boldsymbol{\Lambda}\boldsymbol{P}^{-1}\boldsymbol{X} = \boldsymbol{\Lambda}\boldsymbol{Y}.$$

Write $\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{Y}_1 \\ \boldsymbol{Y}_2 \\ \vdots \\ \boldsymbol{Y}_\ell \end{bmatrix}$, where $\boldsymbol{Y}_k$ is a $n_k \times 1$ column vector if $\boldsymbol{J}_k$ is $n_k \times n_k$ matrix. Then for each $1 \leqslant k \leqslant \ell$ we have $\boldsymbol{Y}'_k = \boldsymbol{J}_k\boldsymbol{Y}_k$, so it suffices to solve the linear system $\boldsymbol{Y}' = \boldsymbol{J}\boldsymbol{Y}$, where $\boldsymbol{J}$ is in diagonal form or $\boldsymbol{J}$ takes the form (8.27). Once we obtain a solution $\boldsymbol{Y}$ to the equation above, the solution $\boldsymbol{X}$ to (8.11) is then obtained by letting $\boldsymbol{X} = \boldsymbol{P}\boldsymbol{Y}$.

Example 8.32 (Example 8.28 Revisit)**.** Consider the linear system

$$\boldsymbol{X}' = \boldsymbol{A}\boldsymbol{X} = \begin{bmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix} \boldsymbol{X}$$

that we discussed in Example 8.28. Note that

$$\begin{bmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 5 & 6 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1},$$

from the discussion above we conclude that the general solution to the given linear system is given by

$$\begin{aligned} \boldsymbol{X}(t) &= \begin{bmatrix} 5 & 6 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \exp\left(\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} t\right) \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 6 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & te^{2t} & \frac{t^2}{2}e^{2t} \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 6 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_1 e^{2t} + C_2 te^{2t} + C^3 \frac{t^2}{2} e^{2t} \\ C_2 e^{2t} + C_3 te^{2t} \\ C_3 e^{2t} \end{bmatrix} \\ &= \Big(C_1 + C_2 t + \frac{C_3 t^2}{2}\Big) \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} e^{2t} + \big(C_2 + C_3 t\big) \begin{bmatrix} 6 \\ 5 \\ 0 \end{bmatrix} e^{2t} + C_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t}. \end{aligned}$$

In terms of the fundamental set $\{\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3\}$ given in Example 8.28, the general solution above is

$$\boldsymbol{X}(t) = (5C_1 + C_2)\boldsymbol{X}_1 + 5C_2 \boldsymbol{X}_2(t) + C_3 \boldsymbol{X}_3(t).$$

Example 8.33. Consider the linear system $\boldsymbol{X}' = \boldsymbol{A}\boldsymbol{X}$, where

$$\boldsymbol{A} = \begin{bmatrix} 4 & -2 & 0 & 2 \\ 0 & 6 & -2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -2 & 0 & 6 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 & 1 \\ -2 & 1 & 1 & 0 \\ -2 & 2 & 1 & 0 \\ -2 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 & 1 \\ -2 & 1 & 1 & 0 \\ -2 & 2 & 1 & 0 \\ -2 & 0 & 1 & 1 \end{bmatrix}^{-1}.$$

From the discussion above, we find that the general solution to the given linear system is

$$\begin{aligned} \boldsymbol{X}(t) &= \begin{bmatrix} 1 & -1 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \exp\left(\begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix} t\right) \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{4t} & te^{4t} & 0 & 0 \\ 0 & e^{4t} & 0 & 0 \\ 0 & 0 & e^{4t} & 0 \\ 0 & 0 & 0 & e^{6t} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}. \end{aligned}$$

Example 8.34. Consider the linear system $\boldsymbol{X}' = \boldsymbol{A}\boldsymbol{X}$, where

$$\boldsymbol{A} = \begin{bmatrix} 6 & 5 & 9 & 4 \\ -8 & -6 & -11 & -8 \\ 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 & 1 \\ -2 & 1 & 1 & 0 \\ -2 & 2 & 1 & 0 \\ -2 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}^{-1}.$$

From the discussion above, we find that the general solution to the given linear system is

$$\begin{aligned} \boldsymbol{X}(t) &= \begin{bmatrix} -2 & 0 & 0 & 1 \\ -2 & 1 & 1 & 0 \\ -2 & 2 & 1 & 0 \\ -2 & 0 & 1 & 1 \end{bmatrix} \exp\left(\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} t \right) \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 0 & 0 & 1 \\ -2 & 1 & 1 & 0 \\ -2 & 2 & 1 & 0 \\ -2 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & te^{2t} & 0 & 0 \\ 0 & e^{2t} & 0 & 0 \\ 0 & 0 & e^{-t} & te^{-t} \\ 0 & 0 & 0 & e^{-t} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}. \end{aligned}$$

- **The construction of Jordan decompositions**: Let $\boldsymbol{A} \in \mathcal{M}_{n\times n}$ be given.

Step 1: Let λ_1, λ_2, $\cdots$, λ_k be the distinct eigenvalues of $\boldsymbol{A}$ with multiplicity m_1, m_2, $\cdots$, m_k. We first focus on how to determine the block

$$\boldsymbol{\Lambda}_j = \begin{bmatrix} \boldsymbol{\Lambda}_j^{(1)} & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{\Lambda}_j^{(2)} & \ddots & \boldsymbol{O} \\ \vdots & \ddots & \ddots & \vdots \\ \boldsymbol{O} & \cdots & \boldsymbol{O} & \boldsymbol{\Lambda}_j^{(r_j)} \end{bmatrix},$$

whose diagonal is a fixed eigenvalue λ_j with multiplicity m_j for some $j \in \{1, 2, \cdots, k\}$, and the size of $\boldsymbol{\Lambda}_j^{(i)}$ is not smaller than the size of $\boldsymbol{\Lambda}_j^{(i+1)}$ for $i = 1, \cdots, r_j - 1$. Once all $\boldsymbol{\Lambda}_j's$ are obtained, then

$$\boldsymbol{\Lambda} = \begin{bmatrix} \boldsymbol{\Lambda}_1 & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{\Lambda}_2 & \ddots & \boldsymbol{O} \\ \vdots & \ddots & \ddots & \vdots \\ \boldsymbol{O} & \cdots & \boldsymbol{O} & \boldsymbol{\Lambda}_k \end{bmatrix}.$$

Step 2: Let $\mathbf{E}_j$ and $\mathbf{K}_j$ denote the eigenspace and the generalized eigenspace associated with λ_j, respectively. Then $r_j = \dim(\mathbf{E}_j)$ and $m_j = \dim(\mathbf{K}_j)$. Determine the smallest integer n_j such that

$$m_j = \dim\left(\ker(\boldsymbol{A} - \lambda_j \mathbf{I})^{n_j}\right).$$

Find the value

$$p_j^{(\ell)} = \dim(\ker(\boldsymbol{A} - \lambda_j \mathbf{I})^\ell) \quad \text{for} \quad \ell \in \{1, 2, \cdots, n_j\}$$

and set $p_j^{(0)} = 0$. Construct an $r_j \times n_j$ matrix whose entries only takes the value 0 or 1 and for each $\ell \in \{1, \cdots, n_j\}$ only the first $p_j^{(\ell)} - p_j^{(\ell-1)}$ components takes value 1 in the ℓ-th column of this matrix. Let $s_j^{(i)}$ be the sum of the i-th row of the matrix just obtained. Then $\boldsymbol{\Lambda}_j^{(i)}$ is a $s_j^{(i)} \times s_j^{(i)}$ matrix.

Step 3: Next, let us determine matrix $\boldsymbol{P}$. Suppose that

$$\boldsymbol{P} = \left[\boldsymbol{u}_1^{(1)} \vdots \cdots \vdots \boldsymbol{u}_1^{(m_1)} \vdots \boldsymbol{u}_2^{(1)} \vdots \cdots \vdots \boldsymbol{u}_2^{(m_2)} \vdots \boldsymbol{u}_3^{(1)} \vdots \cdots \vdots \boldsymbol{u}_k^{(n)}\right].$$

Then $\boldsymbol{A}\left[\boldsymbol{u}_j^{(1)} \vdots \cdots \vdots \boldsymbol{u}_j^{(m_j)}\right] = \left[\boldsymbol{u}_j^{(1)} \vdots \cdots \vdots \boldsymbol{u}_j^{(m_j)}\right]\boldsymbol{\Lambda}_j$. Divide $\left\{\boldsymbol{u}_j^{(1)}, \cdots, \boldsymbol{u}_j^{(m_j)}\right\}$ into r_j groups

$$\left\{\boldsymbol{u}_j^{(1)}, \cdots, \boldsymbol{u}_j^{(s_j^{(1)})}\right\}, \left\{\boldsymbol{u}_j^{(s_j^{(1)}+1)}, \cdots, \boldsymbol{u}_j^{(s_j^{(1)}+s_j^{(2)})}\right\}, \cdots, \text{ and } \left\{\boldsymbol{u}_j^{(s_j^{(1)}+\cdots+s_j^{(r_j-1)}+1)}, \cdots, \boldsymbol{u}_j^{(m_j)}\right\}$$

so that for each $\ell \in \{1, \cdots, r_j\}$, we let the ℓ-th group refer to the group of vectors

$$\left\{\boldsymbol{u}_j^{(s_j^{(1)}+\cdots+s_j^{(\ell-1)}+1)}, \boldsymbol{u}_j^{(s_j^{(1)}+\cdots+s_j^{(\ell-1)}+2)}, \cdots, \boldsymbol{u}_j^{(s_j^{(1)}+\cdots+s_j^{(\ell)})}\right\}.$$

We then set up the first group by picking up an arbitrary non-zero vectors $\boldsymbol{v}_1 \in \ker\left((\boldsymbol{A} - \lambda_j \mathbf{I})^{s_j^{(1)}}\right) \backslash \ker\left((\boldsymbol{A} - \lambda_j \mathbf{I})^{s_j^{(1)}-1}\right)$ and let

$$\boldsymbol{u}_j^{(i)} = (\boldsymbol{A} - \lambda_j \mathbf{I})^{s_j^{(1)}-i}\boldsymbol{v}_1 \quad \text{for } i \in \{1, \cdots, s_j^{(1)} - 1\}.$$

Inductively, once the first ℓ groups of vectors are set up, pick up an arbitrary non-zero vector $\boldsymbol{v}_{\ell+1} \in \ker\left((\boldsymbol{A} - \lambda_j \mathbf{I})^{s_j^{(\ell+1)}}\right) \backslash \ker\left((\boldsymbol{A} - \lambda_j \mathbf{I})^{s_j^{(\ell+1)}-1}\right)$ such that $\boldsymbol{v}_{\ell+1}$ is not in the span of the vectors from the first ℓ groups, and define

$$\boldsymbol{u}_j^{(s_j^{(1)}+\cdots+s_j^{(\ell)}+i)} = (\boldsymbol{A} - \lambda_j \mathbf{I})^{s_j^{(\ell+1)}-i}\boldsymbol{v}_{\ell+1} \quad \text{for } i \in \{1, \cdots, s_j^{(\ell+1)} - 1\}.$$

This defines the $(\ell+1)$-th group. Keep on doing so for all $\ell \leqslant r_j$ and for $j \in \{1, \cdots, k\}$, we complete the construction of $\boldsymbol{P}$.

Example 8.35. Find the Jordan decomposition of the matrix $\boldsymbol{A} = \begin{bmatrix} 4 & -2 & 0 & 2 \\ 0 & 6 & -2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -2 & 0 & 6 \end{bmatrix}$.

If λ is an eigenvalue of $\boldsymbol{A}$, then λ satisfies

$$\begin{aligned}
0 = \det(\boldsymbol{A} - \lambda \mathbf{I}) &= \begin{vmatrix} 4-\lambda & -2 & 0 & 2 \\ 0 & 6-\lambda & -2 & 0 \\ 0 & 2 & 2-\lambda & 0 \\ 0 & -2 & 0 & 6-\lambda \end{vmatrix} = (4-\lambda) \begin{vmatrix} 6-\lambda & -2 & 0 \\ 2 & 2-\lambda & 0 \\ -2 & 0 & 6-\lambda \end{vmatrix} \\
&= (4-\lambda)\big[(6-\lambda)^2(2-\lambda) + 4(6-\lambda)\big] = (6-\lambda)(4-\lambda)\big[(6-\lambda)(2-\lambda)+4\big] \\
&= (\lambda-4)^3(\lambda-6)\,.
\end{aligned}$$

Let $\lambda_1 = 4$, $\lambda_2 = 6$, $m_1 = 3$ and $m_2 = 1$. Note that

$$\dim\big(\ker(\boldsymbol{A} - 4\mathbf{I})\big) = 2 \quad \text{and} \quad \dim\big(\ker(\boldsymbol{A} - 4\mathbf{I})^2\big) = 3\,.$$

Therefore, $n_1 = 2$ and $p_1^{(1)} = 2$, $p_1^{(2)} = 3$. We then construct the matrix according to Step 2 above, and the matrix is a 2×2 matrix given by $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. This matrix provides that $s_1 = 2$ and $s_2 = 1$; thus the block associated with the eigenvalue $\lambda = 4$, is $\begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Therefore,

$$\boldsymbol{\Lambda} = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}.$$

First, we note that the eigenvector associated with $\lambda = 6$ can be chosen as $(1, 0, 0, 1)^{\mathrm{T}}$. Computing $\ker\big((\boldsymbol{A} - 4\mathbf{I})\big)$ and $\ker\big((\boldsymbol{A} - 4\mathbf{I})^2\big)$, we find that

$$\begin{aligned}
\ker\big((\boldsymbol{A} - 4\mathbf{I})\big) &= \operatorname{span}\big((1,0,0,0)^{\mathrm{T}}, (0,1,1,1)^{\mathrm{T}}\big)\,, \\
\ker\big((\boldsymbol{A} - 4\mathbf{I})^2\big) &= \operatorname{span}\big((1,0,0,0)^{\mathrm{T}}, (0,1,0,2)^{\mathrm{T}}, (0,1,2,0)^{\mathrm{T}}\big)\,.
\end{aligned}$$

We note that either $(0,1,0,2)^{\mathrm{T}}$ or $(0,1,2,0)^{\mathrm{T}}$ is in $\ker\big((\boldsymbol{A} - 4\mathbf{I})\big)$, we can choose $\boldsymbol{v} = (0,1,0,2)^{\mathrm{T}}$. Then $(\boldsymbol{A} - 4\mathbf{I})\boldsymbol{v} = (2,2,2,2)^{\mathrm{T}}$. Finally, for the third column of $\boldsymbol{P}$ we can choose either $(1,0,0,0)^{\mathrm{T}}$ or $(0,1,1,1)^{\mathrm{T}}$ (or even their linear combination) since these vectors are not in the span of $(2,2,2,2)^{\mathrm{T}}$ and $(0,1,0,2)$. Therefore,

$$\boldsymbol{P} = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \boldsymbol{P} = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 1 \end{bmatrix}$$

satisfies $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1}$.

Example 8.36. Let $\boldsymbol{A} = \begin{bmatrix} a & 0 & 1 & 0 & 0 \\ 0 & a & 0 & 1 & 0 \\ 0 & 0 & a & 0 & 1 \\ 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & a \end{bmatrix}$. Then the characteristic equation of $\boldsymbol{A}$ is $(a-\lambda)^5$; thus $\lambda = a$ is the only eigenvalue of $\boldsymbol{A}$. First we compute the kernel of $(\boldsymbol{A} - a\mathbf{I})^p$ for various p. With $\boldsymbol{e}_i = (\underbrace{0, \cdots, 0}_{(i-1)\text{-slots}}, 1, 0, \cdots, 0)^{\mathrm{T}}$ denoting the i-th vector in the standard basis of $\mathbb{R}^5$, we find that

$$\begin{aligned} \ker((\boldsymbol{A} - a\mathbf{I})) &= \left\{(x_1, x_2, 0, 0)^{\mathrm{T}} \,\middle|\, x_1, x_2 \in \mathbb{R}\right\} = \operatorname{span}(\boldsymbol{e}_1, \boldsymbol{e}_2)\,, \\ \ker((\boldsymbol{A} - a\mathbf{I})^2) &= \left\{(x_1, x_2, x_3, x_4, 0)^{\mathrm{T}} \,\middle|\, x_1, x_2, x_3, x_4 \in \mathbb{R}\right\} = \operatorname{span}(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4)\,, \\ \ker((\boldsymbol{A} - a\mathbf{I})^3) &= \mathbb{R}^5 = \operatorname{span}(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4, \boldsymbol{e}_5)\,. \end{aligned}$$

The matrix obtained by Step 2 is $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ which implies that the two Jordan blocks is of size 3×3 and 2×2. Therefore,

$$\Lambda = \begin{bmatrix} a & 1 & 0 & 0 & 0 \\ 0 & a & 1 & 0 & 0 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & a & 1 \\ 0 & 0 & 0 & 0 & a \end{bmatrix}.$$

We note that $\boldsymbol{e}_5 \in \ker\big((\boldsymbol{A} - a\mathbf{I})^3\big) \backslash \ker\big((\boldsymbol{A} - a\mathbf{I})^2\big)$; thus the first three column of $\boldsymbol{P}$ can be chosen as

$$\boldsymbol{P}(1:3) = \big[(\boldsymbol{A} - a\mathbf{I})^2 \boldsymbol{e}_5 \,\vdots\, (\boldsymbol{A} - a\mathbf{I})\boldsymbol{e}_5 \,\vdots\, \boldsymbol{e}_5\big] = \big[\boldsymbol{e}_1 \,\vdots\, \boldsymbol{e}_3 \,\vdots\, \boldsymbol{e}_5\big]\,.$$

To find the last two columns, we try to find a vector $\boldsymbol{w} \in \ker\big((\boldsymbol{A} - a\mathbf{I})^2\big) \backslash \ker\big((\boldsymbol{A} - a\mathbf{I})\big)$ so that $\boldsymbol{w}$ is not in the span of $\{\boldsymbol{e}_1, \boldsymbol{e}_3, \boldsymbol{e}_5\}$. Therefore, we may choose $\boldsymbol{w} = \boldsymbol{e}_4$; thus the last two columns of $\boldsymbol{P}$ is

$$\boldsymbol{P}(4:5) = \big[(\boldsymbol{A} - a\mathbf{I})\boldsymbol{e}_4 \,\vdots\, \boldsymbol{e}_4\big] = \big[\boldsymbol{e}_2 \,\vdots\, \boldsymbol{e}_4\big]$$

which implies that

$$\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Chapter 10

System of Nonlinear First-Order Differential Equations

10.1 Autonomous Systems

Definition 10.1. A system of first-order differential equations is said to be autonomous if the system can be written in the form

$$\begin{aligned}
\frac{dx_1}{dt} &= g_1(x_1, x_2, \cdots, x_n)\,,\\
\frac{dx_2}{dt} &= g_2(x_1, x_2, \cdots, x_n)\,,\\
\vdots \quad & \qquad \vdots\\
\frac{dx_n}{dt} &= g_n(x_1, x_2, \cdots, x_n)\,,
\end{aligned} \tag{10.1}$$

Example 10.2. The most famous autonomous system is obtained the pendulum system

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0 :$$

letting $x = \theta$ and $y = \theta'$, then

$$\begin{aligned}
x' &= y\,,\\
y' &= -\frac{g}{L}\sin x\,.
\end{aligned}$$

Notation: Let $\boldsymbol{X}$ and $\boldsymbol{g}(\boldsymbol{X})$ denote the respective column vectors

$$\boldsymbol{X}(t) = \begin{bmatrix} x_1(t)\\ x_2(t)\\ \vdots\\ x_n(t) \end{bmatrix}, \qquad \boldsymbol{g}(\boldsymbol{X}) = \begin{bmatrix} g_1(x_1, x_2, \cdots, x_n)\\ g_2(x_1, x_2, \cdots, x_n)\\ \vdots\\ g_n(x_1, x_2, \cdots, x_n) \end{bmatrix},$$

then the autonomous system (10.1) can be written in the form $\boldsymbol{X}' = \boldsymbol{g}(\boldsymbol{X})$.

When the independent variable t is interpreted as time, the autonomous system (10.1) is also called a dynamical system. When $n = 2$, the system in (10.1) is called a plane autonomous system, and we write the system as

$$\begin{aligned}\frac{dx}{dt} &= P(x, y)\,,\\ \frac{dy}{dt} &= Q(x, y)\,.\end{aligned} \tag{10.2}$$

If $P(x, y)$ and $Q(x, y)$ and the partial derivatives P_x, P_y, Q_x, Q_y are continuous in a region R of the plane, then a solution of the plane autonomous system (10.2) is unique and of one of the three basic types:

1. A constant solution $x(t) = x_0$, $y(t) = y_0$ (or $\boldsymbol{X}(t) = \boldsymbol{X}_0$ for all t). A constant solution is called a critical or stationary point. When the particle is placed at a critical point $\boldsymbol{X}_0$, it remains there indefinitely. For this reason a constant solution is also called an equilibrium solution or simply equilibrium. Note that $\boldsymbol{X}_0 = [x_0, y_0]^{\mathrm{T}}$ is an equilibrium if $P(x_0, y_0) = Q(x_0, y_0) = 0$.

2. A periodic solution $x = x(t)$, $y = y(t)$. A periodic solution is called a cycle. If p is the period of the solution, then $\boldsymbol{X}(t+p) = \boldsymbol{X}(t)$ for all t and a particle placed on the curve at $\boldsymbol{X}_0$ will cycle around the curve and return to $\boldsymbol{X}_0$ in p units of time.

3. A solution $x = x(t)$, $y = y(t)$ defines an arc - a plane curve that does not cross itself (by the uniqueness of the solution to initial-value problems).

Example 10.3. Consider the autonomous system

$$\begin{aligned}x' &= 2x + 8y\,,\\ y' &= -x - 2y\,.\end{aligned}$$

The general solution to the linear system above is

$$\begin{aligned}x(t) &= c_1(2\cos 2t - 2\sin 2t) + c_2(2\cos 2t + 2\sin 2t)\,,\\ y(t) &= -c_1\cos 2t - c_2\sin 2t\,;\end{aligned}$$

thus every solution is periodic with period $p = \pi$.

Example 10.4. Consider the autonomous system

$$\begin{aligned} x' &= -y - x\sqrt{x^2+y^2}\,, \\ y' &= x - y\sqrt{x^2+y^2}\,. \end{aligned}$$

Let us rewrite the differential equations above in polar coordinate. Let $r = \sqrt{x^2+y^2}$ and $\theta = \arctan\dfrac{y}{x}$. Then

$$\frac{dr}{dt} = \frac{1}{r}\Big(x\frac{dx}{dt} + y\frac{dy}{dt}\Big) \qquad \text{and} \qquad \frac{d\theta}{dt} = \frac{1}{r^2}\Big(-y\frac{dx}{dt} + x\frac{dy}{dt}\Big)$$

so that r, θ satisfy $r' = -r^2$ and $\theta' = 1$. Therefore, $r(t) = \dfrac{1}{t+c_1}$ and $\theta(t) = t + c_2$ for some constants c_1, c_2; thus

$$\begin{aligned} x(t) &= r(t)\cos\theta(t) = \frac{\cos(t+c_2)}{t+c_1}\,, \\ y(t) &= r(t)\sin\theta(t) = \frac{\sin(t+c_2)}{t+c_1}\,. \end{aligned}$$

The trajectory is given by the polar graph of the polar equation $r = \dfrac{1}{\theta + c}$ for some constant $c(= c_1 - c_2)$.

If an initial condition $\boldsymbol{X}(0) = (3,3)$ is imposed to the autonomous system, the solution is then given by

$$x(t) = \frac{\cos(t+\pi/4)}{t+\sqrt{2}/6}\,, \qquad y(t) = \frac{\sin(t+\pi/4)}{t+\sqrt{2}/6}$$

whose trajectory is an arc $r = \dfrac{1}{\theta + \sqrt{2}/6 - \pi/4}$.

Example 10.5. Suppose in terms of polar coordinate an autonomous system is written as

$$\begin{aligned} \frac{dr}{dt} &= 0.5(3-r)\,, \\ \frac{d\theta}{dt} &= 1\,. \end{aligned}$$

The general solution is given by $r(t) = 3 + c_1 e^{-0.5t}$, $\theta(t) = t + c_2$.

1. If an initial condition $(r,\theta)(0) = \big(1, \frac{\pi}{2}\big)$ is imposed, the trajectory is

   $$r = 3 - 2e^{-0.5(\theta - \frac{\pi}{2})}$$

   so that $r \to 3$ as $t \to \infty$.

2. If an initial condition $(r,\theta)(0) = (3,0)$ is imposed, the solution is $(x,y) = (3\cos t, 3\sin t)$ so that the trajectory is a circle with radius 3.

10.2 Stability of Autonomous Systems

Suppose that $\boldsymbol{X}_e$ is an equilibrium of a plane autonomous system, and $\boldsymbol{X} = \boldsymbol{X}(t)$ is a solution of the system that satisfies the initial condition $\boldsymbol{X}(0) = \boldsymbol{X}_0 \neq \boldsymbol{X}_e$. Some fundamental question that we would like to answer in the study of autonomous system are:

1. Will the particle return to the equilibrium; that is, $\lim\limits_{t\to\infty} \boldsymbol{X}(t) = \boldsymbol{X}_1$.
2. If the particle does not return to the equilibrium, does it remain close to the equilibrium or move away from the equilibrium? In mathematical terms, we would like to know if there exists (small) $\delta > 0$ such that $|\boldsymbol{X}(t) - \boldsymbol{X}_e| < \delta$ for large enough t or there exists an increasing sequence $\{t_n\}_{n=1}^{\infty}$ such that $|\boldsymbol{X}(t_n) - \boldsymbol{X}_e| \geqslant \delta$.

Definition 10.6. Let $\boldsymbol{X}_e$ be an equilibrium of an autonomous system, and $\boldsymbol{X} = \boldsymbol{X}(t)$ is a solution of the system that satisfies the initial condition $\boldsymbol{X}(0) = \boldsymbol{X}_0 \neq \boldsymbol{X}_e$.

