Differential Equations MA2041-A Midterm Exam 1

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1	2	3	4(1)(2)	4(3)	4(4)	5(1)	5(2)	6	7	Total

Problem 1. (5pts) Sketch the direction field at the grid points $\{(m,n) \in \mathbb{N}^2 \mid 1 \leqslant m \leqslant 5, 1 \leqslant n \leqslant 5\}$ of the ODE

$$\frac{dy}{dx} = 2 - y + \frac{1}{x^2}\,.$$

Problem 2. (15pts) Find the integral curve of the vector field $F(x, y) = \big(-x^3, 3x^2y + y^3\big)$ passing through the point $(x, y) = (1, 1)$.

Solution: Suppose that the integral curve of the vector field F can be parameterized by $\big(x(t), y(t)\big)$ for some x, y and $t \in I$. Then

$$\begin{aligned} \frac{dx}{dt} &= -x^3\,, \\ \frac{dy}{dt} &= 3x^2y + y^3\,. \end{aligned}$$

Therefore, at each point (x, y) on the integral curve,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{3x^2y + y^3}{x^3} = -\frac{y}{x} - \Big(\frac{y}{x}\Big)^3\,.$$

Let $v = \dfrac{y}{x}$ and $g(v) = -v^3 - v$. Then the equation above implies that

$$x\frac{dv}{dx} + v = -v^3 - 3v\,;$$

thus using the method of separation of variables,

$$\frac{dv}{v(v^2+4)} = -\frac{dx}{x}\,.$$

Using partial fractions,

$$\frac{1}{4}\Big(\frac{1}{v} - \frac{v}{v^2+4}\Big)dv = -\frac{dx}{x}\,;$$

thus

$$\frac{1}{4}\log v - \frac{1}{8}\log(v^2+4) + \log x = C\,.$$

Substituting $v = \dfrac{y}{x}$ and using the fact that the integral curve passing through $(1, 1)$, we find that $C = -\dfrac{1}{8}\log 5$; thus the desired integral curve is

$$\frac{1}{4}\log\frac{y}{x} - \frac{1}{8}\log\Big(\frac{y^2}{x^2} + 4\Big) + \log x = -\frac{1}{8}\log 5\,.$$

Problem 3. (15pts) Find the general form of the implicit solutions to the equation

$$\big(\sin y - 3ye^{-x}\sin x\big)dx + \big(\cos y + 3e^{-x}\cos x\big)dy = 0\,.$$

Solution: Let $M(x,y) = \sin y - 3ye^{-x}\sin x$ and $N(x,y) = \cos y + 3e^{-x}\cos x$. Then

$$\begin{aligned} M_y(x,y) &= \cos y - 3e^{-x}\sin x\,,\\ N_x(x,y) &= -3e^{-x}\cos x - 3e^{-x}\sin x\,. \end{aligned}$$

Therefore, $(N_x - M_y)(x,y) = -\cos y - 3e^{-x}\cos x = -N(x,y)$; thus there exists an integrating factor $\mu = \mu(x)$ such that

$$\mu'(x) = \frac{M_y - N_x}{N}(x,y)\mu(x) = \mu(x)\,.$$

Solving for μ, we find that $\mu(x) = e^x$ is an integrating factor; thus there exists a scalar function φ such that

$$(\varphi_x, \varphi_y) = (\mu M, \mu N)\,.$$

Since $\varphi_x(x,y) = (\mu M)(x,y) = e^x \sin y - 3y\sin x$, we find that

$$\varphi(x,y) = e^x\sin y + 3y\cos x + \psi(y)$$

for some function ψ of y. Using $\varphi_y = \mu N$, we obtain that

$$e^x\cos y + 3\cos x + \psi'(y) = e^x\cos y + 3\cos x$$

which implies that ψ is a constant. Therefore, the general form of the implicit solution is

$$e^x\sin y + 3y\cos x = C\,.$$

Problem 4. Consider the initial value problem $y' + y = \cos t - \sin t$ with initial condition $y(0) = 1$. Complete the following.

