A Concise Lecture Note on Differential Equations

1 Introduction

1.1 Background

Definition 1.1. A differential equation is a mathematical equation that relates some unknown function with its derivatives. The unknown functions in a differential equations are sometimes called ***dependent variables***, and the variables which the derivatives of the unknown functions are taken with respect to are sometimes called the ***independent variables***. A differential equation is called an ***ordinary differential equation*** (ODE) if it contains an unknown function of one independent variable and its derivatives. A differential equation is called a ***partial differential equation*** (PDE) if it contains unknown multi-variable functions and their partial derivatives.

Example 1.2. The following three differential equations are identical (with different expression):

$$\begin{aligned} y' + y &= x + 3\,, \\ \frac{dy}{dx} + y &= x + 3\,, \\ f'(x) + f(x) &= x + 3\,. \end{aligned}$$

Example 1.3. Let $u : \left\{ \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R} \\ (x,t) & \mapsto & u(x,t) \end{array} \right.$ be an unknown function. The differential equation

$$u_t - u_x = t - x$$

is a partial differential equation.

Definition 1.4. The ***order*** of a differential equation is the order of the highest-order derivatives present in the equation. A differential equation of order 1 is called first order, order 2 second order, etc.

Example 1.5. The differential equations in Example 1.2 and 1.3 are both first order differential equations, while the equation $y'' + xy'^3 = x^7$ and $u_t - u_{xx} = x^3 + t^5$ are second order equations.

Definition 1.6. The ordinary differential equation

$$F(t, y, y', \cdots, y^{(n)}) = 0$$

is said to be ***linear*** if F is linear (or more precise, affine) function of the variable y, y', $\cdots$, $y^{(n)}$. In other words, a linear ordinary differential equation has the form

$$a_n(t)\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t)y = f(t)\,.$$

If an ordinary differential equation said to be ***nonlinear*** if it is not linear.

Similar terminologies are applied to partial differential equations.

1.1.1 Why do we need to study differential equations?

Example 1.7 (Spring with or without Friction)**.**

$$m\ddot{x} = -kx - r\dot{x}\,.$$

Example 1.8 (Oscillating pendulum)**.**

$$mL\ddot{\theta} = -mg\sin\theta$$

Example 1.9 (System of ODEs)**.** Let $p : [0,\infty) \to \mathbb{R}^+$ denote the population of certain species. If there are plenty of resource for the growth of the population, the growth rate (the rate of change of the population) is proportion to the population. In other words, there exists constant $\gamma > 0$ such that

$$\frac{d}{dt}p(t) = \gamma p(t)\,.$$

The Lotka-Volterra equation or the predator-prey equation:

$$\begin{aligned} p' &= \gamma p - \alpha pq\,, \\ q' &= \beta q + \delta pq\,. \end{aligned}$$

Example 1.10. A brachistochrone curve, meaning "shortest time" or curve of fastest descent, is the curve that would carry an idealized point-like body, starting at rest and moving along the curve, without friction, under constant gravity, to a given end point in the shortest time. For given two point $(0,0)$ and (a,b), where $b < 0$, what is the brachistochrone curve connecting $(0,0)$ and (a,b)?

Define

$$\mathcal{A} = \big\{h : [0,b] \to \mathbb{R} \,\big|\, h(0) = 0, h(b) = a,\ h \text{ is differentiable on (0,b)}\big\}$$

and

$$\mathcal{N} = \big\{\varphi : [0,b] \to \mathbb{R} \,\big|\, \varphi(0) = 0, \varphi(b) = 0,\ \varphi \text{ is differentiable on (0,b)}\big\}\,,$$

and suppose that the brachistochrone curve can be expressed as $x = f(y)$ for some $f \in \mathcal{A}$. Then f the minimizer of the functional

$$T(h) = \int_0^b \frac{ds}{v} = -\int_0^b \frac{\sqrt{1 + h'(y)^2}}{\sqrt{-2gy}}\,dy$$

or equivalently,

$$T(f) = \min_{h\in\mathcal{A}} -\int_0^b \frac{\sqrt{1 + h'(y)^2}}{\sqrt{-2gy}}\,dy = -\max_{h\in\mathcal{A}} \int_0^b \frac{\sqrt{1 + h'(y)^2}}{\sqrt{-2gy}}\,dy\,.$$