1. $\boldsymbol{X}_e$ is called a stable equilibrium if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if the initial point $\boldsymbol{X}_0$ satisfies $|\boldsymbol{X}_0 - \boldsymbol{X}_e| < \delta$, then $|\boldsymbol{X}(t) - \boldsymbol{X}_e| < \varepsilon$ for all $t > 0$. If in addition there exists $\delta_0 > 0$ such that $\lim\limits_{t\to\infty} \boldsymbol{X}(t) = \boldsymbol{X}_e$ whenever $|\boldsymbol{X}_0 - \boldsymbol{X}_e| < \delta_0$, $\boldsymbol{X}_e$ is called an asymptotically stable equilibrium.
2. $\boldsymbol{X}_e$ is called an unstable equilibrium if there exists $\rho > 0$ such that for any $r > 0$ there is at least one initial position $\boldsymbol{X}_0$ satisfying $|\boldsymbol{X}_0 - \boldsymbol{X}_e| < r$ and $t > 0$ such that $|\boldsymbol{X}(t) - \boldsymbol{X}_e| \geqslant \rho$.

10.2.1 Stability Analysis for plane autonomous systems

We first investigate the stability of linear plane autonomous system

$$x' = ax + by\,, \tag{10.3a}$$

$$y' = cx + dy\,. \tag{10.3b}$$

Let $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Suppose that $\Delta \equiv \det(\boldsymbol{A}) \neq 0$ so that $\boldsymbol{0}$ is the only equilibrium. Note that the eigenvalues of $\boldsymbol{A}$ is given by $\lambda = \dfrac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$, where $\tau = a + d$.

1. The case $\tau^2 - 4\Delta > 0$: Then the general solution to system (10.3) is given by
$$\boldsymbol{X}(t) = c_1 \boldsymbol{K}_1 e^{\lambda_1 t} + c_2 \boldsymbol{K}_2 e^{\lambda_2 t}\,,$$
where λ_1, λ_2 are the eigenvalues and $\boldsymbol{K}_1, \boldsymbol{K}_2$ are corresponding eigenvectors.

(a) If $\lambda_1, \lambda_2 < 0$ (which corresponds to the case $\tau^2 - 4\Delta > 0, \tau < 0, \Delta > 0$), then $\lim\limits_{t\to\infty} \boldsymbol{X}(t) = \mathbf{0}$ which shows that $\mathbf{0}$ is a stable equilibrium.

(b) If $\lambda_1, \lambda_2 > 0$ (which corresponds to the case $\tau^2 - 4\Delta > 0, \tau > 0, \Delta > 0$), then $\lim\limits_{t\to\infty} \big|\boldsymbol{X}(t)\big| = \infty$ unless $c_1 = c_2 = 0$. In this case, $\mathbf{0}$ is an unstable equilibrium.

(c) If $\lambda_2 < 0 < \lambda_1$ (which corresponds to the case $\tau^2 - 4\Delta > 0, \Delta < 0$), then $\lim\limits_{t\to\infty} \big|\boldsymbol{X}(t)\big| = \infty$ unless $c_1 = 0$. Therefore, even though $\boldsymbol{X}(t)$ still approaches zero along the line determined by $\boldsymbol{K}_2$ if $\boldsymbol{X}_0$ lies on this line, $\mathbf{0}$ is still an unstable equilibrium (and is called a saddle equilibrium).

2. The case $\tau^2 - 4\Delta = 0$: Then $\boldsymbol{A}$ has a repeated eigenvalue $\lambda = \dfrac{\tau}{2}$.

   (a) If $\boldsymbol{A}$ has two linearly independent eigenvectors $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$, then the general solution to system (10.3) is given by

$$\boldsymbol{X}(t) = c_1\boldsymbol{K}_1 e^{\lambda t} + c_2\boldsymbol{K}_2 e^{\lambda t}\,,$$

   Then $\mathbf{0}$ is stable if and only if $\lambda < 0$ (or $\tau < 0$). In this case, $\mathbf{0}$ is called a degenerate stable/unstable node.

   (b) If $\boldsymbol{A}$ has only one linearly independent eigenvector $\boldsymbol{K}_1$, then there exists $\boldsymbol{K}_2 \neq \mathbf{0}$ such that $(\boldsymbol{A}-\lambda\mathbf{I})^2\boldsymbol{K}_2 = \mathbf{0}$ but $(\boldsymbol{A}-\lambda\mathbf{I})\boldsymbol{K}_2 \neq \mathbf{0}$. In this case the general solution to system (10.3) is given by

$$\boldsymbol{X}(t) = c_1\boldsymbol{K}_1 e^{\lambda t} + c_2\Big[(A + \frac{\tau}{2}\mathbf{I})\boldsymbol{K}_2 t e^{\lambda t} + \boldsymbol{K}_2 e^{\lambda t}\Big]\,,$$

   Then $\mathbf{0}$ is stable if and only if $\lambda < 0$ (or $\tau < 0$). In this case, $\mathbf{0}$ is again called a degenerate stable/unstable node.

3. The case $\tau^2 - 4\Delta < 0$: Then $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$ for $\alpha = \dfrac{\tau}{2}$ and $\beta = \dfrac{\sqrt{4\Delta - \tau^2}}{2}$. In this case, the general solution to system (10.3) is given by

$$\boldsymbol{X}(t) = c_1(\boldsymbol{B}_1\cos\beta t - \boldsymbol{B}_2\sin\beta t)e^{\alpha t} + c_2(\boldsymbol{B}_2\cos\beta t + \boldsymbol{B}_1\sin\beta t)e^{\alpha t}\,.$$

   (a) If $\alpha = 0$ (which corresponds to the case $\tau^2 - 4\Delta < 0$ and $\tau = 0$), then $\boldsymbol{A}$ has pure imaginary eigenvalues and the general solution are periodic with period $p = \dfrac{2\pi}{\beta}$. In this case the trajectory of solutions are ellipses centered at the equilibrium $\mathbf{0}$, so $\mathbf{0}$ is also called a center.

(b) If $\alpha \neq 0$ (which corresponds to the case $\tau^2 - 4\Delta < 0$ and $\tau \neq 0$), then $\mathbf{0}$ is stable if and only if $\alpha < 0$ (or $\tau < 0$). In this case, $\mathbf{0}$ is called a stable (or unstable if $\alpha > 0$) spiral equilibrium.

Theorem 10.7. *Let $\boldsymbol{X} = \boldsymbol{X}(t)$ denote the solution to a linear plane autonomous system $\boldsymbol{X}' = \boldsymbol{A}\boldsymbol{X}$ satisfying the initial condition $\boldsymbol{X}(0) = \boldsymbol{X}_0$, where $\det(\boldsymbol{A}) \neq 0$ and $\boldsymbol{X}_0 \neq \mathbf{0}$.*

1. $\lim\limits_{t\to\infty} \boldsymbol{X}(t) = \mathbf{0}$ *if and only if the eigenvalues of A has negative real parts.*
2. $X(t)$ *is periodic if and only if the eigenvalues of $\boldsymbol{A}$ are purely imaginary.*
3. *In all other cases, given any neighborhood of the origin, there is at least one $\boldsymbol{X}_0$ in the neighborhood for which $\boldsymbol{X}(t)$ becomes unbounded as t increases.*

10.3 Linearization and Local Stability

Theorem 10.8. *Let x_e be an equilibrium of the autonomous first-order differential equation $x' = g(x)$ (so $g(x_e) = 0$), where g is continuously differentiable in a neighborhood of x_e.*

1. *If $g'(x_e) < 0$, then x_e is an asymptotically stable equilibrium.*
2. *If $g'(x_e) > 0$, then x_e is an unstable equilibrium.*

Proof. Suppose that $g'(x_e) \neq 0$. Since g' is continuous in a neighborhood of x_e, there exists $\delta > 0$ such that

$$\big|g'(x) - g'(x_e)\big| < \frac{|g'(x_e)|}{2} \qquad \text{whenever} \quad |x - x_e| < \delta\,.$$

Note that for every $t > 0$, the mean value theorem implies that

$$x'(t) = g(x(t)) = g(x_e) + g'(\xi(t))(x(t) - x_e) = g'(\xi(t))(x(t) - x_e) \tag{10.4}$$

for some $\xi(t)$ between x_e and $x(t)$.

1. If $g'(x_e) < 0$, then

   $$\frac{3g'(x_e)}{2} < g'(x) < \frac{g'(x_e)}{2} < 0 \qquad \text{whenever} \quad |x - x_e| < \delta\,.$$

   Let $x_0 \in [x_e - \delta/2, x_e + \delta/2]$. We claim that $x(t) \in [x_e - \delta/2, x_e + \delta/2]$ for all $t > 0$. Suppose the contrary that $\big\{t > 0 \,\big|\, |x(t) - x_e| > \delta/2\big\}$ is non-empty so that $t_* = \inf\big\{t > 0 \,\big|\, |x(t) - x_e| > \delta/2\big\} \in \mathbb{R}$. The continuity of x then implies that $|x(t_*) - x_e| = \delta/2$ and $|x(t) - x_e| \leqslant \delta/2$ if $t < t_*$.

(a) If $x(t_*) = x_e + \delta/2$, then there exists $\delta_1 > 0$ such that $x(t) \in (x_e, x_e + \delta)$ for all $t \in (t_* - \delta_1, t_* + \delta_1)$. However, (10.4) implies that $x'(t) < 0$ for all $t \in (t_* - \delta_1, t_* + \delta_1)$. This then implies that $|x(t) - x_e| \leqslant \delta/2$ for all $t \in [0, t_* + \delta_1)$, a contradiction to that $t_* = \inf\{t > 0 \,\big|\, |x(t) - x_e| \geqslant \delta/2\}$.

(b) If $x(t_*) = x_e - \delta/2$, then there exists $\delta_2 > 0$ such that $x(t) \in (x_e - \delta, x_e)$ for all $t \in (t_* - \delta_2, t_* + \delta_2)$. However, (10.4) implies that $x'(t) > 0$ for all $t \in (t_* - \delta_2, t_* + \delta_2)$. This then implies that $|x(t) - x_e| \leqslant \delta/2$ for all $t \in [0, t_* + \delta_2)$, a contradiction to that $t_* = \inf\{t > 0 \,\big|\, |x(t) - x_e| \geqslant \delta/2\}$.

Having established that $x(t) \in [x_e - \delta/2, x_e + \delta/2]$ for all $t > 0$, we find that

$$x'(t) \leqslant \frac{g'(x_e)}{2}\big(x(t) - x_e\big) \quad \text{if } x_e < x(t) < x_e + \delta\,,$$
$$x'(t) \geqslant \frac{g'(x_e)}{2}\big(x(t) - x_e\big) \quad \text{if } x_e - \delta < x(t) < x_e\,.$$

Therefore,

$$\frac{d}{dt}\ln|x(t) - x_e| \leqslant \frac{g'(x_e)}{2} \qquad \forall\, t > 0$$

which implies that

$$|x(t) - x_e| \leqslant |x_0 - x_e| \exp\Big(\frac{g'(x_e)}{2}t\Big) \qquad \forall\, t > 0\,.$$

Therefore, $\lim\limits_{t\to\infty} x(t) = x_e$ which shows that x_e is an asymptotically stable equilibrium.

2. If $g'(x_e) > 0$, then there exists $\delta > 0$ such that

$$0 < \frac{g'(x_e)}{2} < g'(x) < \frac{3g'(x_e)}{2} \qquad \text{whenever} \quad |x - x_e| < \delta\,,$$

and (10.4) further implies that

$$x'(t) \geqslant \frac{g'(x_e)}{2}\big(x(t) - x_e\big) \quad \text{if } x_e < x(t) < x_e + \delta\,,$$
$$x'(t) \leqslant \frac{g'(x_e)}{2}\big(x(t) - x_e\big) \quad \text{if } x_e - \delta < x(t) < x_e\,.$$

or equivalently,

$$\frac{d}{dt}\ln|x(t) - x_e| \geqslant \frac{g'(x_e)}{2} \qquad \text{if } 0 < |x(t) - x_e| < \delta\,.$$

Therefore, if $0 < |x(t) - x_e| < \delta$ for $t \in [0, T]$,

$$|x(t) - x_e| \geqslant |x_0 - x_e| \exp\Big(\frac{g'(x_e)}{2}t\Big) \qquad \forall\, t \in [0, T]$$

which shows that $|x(t) - x_e| \geqslant \dfrac{\delta}{2}$ for some $t \in \mathbb{R}$ no matter how small $|x_0 - x_e|$ is. Therefore, x_e is an unstable equilibrium if $g'(x_e) > 0$. □

Example 10.9. Analyze the stability of equilibria of the logistic differential equation $x' = \dfrac{r}{K}x(K-x)$, where r and K are positive constants.

Let $g(x) = \dfrac{r}{K}x(K-x)$, and x_e be an equilibrium. Then $x_e = 0$ or $x_e = K$. Since $g'(0) = r$ and $g'(K) = -r$, by Theorem 10.8 we conclude that K is an asymptotically stable equilibrium but 0 is an unstable equilibrium.

Theorem 10.10. *Let $\boldsymbol{X}_e$ be an equilibrium of the autonomous first-order differential equation $\boldsymbol{X}' = \boldsymbol{g}(\boldsymbol{X})$, where $\boldsymbol{g}$ is continuously differentiable in a neighborhood of $\boldsymbol{X}_e$ (that is, each component of $\mathbf{g}$ has continuous first partial derivatives in a neighborhood of $\boldsymbol{X}_e$), and $\boldsymbol{A} = (D\boldsymbol{g})(\boldsymbol{X}_e)$ be the Jacobian matrix of $\boldsymbol{g}$ at $\boldsymbol{X}_e$.*

1. *If every eigenvalue of $\boldsymbol{A}$ has negative real part, then $\boldsymbol{X}_e$ is an asymptotically stable equilibrium.*
2. *If one of the eigenvalue of $\boldsymbol{A}$ has positive real part, then $\boldsymbol{X}_e$ is an unstable stable equilibrium.*

Example 10.11. Classify (if possible) the stability of the equilibria of each of the plane autonomous system $\boldsymbol{X}' = \boldsymbol{g}(\boldsymbol{X})$ as stable or unstable.

1. $\boldsymbol{g}(x,y) = \begin{bmatrix} x^2+y^2-6 \\ x^2-y \end{bmatrix}$. In this case $\boldsymbol{g}(x,y) = \mathbf{0}$ if and only if $(x,y) = (\pm\sqrt{2}, 2)$. The Jacobian matrix of $\boldsymbol{g}$ is
$$(D\boldsymbol{g})(x,y) = \begin{bmatrix} 2x & 2y \\ 2x & -1 \end{bmatrix}$$
so that
$$(D\boldsymbol{g})(\sqrt{2},2) = \begin{bmatrix} 2\sqrt{2} & 4 \\ 2\sqrt{2} & -1 \end{bmatrix} \qquad \text{and} \qquad (D\boldsymbol{g})(-\sqrt{2},2) = \begin{bmatrix} -2\sqrt{2} & 4 \\ -2\sqrt{2} & -1 \end{bmatrix}$$
Therefore, $(\sqrt{2},2)$ is an unstable equilibrium since $\det(D\boldsymbol{g})(\sqrt{2},2) < 0$ (which implies that one of the eigenvalues is positive). On the other hand, $(D\boldsymbol{g})(-\sqrt{2},2)$ is negative definite so that $(-\sqrt{2},2)$ is a stable equilibrium.

2. $\boldsymbol{g}(x,y) = \begin{bmatrix} 0.01x(100-x-y) \\ 0.05y(60-y-0.2x) \end{bmatrix}$. In this case $\boldsymbol{g}(x,y) = \mathbf{0}$ if and only if $(x,y) = (0,0)$, $(0,60)$, $(100,0)$ and $(50,50)$. The Jacobian matrix of $\boldsymbol{g}$ is
$$(D\boldsymbol{g})(x,y) = \begin{bmatrix} 0.01(100-2x-y) & -0.01x \\ -0.01y & 0.05(60-2y-0.2x) \end{bmatrix}$$

so that

$$(D\boldsymbol{g})(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \qquad (D\boldsymbol{g})(0,60) = \begin{bmatrix} 0.4 & 0 \\ -0.6 & -3 \end{bmatrix},$$
$$(D\boldsymbol{g})(100,0) = \begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}, \qquad (D\boldsymbol{g})(50,50) = \begin{bmatrix} -0.5 & -0.5 \\ -0.5 & -2.5 \end{bmatrix}.$$

Therefore, $(0,0)$, $(0,60)$, $(100,0)$ are unstable equilibrium, and $(50,50)$ is a stable equilibrium.

10.3.1 The stability for the plane autonomous system

- **Classification of Equilibria**

Let $\boldsymbol{X}_e$ be an equilibrium of the autonomous system $\boldsymbol{X}' = \boldsymbol{g}(\boldsymbol{X})$ for some continuously differentiable function $\boldsymbol{g}$, and $\boldsymbol{A} = (D\boldsymbol{g})(\boldsymbol{X}_e)$ be the Jacobian matrix of $\boldsymbol{g}$ at $\boldsymbol{X}_e$ with $\tau = \mathrm{tr}(\boldsymbol{A})$ and $\Delta = \det(\boldsymbol{A})$. We can obtain some additional "geometric" information from the corresponding linear system when considering plane autonomous system:

1. In five separate cases (stable equilibrium, stable spiral equilibrium, unstable spiral equilibrium, unstable equilibrium and saddle) the equilibrium may be categorized like the equilibrium in the corresponding linear system.

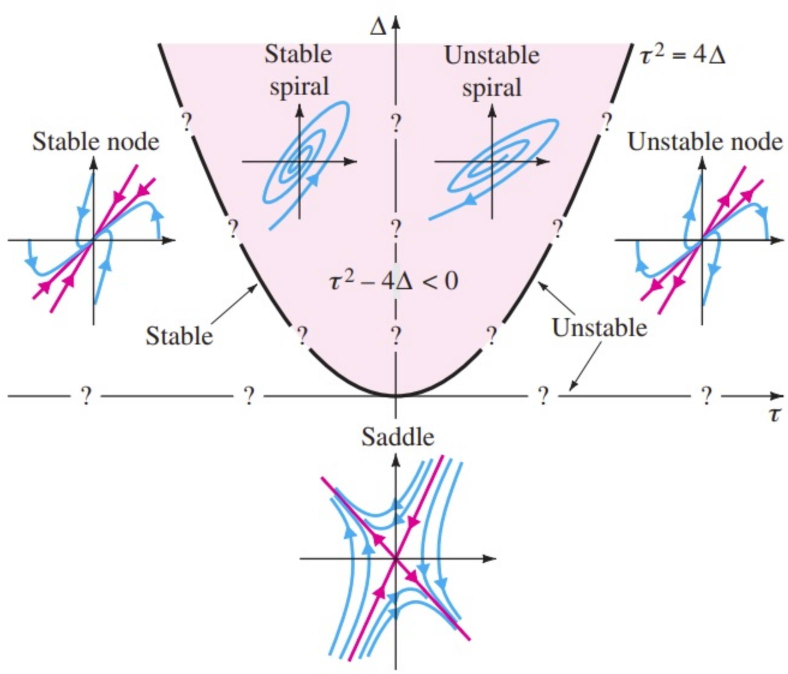


Figure 10.1: Geometric summary of some conclusions

2. If $\tau^2 = 4\Delta$ and $\tau > 0$, the equilibrium is unstable, but we are not able to determine whether $\boldsymbol{X}_e$ is an unstable spiral, unstable node, or degenerate unstable node. Similarly, if $\tau^2 = 4\Delta$ and $\tau < 0$, the equilibrium is stable but may be either a stable spiral, a stable node or a degenerate stable node.

3. If $\tau = 0$ and $\Delta > 0$, the eigenvalues of $(D\boldsymbol{g})(\boldsymbol{X}_e)$ are pure imaginary and in this case $\boldsymbol{X}_e$ may be either a stable spiral, an unstable spiral or a center.

Example 10.12. Consider the differential equation $mx'' + kx + k_1x^3 = 0$ for $k > 0$ which represents a general model for the free, undamped oscillations of a mass m attached to a nonlinear spring. If $k = 1$ and $k_1 = -1$, the spring is called soft, and the plane autonomous system corresponding to the nonlinear equation is

$$\begin{aligned} \frac{dx}{dt} &= y = g_1(x,y)\,, \\ \frac{dy}{dt} &= x^3 - x = g_2(x,y)\,. \end{aligned}$$

There are three equilibria of the system: $(0,0)$, $(1,0)$ and $(-1,0)$. Let $\boldsymbol{g}(x,y) = \begin{bmatrix} g_1(x,y) \\ g_2(x,y) \end{bmatrix}$. Then $(D\boldsymbol{g})(x,y) = \begin{bmatrix} 0 & 1 \\ 3x^2-1 & 0 \end{bmatrix}$.

1. $(D\boldsymbol{g})(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ whose eigenvalues are $\pm i$. In this case $(0,0)$ is a center of the corresponding linear system, but we do not know if $(0,0)$ is a stable or unstable equilibrium of the original nonlinear system.

2. $(D\boldsymbol{g})(1,0) = (D\boldsymbol{g})(-1,0) = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ whose eigenvalues are $\pm\sqrt{2}$. Therefore, the equilibrium $(1,0)$ and $(-1,0)$ are saddle points of the corresponding linear system and the original nonlinear system.

- **The Phase-Plane Method**

Plotting the vector field $\boldsymbol{V}(x,y) = \big(P(x,y), Q(x,y)\big)$ (near an equilibrium) will help us determine the stability of the equilibria of the autonomous system

$$\begin{aligned} \frac{dx}{dt} &= P(x,y)\,, \\ \frac{dy}{dt} &= Q(x,y)\,; \end{aligned}$$

however, it is in general not possible for us to plot "all" the behavior of the vector field. On the other hand, the solution of the first-order equation

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}$$

provides almost as much information as the vector field since each initial data (x_0, y_0) corresponding to an integral curves $(x(t), y(t))$ of solutions to the differential equation above.

Example 10.13. Use the phase-plane method to classify the equilibrium $(0,0)$ of the plane autonomous system

$$\begin{aligned} x' &= y^2\,, \\ y' &= x^2\,. \end{aligned}$$

We note that the Jacobian matrix at $(0,0)$ is the zero matrix; thus we does not know the stability of the equilibrium $(0,0)$ from the previous method. Nevertheless, we solve the differential equation

$$\frac{dy}{dx} = \frac{x^2}{y^2}$$

and find that the integral curves are given by $y^3 = x^3 + C$. Therefore, if we start from the initial data (x_0, y_0), the trajectory is the curve $y^3 = x^3 + y_0^3 - x_0^3$ so that $(x(t), y(t))$ moves beyond any bound as t increases. Therefore, no matter how close the initial data (x_0, y_0) to the equilibrium $(0,0)$ is, $\boldsymbol{X}(t) = (x(t), y(t))$ moves away from the equilibrium as t increases.

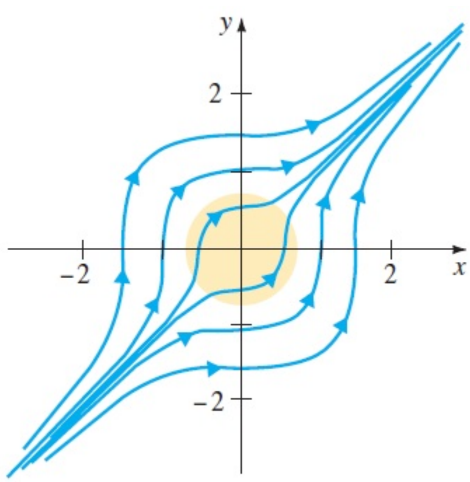


Figure 10.2: Phase portrait of nonlinear system in Example 10.13

Example 10.14. In this example we try to determine the stability of the equilibrium $(0,0)$ of the differential equation

$$\begin{aligned} x' &= y\,, \\ y' &= x^3 - x\,. \end{aligned}$$

obtained from considering the soft spring in Example 10.12. We solve the differential equation $\dfrac{dy}{dx} = \dfrac{x^3 - x}{y}$ and obtain that the integral curve are given by

$$\frac{y^2}{2} = \frac{x^4}{4} - \frac{x^2}{2} + C$$

or equivalently,

$$y^2 = \frac{1}{2}(x^2-1)^2 + C_0\,,$$

where $C_0 = y_0^2 - \dfrac{1}{2}(x_0^2 - 1)^2$ if the integral curve passes (x_0, y_0).

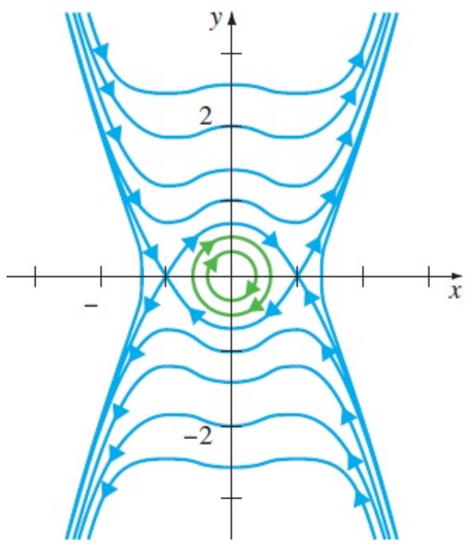


Figure 10.3: Phase portrait of nonlinear system in Example 10.14

Now we try to determine the stability of the equilibrium $(0,0)$. Suppose that (x_0, y_0) are very closed to $(0,0)$, say $(x_0, y_0) = (x_0, 0)$ with $x_0 \ll 1$, then $C_0 = -\dfrac{1}{2}(x_0^2 - 1)^2$ so that the integral curve passing through (x_0, y_0) is

$$y^2 = \frac{1}{2}(x^2-1)^2 - \frac{1}{2}(x_0^2-1)^2 = \frac{1}{2}(x^2 + x_0^2 - 2)(x^2 - x_0^2)\,.$$

The right-hand side is positive if $-x_0 < x < x_0$, and in this case each x corresponds to two y; thus the trajectory of the solution $\boldsymbol{X} = \boldsymbol{X}(t)$ satisfying the initial condition $\boldsymbol{X}(0) = (x_0, 0)$ is a closed curve. Therefore, $(0,0)$ is a center.

