Differential Equations MA2042-* Final Exam

National Central University, Jun. 23 2021

在下面問題中，當提到 Fourier series 時，是包含了 Fourier series、Fourier cosine series 與 Fourier sine series；而當提到 Fourier transform 時，是包含了 Fourier transform、Fourier cosine transform 與 Fourier sine transform。

Problem 1. Consider the Laplace equation

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0 \qquad & \forall\, (x,y) \in B(0,1)\,, \\ u(x,y) = f(x,y) \qquad & \forall\, (x,y) \in \partial B(0,1)\,, \end{aligned}$$

where $B(0,1) \equiv \{(x,y) \,|\, x^2 + y^2 < 1\}$ is the unit disk centered at the origin with radius 1, and f is a given function. Complete the following.

1. (5%) Since the domain of interest is a disk, it is nature to introduce the polar coordinate. Let $v(r,\theta) = u(r\cos\theta, r\sin\theta)$. Show that v satisfies the PDE
$$\frac{\partial^2 v}{\partial r^2}(r,\theta) + \frac{1}{r}\frac{\partial v}{\partial r}(r,\theta) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}(r,\theta) = 0\,.$$

2. (5%) For each fixed $r > 0$, the function $v(r,\cdot)$ can be viewed as a periodic function with period 2π so that $v(r,\cdot)$ can be expressed as
$$v(r,\theta) = \sum_{n=0}^{\infty} \big[A_n(r)\cos(n\theta) + B_n(r)\sin(n\theta)\big]\,.$$
Assume that $\frac{\partial}{\partial r}\sum\limits_{n=0}^{\infty} = \sum\limits_{n=0}^{\infty}\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}\sum\limits_{n=0}^{\infty} = \sum\limits_{n=0}^{\infty}\frac{\partial}{\partial \theta}$. Show that A_n and B_n satisfy
$$\begin{aligned} A_n''(r) + \frac{1}{r}A_n'(r) - \frac{n^2}{r^2}A_n(r) = 0\,, \\ B_n''(r) + \frac{1}{r}B_n'(r) - \frac{n^2}{r^2}B_n(r) = 0\,. \end{aligned}$$

3. (10%) Use the power series method to find the general form of A_n and B_n.

4. (10%) Suppose that
$$f(\cos\theta, \sin\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} \big[c_n\cos(n\theta) + s_n\sin(n\theta)\big]\,.$$
Find $\{c_n\}_{n=0}^{\infty}$ and $\{s_n\}_{n=1}^{\infty}$, and the expression of v in terms of $\{c_n\}_{n=0}^{\infty}$ and $\{s_n\}_{n=1}^{\infty}$.

Problem 2. (15%) Show, using the method of the Laplace transform, that a solution to the following PDE

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) - hu(x,t) \qquad & x > 0, t > 0\,, \\ u(0,t) = u_0\,,\ \lim_{x\to\infty} u_x(x,t) = 0 \qquad & t > 0\,, \\ u(x,0) = 0 \qquad & x > 0\,, \end{aligned}$$

where h and u_0 are constants, is given by

$$u(x,t) = \frac{u_0 x}{2\sqrt{\pi}} \int_0^t \tau^{-\frac{3}{2}} \exp\Big(-h\tau - \frac{x^2}{4\tau}\Big)\, d\tau\,.$$

Problem 3. Consider the wave equation

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(x,t) &= \frac{\partial^2 u}{\partial x^2}(x,t) && -\infty < x < \infty, t > 0\,, \\ \lim_{|x|\to\infty} u(x,t) &= 0 && t > 0\,, \\ u(x,0) = f(x)\,,\ u_t(x,0) &= g(x) && -\infty < x < \infty\,, \end{aligned}$$

where f, g are functions satisfying that $f, g, \widehat{f}, \widehat{g}$ are integrable on $\mathbb{R}$. Complete the following.

1. (5%) Let $U(\xi,t) = \mathscr{F}[u(\cdot,t)](\xi) = \int_{-\infty}^{\infty} u(x,t)e^{-ix\xi}\,dx$, and suppose that

$$\int_{-\infty}^{\infty} \frac{\partial^k u}{\partial t^k}(x,t)e^{-ix\xi}\,dx = \frac{\partial^k}{\partial t^k}\int_{-\infty}^{\infty} u(x,t)e^{-ix\xi}\,dx \qquad \text{for } k = 1 \text{ and } 2\,.$$

   Show that

$$U(\xi,t) = \widehat{f}(\xi)\cos(\xi t) + \widehat{g}(\xi)\frac{\sin(\xi t)}{\xi}\,.$$

2. (12%) Show that if ϕ and $\widehat{\phi}$ are both integrable on $\mathbb{R}$, then

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \widehat{\phi}(\xi)\frac{\sin(\xi t)}{\xi}e^{ix\xi}\,d\xi = \int_{-t}^{t} \phi(x-z)\,dz\,.$$

3. (10%) Suppose that f satisfies that

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} \widehat{f}(\xi)\frac{\sin(\xi t)}{\xi}e^{ix\xi}\,d\xi = \int_{-\infty}^{\infty} \frac{\partial}{\partial t}\Big[\widehat{f}(\xi)\frac{\sin(\xi t)}{\xi}e^{ix\xi}\Big]\,d\xi\,.$$

   Find the solution $u(x,t)$ in terms of f and g.

Hint of 2: Show that for each fixed $t > 0$ the Fourier transform of the function $y = \int_{-t}^{t} \phi(x-z)\,dz$ is $\widehat{\phi}(\xi)\frac{\sin(\xi t)}{\xi}$, and conclude from this fact.

Hint of 3: Using the conclusion in 2 with $\phi = f$ and g.

Problem 4. Consider the Laplace equation

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) &= 0 && x > 0\,, 0 < y < \frac{\pi}{2}\,, \\ u(x,0) = 0\,,\ u\big(x,\frac{\pi}{2}\big) &= e^{-3x} && x > 0\,, \\ u(0,y) &= 0 && 0 < y < \frac{\pi}{2}\,. \end{aligned}$$

1. (14%) Solve the Laplace equation above using the Fourier transform.

2. (14%) Solve the Laplace equation above using the Fourier series.