1. (5pts) Using the fundamental theorem of ODE to explain why there is a unique solution defined for t in an interval containing 0.

2. (5pts) Let $\{\varphi_n\}_{n=0}^{\infty}$ be the sequence of functions produces by the Picard iteration. Find φ_1 and φ_2.

3. (10pts) Show that that φ_n has the form

$$\varphi_n(t) = A_n \sin t + B_n \cos t + \sum_{k=0}^{n} c_{n,k} t^k \,.$$

4. (10pts) Find the limit of the sequence $\{\varphi_{4n}\}_{n=0}^{\infty}$.

Solution:

1. Let $f(t, y) = \cos t - \sin t - y$. Clearly f and f_y are both continuous on $\mathbb{R}^2$; thus f and f_y are continuous on a rectangle containing $(0, 1)$ as an interior point. Therefore, the fundamental theorem of ODE guarantees that for some $h > 0$, there exists a unique solution y to the initial value problem $y' = f(t, y)$, $y(0) = 1$, in the time interval $(-h, h)$.

2. Picard iteration provides

$$\varphi_{n+1}(t) = 1 + \int_0^t \big[\cos s - \sin s - \varphi_n(s)\big]\, ds = \sin t + \cos t - \int_0^t \varphi_n(s)\, ds\,, \qquad \varphi_0(t) = 1\,.$$

Since $\varphi_0(t) = 1$,

$$\varphi_1(t) = \sin t + \cos t - t\,;$$

thus

$$\varphi_2(t) = \sin t + \cos t - \int_0^t (\sin s + \cos s - s)\, ds = 2\cos t - 1 + \frac{t^2}{2}\,.$$

3. It is clear that φ_1 has the form specified above. If $\varphi_n(t) = A_n \sin t + B_n \cos t + \sum\limits_{k=0}^{n} c_{n,k} t^k$, then

$$\begin{aligned}
\varphi_{n+1}(t) &= \sin t + \cos t - \int_0^t \Big[A_n \sin s + B_n \cos s + \sum_{k=0}^{n} c_k^{(n)} s^k\Big]\, ds \\
&= \sin t + \cos t + A_n(\cos t - 1) - B_n \sin t - \sum_{k=0}^{n} \frac{c_{n,k}}{k+1} t^{k+1} \\
&= (1 - B_n)\sin t + (1 + A_n)\cos t - A_n - \sum_{k=1}^{n+1} \frac{c_{n,k-1}}{k} t^k
\end{aligned}$$

which has the form $A_{n+1} \sin t + B_{n+1} \cos t + \sum\limits_{k=0}^{n+1} c_{n+1,k} t^k$, where $A_n, B_n, c_{n,k}$ satisfy

$$A_{n+1} = 1 - B_n\,, \quad B_{n+1} = 1 + A_n\,, \quad c_{n+1,0} = -A_n\,, \quad c_{n+1,k} = -\frac{c_{n,k-1}}{k} \text{ for } k \in \mathbb{N}\,.$$

Therefore, by induction φ_n has the form $\varphi_n(t) = A_n \sin t + B_n \cos t + \sum\limits_{k=0}^{n} c_{n,k} t^k$.

4. We have proved in class that the limit of the Picard iteration is the solution to the initial value problem, so we solve the initial value problem in order to find the limit. Using the method of integrating factor,

$$\frac{d}{dt}(e^t y) = e^t(\cos t - \sin t)\,. \tag{0.1}$$

In order to solve for y, we need to ind the integral of $e^t(\cos t - \sin t)$. Integrating by parts,

$$\begin{aligned}\int e^t \cos t\, dt &= \int e^t d\sin t = e^t \sin t - \int e^t \sin t\, dt = e^t \sin t + \int e^t d\cos t \\ &= e^t(\sin t + \cos t) - \int e^t \cos t\, dt\,;\end{aligned}$$