10.4 Autonomous Systems as Mathematical Models

Example 10.15. Consider the pendulum differential equation $\dfrac{d^2\theta}{dt^2} + \dfrac{g}{L}\sin\theta = 0$. Let $x = \theta$, $y = \theta'$, and $\boldsymbol{X} = (x, y)$. Then

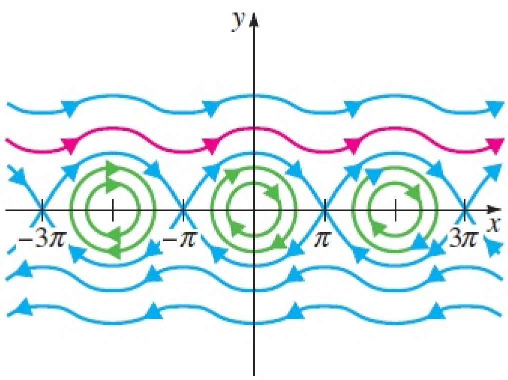
$$x' = y\,,$$
$$y' = -\frac{g}{L}\sin x\,.$$

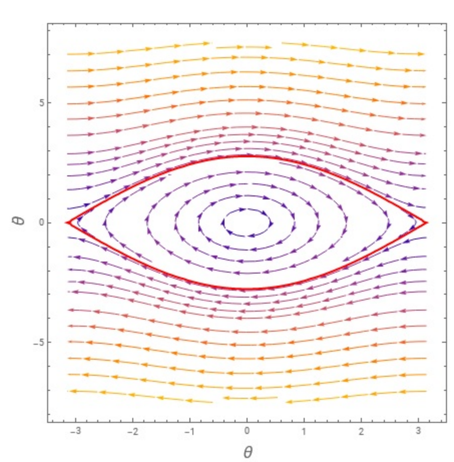
We would like to examine the solution satisfying $\boldsymbol{X}(0) = (\theta_0, 0)$ and $\boldsymbol{X}(0) = (0, \omega_0)$. Note that the trajectory of the solution satisfies that

$$\frac{dy}{dx} = -\frac{g}{L}\frac{\sin x}{y}$$

so that with the initial condition $\boldsymbol{X}(0) = (x_0, y_0)$, the trajectory satisfies

$$y^2 = \frac{2g}{L}\Big(\cos x - \cos x_0 + \frac{L}{2g}y_0^2\Big)\,.$$





1. For the case $\boldsymbol{X}(0) = (\theta_0, 0)$: In this case we have

$$y^2 = \frac{2g}{L}(\cos x - \cos\theta_0)\,.$$

As long as $\theta_0 \neq (2n+1)\pi$ for some integer n, from the phase portrait we see that the solution is periodic. We also note that $((2n+1)\pi, 0)$ is an unstable equilibrium/saddle node of the system since the eigenvalues of the Jacobian matrix at $((2n+1)\pi, 0)$ are ± 1.

2. For the case $\boldsymbol{X}(0) = (0, \omega_0)$: In this case we have

$$y^2 = \frac{2g}{L}\Big(\cos x - 1 + \frac{L}{2g}\omega_0^2\Big)\,.$$

(a) If $|\omega_0| < \sqrt{\frac{4g}{L}}$, then letting $\theta_0 \in (0, \pi)$ satisfying $\cos\theta_0 = 1 - \frac{L}{2g}\omega_0^2$ so that the trajectory satisfies that
$$y^2 = \frac{2g}{L}(\cos x - \cos\theta_0)\,.$$
In this case, the dynamics is the same as the one with initial data $(\theta_0, 0)$ so that every $x \in (-\theta_0, \theta_0)$ corresponds to two y's; thus the trajectory is closed and we obtain periodic solutions.

(b) If $|\omega_0| = \sqrt{\frac{4g}{L}}$, then the trajectory is an arc of finite length and does not form a closed curve (so that the solution is not periodic).

(c) If $|\omega_0| > \sqrt{\frac{4g}{L}}$, then the trajectory is an arc of infinite length and does not form a closed curve (so that the solution is not periodic).

Example 10.16 (Nonlinear Oscillations: the Sliding Bead)**.** Consider a bead with mass m slides along a thin wire whose shape is described by $z = f(x)$, and we are interested in the dynamics of the x-coordinate of the bead. Under the effect of gravity, the tangential force $\boldsymbol{F}$ due to the gravity g has magnitude $mg\sin\theta$ so that the x-component of $\boldsymbol{F}$ is $F_x = -mg\sin\theta\cos\theta$. Since $\tan\theta = f'(x)$, we find that
$$F_x = -mg\sin\theta\cos\theta = -mg\frac{f'(x)}{1+f'(x)^2}\,.$$
Assume the existence of a damping force $\boldsymbol{D}$, acting in the direction opposite to the motion, is a constant multiple of the velocity of the bead. The x-component of $\boldsymbol{D}$ is then $D_x = -\beta x'$; thus the Newton second law shows that x satisfies that
$$x'' = -mg\frac{f'(x)}{1+f'(x)^2} - \beta x'\,,$$
and with $y = x'$, we obtain the corresponding plane autonomous system
$$\begin{aligned} x' &= y \\ y' &= -g\frac{f'(x)}{1+f'(x)^2} - \frac{\beta}{m}y\,. \end{aligned}$$
We note that the equilibrium (x_1, y_1) of the system above satisfies that $f'(x_1) = 0$ and $y_1 = 0$. At such a point (x_1, y_1), the Jacobian matrix of the right-hand side function is
$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -gf''(x_1) & -\beta/m \end{bmatrix}.$$
Therefore,

1. if $f''(x_1) < 0$, then f attains a relative maximum at x_1 so that we can expect that (x_1, y_1) is an unstable equilibrium. In fact, (x_1, y_1) is an unstable saddle point since $\Delta = \det(\boldsymbol{A}) < 0$.

2. if $f''(x_1) > 0$, then f attains a relative minimum at x_1 so that we can expect that (x_1, y_1) is a stable equilibrium.

   (a) if $\beta > 0$, then $\mathrm{tr}(\boldsymbol{A}) < 0$ so that (x_1, y_1) is a stable equilibrium. Moreover, if $\beta^2 > 4gm^2 f''(x_1)$ (which corresponds to the "overdamped" system), then (x_1, y_1) is a stable node, while if $\beta^2 < 4gm^2 f''(x_1)$ (which corresponds to the "underdamped" system), then (x_1, y_1) is a stable sspiral equilibrium.

   (b) if $\beta = 0$, then the eigenvalues of $\boldsymbol{A}$ is purely imaginary. We use the phase-plane method and find that the trajectory $(x(t), y(t))$ satisfies

$$\frac{dy}{dx} = -\frac{gf'(x)}{(1 + f'(x)^2)y} - \frac{\beta}{m}$$

   which shows that

$$y^2 = -2ge^{\frac{-2\beta x}{m}} \int \frac{e^{\frac{2\beta x}{m}} f'(x)}{(1 + f'(x)^2)}\, dx$$

   or

$$y^2 = y_0^2 - 2ge^{\frac{-2\beta(x-x_0)}{m}} \int_{x_0}^{x} \frac{e^{\frac{2\beta x'}{m}} f'(x')}{(1 + f'(x')^2)}\, dx'$$

   if the initial condition $(x(0), y(0)) = (x_0, y_0)$ is imposed. This "shows" that the solution $(x(t), y(t))$ is periodic if $(x_0, y_0) \approx (x_1, y_1)$ so that (x_1, y_1) is a center.

Example 10.17 (Lotka-Volterra Predator-Prey Model)**.** Let x, y denotes the population/ number of the predators and the prey, respectively. The Lotka-Volterra model takes the form

$$\begin{aligned} x' &= x(-a + by) = -ax + bxy = g_1(x, y)\,, \\ y' &= y(-cx + d) = -cxy + dy = g_2(x, y)\,, \end{aligned}$$

where a, b, c, d are positive constants. Let $\boldsymbol{g} = [g_1, g_2]^{\mathrm{T}}$.

We first note that the equilibrium of this plane autonomous system are $(0, 0)$ and $(d/c, a/b)$, and the corresponding Jacobian matrices are

$$\boldsymbol{A}_1 \equiv (D\boldsymbol{g})(0, 0) = \begin{bmatrix} -a & 0 \\ 0 & d \end{bmatrix} \quad \text{and} \quad \boldsymbol{A}_2 \equiv (D\boldsymbol{g})(d/c, a/b) = \begin{bmatrix} 0 & bd/c \\ -ac/b & 0 \end{bmatrix};$$

thus $(0,0)$ is a saddle point.

Since $\boldsymbol{A}_2$ has purely imaginary eigenvalues $\lambda = \pm\sqrt{ad}i$, we need more information to determine the nature of the equilibrium $(d/c, a/b)$. Using the phase-plane method, we solve

$$\frac{dy}{dx} = \frac{y(-cx+d)}{x(-a+by)}$$

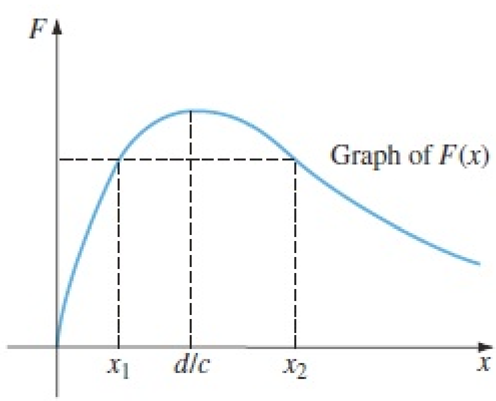
and obtain that

$$by - a\ln y = -cx + d\ln x + c_1$$

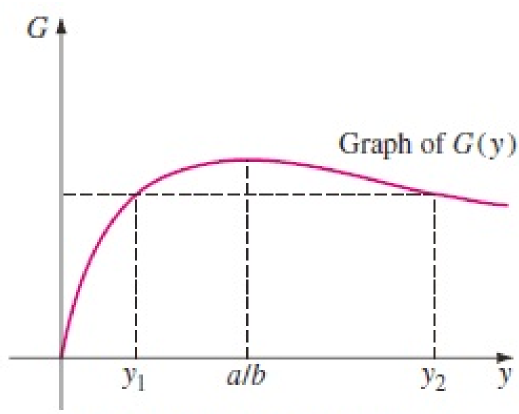
for some constant c_1 or equivalently,

$$x^d e^{-cx} y^a e^{-by} = c_0$$

for some constant c_0. Let $F(x) = x^d e^{-cx}$ and $G(y) = y^a e^{-by}$. Then F and G attain their maximum at $x = d/c$ and $y = a/b$, respectively. Note that with the exception of 0 and the absolute maximum, F and G each take on all values in their range precisely twice.



(a) Maximum of F at $x = d/c$



(b) Maximum of G at $y = a/b$

These graphs can be used to establish the following properties of a solution curve that originates at a point $\boldsymbol{X}_0 = (x_0, y_0)$ in the first quadrant but $\boldsymbol{X}_0$ is not an equilibrium. We note that in this case $c_0 = F(x_0)G(y_0)$ must satisfy that $F(d/c)G(a/b) > c_0$.

1. If $y = a/b$, the equation $F(x)G(y) = c_0$ has exactly two solution x_m and x_M satisfying $x_m < d/c < x_M$ since
$$0 < \frac{c_0}{G(a/b)} < F(d/c)$$
which implies that $F(x) = \dfrac{c_0}{G(a/b)}$ has precisely two solutions x_m and x_M that satisfy $x_m < d/c < x_M$.

2. If $x_m < x_1 < x_M$, then $F(x_1)G(y) = c_0$ has exactly two solutions y_1 and y_2 that satisfy $y_1 < a/b < y_2$ since
$$0 < \frac{c_0}{F(x_1)} < \frac{c_0}{F(d/c)} < G(a/b)\,.$$

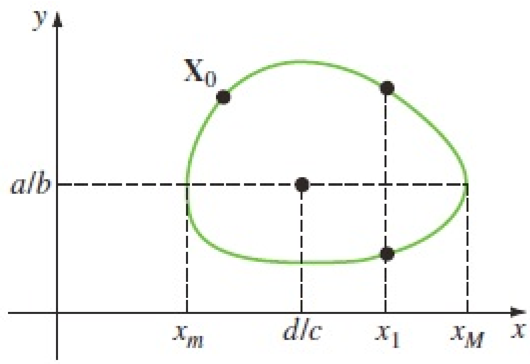


Figure 10.4: Periodic solution of the Lotka-Volterra model

3. If x is outside the interval $[x_m, x_M]$, then $F(x) < \dfrac{c_0}{G(a/b)}$; thus $F(x)G(y) = c_0$ has no solution since such y must satisfy
$$G(y) = \frac{c_0}{F(x)} > G(a/b)\,.$$

From the discussion above, we also conclude that the equilibrium $(d/c, a/b)$ is a center.

We note that similar argument can be applied to obtain y_m and y_M such that

1. $G(y_m) = G(y_M) = \dfrac{c_0}{F(a/b)}$.
2. Each $y \in (y_m, y_M)$ corresponds to two x, called x_1, x_2, such that $F(x_1) = F(x_2) = \dfrac{c_0}{G(y)}$.
3. If $y \notin [y_m, y_m]$, there is no x satisfying $F(x)G(y) = c_0$.

This implies that the solution curve originates from $\boldsymbol{X}_0$ looks more like a "circle".

Example 10.18 (Lotka-Volterra Competition Model)**.** The Lotka-Volterra competition model takes the form
$$\begin{aligned} x' &= \frac{r_1}{K_1}x(K_1 - x - \alpha_{12}y)\,,\\ y' &= \frac{r_2}{K_2}y(K_2 - y - \alpha_{21}x)\,, \end{aligned}$$

where $\alpha_{12}, \alpha_{21} \geqslant 0$, $r_1, r_2, K_1, K_2 > 0$. The numbers K_1 and $K_2 > 0$ are the maximum population of the two competitors that the environment can support, respectively.

The points $(0,0)$, $(K_1, 0)$ and $(0, K_2)$ are equilibria of this plane autonomous system. Moreover, if $\alpha_{12}\alpha_{21} \neq 1$, the lines $K_1 - x - \alpha_{12} y = 0$ and $K_2 - y - \alpha_{21} x = 0$ intersect to produce a fourth equilibrium $\widehat{\boldsymbol{X}} = (\widehat{x}, \widehat{y}) = \big(\frac{K_1 - \alpha_{12} K_2}{1 - \alpha_{12}\alpha_{21}}, \frac{K_2 - \alpha_{21} K_1}{1 - \alpha_{12}\alpha_{21}}\big)$. Since the Jacobian matrix of the right-hand side function is

$$J(x,y) = \left[\begin{array}{cc} \frac{r_1}{K_1}(K_1 - 2x - \alpha_{12} y) & -\frac{r_1 \alpha_{12}}{K_1} x \\ -\frac{r_2 \alpha_{21}}{K_2} y & \frac{r_2}{K_2}(K_2 - 2y - \alpha_{21} x) \end{array}\right],$$

by Theorem 10.10 we find that

1. Since the eigenvalues of $J(0,0)$ are r_1, r_2, we conclude that $(0,0)$ is an unstable equilibrium.

2. Since the eigenvalues of $J(K_1, 0)$ are $-r_1$ and $\frac{r_2}{K_2}(K_2 - \alpha_{21} K_1)$, we conclude that $(K_1, 0)$ is an asymptotically stable equilibrium if $K_2 - \alpha_{21} K_1 < 0$ and a saddle point if $K_2 - \alpha_{21} K_1 > 0$.

3. Since the eigenvalues of $J(0, K_2)$ are $-r_2$ and $\frac{r_1}{K_1}(K_1 - \alpha_{12} K_2)$, we conclude that $(0, K_2)$ is an asymptotically stable equilibrium if $K_1 - \alpha_{12} K_2 < 0$ and a saddle point if $K_1 - \alpha_{12} K_2 > 0$.

4. If $(\widehat{x}, \widehat{y})$ is in the first quadrant, by the fact that the trace and the determinant of $J(\widehat{x}, \widehat{y})$ are

$$\tau = -\widehat{x}\frac{r_1}{K_1} - \widehat{y}\frac{r_2}{K_2} \qquad \text{and} \qquad \Delta = (1 - \alpha_{12}\alpha_{21})\widehat{x}\widehat{y}\frac{r_1}{K_1}\frac{r_2}{K_2},$$

respectively, we find that $(\widehat{x}, \widehat{y})$ is an asymptotically stable equilibrium (but not stable spiral equilibrium) if $\alpha_{12}\alpha_{21} < 1$ since

$$\tau^2 - 4\Delta = \big(\widehat{x}\frac{r_1}{K_1} - \widehat{y}\frac{r_2}{K_2}\big)^2 + 4\alpha_{12}\alpha_{21}\frac{r_1}{K_1}\frac{r_2}{K_2} > 0$$

and $(\widehat{x}, \widehat{y})$ is a saddle point if $\alpha_{12}\alpha_{21} > 1$.

Chapter 11

Fourier Series

To begin the story, let us first consider the 1-dimensional heat equation

$$u_t(x,t) = \alpha^2 u_{xx}(x,t) \qquad \forall\, t > 0\,, x \in (0,\pi)\,, \tag{11.1a}$$

$$u(x,0) = u_0(x) \qquad \forall\, x \in [0,\pi]\,, \tag{11.1b}$$

$$u(0,t) = u(\pi,t) = 0 \qquad \forall\, t > 0\,, \tag{11.1c}$$

where $\alpha^2 > 0$ is a constant, and u is an unknown function of x and t with u_t and u_{xx} denote $\dfrac{\partial u}{\partial t}$ and $\dfrac{\partial^2 u}{\partial x^2}$, respectively. The conditions (11.1b) and (11.1c) are called the initial condition and boundary condition of this heat equation, respectively, and u_0 is called the initial data. We will not explain why we want to study this equation until next chapter, but instead we will use the procedure of solving this problem to motivate some important ideas in mathematics.

In order to solve (11.1) using what we have learned from the last chapter, we discretize the interval $(0,\pi)$ by $\big\{0 = x_0 < x_1 < \cdots < x_{n+1} = \pi\big\}$, where $x_j = \dfrac{j\pi}{n+1}$. Let $h = \pi/(n+1)$, and define $\varphi_j(t) = u(x_j,t)$. Then under the assumption that u is four times continuously differentiable, (11.1) implies that

$$\begin{aligned}
\frac{d\varphi_j}{dt} - \frac{\alpha^2}{h^2}(\varphi_{j+1} - 2\varphi_j + \varphi_{j-1}) = \mathcal{O}(h^2) \qquad & \text{for all } 1 \leqslant j \leqslant n \text{ and } t > 0\,, \\
\varphi_j(0) = u_0(x_j) \qquad & \text{for all } 1 \leqslant j \leqslant n\,, \\
\varphi_0(t) = \varphi_{n+1}(t) = 0 \qquad & \text{for all } t > 0\,,
\end{aligned}$$

where we have used the central difference approximation

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + \mathcal{O}(h^2)$$

for the second derivative (if f is four-times continuously differentiable). Therefore, naively we look for the solution to the ODE

$$\frac{d}{dt}\begin{bmatrix}\phi_1(t)\\ \phi_2(t)\\ \phi_3(t)\\ \vdots\\ \vdots\\ \phi_{n-2}(t)\\ \phi_{n-1}(t)\\ \phi_n(t)\end{bmatrix} = \frac{\alpha^2}{h^2}\begin{bmatrix}-2 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0\\ 1 & -2 & 1 & 0 & \cdots & \cdots & \cdots & 0\\ 0 & 1 & -2 & 1 & 0 & \cdots & \cdots & 0\\ \vdots & 0 & \ddots & \ddots & \ddots & 0 & \cdots & 0\\ \vdots & & \ddots & \ddots & \ddots & \ddots & \cdots & 0\\ 0 & \cdots & \cdots & 0 & 1 & -2 & 1 & 0\\ 0 & \cdots & \cdots & \cdots & 0 & 1 & -2 & 1\\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & -2\end{bmatrix}\begin{bmatrix}\phi_1(t)\\ \phi_2(t)\\ \phi_3(t)\\ \vdots\\ \vdots\\ \phi_{n-2}(t)\\ \phi_{n-1}(t)\\ \phi_n(t)\end{bmatrix} \tag{11.2}$$

with initial condition

$$\begin{bmatrix}\phi_1(0) & \phi_2(0) & \cdots & \phi_n(0)\end{bmatrix}^{\mathrm{T}} = \begin{bmatrix}u_0(x_1) & u_0(x_2) & \cdots & u_0(x_n)\end{bmatrix}^{\mathrm{T}}, \tag{11.3}$$

and treat $\phi_i(t)$ as an approximated value of $\varphi_i(t)$ (and expect that as $h \to 0$ we can obtain information about u from these values of ϕ). Now you see why what we learn from solving a linear system can help us. Let

$$\boldsymbol{X} = \begin{bmatrix}\phi_1(t)\\ \phi_2(t)\\ \phi_3(t)\\ \vdots\\ \vdots\\ \phi_{n-2}(t)\\ \phi_{n-1}(t)\\ \phi_n(t)\end{bmatrix}, \quad \boldsymbol{X}_0 = \begin{bmatrix}u_0(x_1)\\ u_0(x_2)\\ u_0(x_3)\\ \vdots\\ \vdots\\ u_0(x_{n-2})\\ u_0(x_{n-1})\\ u_0(x_n)\end{bmatrix} \quad \text{and} \quad \boldsymbol{A} = \frac{\alpha^2}{h^2}\begin{bmatrix}-2 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0\\ 1 & -2 & 1 & 0 & \cdots & \cdots & \cdots & 0\\ 0 & 1 & -2 & 1 & 0 & \cdots & \cdots & 0\\ \vdots & 0 & \ddots & \ddots & \ddots & 0 & \cdots & 0\\ \vdots & & \ddots & \ddots & \ddots & \ddots & \cdots & 0\\ 0 & \cdots & \cdots & 0 & 1 & -2 & 1 & 0\\ 0 & \cdots & \cdots & \cdots & 0 & 1 & -2 & 1\\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & -2\end{bmatrix},$$

then (11.2) and (11.3) provide the initial-value problem

$$\boldsymbol{X}' = \boldsymbol{A}\boldsymbol{X}\,, \qquad \boldsymbol{X}(0) = \boldsymbol{X}_0\,. \tag{11.4}$$

In order to solve the linear system above, let us find the eigenvalue of $\boldsymbol{A}$. Let $\boldsymbol{B} = \dfrac{1}{\alpha^2}\boldsymbol{A}$. Since $\boldsymbol{B}$ is used to approximate the differential operator $\dfrac{d^2}{dx^2}$, we first try to find functions, the so-called eigen-functions of the differential operator $\dfrac{d^2}{dx^2}$ (subject to the zero boundary condition). This is to find non-trivial functions v satisfying

$$v''(x) = \lambda v(x)\,, \qquad v(0) = v(\pi) = 0\,.$$

We note that λ has to be negative since if $\lambda \geqslant 0$ we must have $v = 0$. If $\lambda < 0$, then the general solution to the ODE $v'' = \lambda v$ is

$$v(x) = c_1 \cos \sqrt{-\lambda} x + c_2 \sin \sqrt{-\lambda} x$$

and the boundary condition $v(0) = v(\pi) = 0$ implies that $\lambda = -\ell^2$ for some $\ell \in \mathbb{N}$ and $v(x)$ is a constant multiple of $\sin(\ell x)$. Therefore, we conjecture that the vector

$$\begin{bmatrix} \sin(\ell x_1) & \sin(\ell x_2) & \cdots & \sin(\ell x_n) \end{bmatrix}^{\mathrm{T}}$$

is an eigenvector of $\boldsymbol{B}$. To see this is the case, we let $\theta = \ell h = \dfrac{\ell\pi}{n+1}$. Then by the identity

$$\sin((j-1)\theta) + \sin((j+1)\theta) - 2\sin(j\theta) = -2\sin(j\theta)(1-\cos\theta)\,,$$

we find that

$$\begin{aligned}
&\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix} \begin{bmatrix} \sin\theta \\ \sin(2\theta) \\ \sin(3\theta) \\ \vdots \\ \sin(n\theta) \end{bmatrix} \\
&= \begin{bmatrix} \sin(2\theta) + \sin(0\theta) - 2\sin\theta \\ \sin\theta + \sin(3\theta) - 2\sin(2\theta) \\ \sin(2\theta) + \sin(4\theta) - 2\sin(3\theta) \\ \vdots \\ \sin(n\theta) + \sin((n-2)\theta) - 2\sin((n-1)\theta) \\ \sin((n-1)\theta) + \sin((n+1)\theta) - 2\sin(n\theta) \end{bmatrix} = -2(1-\cos\theta) \begin{bmatrix} \sin\theta \\ \sin(2\theta) \\ \sin(3\theta) \\ \vdots \\ \sin((n-1)\theta) \\ \sin(n\theta) \end{bmatrix}.
\end{aligned}$$

The computation above shows that $\boldsymbol{A}$ has eigenvalues $\lambda_\ell = -\dfrac{2\alpha^2(1-\cos(\ell h))}{h^2}$ for $1 \leqslant \ell \leqslant n$ and the corresponding eigenvectors are $\boldsymbol{K}_\ell = \begin{bmatrix} \sin(\ell x_1) & \sin(\ell x_2) & \cdots & \sin(\ell x_n) \end{bmatrix}^{\mathrm{T}}$. The solution $\boldsymbol{X}$ of the initial-value problem (11.4) is then

$$\boldsymbol{X}(t) = \sum_{\ell=1}^{n} c_\ell \boldsymbol{K}_\ell e^{\lambda_\ell t}\,,$$

where $c_1, \cdots, c_n$ are chosen so that $\sum\limits_{\ell=1}^{n} c_\ell \boldsymbol{K}_\ell = \boldsymbol{X}_0$. Note that the symmetry of $\boldsymbol{A}$ implies that $\boldsymbol{K}_\ell \cdot \boldsymbol{K}_j = 0$ if $\ell \neq j$; thus by the fact that

$$K_\ell \cdot K_\ell = \sum_{j=1}^{n} \sin^2(\ell x_j) = \sum_{j=1}^{n} \frac{1-\cos(2\ell x_j)}{2} = \frac{n}{2}\,,$$

we conclude that $c_\ell = \dfrac{2}{n}\boldsymbol{X}_0 \cdot \boldsymbol{K}_\ell$ so that

$$\boldsymbol{X}(t) = \sum_{\ell=1}^{n} \frac{2}{n}(\boldsymbol{X}_0 \cdot \boldsymbol{K}_\ell) e^{\lambda_\ell t} \boldsymbol{K}_\ell \,.$$

Since each component of $\boldsymbol{X}$ correspond to an approximated value of the true solution u at some x_j, passing to the limit as $n \to \infty$ we conjecture that the solution to the 1-d heat equation (11.1) is

$$u(x,t) = \sum_{\ell=1}^{\infty} C_\ell e^{-\alpha^2 \ell^2 t} \sin(\ell x)\,,$$

where $C_\ell = \lim\limits_{n\to\infty} \dfrac{2}{n}\boldsymbol{X}_0 \cdot \boldsymbol{K}_\ell = \dfrac{2}{\pi}\displaystyle\int_0^\pi u_0(x)\sin(\ell x)\,dx$.

