Differential Equations Recommended Exercise

Problem 1. Consider the initial value problem $y' = 1 + y$ with $y(0) = 0$.

1. Find the local truncation error $\tau_k(h)$ for the improved Euler's method.

2. Find the local truncation error $\tau_k(h)$ for the Taylor's method of order 2.

3. Use the Runge-Kutta method of order 4 (given by (3.11) in the lecture note) with $h = 0.1$ to find an approximated value of $y(0.1)$.

4. Show that the numerical method
$$y_{k+1} = y_k + \frac{h}{6}(1 + y_k)(6 + 3h + h^2)$$
is a third order numerical method; that is, show that the global truncation error $e_k(h)$ satisfies
$$\big|e_k(h)\big| \leqslant Ch^3 \qquad \forall\, 1 \leqslant k \leqslant \frac{T}{h}\,.$$
for some constant $C > 0$.

Problem 2. Consider the initial value problem $y' = \sin(t^2 + y)$ with $y(0) = 0$.

1. Write the improved Euler method in the form
$$y_{k+1} = y_k + h\Phi(h, t_k, y_k)\,.$$
In other words, find the function Φ such that the iterative scheme above is equivalent to the improved Euler method.

2. Show that $\big|\Phi_y(h, t, y)\big| \leqslant \frac{3}{2}$ if $h < 1$.

3. Show that the local truncation error $\tau_k(h)$ satisfies
$$\big|\tau_k(h)\big| \leqslant 7h^2 \qquad \forall\, h \leqslant \frac{1}{k}\,.$$

4. Use Theorem 3.8 in the lecture note to find h in order to have an approximated value of $y(1)$ which is accurate to the six decimal places（到小數點以下六位是正確的）

Problem 3. Consider the initial value problem $y' = f(t, y)$ with $y(t_0) = y_0$. Show that if f is four times continuously differentiable and $f^{(j)}$ is bounded for $j = 0, 1, 2, 3, 4$, the numerical method

$$\begin{aligned}
r_1 &= hf(t_k, y_k)\,,\\
r_2 &= hf(t_k + \frac{h}{2}, y_k + \frac{r_1}{2})\,,\\
r_3 &= hf(t_k + h, y_k - r_1 + 2r_2)\,,\\
y_{k+1} &= y_k + \frac{1}{6}(r_1 + 4r_2 + r_3)
\end{aligned}$$

is a third order numerical method; that is, show that the global truncation error $e_k(h)$ satisfies

$$\big|e_k(h)\big| \leqslant Ch^3$$

for some constant $C > 0$.

Problem 4. Mimic the proof of Theorem 4.1 in the lecture note to show that if $g : I \to \mathbb{R}$ is continuous, where I is an interval containing t_0 as an interior point, $b, c \in \mathbb{R}$ be constants, the initial value problem

$$y'' + by' + cy = g(t)\,, \qquad y(t_0) = y_0, \quad y'(t_0) = y_1$$

has a unique solution $y : I \to \mathbb{R}$.