thus $\displaystyle\int e^t \cos t\, dt = \frac{1}{2}e^t(\sin t + \cos t)$. Similarly,

$$\int e^t \sin t\, dt = \frac{1}{2}e^t(\sin t - \cos t)\,.$$

Therefore, $\displaystyle\int e^t(\cos t - \sin t)\, dt = e^t \cos t$; thus (0.1) implies that

$$e^t y = e^t \cos t + C$$

which further implies that $y(t) = \cos t + Ce^{-t}$. Using the initial condition $y(0) = 1$, we find that $C = 0$; thus the solution to the IVP is $y(t) = \cos t$. Therefore, the limit of the sequence $\{\varphi_{4n}\}_{n=1}^{\infty}$ is $y(t) = \cos t$.

Problem 5. (20pts) Let y be a continuous solution to the initial value problem

$$\frac{dy}{dt} + p(t)y = q(t)\,, \qquad y(0) = 0\,,$$

where

$$p(t) = \begin{cases} -1 & \text{if } 0 \leqslant t \leqslant \pi\,, \\ 1 & \text{if } t > \pi\,, \end{cases} \quad \text{and} \quad q(t) = \begin{cases} \cos t - \sin t & \text{if } 0 \leqslant t \leqslant \pi\,, \\ -1 & \text{if } t > \pi\,. \end{cases}$$

1. Evaluate $y(2\pi)$.

2. If y is differentiable at $t = \pi$, find $y'(\pi)$, otherwise explain why y is not differentiable at $t = \pi$.

Solution: First we find the anti-derivative of $e^{-t}\cos t$. Integrating by parts,

$$\int e^{-t}\cos t\,dt = \int e^{-t} d\sin t = e^{-t}\sin t + \int e^{-t}\sin t\,dt = e^{-t}\sin t - \int e^{-t} d\cos t$$
$$= e^{-t}(\sin t - \cos t) - \int e^{-t}\cos t\,dt\,;$$

thus $\displaystyle\int e^{-t}\cos t\,dt = \frac{1}{2}e^{-t}(\sin t - \cos t)$. Similarly,

$$\int e^{-t}\sin t\,dt = -\frac{1}{2}e^{-t}(\sin t + \cos t)\,.$$

As a consequence, for $0 \leqslant t \leqslant \pi$,

$$(e^{-t}y)' = e^{-t}(\cos t - \sin t) \Rightarrow y(t) = \sin t + Ce^{t}\,.$$

Since $y(0) = 0$, we must have $C = 0$; thus $y(t) = \sin t$ for $t \in [0, \pi]$. In particular, $y(\pi) = 0$.

For $t > \pi$, we use the integrating factor e^t and find that for $t > \pi$,

$$(e^t y)' = -e^t \Rightarrow y(t) = Ce^{-t} - 1$$

Since $y(\pi) = 0$, we must have $Ce^{-\pi} = 1$; thus $C = e^{\pi}$ which implies that for $t > \pi$,

$$y(t) = e^{\pi - t} - 1\,.$$

Therefore,

$$y(t) = \begin{cases} \sin t & \text{if } 0 \leqslant t \leqslant \pi\,, \\ e^{\pi - t} - 1 & \text{if } t > \pi\,. \end{cases}$$

1. $y(2\pi) = e^{-\pi} - 1$.

2. y is differentiable at $t = \pi$ since

$$\lim_{h \to 0^+} \frac{y(\pi + h) - y(\pi)}{h} = e^{t-\pi}\Big|_{t=\pi} = 1 \quad \text{and} \quad \lim_{h \to 0^-} \frac{y(\pi + h) - y(\pi)}{h} = \cos t\Big|_{t=\pi} = -1\,.$$

   Moreover, the computation above shows that $y'(\pi) = -1$.

Problem 6. (15%) Find the necessary and sufficient conditions for that there exists an integrating factor μ of the form $\mu = \mu(x^2 + y^2)$ for the equation $M(x, y)dx + N(x, y)dy = 0$.