11.1 Orthogonal Functions

Definition 11.1. 1. The "inner product" of two real-valued functions f and g on an interval $[a,b]$ is the number

$$(f,g)_{L^2(a,b)} = \int_a^b f(x)g(x)\,dx\,,$$

and the "square norm" of a function f on $[a,b]$ is the number

$$\|f\|_{L^2(a,b)} = \Big(\int_a^b f(x)^2\,dx\Big)^{\frac{1}{2}}\,.$$

2. Two real-valued functions f and g are said to be orthogonal on an interval $[a,b]$ if

$$(f,g)_{L^2(a,b)} = \int_a^b f(x)g(x)\,dx = 0\,.$$

3. A set of real-valued functions $\{\phi_0, \phi_1, \phi_2, \cdots\}$ is said to be orthogonal on an interval $[a,b]$ if

$$(\phi_m,\phi_n)_{L^2(a,b)} = \int_a^b \phi_m(x)\phi_n(x)\,dx = 0 \quad \text{if } m \neq n\,.$$

It is called an orthonormal set on $[a,b]$ if $\|\phi_k\|_{L^2(a,b)} = 1$ for all $k \in \mathbb{N} \cup \{0\}$.

Example 11.2. The set $\big\{\cos(\ell x) \,\big|\, \ell \in \mathbb{N} \cup \{0\}\big\} = \{1, \cos x, \cos 2x, \cdots,\}$ is an orthogonal set on $[-\pi,\pi]$ since if $k \neq \ell$,

$$\int_{-\pi}^{\pi} \cos(kx)\cos(\ell x)\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \Big[\cos(k-\ell)x + \cos(k+\ell)x\Big]\,dx = 0\,.$$

Moreover, since

$$\int_{-\pi}^{\pi} \cos^2(\ell x)\, dx = \frac{1}{2}\int_{-\pi}^{\pi} \big[1+\cos(2\ell x)\big]\, dx = \begin{cases} \pi & \text{if } \ell \neq 0\,, \\ 2\pi & \text{if } \ell = 0\,, \end{cases}$$

we conclude that $\Big\{\dfrac{1}{\sqrt{2\pi}}\Big\} \cup \Big\{\dfrac{\cos(\ell x)}{\sqrt{\pi}} \,\Big|\, \ell \in \mathbb{N}\Big\}$ is an orthonormal set on $[-\pi,\pi]$.

Given a function f defined on $[a,b]$ and an infinite orthogonal set $\big\{\phi_0,\phi_1,\phi_2,\cdots\big\}$ on $[a,b]$, we would like to know if f can be expressed as

$$f = \sum_{\ell=0}^{\infty} c_\ell \phi_\ell$$

for some sequence $\{c_\ell\}_{\ell=0}^{\infty}$, where the sum converges in some sense. An orthogonal set which makes the decomposition above possible for "all" f defined on $[a,b]$ is called a complete orthogonal set. We note that every "square integrable" functions f on $[0,\pi]$ (that is, $\|f\|_{L^2(0,\pi)} < \infty$) can be written as

$$f = \sum_{\ell=1}^{\infty} s_\ell \phi_\ell\,,$$

for some sequence $\{s_\ell\}_{\ell=1}^{\infty}$, where $\phi_\ell(x) = \sin(\ell x)$ for all $\ell \in \mathbb{N}$ and the sum converges in the sense

$$\lim_{n\to\infty} \Big\| f - \sum_{\ell=1}^{n} s_\ell \phi_\ell \Big\|_{L^2(0,\pi)} = 0\,.$$

In other words, $\big\{\sin(\ell x) \,\big|\, \ell \in \mathbb{N}\big\}$ is a complete orthogonal set on $[0,\pi]$.

A non-rigorous reason: For each $n \in \mathbb{N}$, the matrix

$$\boldsymbol{B} = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 1 & -2 & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & -2 & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & -2 \end{bmatrix},$$

a discretization of $\dfrac{d^2}{dx^2}$, is symmetric so that the eigenspace is identical to $\mathbb{R}^n$. Note that the eigenspace of B is $\text{span}\big\{\boldsymbol{K}_1, \boldsymbol{K}_2, \cdots, \boldsymbol{K}_n\big\}$, where $\boldsymbol{K}_\ell$ is the evaluation of $\sin(\ell x)$ on the set $\big[\frac{\pi}{n+1}, \frac{2\pi}{n+1}, \cdots, \frac{n\pi}{n+1}\big]$ so that if f is a function defined on $[0,\pi]$, then

$$\boldsymbol{F}_n \equiv \Big[f\big(\frac{\pi}{n+1}\big), f\big(\frac{2\pi}{n+1}\big), \cdots, f\big(\frac{n\pi}{n+1}\big)\Big] = \sum_{\ell=1}^{n} s_\ell^{(n)} \boldsymbol{K}_\ell \tag{11.5}$$

for some real numbers $s_1^{(n)}, s_2^{(n)}, \cdots, s_n^{(n)}$. Passing to the limit as $n \to \infty$, we expect that

$$f(x) = \sum_{\ell=1}^{\infty} s_\ell \sin(\ell x) \tag{11.6}$$

for some sequence $\{s_\ell\}_{\ell=1}^{\infty}$ (here each s_ℓ should be the limit of $s_\ell^{(n)}$ as $n \to \infty$). We also note that the coefficients $\boldsymbol{s}_\ell^{(n)}$ in (11.5) is given by

$$s_\ell^{(n)} = \frac{\boldsymbol{F}_n \cdot \boldsymbol{K}_\ell}{\boldsymbol{K}_\ell \cdot \boldsymbol{K}_\ell}$$

and again with $\phi_\ell(x) = \sin(\ell x)$,

$$\lim_{n\to\infty} s_\ell^{(n)} = \lim_{n\to\infty} \frac{\frac{\pi}{n+1}\boldsymbol{F}_n \cdot \boldsymbol{K}_\ell}{\frac{\pi}{n+1}\boldsymbol{K}_\ell \cdot \boldsymbol{K}_\ell} = \frac{\lim\limits_{n\to\infty} \frac{\pi}{n+1}\boldsymbol{F}_n \cdot \boldsymbol{K}_\ell}{\lim\limits_{n\to\infty} \frac{\pi}{n+1}\boldsymbol{K}_\ell \cdot \boldsymbol{K}_\ell} = \frac{(f,\phi_\ell)_{L^2(0,\pi)}}{\|\phi_\ell\|^2_{L^2(0,\pi)}}$$

so that (11.6) becomes

$$f = \sum_{\ell=1}^{\infty} \frac{(f,\phi_\ell)_{L^2(0,\pi)}}{\|\phi_\ell\|^2_{L^2(0,\pi)}} \phi_\ell \,.$$

Bottom line: The collection of eigenfunctions of $\dfrac{d^2}{dx^2}$ (with certain homogeneous boundary condition) forms an "orthogonal basis" on $[0,\pi]$.

11.2 Fourier Cosine and Sine Series

From the introduction above, we have some ideas about why $\big\{\sin(\ell x) \,\big|\, \ell \in \mathbb{N}\big\}$ forms a complete orthogonal set. In this section, we investigate further these concepts. We recall that an $n \times n$ real matrix $\boldsymbol{A}$ is said to be symmetric if $\boldsymbol{A} = \boldsymbol{A}^{\mathrm{T}}$. To understand the concept even deeply, let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Then L is said to symmetric if

$$(L\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}, L\boldsymbol{y}) \qquad \forall\, \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n \,,$$

here $(\cdot,\cdot)$ is the inner product on $\mathbb{R}^n$. In general, if $(\mathcal{V}, \langle\cdot,\cdot\rangle)$ is an inner product space, a linear map $L : \mathcal{V} \to \mathcal{V}$ is said to be self-adjoint (an analogy of symmetry) if

$$\langle L\boldsymbol{x}, \boldsymbol{y}\rangle = \langle \boldsymbol{x}, L\boldsymbol{y}\rangle \qquad \forall\, \boldsymbol{x}, \boldsymbol{y} \in \mathcal{V} \,.$$

Since we have defined the inner product of functions defined on an interval, we can talk about the "symmetry" of the linear map $\frac{d^2}{dx^2}$. For f, g being twice continuously differentiable on $[a, b]$, integrating by parts we obtain that

$$\begin{aligned}\int_a^b f''(x)g(x)\,dx &= f'(x)g(x)\Big|_{x=a}^{x=b} - \int_a^b f'(x)g'(x)\,dx \\ &= \big[f'(x)g(x) - f(x)g'(x)\big]\Big|_{x=a}^{x=b} + \int_a^b f(x)g''(x)\,dx\,.\end{aligned}$$

Therefore, if $f'(x)g(x) - f(x)g'(x)$ vanish at the end-points $x = a$ and $x = b$, we have

$$\Big(\frac{d^2}{dx^2}f, g\Big)_{L^2(a,b)} = \int_a^b f''(x)g(x)\,dx = \int_a^b f(x)g''(x)\,dx = \Big(f, \frac{d^2}{dx^2}g\Big)_{L^2(a,b)}. \tag{11.7}$$

In other words, if certain boundary conditions (such as vanishes at the end-points) are imposed, then $\frac{d^2}{dx^2}$ is "symmetric".

Example 11.3. Identity (11.7) holds if f, g are twice continuously differentiable on $[a, b]$ and f, g satisfy on of the following boundary conditions:

1. f, g vanish at the end-points $x = a$ and $x = b$,
2. f', g' vanish at the end-points $x = a$ and $x = b$,
3. f, g vanish at the left end-points $x = a$ and f', g' vanish at the right end-point $x = b$,
4. f', g' vanish at the left end-points $x = a$ and f, g vanish at the right end-point $x = b$,

Definition 11.4. A function $v : [a, b] \to \mathbb{R}$ is called an eigenfunction of the differentiable operator $\frac{d^2}{dx^2}$ on $[a, b]$ if v is not a zero function and there exists a constant λ such that

$$\frac{d^2}{dx^2}v(x) = \lambda v(x) \qquad \forall\, x \in [a, b]\,.$$

We also note that if f, g are eigenfunctions of the symmetric differential operator $\frac{d^2}{dx^2}$ corresponding to eigenvalues λ and μ, respectively; that is,

$$\frac{d^2}{dx^2}f(x) = \lambda f(x) \quad \text{and} \quad \frac{d^2}{dx^2}g(x) = \mu g(x) \qquad \forall\, x \in [a, b]\,,$$

then

$$\lambda(f, g)_{L^2(a,b)} = (f'', g)_{L^2(a,b)} = (f, g'')_{L^2(a,b)} = \mu(f, g)_{L^2(a,b)}\,.$$

Since $\lambda \neq \mu$, we have $(f,g)_{L^2(a,b)} = 0$. In other words, eigenfunctions corresponding to different eigenvalues of the symmetric differential operator $\dfrac{d^2}{dx^2}$ are orthogonal.

Recall that a symmetric $n\times n$ real matrix has n linearly independent eigenvectors that are mutually orthogonal and these n eigenvectors form a basis of $\mathbb{R}^n$. Similarly, the collection of "maximal" mutually orthogonal eigenfunctions of $\dfrac{d^2}{dx^2}$ on $[a,b]$ form a complete orthogonal set in the sense that if $\{\phi_k\}_{k=1}^{\infty}$ is a maximal mutually orthogonal eigenfunctions of $\dfrac{d^2}{dx^2}$ on $[a,b]$, then for every "square integrable" f defined on $[a,b]$,

$$\lim_{n\to\infty} \Big\| f - \sum_{k=1}^{n} f_k \phi_k \Big\|_{L^2(a,b)} = 0\,,$$

where $f_k = \dfrac{(f,\phi_k)_{L^2(a,b)}}{\|\phi_k\|^2_{L^2(a,b)}}$.

Example 11.5. We have "shown" that $\big\{ \sin(\ell x) \,\big|\, \ell \in \mathbb{N} \big\}$ forms a complete orthogonal set on $[0,\pi]$. Now suppose that we look for a complete orthogonal set on $[0,p]$ with the boundary condition $v'(x) = 0$ at the end-point $x = 0$ and $x = p$ for eigenfunction v of $\dfrac{d^2}{dx^2}$. Then there exists λ such that

$$v''(x) = \lambda v(x) \quad \forall\, x \in [0,p]\,, \qquad v'(0) = v'(p) = 0\,. \tag{11.8}$$

1. If $\lambda > 0$, then the general solution of the differential equation is $v(x) = C_1 e^{\sqrt{\lambda}x} + C_2 e^{-\sqrt{\lambda}x}$ and the boundary condition implies that $C_1 = C_2 = 0$.

2. If $\lambda = 0$, then the solution to the boundary-value problem (11.8) is constant.

3. If $\lambda < 0$, then the general solution of the differential equation is $v(x) = C_1 \cos(\sqrt{-\lambda}x) + C_2 \sin(\sqrt{-\lambda}x)$. To satisfies the boundary condition, we must have

$$0 = v'(0) = C_2\sqrt{-\lambda}\cos 0$$

so that $C_2 = 0$, and

$$0 = v'(p) = C_1\sqrt{-\lambda}\sin(\sqrt{-\lambda}p)$$

which implies that $\sqrt{-\lambda}p = \ell\pi$ for $\ell \in \mathbb{N}$. Therefore, the collection of eigenvalues are $\Big\{ \lambda_\ell \,\Big|\, \lambda_\ell = -\dfrac{\ell^2\pi^2}{p^2}\,, \ell \in \mathbb{N}\cup\{0\} \Big\}$ with corresponding eigenfunctions $\phi_\ell = \cos\dfrac{\ell\pi x}{p}$ so that $\Big\{ \cos\dfrac{\ell\pi x}{p} \,\Big|\, \ell \in \mathbb{N}\cup\{0\} \Big\}$ forms a complete orthogonal set on $[0,p]$.

Similar argument can be applied to obtain that $\left\{\sin\frac{\ell\pi x}{p}\,\Big|\,\ell\in\mathbb{N}\right\}$ is a complete orthogonal set on $[0,p]$.

Because of the identities

$$\int_0^p \cos^2\frac{k\pi x}{p}\,dx = \int_0^p \sin^2\frac{k\pi x}{p}\,dx = \frac{p}{2} \qquad \forall\, k\in\mathbb{N}\,,$$

the fact that $\left\{\sin\frac{\ell\pi x}{p}\,\Big|\,\ell\in\mathbb{N}\right\}$ and $\left\{\cos\frac{\ell\pi x}{p}\,\Big|\,\ell\in\mathbb{N}\cup\{0\}\right\}$ are both complete orthogonal sets on $[0,p]$ motivates the following

Definition 11.6. 1. The Fourier cosine series of a function f defined on the interval $(0,p)$ is

$$\frac{c_0}{2}+\sum_{k=1}^{\infty} c_k\cos\frac{k\pi x}{p}$$

whenever the sum makes sense, where $c_k=\frac{2}{p}\int_0^p f(x)\cos\frac{k\pi x}{p}\,dx$.

2. The Fourier sine series of an odd function f defined on the interval $(0,p)$ is

$$\sum_{k=1}^{\infty} s_k\sin\frac{k\pi x}{p}$$

whenever the sum makes sense, where $s_k=\frac{2}{p}\int_0^p f(x)\sin\frac{k\pi x}{p}\,dx$.

Example 11.7. Consider the function $f(x)=x^2$ on the interval $(0,L)$.

1. We first expand f in a cosine series. Integrating by parts, for $k\in\mathbb{N}$ we have

$$\begin{aligned}c_k &= \frac{2}{L}\int_0^L x^2\cos\frac{k\pi x}{L}\,dx = \frac{2}{L}\Big(\frac{Lx^2}{k\pi}\sin\frac{k\pi x}{L}\Big|_{x=0}^{x=L} - \frac{2L}{k\pi}\int_0^L x\sin\frac{k\pi x}{L}\,dx\Big)\\ &= -\frac{4}{k\pi}\Big(\frac{-Lx}{k\pi}\cos\frac{k\pi x}{L}\Big|_{x=0}^{x=L} + \frac{L}{k\pi}\int_0^L \cos\frac{k\pi x}{L}\,dx\Big) = \frac{4L^2\cos(k\pi)}{k^2\pi^2} = \frac{4L^2(-1)^k}{k^2\pi^2}\end{aligned}$$

while

$$c_0=\frac{2}{L}\int_0^L x^2\,dx=\frac{2L^2}{3}\,.$$

Therefore, the cosine series of f is

$$\frac{L^2}{3}+\sum_{k=1}^{\infty}\frac{4L^2(-1)^k}{k^2\pi^2}\cos\frac{k\pi x}{L}\,.$$

2. Next we expand f in a sine series. Integrating by parts, for all $k \in \mathbb{N}$ we have

$$\begin{aligned} s_k &= \frac{2}{L}\int_0^L x^2 \sin\frac{k\pi x}{L}\,dx = \frac{2}{L}\Big(\frac{-Lx^2}{k\pi}\cos\frac{k\pi x}{L}\Big|_{x=0}^{x=L} + \frac{2L}{k\pi}\int_0^L x\cos\frac{k\pi x}{L}\,dx\Big) \\ &= \frac{2}{L}\Big[\frac{L^3(-1)^{k+1}}{k\pi} + \frac{2L}{k\pi}\Big(\frac{Lx}{k\pi}\sin\frac{k\pi x}{L}\Big|_{x=0}^{x=L} - \frac{L}{k\pi}\int_0^L \sin\frac{k\pi x}{L}\,dx\Big)\Big] \\ &= \frac{2L^2(-1)^{k+1}}{k\pi} + \frac{4L^2\big[(-1)^k - 1\big]}{k^3\pi^3}\,. \end{aligned}$$

Therefore, the sine series of f is

$$\sum_{k=1}^{\infty}\Big(\frac{2L^2(-1)^{k+1}}{k\pi} + \frac{4L^2\big[(-1)^k - 1\big]}{k^3\pi^3}\Big)\sin\frac{k\pi x}{L}\,.$$

Since $\Big\{\phi_\ell \,\Big|\, \phi_\ell(x) = \sin\frac{\ell\pi x}{p}\ell \in \mathbb{N}\Big\}$ are collection of odd functions and forms a complete orthogonal set on $[0,p]$, for an odd function f defined on $(-p,p)$, the sine series $\sum\limits_{k=1}^{n} s_k\phi_k$ of the restriction of f on $(0,p)$ has the properties that

$$\lim_{n\to\infty}\Big\|f - \sum_{k=1}^{n} s_k\phi_k\Big\|_{L^2(0,\pi)} = 0\,;$$

thus the fact that $\big\|f - \sum\limits_{k=1}^{n} s_k\phi_k\big\|_{L^2(-\pi,\pi)} = 2\big\|f - \sum\limits_{k=1}^{n} s_k\phi_k\big\|_{L^2(0,\pi)}$ implies that

$$\lim_{n\to\infty}\Big\|f - \sum_{k=1}^{n} s_k\phi_k\Big\|_{L^2(-\pi,\pi)} = 0\,.$$

In other words, the sine series can be used to approximate odd functions on a symmetric interval. Similarly, the cosine series can be used to approximate even functions on a symmetric interval; thus we have the following

Definition 11.8. 1. The Fourier cosine series of an even function f defined on the interval $(-p,p)$ is

$$\frac{c_0}{2} + \sum_{k=1}^{\infty} c_k\cos\frac{k\pi x}{p}$$

whenever the sum makes sense, where $c_k = \dfrac{2}{p}\displaystyle\int_0^p f(x)\cos\frac{k\pi x}{p}\,dx$.

2. The Fourier sine series of an odd function f defined on the interval $(-p,p)$ is

$$\sum_{k=1}^{\infty} s_k\sin\frac{k\pi x}{p}$$

whenever the sum makes sense, where $s_k = \dfrac{2}{p}\displaystyle\int_0^p f(x)\sin\frac{k\pi x}{p}\,dx$.

Even though the convergence behavior of the cosine and sine series is in the sense of L^2-norm, we still have some pointwise convergence result.

Theorem 11.9. *Let $f:(0,p)\to\mathbb{R}$ be a piecewise continuous function such that f', which exists everywhere except possibly at finitely many points, be piecewise continuous on $(0,p)$. Then*

$$\frac{f(x^+)+f(x^-)}{2}=\frac{c_0}{2}+\sum_{k=1}^{\infty}c_k\cos\frac{k\pi x}{p}=\sum_{k=1}^{\infty}s_k\sin\frac{k\pi x}{p}\qquad\forall\,x\in(0,p)\,,$$

where $\frac{c_0}{2}+\sum\limits_{k=1}^{\infty}c_k\cos\frac{k\pi x}{p}$ and $\sum\limits_{k=1}^{\infty}s_k\sin\frac{k\pi x}{p}$ are the cosine and the sine series for f, respectively, and $f(x^{\pm})=\lim\limits_{h\to0^{\pm}}f(x+h)$ is the right/left limit of f at x. Moreover,

$$\frac{c_0}{2}+\sum_{k=1}^{\infty}c_k=\lim_{x\to0^+}f(x)\qquad\text{and}\qquad\frac{c_0}{2}+\sum_{k=1}^{\infty}c_k(-1)^k=\lim_{x\to p^-}f(x)\,.$$

Example 11.10. From Example 11.7, the cosine series of $f:(0,L)\to\mathbb{R}$ given by $f(x)=x^2$ is

$$\frac{L^2}{3}+\sum_{k=1}^{\infty}\frac{4L^2(-1)^k}{k^2\pi^2}\cos\frac{k\pi x}{L}\,.$$

Theorem 11.9 then implies that

$$\begin{aligned}L^2&=\lim_{x\to L^-}f(x)=\frac{L^2}{3}+\sum_{k=1}^{\infty}\frac{4L^2(-1)^k}{k^2\pi^2}\cos k\pi=\frac{L^2}{3}+\frac{4L^2}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\,,\\ 0&=\lim_{x\to0^+}f(x)=\frac{L^2}{3}+\sum_{k=1}^{\infty}\frac{4L^2(-1)^k}{k^2\pi^2}=\frac{L^2}{3}+\frac{4L^2}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\,,\end{aligned}$$

and

$$\begin{aligned}\frac{L^2}{4}&=\frac{1}{2}\Big[\lim_{x\to(L/2)^+}f(x)+\lim_{x\to(L/2)^-}f(x)\Big]=\frac{L^2}{3}+\sum_{k=1}^{\infty}\frac{4L^2(-1)^k}{k^2\pi^2}\cos\frac{k\pi}{2}\\ &=\frac{L^2}{3}+\sum_{k=1}^{\infty}\frac{4L^2(-1)^{2k}}{(2k)^2\pi^2}\cos\frac{2k\pi}{2}=\frac{L^2}{3}+\frac{L^2}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\,.\end{aligned}$$

Therefore, $\sum\limits_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$ and $\sum\limits_{k=1}^{\infty}\frac{(-1)^k}{k^2}=-\frac{\pi^2}{12}$.

Example 11.11. Let $f:(0,\pi)\to\mathbb{R}$ be defined by

$$f(x)=\begin{cases}x & \text{if } 0<x<\pi/2\,,\\ 0 & \text{if } \pi/2\leqslant x\leqslant\pi\,.\end{cases}$$

Then if $k \in \mathbb{N}$,

$$\begin{aligned}\frac{2}{\pi}\int_0^{\frac{\pi}{2}} x \sin\frac{k\pi x}{\pi}\,dx &= \frac{2}{\pi}\Big[-\frac{\pi x}{k\pi}\cos\frac{k\pi x}{\pi}\Big|_{x=0}^{x=\frac{\pi}{2}} + \frac{\pi}{k\pi}\int_0^{\frac{\pi}{2}}\cos\frac{k\pi x}{\pi}\,dx\Big] \\ &= -\frac{1}{k}\cos\frac{k\pi}{2} + \frac{2}{k^2\pi}\sin\frac{k\pi}{2}\,.\end{aligned}$$

Therefore, the sine series of f is

$$\sum_{k=1}^{\infty}\Big(-\frac{1}{k}\cos\frac{k\pi}{2} + \frac{2}{k^2\pi}\sin\frac{k\pi}{2}\Big)\sin kx$$

and Theorem 11.9 implies that

$$\begin{aligned}\frac{\pi}{4} &= \frac{1}{2}\Big[\lim_{x\to\frac{\pi}{2}^+} f(x) + \lim_{x\to\frac{\pi}{2}^-} f(x)\Big] = \sum_{k=1}^{\infty}\Big(-\frac{1}{k}\cos\frac{k\pi}{2} + \frac{2}{k^2\pi}\sin\frac{k\pi}{2}\Big)\sin\frac{k\pi}{2} \\ &= \sum_{k=1}^{\infty}\frac{2}{k^2\pi}\sin^2\frac{k\pi}{2} = \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1-\cos k\pi}{k^2} = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\end{aligned}$$

which shows that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{\pi^2}{8}\,.$$

11.3 Fourier Series

In the previous section, we introduce the cosine and sine series of functions defined on the interval $(0,p)$ or the cosine/sine series of even/odd functions define on the interval $(-p,p)$. Since every function $f:(-p,p)\to\mathbb{R}$ can be written as the sum of an even function and an odd function, or to be more precise,

$$f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2} = f_{\text{even}}(x) + f_{\text{odd}}(x)\,,$$

we expect that a general function $f:(-p,p)\to\mathbb{R}$ can be approximated by linear combinations of sine and cosines.

Note that the cosine and the sine series for f_{even} and f_{odd} are given respectively by

$$\frac{c_0}{2} + \sum_{k=1}^{\infty} c_k\cos\frac{k\pi x}{p} \qquad\text{and}\qquad \sum_{k=1}^{\infty} s_k\sin\frac{k\pi x}{p}$$

where

$$c_k = \frac{2}{p}\int_0^p f_{\text{even}}(x)\cos\frac{k\pi x}{p}\,dx = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{k\pi x}{p}\,dx$$

and

$$s_k = \frac{2}{p}\int_0^p f_{\rm odd}(x)\sin\frac{k\pi x}{p}\,dx = \frac{1}{p}\int_{-p}^p f(x)\sin\frac{k\pi x}{p}\,dx\,.$$

The discussion above motivates the following

Definition 11.12. The Fourier series of a function f defined on the interval $(-p,p)$ is

$$\frac{c_0}{2} + \sum_{k=1}^\infty \Big(c_k\cos\frac{k\pi x}{p} + s_k\sin\frac{k\pi x}{p}\Big)$$

whenever the sum makes sense, where

$$c_k = \frac{1}{p}\int_{-p}^p f(x)\cos\frac{k\pi x}{p}\,dx \qquad \text{and} \qquad s_k = \frac{1}{p}\int_{-p}^p f(x)\sin\frac{k\pi x}{p}\,dx$$

are called the Fourier coefficients of f.

