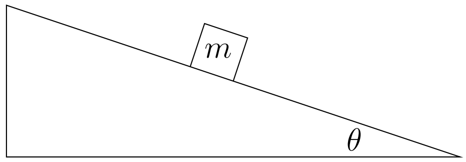
Differential Equations Recommended Exercise 1

Problem 1. 推導放置在角度為 θ 的斜面上、懸於平行於斜面虎克常數為 k 的彈簧上質量為 m 的物體的運動方程式。



Problem 2. In class we derive the differential equation for the brachistochrone curve connecting $(0,0)$ and (a,b), where $b<0$:

$$\Big[\frac{f'(y)}{\sqrt{-2gy}\sqrt{1+f'(y)^2}}\Big]' = 0\,,\quad f(0)=0, f(a)=b\,.$$

Solve this differential equation.

Suppose that the brachistochrone curve connecting $(0,0)$ and (a,b) can be represented as $y=h(x)$ (thus $h(0)=0$ and $h(a)=b$). Use the variational principle to derive the differential equation that h has to satisfy.

Problem 3. Suppose that there exists a twice continuously differentiable minimizer $y=y(t)$ to the following variational problem

$$\min_{\varphi\in\mathcal{A}}\int_0^a L(\varphi,\varphi',t)\,dt\,,\ \text{where } \mathcal{A}=\big\{\varphi:[0,a]\to\mathbb{R}\,\big|\,\varphi(0)=\varphi(a)=0\big\}\,,$$

where $L(p,q,t)$ is differentiable with respect to p and q. Derive the equation that the minimizer y has to satisfy.

Problem 4. §1.2 課本習題 1.(b) Show that $xy^3 - xy^3\sin x = 1$ is an implicit solution to

$$\frac{dy}{dx} = \frac{(x\cos x+\sin x-1)y}{3(x-x\sin x)}\,.$$

on the interval $(0,\pi/2)$.

Problem 5. §1.2 課本習題 9. Determine whether the relation $y-\ln y = x^2+1$ is an implicit solution to the differential equation

$$\frac{dy}{dx} = \frac{2xy}{y-1}\,.$$

Problem 6. §1.2 課本習題 13. Determine whether the relation $\sin y + xy - x^3 = 2$ is an implicit solution to the differential equation

$$y'' = \frac{6xy' + (y')^3\sin y - 2(y')^2}{3x^2-y}\,.$$

Problem 7. §1.2 課本習題 30. **Implicit Function Theorem**. Let $G(x, y)$ have continuous first partial derivatives in the rectangle $R = \{(x, y) \,|\, a < x < b, c < y < d\}$ containing the point (x_0, y_0). If $G(x_0, y_0) = 0$ and the partial derivative $G_y(x_0, y_0) \neq 0$, then there exists a differentiable function $y = \phi(x)$, defined in some interval $I = (x_0 - \delta, x_0 + \delta)$, that satisfies $G(x, \phi(x)) = 0$ for all $x \in I$.

The implicit function theorem gives conditions under which the relationship $G(x, y) = 0$ defines y implicitly as a function of x. Use the implicit function theorem to show that the relationship $x + y + e^{xy} = 0$ defines y implicity as a function of x near the point $(0, -1)$.