Solution: Note that μ is an integrating factor of $Mdx + Ndy = 0$, then $(\mu M)_y = (\mu N)_x$ which can be rewritten as

$$\mu_y M - \mu_x N = \mu(N_x - M_y)\,.$$

If $\mu = \mu(x^2 + y^2)$ is an integrating factor, then

$$\mu'(x^2 + y^2)\big(2yM(x, y) - 2xN(x, y)\big) = \mu(x^2 + y^2)\big(N_x(x, y) - M_y(x, y)\big)\,;$$

thus

$$\frac{\mu'(x^2 + y^2)}{\mu(x^2 + y^2)} = \frac{N_x(x, y) - M_y(x, y)}{2yM(x, y) - 2xN(x, y)}$$

which implies that the right-hand side has to be a function of x^2+y^2. Therefore, a necessary condition for that there exists an integrating factor μ of the form $\mu = \mu(x^2 + y^2)$ is

$$\frac{N_x(x, y) - M_y(x, y)}{2yM(x, y) - 2xN(x, y)} = H(x^2 + y^2)$$

for some function H.

On the other hand, if $\frac{N_x(x, y) - M_y(x, y)}{2yM(x, y) - 2xN(x, y)} = H(x^2 + y^2)$ for some function H, by defining μ as the solution to the ODE $\mu'(z) = H(z)\mu(z)$, the computation above shows that we must have $(\mu M)_y = (\mu N)_x$. Therefore, $(\mu M, \mu N)$ is conservative in a simply connected domain $\mathcal{D}$ which implies that $(\mu M)dx + (\mu N)dy$ is exact on $\mathcal{D}$. Therefore, the condition

$$\frac{N_x(x, y) - M_y(x, y)}{2yM(x, y) - 2xN(x, y)} = H(x^2 + y^2)$$

for some function H is also a sufficient condition for that $Mdx + Ndy = 0$ has an integrating fact μ of the form $\mu = \mu(x^2 + y^2)$.

Problem 7. Solve the initial value problem

$$t^3 z'(t) + t^2 z(t) = z(t)^4\,, \qquad z(1) = 1$$

by the following procedure.

1. (10pts) Let $y(x) = z(e^x)$. Find the equation that y satisfies.

2. (10pts) Find the solution z to the initial value problem above by solving for y first.

Solution:

1. Note that the ODE can be rewritten as

$$z'(t) = -\frac{z(t)}{t} + \frac{z(t)^4}{t^3}\,.$$

If $y(x) = z(e^x)$, then

$$y'(x) = z'(e^x)e^x = e^x\Big[-\frac{z(e^x)}{e^x} + \frac{z(e^x)^4}{e^{3x}}\Big] = -y(x) + e^{-2x}y(x)^4\,;$$

thus y is the solution to the initial value problem

$$y' + y = e^{-2x}y^4\,, \qquad y(0) = 1\,.$$

2. We note that the equation for y is a Bernoulli equation with $p(x) = 1, q(x) = e^{-2x}$ and $r = 4$. Let $v = y^{-3}$. Then v satisfies

$$\frac{dv}{dx} = -3(e^{-2x} - v) = 3v - 3e^{-2x}\,.$$

Using the method of integrating factor with integrating factor e^{-3x}, we find that

$$(e^{-3x}v)' = -3e^{-5x}$$

Therefore, $e^{-3x}v = \frac{3}{5}e^{-5x} + C$ or $y(x)^{-3} = \frac{3}{5}e^{-2x} + Ce^{3x}$. The initial condition $y(0) = 1$ implies that $C = \frac{2}{5}$; thus

$$y(x) = \Big[\frac{3}{5}e^{-2x} + \frac{2}{5}e^{3x}\Big]^{-\frac{1}{3}}\,.$$

Finally, since $z(t) = y(\log t)$, we conclude that

$$z(t) = \Big[\frac{3}{5}t^{-2} + \frac{2}{5}t^{-3}\Big]^{-\frac{1}{3}}\,.$$