Suppose that f is defined on $[0,p)$. The periodic extension of f is the function $F:\mathbb{R}\to\mathbb{R}$ satisfying $F=f$ on I and $F(x+p)=F(x)$ for all $x\in\mathbb{R}$. The Fourier series of F is called the Fourier series of f. Since

$$\begin{aligned}
\frac{1}{p}\int_{-p}^p F(x)\cos\frac{k\pi x}{p}\,dx &= \frac{1}{p}\Big[\Big(\int_{-p}^0 + \int_0^p\Big)F(x)\cos\frac{k\pi x}{p}\,dx\Big] \\
&= \frac{1}{p}\int_0^p F(x-p)\cos\frac{k\pi(x-p)}{p}\,dx + \frac{1}{p}\int_0^p F(x)\cos\frac{k\pi x}{p}\,dx \\
&= \frac{1+(-1)^k}{p}\int_0^p f(x)\cos\frac{k\pi x}{p}\,dx
\end{aligned}$$

and similarly,

$$\frac{1}{p}\int_{-p}^p F(x)\sin\frac{k\pi x}{p}\,dx = \frac{1+(-1)^k}{p}\int_0^p f(x)\sin\frac{k\pi x}{p}\,dx\,,$$

we find that c_k and s_k vanish if k is odd. This induces the following

Definition 11.13. The Fourier series of a function f defined on the interval $(0,p)$ is

$$\frac{c_0}{2} + \sum_{k=1}^\infty \Big(c_k\cos\frac{2k\pi x}{p} + s_k\sin\frac{2k\pi x}{p}\Big)\,,$$

whenever the sum makes sense, where

$$c_k = \frac{2}{p}\int_0^p f(x)\cos\frac{2k\pi x}{p}\,dx \qquad \text{and} \qquad s_k = \frac{2}{p}\int_0^p f(x)\sin\frac{2k\pi x}{p}\,dx\,.$$

We note that the Fourier series of f is the same as the Fourier series of the restriction of the periodic extension of f on $[-p/2, p/2)$.

Example 11.14. In Example 11.7 we compute the cosine and sine series of the function $f : [0, L) \to \mathbb{R}$ given by $f(x) = x^2$. Now we compute the Fourier series of f. From the computations in Example 11.7, we find that

$$c_0 = \frac{2L^2}{3} \qquad \text{and} \qquad c_k = \frac{L^2}{k^2\pi^2}\,,\ s_k = -\frac{L^2}{k\pi} \quad \text{if } k \in \mathbb{N}\,;$$

thus the Fourier series of f is given by

$$\frac{L^2}{3} + \frac{L^2}{\pi}\sum_{k=1}^{\infty}\Big(\frac{1}{k^2\pi}\cos\frac{2k\pi x}{L} - \frac{1}{k}\sin\frac{2k\pi x}{L}\Big)\,.$$

Similar to Theorem 11.9, we have the following

Theorem 11.15. *Let $f : [-p, p] \to \mathbb{R}$ be a piecewise continuous function such that f', which exists everywhere except possibly at finitely many points, be piecewise continuous on $[-p, p]$. Then*

$$\frac{f(x^+) + f(x^-)}{2} = \frac{c_0}{2} + \sum_{k=1}^{\infty}\Big(c_k\cos\frac{k\pi x}{p} + s_k\sin\frac{k\pi x}{p}\Big) \qquad \forall\, x \in (-p, p)\,,$$

where $f(x^+) = \lim\limits_{h\to 0^+} f(x+h)$ and $f(x^-) = \lim\limits_{h\to 0^-} f(x+h)$ are the right limit and the left limit of f at x, respectively. Moreover,

$$\frac{f(p^-) + f((-p)^+)}{2} = \frac{c_0}{2} + \sum_{k=1}^{\infty}(-1)^k c_k\,;$$

that is, the evaluation of the Fourier series at p and $-p$ are the same and is the average of the left limit of f at p and the right limit of f at $-p$.

Example 11.16. Consider the function

$$f(x) = \begin{cases} 0 & -\pi < x < 0\,, \\ \pi - x & 0 \leqslant x < \pi\,. \end{cases}$$

We compute the Fourier coefficients as follows. For $k \in \mathbb{N}$,

$$s_k = \frac{1}{\pi}\int_0^{\pi}(\pi - x)\sin(kx)\,dx = \frac{1}{\pi}\Big[\frac{-(\pi - x)\cos(kx)}{k}\Big|_{x=0}^{x=\pi} - \frac{1}{k}\int_0^{\pi}\cos(kx)\,dx\Big] = \frac{1}{k}$$

and

$$\begin{aligned} c_k &= \frac{1}{\pi}\int_0^\pi (\pi - x)\cos(kx)\,dx = \frac{1}{\pi}\Big[\frac{(\pi - x)\sin(kx)}{k}\Big|_{x=0}^{x=\pi} + \frac{1}{k}\int_0^\pi \sin(kx)\,dx\Big] \\ &= \frac{-\cos(kx)}{k^2\pi}\Big|_{x=0}^{x=\pi} = \frac{1-(-1)^k}{k^2\pi}\,, \end{aligned}$$

while

$$c_0 = \frac{1}{\pi}\int_0^\pi (\pi - x)\,dx = \frac{\pi}{2}\,.$$

Therefore, Theorem 11.15 implies that

$$\frac{\pi}{4} + \sum_{k=1}^\infty \Big(\frac{1-(-1)^k}{k^2\pi}\cos(kx) + \frac{1}{k}\sin(kx)\Big) = \left\{\begin{array}{cl} 0 & \text{if } -\pi \leqslant x < 0\,, \\ \pi - x & \text{if } 0 < x \leqslant \pi\,, \\ \dfrac{\pi}{2} & \text{if } x = 0\,. \end{array}\right.$$

We note that the case $x = 0$ implies that

$$\frac{\pi}{2} = \frac{\pi}{4} + \sum_{k=1}^\infty \frac{1-(-1)^k}{k^2\pi}$$

which shows the identity

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \sum_{k=1}^\infty \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}\,.$$

We also note that the identity above can be obtained by

$$\sum_{k=1}^\infty \frac{1}{k^2} = \sum_{k=1}^\infty \frac{1}{(2k-1)^2} + \sum_{k=1}^\infty \frac{1}{(2k)^2} = \sum_{k=1}^\infty \frac{1}{(2k-1)^2} + \frac{1}{4}\sum_{k=1}^\infty \frac{1}{k^2}$$

so that

$$\sum_{k=1}^\infty \frac{1}{(2k-1)^2} = \frac{3}{4}\sum_{k=1}^\infty \frac{1}{k^2} = \frac{3}{4}\cdot\frac{\pi^2}{6} = \frac{\pi^2}{8}\,.$$

• **Gibbs phenomenon**: When f has a jump discontinuity at some point x_0, the Fourier series of f behaves "strangely" near x_0. In fact, under the condition in Theorem 11.15, for a jump discontinuity x_0 of f (which means $f(x_0^+) \neq f(x_0^-)$) we have

$$\begin{aligned} \lim_{n\to\infty} S_n(x_0 + \frac{p}{n}) &= f(x_0^+) + c\big[f(x_0^+) - f(x_0^-)\big]\,, \\ \lim_{n\to\infty} S_n(x_0 - \frac{p}{n}) &= f(x_0^-) - c\big[f(x_0^+) - f(x_0^-)\big]\,, \end{aligned}$$

where $S_n(x)$ is the n-th partial sum of the Fourier series of f; that is,

$$S_n(x) = \frac{c_0}{2} + \sum_{k=1}^{n} \big(c_k \cos kx + s_k \sin kx\big)\,,$$

and c is the constant $c = \dfrac{1}{\pi}\displaystyle\int_0^\pi \frac{\sin x}{x}\,dx - \frac{1}{2} \approx 0.089490....$

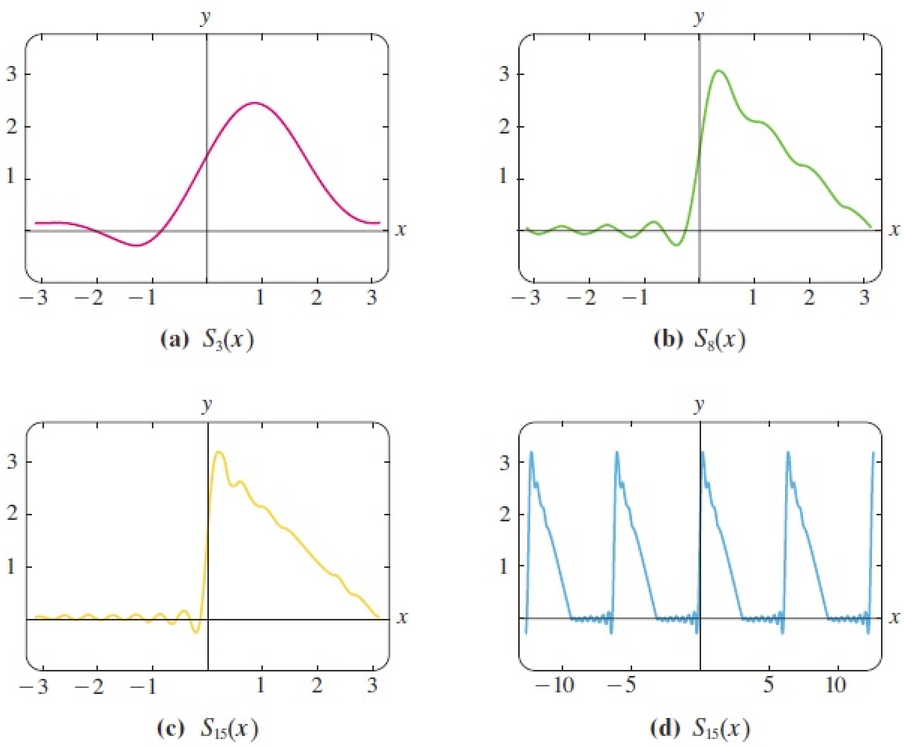


Figure 11.1: Partial sums of the Fourier series of f given in Example 11.16.

We note that the presence of the Gibbs phenomenon does not violate the fact that

$$\lim_{n\to\infty} \|f - S_n\|_{L^2(-p,p)} = 0\,.$$

11.4 Sturm-Liouville Problem

The regular Sturm-Liouville problem is the eigenvalue problem

$$\frac{d}{dx}\big[r(x)y'\big] + \big[q(x) + \lambda p(x)\big]y = 0 \qquad x \in (a,b)\,, \tag{11.9a}$$

$$A_1 y(a) + B_1 y'(a) = 0\,, \tag{11.9b}$$

$$A_2 y(b) + B_2 y'(b) = 0\,, \tag{11.9c}$$

where $(A_1, B_1) \neq (0,0)$, $(A_2, B_2) \neq (0,0)$ are given constant vectors, p, q, r are given real-valued continuous functions on $[a, b]$ satisfying

1. $p, r > 0$ on $[a, b]$, 2. r' is continuous on $[a, b]$,

and λ and y are unknowns to be solved. If there exists $\lambda \in \mathbb{R}$ and non-trivial y satisfying (11.9), λ is called an eigenvalue and y is called an eigenfunction corresponding to th eigenvalue λ.

Theorem 11.17. *Consider the regular Sturm-Liouville problem.*

1. *There exist an infinite number of real eigenvalues that can be arranged in increasing order* $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ *such that* $\lambda_n \to \infty$ *as* $n \to \infty$
2. *Each eigenvalue corresponds to only one eigenfunction (except for nonzero constant multiples).*
3. *Eigenfunctions corresponding to different eigenvalues are linearly independent.*
4. *The set of eigenfunctions corresponding to the set of eigenvalues is orthogonal with respect to the weighted inner product*

$$\langle f, g \rangle \equiv \int_a^b p(x) f(x) g(x)\, dx \,.$$

Proof of 4. Let y_m and y_n be eigenfunctions corresponding to eigenvalues λ_m and λ_n, respectively. Then

$$\frac{d}{dx}\big[r(x) y_m'\big] + \big[q(x) + \lambda_m p(x)\big] y_m = 0 \,, \tag{11.10a}$$

$$\frac{d}{dx}\big[r(x) y_n'\big] + \big[q(x) + \lambda_n p(x)\big] y_n = 0 \,. \tag{11.10b}$$

Multiplying (11.10a) by y_n and (11.10b) by y_m and subtracting the two resulting equations, we find that

$$(\lambda_m - \lambda_n) p(x) y_m y_n = y_m \frac{d}{dx}\big[r(x) y_n'\big] - y_n \frac{d}{dx}\big[r(x) y_m'\big] \,;$$

thus integrating by parts implies that

$$\begin{aligned} &(\lambda_m - \lambda_n) \int_a^b p(x) y_m(x) y_n(x)\, dx \\ &= \int_a^b \Big(y_m(x) \frac{d}{dx}\big[r(x) y_n'(x)\big] - y_n(x) \frac{d}{dx}\big[r(x) y_m'(x)\big] \Big)\, dx \\ &= r(x)\big[y_m(x) y_n'(x) - y_n(x) y_m'(x)\big]\Big|_{x=a}^{x=b} \,. \end{aligned} \tag{11.11}$$

Since y_m and y_n satisfy the boundary conditions

$$\begin{aligned} A_1 y_m(a) + B_1 y_m'(a) = 0\,, \\ A_1 y_n(a) + B_1 y_n'(a) = 0\,, \end{aligned}$$

or equivalently,

$$\begin{bmatrix} y_m(a) & y_m'(a) \\ y_n(a) & y_n'(a) \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

by the fact $(A_1, B_1) \neq (0,0)$ we must have $y_m(a)y_n'(a) - y_n(a)y_m'(a) = 0$. Similary, $y_m(b)y_n'(b) - y_n(b)y_m'(b) = 0$; thus (11.11) implies that

$$(\lambda_m - \lambda_n)\int_a^b p(x) y_m(x) y_n(x)\,dx = 0\,. \qquad \square$$

Chapter 12

Boundary-Value Problems in Rectangular Coordinates

Definition 12.1. A partial differential equation (PDE) is an equation which imposes relations between the various partial derivatives of a multi-variable function. A PDE is said to be linear if it is linear in the unknown and its derivatives, and is said to be nonlinear if it is not linear.

In the following, we focus on linear PDEs with two independent variables (sometimes two spatial variables x, y, and sometimes one temporal variable t and one spatial variable x).

12.1 Separable Partial Differential Equations

In this section, we are interested in finding solutions of a linear second-order PDE given by

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G\,,$$

where A, B, C, D, E, F, G are given constants. The PDE above is said to be homogeneous if $G \equiv 0$.

12.1.1 Separation of variables

In the method of separation variables we look for a particular solution of the form of a product of a function of x and y so that $u(x, y) = X(x)Y(y)$. With this assumption it is sometimes possible to reduce a linear PDE in two variables to two ODEs.

Example 12.2. Find product solutions of $\dfrac{\partial^2 u}{\partial x^2} = 4\dfrac{\partial u}{\partial y}$.

Suppose that $u(x, y) = X(x)Y(y)$. Then $X''Y = 4XY'$ so that

$$\frac{X''}{4X} = \frac{Y'}{Y}\,.$$

Since the left-hand side is a function of x and independent of y, while the right-hand side is a function of y and is independent of x, we must have

$$\frac{X''}{4X} = \frac{Y'}{Y} = -\lambda$$

for some constant $\lambda \in \mathbb{R}$. Therefore, $X'' + 4\lambda X = 0$ and $Y' + \lambda Y = 0$.

1. If $\lambda = 0$, then $X(x) = ax + b$ and $Y(y) = c$ for some constants a, b, c. Therefore, $u(x, y) = Ax + B$ for some constants A, B.

2. If $\lambda < 0$, then $\lambda = -\alpha^2$ for some $\alpha \in \mathbb{R}$ so that
$$X(x) = C_1 e^{2\alpha x} + C_2 e^{-2\alpha x}\,, \quad Y(y) = C_3 e^{\alpha^2 y}\,.$$
Therefore, $u(x, y) = Ae^{2\alpha x + \alpha^2 y} + Be^{-2\alpha x + \alpha^2 y}$ for some constants A, B.

3. If $\lambda > 0$, then $\lambda = \alpha^2$ for some $\alpha \in \mathbb{R}$ so that
$$X(x) = C_1 \cos(2\alpha x) + C_2 \sin(2\alpha x)\,, \quad Y(y) = C_3 e^{-\alpha^2 y}\,.$$
Therefore, $u(x, y) = e^{-\alpha^2 y}\big[A\cos(2\alpha x) + B\sin(2\alpha x)\big]$ for some constants A, B.

Theorem 12.3 (Superposition Principle)**.** *If $u_1, u_2, \cdots, u_k$ are solutions of a homogeneous linear PDE, then the linear combination*

$$u = c_1 u_1 + c_2 u_2 + \cdots + c_k u_k\,,$$

where $c_k' s$ are constants, is also a solution.

Throughout the remainder of the chapter we shall assume that whenever we have an infinite set $u_1, u_2, \cdots$ of solutions of a homogeneous linear equation, we can construct another solution u by forming the infinite series

$$u = \sum_{k=1}^{\infty} c_k u_k\,,$$

where $c_k' s$ are constants.

12.1.2 Classification of equations

Definition 12.4. The linear second-order PDE

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G\,,$$

where A, B, C, D, E, F, G are real constants, is said to be

1. hyperbolic if $B^2 - 4AC > 0$,

2. parabolic if $B^2 - 4AC = 0$,

3. elliptic if $B^2 - 4AC < 0$.

12.2 Classical PDEs and Boundary-Value Problems

Let $\Omega \subseteq \mathbb{R}^n$ be an open set (with smooth boundary). For a real-valued function $u : \Omega \to \mathbb{R}$ and a vector-valued function $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ (so that $\boldsymbol{u} = (u_1, u_2, \cdots, u_n)$ for some real-valued functions $u_1, u_2, \cdots, u_n$), we define some important differential operators as follows:

$$\begin{aligned}
\nabla u &= \Big(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \cdots, \frac{\partial u}{\partial x_n}\Big)\,,\\
\mathrm{div}\boldsymbol{u} = \nabla \cdot \boldsymbol{u} &= \Big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n}\Big) \cdot (u_1, u_2, \cdots, u_n) = \sum_{k=1}^{n} \frac{\partial u_k}{\partial x_k}\,,\\
\Delta u = \mathrm{div}\nabla u = \nabla \cdot \nabla u &= \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_n^2}\,,\\
\Delta \boldsymbol{u} = \frac{\partial^2 \boldsymbol{u}}{\partial x_1^2} + \frac{\partial^2 \boldsymbol{u}}{\partial x_2^2} + \frac{\partial^2 \boldsymbol{u}}{\partial x_n^2} &= (\Delta u_1, \Delta u_2, \cdots, \Delta u_n)\,.
\end{aligned}$$

We remark that Δ is also denoted by ∇^2 in engineering applications (since $\Delta = \nabla \cdot \nabla$).

12.2.1 Heat equation

Suppose that you are interested in the temperature distribution of a body in space. Let Ω be the region that the body occupies, and $u(\boldsymbol{x}, t)$ be the temperature of the body at location $\boldsymbol{x}$ (in Ω and in Cartesian coordinate) and time t (which is always assume to larger than the initial time). With $\varrho(\boldsymbol{x})$ and $s(\boldsymbol{x})$ denoting the density and the specific heat of the body at location $\boldsymbol{x}$, respectively, and $\kappa(\boldsymbol{x}, t)$ denoting the thermal diffusivity at location $\boldsymbol{x}$ and time

t (the higher the thermal diffusivity, the faster the heat propagation), u (which is assumed to be quite smooth) must satisfy the heat equation

$$\varrho(\boldsymbol{x})s(\boldsymbol{x})\frac{\partial u}{\partial t}(\boldsymbol{x},t) = \operatorname{div}\big[\kappa(\boldsymbol{x},t)\nabla u(\boldsymbol{x},t)\big] \qquad \forall\, \boldsymbol{x}\in\Omega \text{ and } t>0\,.$$

With the presence of a heat source Q in the body $\Big($here $\displaystyle\int_U Q(\boldsymbol{x},t)\,dx$ denotes the rate of heat energy flows into the body through the region U for all $U \subseteq \Omega\Big)$, the heat equation is modified as

$$\varrho(\boldsymbol{x})s(\boldsymbol{x})\frac{\partial u}{\partial t}(\boldsymbol{x},t) = \operatorname{div}\big[\kappa(\boldsymbol{x},t)\nabla u(\boldsymbol{x},t)\big] + Q(\boldsymbol{x},t) \qquad \forall\, \boldsymbol{x}\in\Omega \text{ and } t>0\,.$$

To fully determine the temperature, an initial condition and some type of boundary conditions have to be imposed:

1. Initial condition: this describes the temperature distribution at a certain time $t = t_0$ (which is usually assume to be 0). In mathematical terms, it is

   $$u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) \qquad \forall\, \boldsymbol{x}\in\Omega$$

   for some given function u_0.

2. Boundary condition: the boundary conditions introduces some physical constraints that is imposed to the system.

   (a) Dirichlet boundary condition: the temperature on the boundary of Ω is given. In mathematical terms, it is

   $$u(\boldsymbol{x},t) = g(\boldsymbol{x},t) \qquad \forall\, \boldsymbol{x}\in\partial\Omega \text{ and } t>0$$

   for some given function g.

   (b) Neumann boundary condition: the normal derivative of the temperature on the boundary of Ω is given. In mathematical terms, it is

   $$\frac{\partial u}{\partial \mathbf{N}}(\boldsymbol{x},t) = g(x,t) \qquad \forall\, \boldsymbol{x}\in\partial\Omega \text{ and } t>0$$

   for some given function g, where $\dfrac{\partial u}{\partial \mathbf{N}}$ is the directional derivative of u in the outward pointing direction $\mathbf{N}$ $\Big($so that $\dfrac{\partial u}{\partial \mathbf{N}} = \nabla u\cdot\mathbf{N}\Big)$. When there is not heat energy that can flow in and out of the body (the case of insulation, $g \equiv 0$).

(c) Robin boundary condition: the normal derivative of the temperature is proportion to the difference of the temperature on the boundary. In mathematical terms, it is

$$\frac{\partial u}{\partial \mathbf{N}}(\boldsymbol{x},t) + hu(\boldsymbol{x},t) = g \qquad \forall\, \boldsymbol{x} \in \partial\Omega \text{ and } t > 0$$

for some constant $h > 0$ and g.

(d) Mixed type boundary condition: Different type of boundary conditions are imposed on different portion of the boundary. In mathematical terms, if $\partial\Omega$ is the disjoint union of Γ_1, Γ_2 and Γ_3, we have

$$\begin{aligned} u(\boldsymbol{x},t) &= g_1(\boldsymbol{x},t) \qquad \forall\, \boldsymbol{x} \in \Gamma_1 \text{ and } t > 0\,, \\ \frac{\partial u}{\partial \mathbf{N}}(\boldsymbol{x},t) &= g_2(\boldsymbol{x},t) \qquad \forall\, \boldsymbol{x} \in \Gamma_2 \text{ and } t > 0\,, \\ \frac{\partial u}{\partial \mathbf{N}}(\boldsymbol{x},t) + hu(\boldsymbol{x},t) &= g_3(\boldsymbol{x},t) \qquad \forall\, \boldsymbol{x} \in \Gamma_3 \text{ and } t > 0\,. \end{aligned}$$

The heat equation is the prototype of the parabolic equations.

12.2.2 Wave equation

Suppose that you are interested in the vibration of a string or a membrane (of a drum). Let $\Omega \subseteq \mathbb{R}^n$ be the region of interests, and $u(\boldsymbol{x},t)$ denotes the displacement of the point labeled $\boldsymbol{x}$ (which means we choose some way to "identify" the particles with points in Ω) and time t. With $\varrho(\boldsymbol{x})$ denoting the density of the particle labelled $\boldsymbol{x}$, under certain circumstances u (which is assumed to be quite smooth) satisfies

$$\varrho(\boldsymbol{x})\frac{\partial^2 u}{\partial t^2}(\boldsymbol{x},t) = \operatorname{div}\Big[\frac{T(\boldsymbol{x})}{\sqrt{1+|\nabla u(\boldsymbol{x},t)|^2}}\nabla u(\boldsymbol{x},t)\Big] \qquad \forall\, \boldsymbol{x} \in \Omega \text{ and } t > 0\,,$$

where ∇u is replaced by u_x is Ω is 1-dimensional (that is, in the case of string). If some force f that can affect the vibration is introduced into the system, the equation is modified as

$$\varrho(\boldsymbol{x})\frac{\partial^2 u}{\partial t^2}(\boldsymbol{x},t) = \operatorname{div}\Big[\frac{T(\boldsymbol{x})}{\sqrt{1+|\nabla u(\boldsymbol{x},t)|^2}}\nabla u(\boldsymbol{x},t)\Big] + f(\boldsymbol{x},t) \qquad \forall\, \boldsymbol{x} \in \Omega \text{ and } t > 0\,,$$

In the real world application, $|\nabla u| \ll 1$, and we assume that ϱ and T are independent of $\boldsymbol{x}$ so that the equation becomes

$$\varrho\frac{\partial^2 u}{\partial t^2}(\boldsymbol{x},t) = T\Delta u(\boldsymbol{x},t) + f(\boldsymbol{x},t) \qquad \forall\, \boldsymbol{x} \in \Omega \text{ and } t > 0\,.$$

To fully determine the displacement, two initial conditions and some type of boundary conditions have to be imposed.

1. Initial condition: this describes the displacement and the velocity at a certain time $t = t_0$ (which is usually assume to be 0). In mathematical terms, it is

$$u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), u_t(\boldsymbol{x}, 0) = u_1(\boldsymbol{x}) \qquad \forall\, \boldsymbol{x} \in \Omega$$

for some given functions u_0 and u_1.

2. Boundary condition: similar to the boundary conditions in the heat equations, one type of boundary conditions is imposed. We note that the Neumann boundary condition $\frac{\partial u}{\partial \mathbf{N}} = 0$ on $\partial\Omega$ corresponds to the unconstraint case; that is, the boundary of the string (or the membrane) is not fixed.

The wave equation is the prototype of the hyperbolic equations.

12.2.3 Laplace's equation/Poisson's equation

Laplace's equation in two and three dimensions occurs in time-independent problems involving potentials such as electrostatic, gravitational, and velocity in fluid mechanics. Moreover, a solution of Laplace's equation can also be interpreted as a steady-state temperature distribution of the heat equation. The Laplace/Poisson equation takes the form

$$\Delta u(\boldsymbol{x}) = \operatorname{div}\big[\nabla u(\boldsymbol{x})\big] = \nabla^2 u(\boldsymbol{x}) = f(\boldsymbol{x}) \qquad \forall\, \boldsymbol{x} \in \Omega\,,$$

where f is a given function. The equation above is called Laplace's equation if $f \equiv 0$ and is called Poisson's equation if f is not the zero function.

To fully determine the unknown function u, one type of boundary conditions has to be imposed. However, it does not require the initial condition since the problem is time-independent. The Laplace/Poisson equation is the prototype of the elliptic equations.

12.3 Heat Equation

In this section, we focus on solving the 1-d heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < L\,, t > 0\,, \tag{12.1a}$$

$$u(0, t) = u(L, t) = 0 \qquad t > 0\,, \tag{12.1b}$$

$$u(x, 0) = f(x) \qquad 0 < x < L\,. \tag{12.1c}$$

First, we use the method of separation of variables to find product solutions to $\dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}$. Suppose that $u(x,t) = X(x)T(t)$. Then

$$T'(t)X(x) = \alpha^2 T(t)X''(x)$$

which shows that there exists $\lambda \in \mathbb{R}$ such that

$$\frac{X''}{X} = \frac{T'}{\alpha^2 T} = -\lambda\,.$$

Clearly we have $T(t) = e^{-\lambda\alpha^2 t}$ (we ignore the constant). On the other hand, in order to solve X, we need boundary conditions for X. Because of (12.1b), we choose the boundary condition $X(0) = X(L) = 0$ so that in order to obtain non-trivial X, the discussion from the previous chapter implies that $\lambda = \dfrac{n^2\pi^2}{L^2}$ and $X(x) = \sin\dfrac{n\pi x}{L}$ for some $n \in \mathbb{N}$. The discussion above provides a product solution

$$u_n(x,t) = e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t} \sin\frac{n\pi x}{L}\,.$$

Next we look for a solution of the form $u(x,t) = \sum\limits_{n=1}^{\infty} A_n u_n(x,t)$ satisfying (12.1c). We note that this amounts to find a sequence $\{A_n\}_{n=1}^{\infty}$ such that

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\,.$$

We note that this is the same as finding the sine series of f so that we find that

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \qquad \forall\, n \in \mathbb{N}\,. \tag{12.2}$$

Therefore, the solution u of the 1-d heat equation (12.1) is

$$u(x,t) = \sum_{n=1}^{\infty}\Big(\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx\Big)e^{-\frac{\alpha^2 n^2\pi^2}{L^2}t}\sin\frac{n\pi x}{L}\,.$$

Remark 12.5. 1. The procedure above is also called solving the heat equation using the method of separation of variables.

2. The procedure above is not rigorous even if we assume that the infinite linear combination of u_n is also a solution. A more rigorous approach is stated as follows. For each

$t > 0$, the temperature $u(x,t)$ is a function of x, vanishes on the boundary $x = 0$ and $x = L$, so that $u(x,t)$ can be represented (pointwise) using the sine series (the cosine series and the Fourier series of u do not satisfy the vanishing boundary condition). Therefore, for each $t > 0$ the coefficients $\{s_n\}_{n=1}^{\infty}$ used to represent the function u is a function of t so that

$$u(x,t) = \sum_{n=1}^{\infty} A_n(t) \sin \frac{n\pi x}{L} .$$

Assume that the derivatives of u can be carried inside the infinite sum. Then

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} A_n'(t) \sin \frac{n\pi x}{L} , \qquad \frac{\partial^2 u}{\partial x^2}(x,t) = -\sum_{n=1}^{\infty} A_n(t) \frac{n^2\pi^2}{L^2} \sin \frac{n\pi x}{L}$$

so that

$$0 = \frac{\partial u}{\partial t}(x,t) - \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t) = \sum_{n=1}^{\infty} \Big[s_n'(t) - \frac{\alpha^2 n^2 \pi^2}{L^2} A_n(t) \Big] \sin \frac{n\pi x}{L} .$$

This implies that A_n satisfies the differential equation

$$A_n'(t) - \frac{\alpha^2 n^2 \pi^2}{L^2} A_n(t) = 0 , \tag{12.3}$$

and the initial condition for the differential equation above should satisfy

$$f(x) = \sum_{n=1}^{\infty} A_n(0) \sin \frac{n\pi x}{L} .$$

This implies that

$$A_n(0) = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx . \tag{12.4}$$

Solving ODE (12.3) with initial condition (12.4), we obtain $A_n(t)$ so that the solution u to (12.1) is determine.

3. For 1-d heat equation (12.1a) with other type of homogeneous boundary conditions, one should choose different complete orthogonal set to represent the solution u. The complete orthogonal set that we should use should obey the boundary conditions.

4. For the case with heat source, one should write the heat source in terms of the complete orthogonal basis and then group all the terms together. For example, if (12.1a) is modified as

$$\frac{\partial u}{\partial t}(x,t) = \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t) + q(x,t) , \qquad 0 < x < L , t > 0 , \tag{12.1a'}$$

here $q(x,t) = \sum\limits_{n=1}^{\infty} B_n(t)\sin\frac{n\pi x}{L}$, then we repeat the procedure in 2 and find that A_n should satisfy

$$A_n'(t) - \frac{\alpha^2 n^2 \pi^2}{L^2} A_n(t) = B_n(t)\,, \qquad A_n(0) = A_n\,.$$

The solution A_n can be found by the method of integrating factor. We note that with the presence of q, one cannot find a product solution satisfying (12.1a).

5. For the case with inhomogeneous boundary condition, one should first find a function $g(x,t)$ that satisfies the boundary condition, then form a new unknown function $v(x,t) = u(x,t) - g(x,t)$. Then v satisfies a heat equation (with a heat source) but v satisfies a homogeneous boundary condition (and probably a different initial condition). For example, consider the heat equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \alpha^2 \frac{\partial^2 u}{\partial x^2} && 0 < x < L\,, t > 0\,, \\ u(0,t) = a(t)\,, u(L,t) &= b(t) && t > 0\,, \\ u(x,0) &= f(x) && 0 < x < L\,. \end{aligned}$$

Let $g(x,t) = \frac{L-x}{L}a(t) + \frac{x}{L}b(t)$. Then $g(0,t) = a(t)$ and $g(L,t) = b(t)$ so that $v(x,t) = u(x,t) - g(x,t)$ satisfies

$$\begin{aligned} \frac{\partial v}{\partial t} &= \alpha^2 \frac{\partial^2 v}{\partial x^2} + g_t(x,t) && 0 < x < L\,, t > 0\,, \\ v(0,t) = 0\,, v(L,t) &= 0 && t > 0\,, \\ v(x,0) &= f(x) - g(x,0) && 0 < x < L\,. \end{aligned}$$

The heat equation above can be solved by procedure stated in 4.

12.4 Wave Equation

In this section, we focus on solving the 1-d wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < L\,, t > 0\,, \tag{12.5a}$$

$$u(0,t) = u(L,t) = 0 \qquad t > 0\,, \tag{12.5b}$$

$$u(x,0) = f(x)\,,\ \frac{\partial u}{\partial t}(x,0) = g(x) \qquad 0 < x < L\,. \tag{12.5c}$$

We first use the method of separation of variables to find product solutions to $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$. Suppose that $u(x,t) = X(x)T(t)$. Then

$$T''(t)X(x) = c^2 T(t)X''(x)$$

which shows that there exists $\lambda \in \mathbb{R}$ such that

$$\frac{X''}{X} = \frac{T''}{c^2 T} = -\lambda\,.$$

Because of (12.1b), we choose the boundary condition $X(0) = X(L) = 0$ so that in order to obtain non-trivial X, we must have $\lambda = \alpha^2 > 0$ so that

$$X(x) = C_1 \cos(\alpha x) + C_2 \sin(\alpha x)\,, \qquad X(0) = X(L) = 0\,.$$

The condition $X(0) = 0$ implies that $C_1 = 0$, and the condition $X(L) = 0$ further implies that $\alpha L = n\pi$ for some $n \in \mathbb{Z}$. This shows that $\lambda = \dfrac{n^2\pi^2}{L^2}$ and $X(x) = \sin \dfrac{n\pi x}{L}$ for some $n \in \mathbb{N}$. On the other hand, $T_n(t) = T(t)$ satisfies

$$T_n''(t) + \frac{c^2 n^2 \pi^2}{L^2} T_n(t) = 0$$

so that

$$T_n(t) = A_n \cos \frac{cn\pi t}{L} + B_n \sin \frac{cn\pi t}{L}\,.$$

Therefore, the solution $u(x,t) = \sum\limits_{n=1}^{\infty} \Big(A_n \cos \dfrac{cn\pi t}{L} + B_n \sin \dfrac{cn\pi t}{L}\Big) \sin \dfrac{n\pi x}{L}$. The exact value of A_n and B_n should follow from the initial condition (12.5c): since

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}$$

and (assuming that $\dfrac{\partial}{\partial t} \sum\limits_{n=1}^{\infty} = \sum\limits_{n=1}^{\infty} \dfrac{\partial}{\partial t}$)

$$g(x) = \frac{\partial u}{\partial t}(x,0) \text{ “}=\text{” } \sum_{n=1}^{\infty} \frac{cn\pi B_n}{L} \sin \frac{n\pi x}{L}\,,$$

thus

$$A_n = \frac{2}{L} \int_0^L f(y) \sin \frac{n\pi y}{L}\,dy \quad \text{and} \quad B_n = \frac{2}{cn\pi} \int_0^L g(y) \sin \frac{n\pi y}{L}\,dy\,.$$

As a summary, the solution to (12.5) is

$$u(x,t) = \sum_{n=1}^{\infty} \Big(\frac{2}{L} \int_0^L f(y) \sin \frac{n\pi y}{L}\,dy \cos \frac{cn\pi t}{L} + \frac{2}{cn\pi} \int_0^L g(y) \sin \frac{n\pi y}{L}\,dy \sin \frac{cn\pi t}{L} \Big) \sin \frac{n\pi x}{L}\,.$$

Remark 12.6. For inhomogeneous PDE or inhomogeneous boundary condition, see Remark 12.5.

12.5 Laplace's Equation

In this section, we first focus on solving the boundary-value problem

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0 \qquad 0 < x < a\,, 0 < y < b\,, \tag{12.6a}$$

$$\frac{\partial u}{\partial x}(0,y) = \frac{\partial u}{\partial x}(a,y) = 0 \qquad 0 < y < b\,, \tag{12.6b}$$

$$u(x,0) = 0\,,\; u(x,b) = f(x) \qquad 0 < x < a\,. \tag{12.6c}$$

We note that (12.6b) is a Neumann boundary condition $\dfrac{\partial u}{\partial \mathbf{N}} = 0$ on $\{0,a\} \times (0,b)$, while (12.6c) is a (inhomogeneous) Dirichlet boundary $u = 0$ on $(0,a) \times \{0\}$ and $u = f$ on $(0,a) \times \{b\}$. Therefore, (12.6) has mixed type boundary conditions.

Remark 12.7. You may ask if there is a non-trivial solution if f is the zero function. The answer is **No** because of the maximum principle which states that if $\Delta u = 0$ in Ω, then the maximum (and the minimum as well) of u cannot occur in the interior of Ω. That is why we have to consider the Laplace equation with inhomogeneous boundary condition right away (to obtain a non-trivial solution).

We again try to find product solutions $u(x,y) = X(x)Y(y)$ of (12.6). Using (12.6a), we obtain that $X''(x)Y(y) + X(x)Y''(y) = 0$; thus

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda$$

for some constant $\lambda \in \mathbb{R}$. From the boundary condition (12.6b), we first look for non-trivial X satisfying the boundary condition $X'(0) = X'(a) = 0$; that is, X is a solution to the BVP

$$X''(x) + \lambda X(x) = 0\,, \qquad X'(0) = X'(a) = 0\,.$$

From Example 11.5, we find that $\lambda = \dfrac{n^2\pi^2}{a^2}$ and $X(x) = \cos\dfrac{n\pi x}{a}$ for some $n \in \mathbb{N} \cup \{0\}$. The corresponding Y satisfies

$$Y''(y) - \frac{n^2\pi^2}{a^2}Y(y) = 0$$

which produces

$$Y(y) = \begin{cases} A_n \exp\left(\frac{n\pi y}{a}\right) + B_n \exp\left(-\frac{n\pi y}{a}\right) & \text{if } n \in \mathbb{N}\,, \\ A_0 y + B_0 & \text{if } n = 0\,. \end{cases}$$

Due to the boundary condition $u(x,0) = 0$, we expect that $Y(0) = 0$ so that $A_n + B_n = 0$ for $n \in \mathbb{N}$ and $A_0 = 0$; thus $Y(y) = \frac{1}{2}\left[\exp\left(\frac{n\pi y}{a}\right) - \exp\left(-\frac{n\pi y}{a}\right)\right] = \sinh\frac{n\pi y}{a}$ if $n \in \mathbb{N}$ or $Y(y) = A_0 y$ if $n = 0$ so that

$$u_n(x,y) = \begin{cases} \cos\frac{n\pi x}{a}\sinh\frac{n\pi y}{a} & \text{if } n \in \mathbb{N} \\ y & \text{if } n = 0\,. \end{cases}$$

Now we look for solution u to (12.6) of the form

$$u(x,y) = A_0 y + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}\,.$$

We note that such a u "should" have satisfied (12.6b) and $u(x,0) = 0$. Now we determine the coefficient A_n so that $u(x,b) = f(x)$ for $0 < x < a$. This amounts to find the cosine series of f so that f can be expressed as

$$f(x) = A_0 b + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}\,.$$

Therefore,

$$A_0 = \frac{1}{ab}\Big(\int_0^a f(x)\,dx\Big) \quad \text{and} \quad A_n = \frac{2}{a\sinh\frac{n\pi b}{a}}\int_0^a f(x)\cos\frac{n\pi x}{a}\,dx$$

so that

$$u(x,y) = \frac{1}{ab}\Big(\int_0^a f(x)\,dx\Big)y + \frac{2}{a}\sum_{n=1}^{\infty}\Big(\int_0^a f(x)\cos\frac{n\pi x}{a}\,dx\Big)\cos\frac{n\pi x}{a}\frac{\sinh\frac{n\pi y}{a}}{\sinh\frac{n\pi b}{a}}\,.$$

Remark 12.8. 1. For the Laplace equation

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0 \qquad & 0 < x < a\,, 0 < y < b\,, \\ u(0,y) = u(a,y) = 0 \qquad & 0 < y < b\,, \\ u(x,0) = 0\,,\ u(x,b) = f(x) \qquad & 0 < x < a\,. \end{aligned}$$

the method of separation of variables provides the solution

$$u(x,y)=\frac{2}{a}\sum_{n=1}^{\infty}\Big(\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx\Big)\sin\frac{n\pi x}{a}\frac{\sinh\frac{n\pi y}{a}}{\sinh\frac{n\pi b}{a}}\,.$$

2. The boundary-value problem

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2}(x,y)+\frac{\partial^2 u}{\partial y^2}(x,y)&=0 && 0<x<a\,,0<y<b\,, && (12.7a)\\
u(0,y)=F(y)\,,u(a,y)&=G(y) && 0<y<b\,, && (12.7b)\\
u(x,0)=f(x)\,,\ u(x,b)&=g(x) && 0<x<a\,. && (12.7c)
\end{aligned}$$

can be solved using of the superposition principle. First we find solutions to the following two problems

$$\begin{aligned}
\frac{\partial^2 v}{\partial x^2}(x,y)+\frac{\partial^2 v}{\partial y^2}(x,y)&=0 && 0<x<a\,,0<y<b\,, && (12.8a)\\
v(0,y)=v(a,y)&=0 && 0<y<b\,, && (12.8b)\\
v(x,0)=f(x)\,,\ v(x,b)&=g(x) && 0<x<a\,. && (12.8c)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 w}{\partial x^2}(x,y)+\frac{\partial^2 w}{\partial y^2}(x,y)&=0 && 0<x<a\,,0<y<b\,, && (12.9a)\\
w(0,y)=F(y)\,,w(a,y)&=G(y) && 0<y<b\,, && (12.9b)\\
w(x,0)=w(x,b)&=0 && 0<x<a\,. && (12.9c)
\end{aligned}$$

The solution u to (12.7) then can be written as $u=v+w$. Laplace's equation with mixed type boundary conditions can be solved in a similar fashion.

Remark 12.9. Consider the Poisson equation

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2}(x,y)+\frac{\partial^2 u}{\partial y^2}(x,y)&=f(x,y) && 0<x<a\,,0<y<b\,, && (12.10a)\\
u(0,y)=u(a,y)&=0 && 0<y<b\,, && (12.10b)\\
u(x,0)=u(x,b)&=0 && 0<x<a\,. && (12.10c)
\end{aligned}$$

For each $0<y<b$, write

$$u(x,y)=\sum_{n=1}^{\infty}A_n(y)\sin\frac{n\pi x}{a}\qquad\text{and}\qquad f(x,y)=\sum_{n=1}^{\infty}f_n(y)\sin\frac{n\pi x}{a}\,,$$

where $f_n(y) = \dfrac{2}{a}\displaystyle\int_0^a f(x,y)\sin\frac{n\pi x}{a}\,dx$. Then

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = \sum_{n=1}^{\infty}\Big[A_n''(y) - \frac{n^2\pi^2}{a^2}A_n(y)\Big]\sin\frac{n\pi x}{a}$$

which implies that A_n satisfies the differential equation

$$A_n''(y) - \frac{n^2\pi^2}{a^2}A_n(y) = f_n(y)\,,$$

and the variation of parameter formula provides a particular solution

$$\begin{aligned} Y_n(y) &= \frac{a}{n\pi}\Big[-\cosh\frac{n\pi y}{a}\int_0^y f_n(z)\sinh\frac{n\pi z}{a}\,dz + \sinh\frac{n\pi y}{a}\int_0^y f_n(z)\cosh\frac{n\pi z}{a}\,dz\Big] \\ &= \frac{a}{n\pi}\int_0^y f_n(z)\sinh\frac{n\pi(y-z)}{a}\,dz \end{aligned}$$

so that

$$A_n(y) = C_n\cosh\frac{n\pi y}{a} + S_n\sinh\frac{n\pi y}{a} + Y_n(y)\,.$$

The boundary condition (12.10c) implies that $A_n(0) = A_n(b) = 0$; thus $C_n = 0$ and $S_n = -\dfrac{Y_n(b)}{\sinh\frac{n\pi b}{a}}$. The computation above provides the solution

$$u(x,y) = \sum_{n=1}^{\infty}\Bigg[Y_n(y) - Y_n(b)\frac{\sinh\frac{n\pi y}{a}}{\sinh\frac{n\pi b}{a}}\Bigg]\sin\frac{n\pi x}{a} \tag{12.11}$$

Question: Is u given above really a solution to (12.10)?

Answer: No! In general we do not know if $Y_n \to 0$ as $n \to \infty$; thus the series given by (12.11) may not converge.

12.6 Non-homogeneous Boundary-Value Problems

For the heat and wave equations, see Remark 12.5.

12.7 Orthogonal Series Expansions

Consider the heat equation

$$\frac{\partial u}{\partial t}(x,t) = \alpha^2\frac{\partial^2 u}{\partial x^2} + q(x,t) \qquad 0 < x < L\,, t > 0\,, \tag{12.12a}$$

$$u(0,t) = 0\,,\ \frac{\partial u}{\partial x}\Big|_{x=L} = -hu(L,t) \qquad t > 0\,, \tag{12.12b}$$

$$u(x,0) = f(x) \qquad 0 < x < L\,, \tag{12.12c}$$

where $h > 0$ is a constant.

First we look for eigenfunctions v satisfying

$$\frac{d^2}{dx^2}v(x) = \lambda v(x)\,, \qquad v(0) = v'(L) + hv(L) = 0\,. \tag{12.13}$$

We note that with this boundary condition, $\dfrac{d^2}{dx^2}$ is "symmetric" since

$$\begin{aligned}(u'', v)_{L^2(0,L)} &= \int_0^L u''(x)v(x)\,dx = u'(x)v(x)\Big|_{x=0}^{x=L} - \int_0^L u'(x)v'(x)\,dx \\ &= u'(L)v(L) - u(L)v'(L) + \int_0^L u(x)v'(x)\,dx \\ &= -hu(L)v(L) + hu(L)v(L) + (u, v'')_{L^2(0,L)} = (u, v'')_{L^2(0,L)}\,.\end{aligned}$$

1. $\lambda = 0$: the eigenfunction takes the form $v(x) = Ax + B$, and the boundary condition implies that
$$B = 0 \qquad \text{and} \qquad A + h(AL + B) = 0\,;$$
thus $A = B = 0$.

2. $\lambda = \beta^2 > 0$: the eigenfunction takes the form $v(x) = Ae^{\beta x} + Be^{-\beta x}$, and the boundary condition implies that
$$A + B = 0 \qquad \text{and} \qquad A\beta e^{\beta L} - B\beta e^{-\beta L} + h\big(Ae^{\beta L} + Be^{-\beta L}\big) = 0\,;$$
thus $A = B = 0$.

3. $\lambda = -\beta^2 < 0$: the eigenfunction takes the form $v(x) = A\cos\beta x + B\sin\beta x$, and the boundary condition implies that
$$A = 0 \qquad \text{and} \qquad -A\beta\sin\beta L + B\beta\cos\beta L + h\big(A\cos\beta L + B\sin\beta L\big) = 0\,;$$
thus to obtain non-trivial solutions, $\beta\cos\beta L + h\sin\beta L = 0$ or
$$\tan\beta L = -\frac{\beta}{h}\,. \tag{12.14}$$
Equation (12.14) has infinitely many roots. Suppose that the positive roots of (12.14), in increasing order, are given by $\beta_1, \beta_2, \cdots, \beta_n, \cdots$. Then $\lambda = -\beta_n^2$ is an eigenvalue to the eigenvalue problem (12.13) and a corresponding eigenfunction is $v_n(x) = \sin\beta_n x$; thus we obtain a complete orthogonal set $\{v_n\}_{n=1}^\infty$.

We then express the solution to (12.12) as

$$u(x,t) = \sum_{n=1}^{\infty} A_n(t)\sin\beta_n x\,,$$

as well as

$$q(x,t) = \sum_{n=1}^{\infty} B_n(t)\sin\beta_n x\,, \qquad B_n(t) = \frac{1}{\|v_n\|^2_{L^2(0,L)}}\int_0^L q(x,t)\sin\beta_n x\,dx\,.$$

Here we note that

$$\|v_n\|^2_{L^2(0,L)} = \int_0^L \sin^2\beta_n x\,dx = \int_0^L \frac{1-\cos 2\beta_n x}{2}\,dx = \frac{L}{2} - \frac{\sin 2\beta_n L}{4\beta_n} = \frac{L}{2} + \frac{h}{2(h^2+\beta_n^2)}\,.$$

Then (12.12a) implies that A_n satisfies the differential equation

$$A_n'(t) + \alpha^2\beta_n^2 A_n(t) = B_n(t)$$

while (12.12b) implies that A_n satisfies the initial condition

$$A_n(0) = \frac{1}{\|v_n\|^2_{L^2(0,L)}}\int_0^L f(x)\sin\beta_n x\,dx\,.$$

Remark 12.10. When the PDE (especially the heat or wave equations) itself is homogeneous, one can always try the method of separation of variables by looking for production solution $u(x,t) = X(x)T(t)$ first. The procedure of solving for such an X is exactly the same as finding eigenfunctions of $\dfrac{d^2}{dx^2}$ subject to some homogeneous boundary conditions.

12.8 Higher-Dimensional Problems

In this section we consider the heat equation

$$\begin{aligned}
&\frac{\partial u}{\partial t}(x,y) = \alpha^2\Big[\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y)\Big] && 0<x<b\,, 0<y<c\,, t>0\,, && (12.15a)\\
&u(0,y,t) = u(b,y,t) = 0 && 0<y<c, t>0\,, && (12.15b)\\
&u(x,0,t) = u(x,c,t) = 0 && 0<x<b, t>0\,, && (12.15c)\\
&u(x,y,0) = f(x,y) && 0<x<b\,, 0<y<c\,. && (12.15d)
\end{aligned}$$

Let us first try the method of separation of variables; that is, we first look for product solutions of the form $u(x,y,t) = X(x)Y(y)T(t)$. Such X, Y, T must satisfy

$$X(x)Y(y)T'(t) = \alpha^2\Big[X''(x)Y(y)T(t) + X(x)Y''(y)T(t)\Big]$$

or equivalently,

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} + \frac{T'(t)}{\alpha^2 T(t)}\,.$$

Since the left-hand side of the equality above is a function of x, while the right-hand side of the equality above is a function of y and t, we must have

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} + \frac{T'(t)}{\alpha^2 T(t)} = -\lambda$$

for some constant $\lambda \in \mathbb{R}$. This further implies that

$$\frac{Y''(y)}{Y(y)} = \frac{T'(t)}{\alpha^2 T(t)} + \lambda = -\mu$$

for some constant $\mu \in \mathbb{R}$.

Because of boundary condition (12.15b) and (12.15c), we impose the boundary conditions $X(0) = X(b) = 0$ and $Y(0) = Y(c) = 0$; thus

$$\lambda = \frac{m^2\pi^2}{b^2} \quad \text{and} \quad X(x) = \sin\frac{m\pi x}{b} \qquad \forall\, m \in \mathbb{N}\,, \tag{12.16a}$$

$$\mu = \frac{n^2\pi^2}{c^2} \quad \text{and} \quad Y(y) = \sin\frac{n\pi y}{c} \qquad \forall\, n \in \mathbb{N}\,. \tag{12.16b}$$

Moreover,

$$T(t) = \exp\big[-\alpha^2(\lambda+\mu)t\big] = \exp\Big[-\alpha^2\pi^2\big(\frac{m^2}{b^2} + \frac{n^2}{c^2}\big)t\Big]\,.$$

Therefore,

$$u_{m,n}(x,y,t) = \exp\Big[-\alpha^2\pi^2\big(\frac{m^2}{b^2} + \frac{n^2}{c^2}\big)t\Big]\sin\frac{m\pi x}{b}\sin\frac{n\pi y}{c}$$

is a product solution for all $m, n \in \mathbb{N}$.

Remark 12.11. The collection of functions

$$\Big\{v(x,y)\,\Big|\,v(x,y) = \sin\frac{m\pi x}{b}\sin\frac{n\pi y}{c} \text{ for some } m,n \in \mathbb{N}\Big\}$$

are eigenfunctions of $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ on the square $\Omega \equiv [0,b]\times[0,c]$ with boundary condition

$$v = 0 \quad \text{on} \quad \partial\Omega\,.$$

In other words, there exists $\nu \in \mathbb{R}$ (here $\nu = \lambda + \mu$ given above by (12.16)) such that

$$\begin{aligned}
-\Delta v(x,y) &= \nu v(x,y) && \forall\,(x,y) \in \Omega\,,\\
v(x,y) &= 0 && \forall\,(x,y) \in \partial\Omega\,.
\end{aligned}$$

Having obtain product solution, now we assume that the solution u to (12.15) can be written as

$$u(x,y,t) = \sum_{m,n=1}^{\infty} A_{m,n} u_{m,n}(x,y,t) = \sum_{m,n=1}^{\infty} A_{m,n} \exp\Big[-\alpha^2\pi^2\big(\frac{m^2}{b^2} + \frac{n^2}{c^2}\big)t\Big] \sin\frac{m\pi x}{b} \sin\frac{n\pi y}{c}\,.$$

The initial condition (12.15d) implies that $A_{m,n}$ should satisfy

$$f(x,y) = \sum_{m,n=1}^{\infty} A_{m,n} \sin\frac{m\pi x}{b} \sin\frac{n\pi y}{c}$$

which shows that $A_{m,n}$ is given by

$$\begin{aligned}
A_{m,n} &= \frac{4}{bc}\int_{\Omega} f(x,y) \sin\frac{m\pi x}{b} \sin\frac{n\pi y}{c}\,dA\\
&= \frac{4}{bc}\int_0^b \Big(\int_0^c f(x,y) \sin\frac{m\pi x}{b} \sin\frac{n\pi y}{c}\,dy\Big)\,dx\\
&= \frac{4}{bc}\int_0^c \Big(\int_0^b f(x,y) \sin\frac{m\pi x}{b} \sin\frac{n\pi y}{c}\,dx\Big)\,dy\,.
\end{aligned}$$

Chapter 14

Integral Transforms

14.1 Error Function

The error function erf and complementary error function erfc are, respectively, defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt \qquad \text{and} \qquad \operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\,dt\,.$$

We have the following properties:

1. $\lim_{x\to\infty} \operatorname{erf}(x) = 1$, $\lim_{x\to-\infty} \operatorname{erf}(x) = -1$.

2. $\lim_{x\to\infty} \operatorname{erfc}(x) = 0$, $\lim_{x\to-\infty} \operatorname{erfc}(x) = 2$.

3. $\dfrac{1}{\sqrt{2\pi\sigma^2}} \displaystyle\int_p^q \exp\big(-\frac{t^2}{2\sigma^2}\big)\,dt = \frac{1}{2}\Big[\operatorname{erf}\big(\frac{q}{\sqrt{2\sigma^2}}\big) - \operatorname{erf}\big(\frac{p}{\sqrt{2\sigma^2}}\big)\Big]$ for all $p, q \in \mathbb{R}$.

$f(t), a > 0$	$\mathscr{L}[f(t)] = F(s)$	$f(t), a > 0$	$\mathscr{L}[f(t)] = F(s)$
$\dfrac{1}{\sqrt{\pi t}} e^{-\frac{a^2}{4t}}$	$\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}}$	$2\sqrt{\dfrac{t}{\pi}} e^{-\frac{a^2}{4t}} - a\operatorname{erfc}\big(\dfrac{a}{2\sqrt{t}}\big)$	$\dfrac{e^{-a\sqrt{s}}}{s\sqrt{s}}$
$\dfrac{a}{2\sqrt{\pi t^3}} e^{-\frac{a^2}{4t}}$	$e^{-a\sqrt{s}}$	$e^{ab} e^{b^2 t} \operatorname{erfc}\big(b\sqrt{t} + \dfrac{a}{2\sqrt{t}}\big)$	$\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}(\sqrt{s} + b)}$
$\operatorname{erfc}\big(\dfrac{a}{2\sqrt{t}}\big)$	$\dfrac{e^{-a\sqrt{s}}}{s}$	$-e^{ab} e^{b^2 t} \operatorname{erfc}\big(b\sqrt{t} + \dfrac{a}{2\sqrt{t}}\big) + \operatorname{erfc}\big(\dfrac{a}{2\sqrt{t}}\big)$	$\dfrac{b e^{-a\sqrt{s}}}{s(\sqrt{s} + b)}$

Table 14.1: Laplace transform of some functions

Now we verify formula 1-3 given in Table 14.1. Let $f_1(t) = \dfrac{1}{\sqrt{\pi t}} e^{-\frac{a^2}{4t}}$ and $F_1(s) = \mathscr{L}[f_1](s)$. By the substitution of variable $t = \dfrac{au}{2\sqrt{s}}$,

$$\begin{aligned} F_1(s) &= \int_0^\infty \frac{1}{\sqrt{\pi t}} e^{-\frac{a^2}{4t}} e^{-st}\, dt = \int_0^\infty \frac{1}{\sqrt{\pi}} \frac{\sqrt{2}s^{\frac{1}{4}}}{\sqrt{au}} \exp\Big[-\frac{a\sqrt{s}}{2}\big(u+\frac{1}{u}\big)\Big] \frac{adu}{2\sqrt{s}} \\ &= \frac{\sqrt{a}}{\sqrt{2\pi}s^{\frac{1}{4}}} \int_0^\infty \frac{1}{\sqrt{u}} \exp\Big[-\frac{a\sqrt{s}}{2}\big(u+\frac{1}{u}\big)\Big]\, du\,. \end{aligned}$$

Replacing u by $\dfrac{1}{u}$, we find that

$$\int_0^\infty \frac{1}{\sqrt{u}} \exp\Big[-\frac{a\sqrt{s}}{2}\big(u+\frac{1}{u}\big)\Big]\, du = \int_0^\infty \frac{1}{\sqrt{u^3}} \exp\Big[-\frac{a\sqrt{s}}{2}\big(u+\frac{1}{u}\big)\Big]\, du\,;$$

thus

$$\int_0^\infty \frac{1}{\sqrt{u}} \exp\Big[-\frac{a\sqrt{s}}{2}\big(u+\frac{1}{u}\big)\Big]\, du = \frac{1}{2}\int_0^\infty \Big(\frac{1}{\sqrt{u}} + \frac{1}{\sqrt{u^3}}\Big) \exp\Big[-\frac{a\sqrt{s}}{2}\big(u+\frac{1}{u}\big)\Big]\, du\,.$$

By the substitution of variable $\sqrt{u} = v$, we find that

$$\int_0^\infty \frac{1}{\sqrt{u}} \exp\Big[-\frac{a\sqrt{s}}{2}\big(u+\frac{1}{u}\big)\Big]\, du = \int_0^\infty \big(1+\frac{1}{v^2}\big) \exp\Big[-\frac{a\sqrt{s}}{2}\big(v^2+\frac{1}{v^2}\big)\Big]\, dv$$

and further substitution $x = v - \dfrac{1}{v}$ shows that

$$\int_0^\infty \frac{1}{\sqrt{u}} \exp\Big[-\frac{a\sqrt{s}}{2}\big(u+\frac{1}{u}\big)\Big]\, du = \int_{-\infty}^\infty \exp\Big[-\frac{a\sqrt{s}}{2}(x^2+2)\Big]\, dx = e^{-a\sqrt{s}} \int_{-\infty}^\infty e^{-\frac{a\sqrt{s}x^2}{2}}\, dx\,;$$

thus by the fact that $\displaystyle\int_{-\infty}^\infty \exp\Big(-\frac{t^2}{2\sigma^2}\Big)\, dt = \sqrt{2\pi\sigma^2}$, we conclude that

$$\int_0^\infty \frac{1}{\sqrt{u}} \exp\Big[-\frac{a\sqrt{s}}{2}\big(u+\frac{1}{u}\big)\Big]\, du = \int_0^\infty \frac{1}{\sqrt{u^3}} \exp\Big[-\frac{a\sqrt{s}}{2}\big(u+\frac{1}{u}\big)\Big]\, du = \sqrt{\frac{2\pi}{a}} e^{-a\sqrt{s}} s^{-\frac{1}{4}}\,. \tag{14.1}$$

As a consequence,

$$F_1(s) = \frac{\sqrt{a}}{\sqrt{2\pi}s^{\frac{1}{4}}} \sqrt{\frac{2\pi}{a}} e^{-a\sqrt{s}} s^{-\frac{1}{4}} = \frac{e^{-a\sqrt{s}}}{\sqrt{s}}\,.$$

Next we compute the Laplace transform of the function $f_2(t) = \dfrac{a}{2\sqrt{\pi t^3}}\, e^{-\frac{a^2}{4t}}$. Again by the substitution of variable $t = \dfrac{au}{2\sqrt{s}}$,

$$\begin{aligned} \int_0^\infty \frac{a}{2\sqrt{\pi t^3}}\, e^{-\frac{a^2}{4t}} e^{-st}\, dt &= \int_0^\infty \frac{a}{2\sqrt{\pi}} \Big(\frac{2\sqrt{s}}{au}\Big)^{\frac{3}{2}} \exp\Big[-\frac{a\sqrt{s}}{2}\big(u+\frac{1}{u}\big)\Big] \frac{a}{2\sqrt{s}}\, du \\ &= \sqrt{\frac{a}{2\pi}} s^{\frac{1}{4}} \int_0^\infty \frac{1}{\sqrt{u^3}} \exp\Big[-\frac{a\sqrt{s}}{2}\big(u+\frac{1}{u}\big)\Big]\, du\,; \end{aligned}$$

thus (14.1) shows that

$$F_2(s) = \mathscr{L}[f_2](s) = \sqrt{\frac{a}{2\pi}} s^{\frac{1}{4}} \sqrt{\frac{2\pi}{a}} e^{-a\sqrt{s}} s^{-\frac{1}{4}} = e^{-a\sqrt{s}} .$$

Finally we compute the Laplace transform of the function $f_3(t) = \operatorname{erfc}\big(\frac{a}{2\sqrt{t}}\big)$. Integrating by parts,

$$\begin{aligned} F_3(s) = \mathscr{L}[f_3](s) &= \int_0^\infty \operatorname{erfc}\big(\frac{a}{2\sqrt{t}}\big) e^{-st}\,dt = \operatorname{erfc}\big(\frac{a}{2\sqrt{t}}\big)\frac{e^{-st}}{-s}\Big|_{t=0}^{t=\infty} + \frac{1}{s}\int_0^\infty e^{-st}\frac{d}{dt}\operatorname{erfc}\big(\frac{a}{2\sqrt{t}}\big)\,dt \\ &= \frac{1}{s}\int_0^\infty e^{-st}\Big(-\frac{2}{\sqrt{\pi}} e^{-\frac{a^2}{4t}}\Big)\frac{d}{dt}\frac{a}{2\sqrt{t}}\,dt = \frac{1}{s}\int_0^\infty \frac{a}{2\sqrt{\pi t^3}} e^{-\frac{a^2}{4t}} e^{-st}\,dt = \frac{1}{s}\mathscr{L}[f_2](s) = \frac{e^{-a\sqrt{s}}}{s} . \end{aligned}$$

14.2 Laplace Transform

Recall that if $f : [0,\infty) \to \mathbb{R}$ is a function, then the Laplace transform of f, denoted by $\mathscr{L}[f]$, is

$$\mathscr{L}[f](s) = \int_0^\infty f(t) e^{-st}\,dt$$

whenever the integral exists. Often time we write $F = \mathscr{L}[f]$. If u is a function of x (in some interval) and $t > 0$, then we define the Laplace transform of u by

$$U(x,s) = \mathscr{L}[u](x,s) \equiv \int_0^\infty u(x,t) e^{-st}\,dt .$$

Similar to the formula

$$\mathscr{L}[f'](s) = sF(s) - f(0) \qquad \text{and} \qquad \mathscr{L}[f''](s) = s^2F(s) - sf(0) - f'(0) ,$$

we have

$$\mathscr{L}\Big[\frac{\partial u}{\partial t}\Big](x,s) = sU(x,s) - u(x,0) , \tag{14.2a}$$

$$\mathscr{L}\Big[\frac{\partial^2 u}{\partial t^2}\Big](x,s) = s^2U(x,s) - su(x,0) - u_t(x,0) . \tag{14.2b}$$

Moreover,

$$\mathscr{L}\Big[\frac{\partial^2 u}{\partial x^2}\Big](x,s) = \int_0^\infty \frac{\partial^2 u}{\partial x^2}(x,t) e^{-st}\,dt \text{ ``$=$'' } \frac{\partial^2}{\partial x^2}\int_0^\infty u(x,t) e^{-st}\,dt = \frac{\partial^2 U}{\partial x^2}(x,s) . \tag{14.3}$$

Identities (14.2) and (14.3) are the key formula that we will use to solve the initial-boundary value problems.

Example 14.1. Consider the wave equation

$$
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2} && 0 < x < 1, t > 0\,, && (14.4a)\\
u(0,t) = u(1,t) &= 0 && t > 0\,, && (14.4b)\\
u(x,0) = 0\,,\ \frac{\partial u}{\partial t}\Big|_{t=0} &= \sin \pi x && 0 < x < 1\,. && (14.4c)
\end{aligned}
$$

Let $U(x,s) = \mathscr{L}[u(x,\cdot)](s)$. Using (14.2) and (14.3) we find that

$$
\frac{\partial^2}{\partial x^2} U(x,s) - s^2 U(x,s) = -\sin \pi x\,. \tag{14.5}
$$

A particular solution to the "ODE" above is

$$
\begin{aligned}
U_p(x,s) &= -\cosh sx \int_0^x \frac{-\sin \pi y \sinh sy}{W[\cosh sy, \sinh sy]}\, dy + \sinh sx \int_0^x \frac{-\sin \pi y \cosh sy}{W[\cosh sy, \sinh sy]}\, dy \\
&= \frac{\cosh sx}{s} \int_0^x \sin \pi y \sinh sy\, dy - \frac{\sinh sx}{s} \int_0^x \sin \pi y \cosh sy\, dy\,.
\end{aligned}
$$

Integrating by parts,

$$
\begin{aligned}
\int_0^x \sin \pi y \sinh sy\, dy &= \frac{-\cos \pi y \sinh sy}{\pi}\Big|_{y=0}^{y=x} + \frac{s}{\pi} \int_0^x \cos \pi y \cosh sy\, dy \\
&= \frac{-\cos \pi x \sinh sx}{\pi} + \frac{s}{\pi}\Big[\frac{\sin \pi y \cosh sy}{\pi}\Big|_{y=0}^{y=x} - \frac{s}{\pi} \int_0^x \sin \pi y \sinh sy\, dy\Big] \\
&= \frac{-\cos \pi x \sinh sx}{\pi} + \frac{s}{\pi^2} \sin \pi x \cosh sx - \frac{s^2}{\pi^2} \int_0^x \sin \pi y \sinh sy\, dy\,;
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^x \sin \pi y \cosh sy\, dy &= \frac{-\cos \pi y \cosh sy}{\pi}\Big|_{y=0}^{y=x} + \frac{s}{\pi} \int_0^x \cos \pi y \sinh sy\, dy \\
&= \frac{1-\cos \pi x \cosh sx}{\pi} + \frac{s}{\pi}\Big[\frac{\sin \pi y \sinh sy}{\pi}\Big|_{y=0}^{y=x} - \frac{s}{\pi} \int_0^x \sin \pi y \cosh sy\, dy\Big] \\
&= \frac{1-\cos \pi x \cosh sx}{\pi} + \frac{s}{\pi^2} \sin \pi x \sinh sx - \frac{s^2}{\pi^2} \int_0^x \sin \pi y \cosh sy\, dy\,;
\end{aligned}
$$

thus

$$
\begin{aligned}
\int_0^x \sin \pi y \sinh sy\, dy &= \frac{s \sin \pi x \cosh sx - \pi \cos \pi x \sinh sx}{s^2 + \pi^2}\,, \\
\int_0^x \sin \pi y \cosh sy\, dy &= \frac{s \sin \pi x \sinh sx - \pi \cos \pi x \cosh sx + \pi}{s^2 + \pi^2}\,.
\end{aligned}
$$

Therefore, a particular solution is given by

$$U_p(x) = \frac{s\sin \pi x - \pi \sinh sx}{s(s^2+\pi^2)}$$

so that the solution to (14.5) is

$$U(x,s) = C_1(s)\cosh sx + C_2(s)\sinh sx + \frac{\sin \pi x}{s^2+\pi^2}\,,$$

here the term $\dfrac{-\pi\sinh sx}{s(s^2+\pi^2)}$ in U_p is absorbed into $C_2(s)$. Note that the boundary condition (14.4b) implies that $U(0,s) = U(1,s) = 0$; thus $C_1(s) = C_2(s) = 0$. Therefore,

$$\mathscr{L}[u(x,\cdot)](s) = U(x,s) = \frac{\sin \pi x}{s^2+\pi^2}$$

Since $\dfrac{\pi}{s^2+\pi^2}$ is the Laplace transform of the function $f(t) = \sin \pi t$, we conclude that

$$u(x,t) = \frac{1}{\pi}\sin \pi t \sin \pi x\,.$$

Example 14.2. Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} - g \qquad x>0, t>0\,, \tag{14.6a}$$

$$u(0,t) = \lim_{x\to\infty}\frac{\partial u}{\partial x} = 0 \qquad t>0\,, \tag{14.6b}$$

$$u(x,0) = 0\,,\ \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \qquad x>0\,, \tag{14.6c}$$

where g is a constant (denoting the gravitational acceleration).

Let $U(x,s) = \mathscr{L}[u(x,\cdot)](s)$. Using (14.2) and (14.3) we find that

$$s^2 U(x,s) = c^2\frac{\partial^2}{\partial x^2}U(x,s) - \frac{g}{s}$$

so that U satisfies the "ODE"

$$\frac{\partial^2}{\partial x^2}U(x,s) - \frac{s^2}{c^2}U(x,s) = \frac{g}{c^2 s}\,. \tag{14.7}$$

A particular solution to the ODE above is $U_p(x,s) = -\dfrac{g}{s^3}$; thus the general solution to the ODE above is

$$U(x,s) = C_1(s)\exp\big(\frac{s}{c}x\big) + C_2(s)\exp\big(-\frac{s}{c}x\big) - \frac{g}{s^3}\,.$$

The boundary condition (14.6b) implies that $U(0,s)=0$ and $\lim\limits_{x\to\infty}\dfrac{\partial U}{\partial x}(x,s)=0$; thus

$$C_1(s)=\frac{g}{s^3} \qquad \text{and} \qquad C_2(s)=0\,.$$

Therefore, $U(x,s)=\dfrac{g}{s^3}\exp\left(-\dfrac{x}{c}s\right)-\dfrac{g}{s^3}$. Since $\dfrac{2}{s^3}$ is the Laplace transform of the function $f(t)=t^2$, by the fact that

$$\mathscr{L}[\mathbf{1}_{\{t>a\}}(t)f(t-a)](s)=e^{-sa}\mathscr{L}[f](s)$$

we conclude that

$$u(x,t)=\begin{cases} -\dfrac{1}{2}gt^2 & \text{if } 0\leqslant t<\dfrac{x}{c}\,,\\ -\dfrac{g}{2c^2}(2cxt-x^2) & \text{if } t\geqslant\dfrac{x}{c}\,.\end{cases}$$

Example 14.3. Consider the heat equation

$$\frac{\partial u}{\partial t}(x,t)=\frac{\partial^2 u}{\partial x^2}(x,t) \qquad 0<x<1, t>0\,, \tag{14.8a}$$

$$u(0,t)=0\,,\ u(1,t)=u_0 \qquad t>0\,, \tag{14.8b}$$

$$u(x,0)=0 \qquad 0<x<1\,, \tag{14.8c}$$

where u_0 is a given constant.

Let $U(x,s)=\mathscr{L}[u(x,\cdot)](s)$. Using (14.2) and (14.3) we find that

$$\frac{\partial^2}{\partial x^2}U(x,s)-sU(x,s)=0\,. \tag{14.9}$$

Moreover, the boundary condition (14.8b) implies that

$$U(0,s)=0 \qquad \text{and} \qquad U(1,s)=\mathscr{L}[u_0](s)=\frac{u_0}{s}\,. \tag{14.10}$$

Since the Laplace transform is always (assumed to be) defined on $s>\alpha$ for some $\alpha\in\mathbb{R}$, W.L.O.G. we can assume that $\alpha>0$ so that the general solution to the ODE (14.9) is

$$U(x,s)=C_1(s)\cosh\sqrt{s}x+C_2(s)\sinh\sqrt{s}x\,.$$

The boundary condition (14.10) shows that

$$C_1(s)=0 \qquad \text{and} \qquad C_2(s)=\frac{u_0}{s\sinh\sqrt{s}}\,;$$

thus

$$U(x,s) = \frac{u_0 \sinh \sqrt{s}x}{s \sinh \sqrt{s}} = u_0 \frac{e^{(x-1)\sqrt{s}} - e^{-(x+1)\sqrt{s}}}{s(1-e^{-2\sqrt{s}})}\,.$$

Since $s > 0$, the geometric series expansion implies that

$$\frac{1}{1-e^{-2\sqrt{s}}} = \sum_{n=0}^{\infty} e^{-2n\sqrt{s}}\,;$$

thus

$$U(x,s) = u_0 \frac{e^{(x-1)\sqrt{s}} - e^{-(x+1)\sqrt{s}}}{s} \sum_{n=0}^{\infty} e^{-2n\sqrt{s}} = u_0 \sum_{n=0}^{\infty} \Big[\frac{e^{-(2n+1-x)\sqrt{s}}}{s} - \frac{e^{-(2n+1+x)\sqrt{s}}}{s}\Big]\,.$$

Assuming that $\mathscr{L}^{-1}\sum\limits_{n=0}^{\infty} = \sum\limits_{n=0}^{\infty}\mathscr{L}^{-1}$, using formula 3 of Table 14.1 we obtain that

$$\begin{aligned} u(x,t) = \mathscr{L}^{-1}[U(x,\cdot)](t) &= u_0 \sum_{n=0}^{\infty} \Big\{\mathscr{L}^{-1}\Big[\frac{e^{-(2n+1-x)\sqrt{s}}}{s}\Big](t) - \mathscr{L}^{-1}\Big[\frac{e^{-(2n+1+x)\sqrt{s}}}{s}\Big](t)\Big\} \\ &= u_0 \sum_{n=0}^{\infty} \Big[\operatorname{erfc}\Big(\frac{2n+1-x}{2\sqrt{t}}\Big) - \operatorname{erfc}\Big(\frac{2n+1+x}{2\sqrt{t}}\Big)\Big] \\ &= u_0 \sum_{n=0}^{\infty} \Big[\operatorname{erf}\Big(\frac{2n+1+x}{2\sqrt{t}}\Big) - \operatorname{erf}\Big(\frac{2n+1-x}{2\sqrt{t}}\Big)\Big]\,. \end{aligned} \tag{14.11}$$

On the other hand, we can solve (14.8) using the method of separation of variables as follows. Let $g(x,t) = xu_0$ and $v(x,t) = u(x,t) - xu_0$. Then v satisfies that

$$\frac{\partial v}{\partial t}(x,t) = \frac{\partial^2 v}{\partial x^2}(x,t) \qquad 0 < x < 1, t > 0\,, \tag{14.12a}$$

$$v(0,t) = 0\,,\ v(1,t) = 0 \qquad t > 0\,, \tag{14.12b}$$

$$u(x,0) = -xu_0 \qquad 0 < x < 1\,, \tag{14.12c}$$

Because of the boundary condition (14.12b), the solution v can be expressed as

$$v(x,t) = \sum_{n=1}^{\infty} A_n(t)\sin(n\pi x)\,,$$

where A_n satisfies the ODE

$$A_n'(t) + n^2\pi^2 A_n(t) = 0$$

as well as the initial condition

$$A_n(0) = -2u_0 \int_0^1 x\sin(n\pi x)\,dx = -2u_0\Big[x\cdot\frac{\cos(n\pi x)}{-n\pi}\Big|_{x=0}^{x=1} + \int_0^1 \frac{\cos n\pi x}{n\pi}\,dx\Big] = \frac{2u_0(-1)^n}{n\pi}\,.$$

Therefore,

$$A_n(t) = \frac{2u_0(-1)^n}{n\pi} e^{-n^2\pi^2 t}$$

so that

$$v(x,t) = \frac{2u_0}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-n^2\pi^2 t} \sin n\pi x \,.$$

Therefore, the solution of (14.8) is given by

$$u(x,t) = xu_0 + \frac{2u_0}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-n^2\pi^2 t} \sin n\pi x \,. \tag{14.13}$$

Question: Are solutions to (14.8) provided by (14.11) and (14.13) the same? In other words, is there a unique solution to (14.8)?

Answer: The two expressions of solutions to (14.8) represent the unique solution of (14.8). In fact, if u_1 and u_2 are two solutions to (14.8), then $v = u_1 - u_2$ satisfies

$$\frac{\partial v}{\partial t}(x,t) = \frac{\partial^2 v}{\partial x^2}(x,t) \qquad 0 < x < 1, t > 0\,, \tag{14.14a}$$

$$v(0,t) = 0\,,\ v(1,t) = 0 \qquad t > 0\,, \tag{14.14b}$$

$$v(x,0) = 0 \qquad 0 < x < 1\,. \tag{14.14c}$$

Integrating $\dfrac{1}{2}\dfrac{\partial}{\partial t} v(x,t)^2$ on $[0,1]$, using (14.14a) we obtain that

$$\int_0^1 \frac{1}{2}\frac{\partial}{\partial t} v(x,t)^2\,dx = \int_0^1 v(x,t)v_t(x,t)\,dx = \int_0^1 v(x,t)v_{xx}(x,t)\,dx \qquad \forall\, t > 0\,.$$

Integrating by parts, we find that

$$\int_0^1 \frac{1}{2}\frac{\partial}{\partial t} v(x,t)^2\,dx = v(x,t)v_x(x,t)\Big|_{x=0}^{x=1} - \int_0^1 v_x(x,t)^2\,dx \qquad \forall\, t > 0\,,$$

and the boundary condition (14.14b) further shows that

$$\frac{1}{2}\int_0^1 \frac{\partial}{\partial t} v(x,t)^2\,dx = -\int_0^1 v(x,t)^2\,dx\,.$$

Assuming that the time derivative can be pulled out of the integral, we obtain that

$$\frac{1}{2}\frac{\partial}{\partial t}\int_0^1 v(x,t)^2\,dx = -\int_0^1 v(x,t)^2\,dx \leqslant 0\,;$$

thus the function $f(t) = \displaystyle\int_0^1 v(x,t)^2\,dt$ is a non-negative decreasing function. Since the initial condition (14.14c) implies that $f(0) = 0$, we conclude that $f(t) = 0$ for all $t > 0$; thus $v(x,t) = 0$ for all $x \in [0,1]$ and $t > 0$.

14.3 Fourier Transform

14.3.1 The Fourier transform, and the Fourier inversion formula

In this section, we extend the study of the Fourier series. Recall that if $f:(-p,p)\to\mathbb{R}$ is a "good" function, then

$$f(x)=\frac{c_0}{2}+\sum_{n=1}^{\infty}\Big(c_n\cos\frac{n\pi x}{p}+s_n\sin\frac{n\pi x}{p}\Big)\,,$$

where the Fourier coefficient of f is given by

$$c_n=\frac{1}{p}\int_{-p}^{p}f(y)\cos\frac{n\pi y}{p}\,dy\quad\forall\,n\in\mathbb{N}\cup\{0\}\quad\text{and}\quad s_n=\frac{1}{p}\int_{-p}^{p}f(y)\sin\frac{n\pi y}{p}\,dy\quad\forall\,n\in\mathbb{N}\,.$$

Using the Euler identity $e^{i\theta}=\cos\theta+i\sin\theta$,

$$\begin{aligned}
f(x)&=\frac{c_0}{2}+\sum_{n=1}^{\infty}\Big[\frac{c_n}{2}\Big(\exp\big(\frac{in\pi x}{p}\big)+\exp\big(-\frac{in\pi x}{p}\big)\Big)+\frac{s_n}{2i}\Big(\exp\big(\frac{in\pi x}{p}\big)-\exp\big(-\frac{in\pi x}{p}\big)\Big)\Big]\\
&=\frac{1}{2}\Big[c_0+\sum_{n=1}^{\infty}\Big((c_n-is_n)\exp\big(\frac{in\pi x}{p}\big)+(c_n+is_n)\exp\big(-\frac{in\pi x}{p}\big)\Big)\Big]\\
&=\frac{1}{2}\Big[c_0+\sum_{n=1}^{\infty}(c_n-is_n)\exp\big(\frac{in\pi x}{p}\big)+\sum_{n=-\infty}^{-1}(c_{-n}+is_{-n})\exp\big(\frac{in\pi x}{p}\big)\Big]\\
&=\frac{1}{2p}\int_{-p}^{p}f(y)\,dy+\sum_{n=1}^{\infty}\Big(\frac{1}{2p}\int_{-p}^{p}f(y)e^{-\frac{in\pi y}{p}}dy\Big)e^{\frac{in\pi x}{p}}+\sum_{n=-\infty}^{-1}\Big(\frac{1}{2p}\int_{-p}^{p}f(y)e^{-\frac{in\pi y}{p}}dy\Big)e^{\frac{in\pi y}{p}}\Big]\\
&=\frac{1}{2p}\sum_{n=-\infty}^{\infty}\int_{-p}^{p}f(y)e^{\frac{in\pi(x-y)}{p}}dy\,.
\end{aligned}$$

Suppose that $p\gg 1$ and $p\in\mathbb{N}$. Making use of the Riemann sum to approximate the integral (by partition $[-p\pi,p\pi]$ into $2p^2$ intervals), we find that

$$\begin{aligned}
f(x)&=\frac{1}{2p}\sum_{n=-\infty}^{\infty}\int_{-p}^{p}f(y)e^{\frac{in\pi(x-y)}{p}}dy\approx\frac{1}{2p}\sum_{n=-p^2}^{p^2-1}\int_{-p}^{p}f(y)\exp\Big[i\frac{n\pi}{p}(x-y)\Big]dy\\
&=\frac{1}{2\pi}\int_{-p}^{p}\Big(\sum_{n=-p^2}^{p^2-1}f(y)\exp\Big[i\frac{n\pi}{p}(x-y)\Big]\frac{\pi}{p}\Big)\,dy\\
&\approx\frac{1}{2\pi}\int_{-p}^{p}\Big(\int_{-p\pi}^{p\pi}f(y)e^{i\xi(x-y)}\,d\xi\Big)\,dy=\frac{1}{2\pi}\int_{-p\pi}^{p\pi}\Big(\int_{-p}^{p}f(y)e^{i\xi(x-y)}\,dy\Big)\,d\xi\\
&\approx\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty}f(y)e^{-i\xi y}dy\Big)e^{i\xi x}d\xi\,.
\end{aligned}\tag{14.15}$$

This motivates the following

Definition 14.4. Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function; that is, $\displaystyle\int_{-\infty}^{\infty} \big|f(x)\big|\,dx < \infty$.

1. The Fourier transform of f, denoted by $\widehat{f}$ or $\mathscr{F}[f]$, is the function

$$\widehat{f}(\xi) = \mathscr{F}[f](\xi) \equiv \int_{-\infty}^{\infty} f(x)e^{-ix\xi}\,dx\,.$$

Sometimes we will also write $\mathscr{F}_x[f(x)](\xi)$ to denote the Fourier transform of f, where the sub-index x in $\mathscr{F}_x$ means the variable to be integrated.

2. The inverse Fourier transform of f, denoted by $\check{f}$ or $\mathscr{F}^{-1}[f]$, is the function

$$\check{f}(\xi) = \mathscr{F}^{-1}[f](\xi) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{ix\xi}\,dx\,.$$

Theorem 14.5 (Fourier Inversion Formula)**.** *Let $f : \mathbb{R} \to \mathbb{C}$ be an integrable function such that $\widehat{f}$ is also integrable. Then*

$$\check{\widehat{f}}(x) = \widehat{\check{f}}(x) = f(x) \quad \textit{wheneven } f \textit{ is continuous at } x.$$

Remark 14.6. Under the assumptions of Theorem 14.5, if in addition f is real-valued, then if f is continuous at x,

$$\begin{aligned}
f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty} f(y)e^{-iy\xi}\,dy\Big)e^{ix\xi}\,d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty} f(y)e^{i\xi(x-y)}\,dy\Big)\,d\xi \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty} f(y)\cos\xi(x-y)\,dy\Big)\,d\xi \\
&= \frac{1}{\pi}\int_{0}^{\infty}\Big(\int_{-\infty}^{\infty} f(y)\big(\cos\xi x\cos\xi y + \sin\xi x\sin\xi y\big)\,dy\Big)\,d\xi \\
&= \frac{1}{\pi}\int_{0}^{\infty}\Big[\Big(\int_{-\infty}^{\infty} f(y)\cos\xi y\,dy\Big)\cos\xi x + \Big(\int_{-\infty}^{\infty} f(y)\sin\xi y\,dy\Big)\sin\xi x\Big]\,d\xi\,.
\end{aligned}$$

The integral

$$\frac{1}{\pi}\int_{0}^{\infty}\Big[\Big(\int_{-\infty}^{\infty} f(y)\cos\xi y\,dy\Big)\cos\xi x + \Big(\int_{-\infty}^{\infty} f(y)\sin\xi y\,dy\Big)\sin\xi x\Big]\,d\xi$$

is called the Fourier integral of function f, and the Fourier inversion formula says that under suitable conditions the Fourier integral of f is identical to f.

We also note that if in addition f is even, then

$$f(x) = \frac{1}{\pi}\int_0^\infty \Big(\int_{-\infty}^\infty f(y)\cos\xi x\cos\xi y\,dy\Big)\,d\xi = \frac{2}{\pi}\int_0^\infty \Big(\int_0^\infty f(y)\cos\xi x\cos\xi y\,dy\Big)\,d\xi\,.$$

Similarly,

$$f(x) = \frac{1}{\pi}\int_0^\infty \Big(\int_{-\infty}^\infty f(y)\cos\xi x\cos\xi y\,dy\Big)\,d\xi = \frac{2}{\pi}\int_0^\infty \Big(\int_0^\infty f(y)\cos\xi x\cos\xi y\,dy\Big)\,d\xi$$

whenever f is odd (and satisfies the assumption in Theorem 14.5).

Now suppose that $f:(0,\infty)\to\mathbb{R}$ be integrable; that is, $\displaystyle\int_0^\infty \big|f(x)\big|\,dx < \infty$. Let F be the even extension of f; that is, $F(x) = f(|x|)$ for all $x \in \mathbb{R}$. Then F is integrable on $\mathbb{R}$. Then

$$\widehat{F}(\xi) = \int_{-\infty}^\infty F(x)e^{ix\xi}\,dx = \int_{-\infty}^\infty F(x)\cos(x\xi)\,dx = 2\int_0^\infty f(x)\cos(x\xi)\,dx$$

since the imaginary part is an odd function. Moreover, the identity above also shows that $\widehat{F}$ is an even function. Therefore, if $\widehat{F}$ is also integrable, the Fourier Inversion Formula implies that

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^\infty \widehat{F}(\xi)e^{ix\xi}\,d\xi = \frac{1}{2\pi}\int_{-\infty}^\infty \widehat{F}(\xi)\cos(x\xi)\,d\xi \\ &= \frac{2}{\pi}\int_0^\infty \Big(\int_0^\infty f(y)\cos(y\xi)\,dy\Big)\cos(x\xi)\,d\xi \end{aligned}$$

whenever f is continuous at x.

On the other hand, we can consider the odd extension of f. Similar to the discussion above, under the condition that f is integrable (on $(0,\infty)$) and the Fourier transform of the odd extension of f is integrable, then

$$f(x) = \frac{2}{\pi}\int_0^\infty \Big(\int_0^\infty f(y)\sin(y\xi)\,dy\Big)\sin(x\xi)\,d\xi \quad \text{whenever } f \text{ is continuous at } x.$$

The discussion above motivates the following

Definition 14.7. Let $f:(0,\infty)\to\mathbb{R}$ be an integrable function; that is, $\displaystyle\int_0^\infty \big|f(x)\big|\,dx < \infty$.

1. The Fourier cosine transform, denoted by $\mathscr{F}_{\cos}[f]$, is the function

$$\mathscr{F}_{\cos}[f](\xi) = \int_0^\infty f(x)\cos(x\xi)\,dx\,.$$

2. The inverse Fourier cosine transform, denoted by $\mathscr{F}_{\cos}^{-1}[f]$, is the function
$$\mathscr{F}_{\cos}^{-1}[f](\xi) = \frac{2}{\pi}\int_0^\infty f(x)\cos(x\xi)\,dx\,.$$
3. The Fourier sine transform, denoted by $\mathscr{F}_{\sin}[f]$, is the function
$$\mathscr{F}_{\sin}[f](\xi) = \int_0^\infty f(x)\sin(x\xi)\,dx\,.$$
4. The inverse Fourier sine transform, denoted by $\mathscr{F}_{\sin}^{-1}[f]$, is the function
$$\mathscr{F}_{\sin}^{-1}[f](\xi) = \frac{2}{\pi}\int_0^\infty f(x)\sin(x\xi)\,dx\,.$$

Theorem 14.8. *Let $f:(0,\infty)\to\mathbb{R}$ be an integrable function.*

1. *If $\mathscr{F}_{\cos}[f]$ is also integrable, then*
$$\mathscr{F}_{\cos}\big[\mathscr{F}_{\cos}^{-1}\big](x) = \mathscr{F}_{\cos}^{-1}\big[\mathscr{F}_{\cos}\big](x) = f(x) \quad \textit{wheneven } f \textit{ is continuous at } x.$$
2. *If $\mathscr{F}_{\sin}[f]$ is also integrable, then*
$$\mathscr{F}_{\sin}\big[\mathscr{F}_{\sin}^{-1}\big](x) = \mathscr{F}_{\sin}^{-1}\big[\mathscr{F}_{\sin}\big](x) = f(x) \quad \textit{wheneven } f \textit{ is continuous at } x.$$

Remark 14.9. The integrals
$$\frac{2}{\pi}\int_0^\infty\Big(\int_0^\infty f(y)\cos(y\xi)\,dy\Big)\cos(x\xi)\,d\xi$$
and
$$\frac{2}{\pi}\int_0^\infty\Big(\int_0^\infty f(y)\sin(y\xi)\,dy\Big)\sin(x\xi)\,d\xi$$
are called the Fourier cosine integral and the Fourier sine integral of function f, respectively. Therefore, Theorem 14.8 says that under suitable conditions the Fourier cosine and sine integrals of f are identical to f.

Example 14.10. Let $g(x) = \exp\big(-\dfrac{x^2}{2\sigma^2}\big)$. Then
$$\begin{aligned}
\widehat{g}(\xi) &= \int_{-\infty}^\infty e^{-x^2/2\sigma^2}e^{-ix\xi}\,dx = \int_{-\infty}^\infty \exp\Big(-\frac{(x+i\sigma^2\xi)^2}{2\sigma^2} - \frac{\sigma^2\xi^2}{2}\Big)\,dx \\
&= \exp\Big(-\frac{\sigma^2\xi^2}{2}\Big)\int_{-\infty}^\infty \exp\Big(-\frac{(x+i\sigma^2\xi)^2}{2\sigma^2}\Big)\,dx \\
&= e^{-\sigma^2\xi^2/2}\int_{-\infty}^\infty e^{-\frac{y^2}{2\sigma^2}}\,dy = \sqrt{2\pi\sigma^2}e^{-\sigma^2\xi^2/2}\,.
\end{aligned}$$

Next we compute the inverse Fourier transform of $\widehat{g}$. Since we have obtained that

$$\mathscr{F}_x\Big[\exp\big(-\frac{x^2}{2\sigma^2}\big)\Big](\xi) = \sqrt{2\pi\sigma^2}\exp\big(-\frac{\sigma^2\xi^2}{2}\big) \tag{14.16}$$

and note that $\widehat{g}(\xi) = \sqrt{2\pi\sigma^2}\exp\big(-\dfrac{\xi^2}{2(1/\sigma)^2}\big)$, by the fact that $\check{f}(\xi) = \widehat{f}(-\xi)$ we find that

$$\check{\widehat{g}}(x) = \frac{1}{2\pi}\widehat{\widehat{g}}(-x) = \frac{1}{2\pi}\sqrt{2\pi\sigma^2}\sqrt{2\pi\frac{1}{\sigma^2}}\exp\big(-\frac{\frac{1}{\sigma^2}(-x)^2}{2}\big) = \exp\big(-\frac{x^2}{2\sigma^2}\big);$$

thus we establish that $\check{\widehat{g}}(x) = g(x)$.

Example 14.11. Let $f : [0,\infty) \to \mathbb{R}$ be defined by $f(x) = e^{-cx}$ for some $c > 0$. Then using the formula

$$\int e^{ax}\sin(bx)\,dx = \frac{e^{ax}\big[a\sin(bx) - b\cos(bx)\big]}{a^2+b^2} + C\,,$$
$$\int e^{ax}\cos(bx)\,dx = \frac{e^{ax}\big[a\cos(bx) + b\sin(bx)\big]}{a^2+b^2} + C\,,$$

we find that

$$\mathscr{F}_{\sin}[f](\xi) = \int_0^\infty e^{-cx}\sin(x\xi)\,dx = \frac{e^{-cx}\big[-c\sin(x\xi) - \xi\cos(x\xi)\big]}{c^2+\xi^2}\Big|_{x=0}^{x=\infty} = \frac{\xi}{c^2+\xi^2}\,,$$
$$\mathscr{F}_{\cos}[f](\xi) = \int_0^\infty e^{-cx}\cos(x\xi)\,dx = \frac{e^{-cx}\big[-c\cos(x\xi) + \xi\sin(x\xi)\big]}{c^2+\xi^2}\Big|_{x=0}^{x=\infty} = \frac{c}{c^2+\xi^2}\,.$$

14.3.2 Properties of Fourier Transform

1. Suppose that f is continuous and integrable on $\mathbb{R}$ such that f' is piecewise continuous on every finite interval. If $\lim\limits_{|x|\to\infty} f(x) = 0$, then integration by parts implies that

$$\begin{aligned}\mathscr{F}[f'](\xi) &= \int_{-\infty}^\infty f'(x)e^{-ix\xi}\,d\xi = f(x)e^{-ix\xi}\Big|_{x=-\infty}^{x=\infty} - \int_{-\infty}^\infty f(x)\frac{\partial}{\partial x}e^{-ix\xi}\,dx \\ &= i\xi\int_{-\infty}^\infty f(x)e^{ix\xi}\,dx = i\xi\widehat{f}(\xi)\end{aligned}$$

which shows

$$\mathscr{F}[f'](\xi) = i\xi\widehat{f}(\xi)\,. \tag{14.17}$$

If in addition f' is continuous and integrable such that f'' is also piecewise continuous on every finite interval and $\lim\limits_{|x|\to\infty} f'(x) = 0$, we have

$$\mathscr{F}[f''](\xi) = -\xi^2\widehat{f}(\xi)\,. \tag{14.18}$$

2. Suppose that f is continuous and integrable on $[0,\infty)$ such that f' is piecewise continuous on every finite interval. If $\lim\limits_{x\to\infty} f(x) = 0$, then integrating by parts implies that

$$\begin{aligned}\mathscr{F}_{\sin}[f'](\xi) &= \int_0^\infty f'(x)\sin(x\xi)\,dx = f(x)\sin(x\xi)\Big|_{x=0}^{x=\infty} - \int_0^\infty f(x)\frac{\partial}{\partial x}\sin(x\xi)\,dx \\ &= -\xi\mathscr{F}_{\cos}[f](\xi)\,.\end{aligned}$$

Similarly, $\mathscr{F}_{\cos}[f'](\xi) = \xi\mathscr{F}_{\sin}[f](\xi) - f(0)$.

If in addition f' is continuous and integrable such that f'' is also piecewise continuous on every finite interval and $\lim\limits_{x\to\infty} f'(x) = 0$, then

$$\mathscr{F}_{\sin}[f''](\xi) = -\xi\mathscr{F}_{\cos}[f'](\xi) = -\xi^2\mathscr{F}_{\sin}[f](\xi) + \xi f(0)$$

so we obtain that

$$\mathscr{F}_{\sin}[f''](\xi) = -\xi^2\mathscr{F}_{\sin}[f](\xi) + \xi f(0)\,. \tag{14.19}$$

Similarly, under the same assumption

$$\mathscr{F}_{\cos}[f''](\xi) = -\xi^2\mathscr{F}_{\cos}[f](\xi) - f'(0)\,. \tag{14.20}$$

14.3.3 Solving PDE using the Fourier transform

The Fourier transform can be used to study the heat equation and the wave equation when the spatial domain of interests is $\mathbb{R}$, while the Fourier cosine or sine transform can be used when the spatial domain of interests is $(0,\infty)$. In particular, if considering PDEs on $(0,\infty)$ with Dirichlet boundary condition at $x = 0$, the Fourier sine transform can be used, and we use the Fourier cosine transform when a Neumann boundary condition at $x = 0$ is imposed.

Example 14.12. Solve the heat equation

$$\frac{\partial u}{\partial t}(x,t) = \alpha^2\frac{\partial^2 u}{\partial x^2}(x,t) \qquad -\infty < x < \infty, t > 0\,, \tag{14.21a}$$

$$\lim_{|x|\to\infty} u(x,t) = 0 \qquad t > 0\,, \tag{14.21b}$$

$$u(x,0) = f(x) \qquad -\infty < x < \infty\,. \tag{14.21c}$$

Let $U(\xi,t) = \mathscr{F}[u(\cdot,t)](\xi)$, and assume that solution u satisfies

$$\mathscr{F}\big[u_t(\cdot,t)\big](\xi) = \int_{-\infty}^\infty \frac{\partial u}{\partial t}(x,t)e^{-ix\xi}\,dx = \frac{\partial}{\partial t}\int_{-\infty}^\infty u(x,t)e^{-ix\xi}\,dx = \frac{\partial U}{\partial t}(\xi,t)\,.$$

Then (14.21a) and (14.18) imply that U satisfies

$$U_t(\xi,t) + \alpha^2\xi^2 U(\xi,t) = 0\,. \tag{14.22}$$

Moreover, the initial condition (14.21c) implies that

$$U(\xi,0) = \mathscr{F}[f](\xi) \equiv \widehat{f}(\xi)\,. \tag{14.23}$$

Solving the ODE (14.22) subject to the initial condition (14.23), we find that

$$U(\xi,t) = e^{-\alpha^2\xi^2 t}\widehat{f}(\xi)\,.$$

Therefore,

$$\begin{aligned} u(x,t) &= \mathscr{F}^{-1}\big[U(\cdot,t)\big[(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha^2\xi^2 t}\widehat{f}(\xi)e^{ix\xi}\,d\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha^2\xi^2 t}\Big(\int_{-\infty}^{\infty} f(y)e^{-iy\xi}\,dy\Big)e^{ix\xi}\,d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty} e^{-\alpha^2\xi^2 t} f(y)e^{-i(x-y)\xi}\,dy\Big)\,d\xi \end{aligned}$$

and the Fubini Theorem (that we assume that we can apply) further shows that

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty} e^{-\alpha^2\xi^2 t} f(x-z)e^{-iz\xi}\,d\xi\Big)\,dz \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x-z)\mathscr{F}_\xi\big[e^{-\alpha^2\xi^2 t}\big](z)\,dz = \frac{1}{2\pi}\sqrt{2\pi\frac{1}{2\alpha^2 t}}\int_{-\infty}^{\infty} f(x-z)\exp\big(-\frac{z^2}{4\alpha^2 t}\big)\,dz \\ &= \frac{1}{\sqrt{4\pi\alpha^2 t}}\int_{-\infty}^{\infty} f(x-z)\exp\big(-\frac{z^2}{2\alpha^2 t}\big)\,dz = \frac{1}{\sqrt{4\pi\alpha^2 t}}\int_{-\infty}^{\infty} f(y)\exp\big(-\frac{|x-y|^2}{4\alpha^2 t}\big)\,dy\,. \end{aligned}$$

In particular, if f is the function

$$f(x) = \begin{cases} u_0 & \text{if } |x| < 1\,, \\ 0 & \text{if } |x| \geqslant 1\,, \end{cases}$$

where u_0 is a constant, then

$$u(x,t) = \frac{u_0}{\sqrt{4\pi\alpha^2 t}}\int_{-1}^{1}\exp\big(-\frac{|x-y|^2}{4\alpha^2 t}\big)\,dy$$

The substitution of variable $s = \dfrac{x-y}{2\alpha\sqrt{t}}$ (here we assume that $\alpha > 0$) shows that

$$\begin{aligned} u(x,t) &= \frac{u_0}{\sqrt{4\pi\alpha^2 t}}\int_{\frac{x+1}{2\alpha\sqrt{t}}}^{\frac{x-1}{2\alpha\sqrt{t}}} e^{-s^2}\,2\alpha\sqrt{t}(-ds) = \frac{u_0}{\sqrt{\pi}}\int_{\frac{x-1}{2\alpha\sqrt{t}}}^{\frac{x+1}{2\alpha\sqrt{t}}} e^{-s^2}\,ds \\ &= \frac{u_0}{2}\Big[\operatorname{erf}\big(\frac{x+1}{2\alpha\sqrt{t}}\big) - \operatorname{erf}\big(\frac{x-1}{2\alpha\sqrt{t}}\big)\Big]\,. \end{aligned}$$

Remark 14.13. Let $H(x,t) = \dfrac{1}{\sqrt{4\pi\alpha^2 t}} \exp\big(-\dfrac{x^2}{4\alpha^2 t}\big)$, and define the convolution of two functions g and h by

$$(g * h)(x) = \int_{-\infty}^{\infty} g(y)h(x-y)\,dy = \int_{-\infty}^{\infty} g(x-y)h(y)\,dy\,.$$

Then the solution u to (14.21) can be written as

$$u(x,t) = (H(\cdot,t) * f)(x) \equiv (H * f)(x,t) = \int_{-\infty}^{\infty} f(y)H(x-y,t)\,dy\,.$$

The function H is called the heat kernel.

Example 14.14. Consider the Laplace equation

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0 \qquad 0 < x < \pi\,, y > 0\,, \tag{14.24a}$$

$$u(0,y) = 0\,,\ u(\pi,y) = e^{-y} \qquad y > 0\,, \tag{14.24b}$$

$$\frac{\partial u}{\partial y}(x,0) = 0 \qquad 0 < x < \pi\,. \tag{14.24c}$$

Let $U(x,\xi) = \mathscr{F}_{\cos}[u(x,\cdot)](\xi) = \int_0^\infty u(x,y)\cos(y\xi)\,dy$. Assume that

$$\int_0^\infty \frac{\partial^2 u}{\partial x^2}(x,y)\cos(y\xi)\,dy = \frac{\partial^2}{\partial x^2}\int_0^\infty u(x,y)\cos(y\xi)\,dy = \frac{\partial^2 U}{\partial x^2}(x,\xi)\,.$$

Using (14.20), we conclude from (14.24a) and (14.24c) that $U = U(x,\xi)$ satisfies that

$$\frac{\partial^2 U}{\partial x^2}(x,\xi) - \xi^2 U(x,\xi) = 0 \qquad 0 < x < \pi\,, \xi > 0\,;$$

thus

$$U(x,\xi) = C_1(\xi)\cosh(x\xi) + C_2(\xi)\sinh(x\xi)\,.$$

Moreover, (14.24b) implies that

$$U(0,\xi) = 0 \qquad \text{and} \qquad U(\pi,\xi) = \frac{1}{1+\xi^2}\,,$$

where we have used Example 14.11 to conclude the second equality. Therefore,

$$C_1(\xi) = 0 \qquad \text{and} \qquad C_1(\xi)\cosh(\pi\xi) + C_2(\xi)\sinh(\pi\xi) = \frac{1}{1+\xi^2}$$

which shows that

$$U(x,\xi) = \frac{\sinh(x\xi)}{(1+\xi^2)\sinh(\pi\xi)}\,.$$

Therefore,

$$u(x,y) = \frac{2}{\pi}\int_0^\infty U(x,\xi)\cos(y\xi)\,d\xi = \frac{2}{\pi}\int_0^\infty \frac{\sinh(x\xi)\cos(y\xi)}{(1+\xi^2)\sinh(\pi\xi)}\,d\xi\,.$$