If $\varphi \in \mathcal{N}$, then for t in a neighborhood of 0, $f + t\varphi \in \mathcal{A}$; thus

$$F(t) \equiv \int_0^b \frac{\sqrt{1 + (f + t\varphi)'(y)^2}}{\sqrt{-2gy}}\,dy$$

attains its minimum at $t = 0$. Therefore,

$$F'(0) = \frac{d}{dt}\Big|_{t=0} \int_0^b \frac{\sqrt{1 + (f + t\varphi)'(y)^2}}{\sqrt{-2gy}}\,dy = 0 \qquad \forall\, \varphi \in \mathcal{N}\,.$$

By the chain rule,

$$\int_0^b \frac{f'(y)\varphi'(y)}{\sqrt{-2gy}\sqrt{1+f'(y)^2}}\,dy = 0 \qquad \forall\,\varphi \in \mathcal{N}\,.$$

Suppose in addition that f is twice differentiable, then integration-by-parts implies that

$$-\int_0^b \Big[\frac{f'(y)}{\sqrt{-2gy}\sqrt{1+f'(y)^2}}\Big]'\varphi(y)\,dy = 0 \qquad \forall\,\varphi \in \mathcal{N}$$

which further implies that f satisfies the ODE

$$\Big[\frac{f'(y)}{\sqrt{-2gy}\sqrt{1+f'(y)^2}}\Big]' = 0$$

since $\varphi \in \mathcal{N}$ is chosen arbitrarily.

Question: What if we assume that $y = f(x)$ to start with? What equation must f satisfy?

Example 1.11 (Euler-Lagrange equation)**.** In general, we often encounter problems of the type

$$\min_{y\in\mathcal{A}} \int_0^a L(y,y',t)\,dt\,, \text{ where } \mathcal{A} = \big\{y : [0,a] \to \mathbb{R}\,\big|\,y(0) = y(a) = 0\big\}\,.$$

Write $L = L(p,q,t)$. Then the minimizer $y \in \mathcal{A}$ satisfies

$$\frac{d}{dt}L_q(y,y',t) = L_p(y,y',t)\,.$$

The equation above is called the ***Euler-Lagrange equation***.

Example 1.12 (Heat equations)**.** Let $u(x,t)$ defined on $\Omega \times (0,T]$ be the temperature of a material body at point $x \in \Omega$ at time $t \in (0,T]$, and $c(x)$, $\varrho(x)$, $k(x)$ be the specific heat, density, and the inner thermal conductivity of the material body at x. Then by the conservation of heat, for any open set $\mathcal{U} \subseteq \Omega$,

$$\frac{d}{dt}\int_{\mathcal{U}} c(x)\varrho(x)u(x,t)\,dx = \int_{\partial\mathcal{U}} k(x)\nabla u(x,t)\cdot \mathrm{N}(x)\,dS\,, \tag{1.1}$$

where N denotes the outward-pointing unit normal of $\mathcal{U}$. Assume that u is smooth, and $\mathcal{U}$ is a Lipschitz domain. By the divergence theorem, (1.1) implies

$$\int_{\mathcal{U}} c(x)\varrho(x)u_t(x,t)dx = \int_{\mathcal{U}} \mathrm{div}\big(k(x)\nabla u(x,t)\big)dx\,.$$

Since $\mathcal{U}$ is arbitrary, the equation above implies

$$c(x)\varrho(x)u_t(x,t) - \mathrm{div}(k(x)\nabla u(x,t)) = 0 \quad \forall\,x \in \Omega\,, t \in (0,T].$$

If k is constant, then

$$\frac{c\varrho}{k}u_t = \Delta u \equiv \sum_{i=1}^{\mathrm{n}} \frac{\partial^2 u}{\partial x_i^2}\,.$$

If furthermore c and ϱ are constants, then after rescaling of time we have

$$u_t = \Delta u\,. \tag{1.2}$$

This is the standard **heat equation**, the prototype equation of **parabolic** equations.

Example 1.13 (Minimal surfaces)**.** Let Γ be a closed curve in $\mathbb{R}^3$. We would like to find a surface which has minimal surface area while at the same time it has boundary Γ.

Suppose that $\Omega \subseteq \mathbb{R}^2$ is a bounded set with boundary parametrized by $(x(t), y(t))$ for $t \in I$, and Γ is a closed curve parametrized by $(x(t), y(t), f(x(t), y(t)))$. We want to find a surface having C as its boundary with minimal surface area. Then the goal is to find a function u with the property that $u = f$ on $\partial\Omega$ that minimizes the functional

$$\mathscr{A}(w) = \int_\Omega \sqrt{1 + |\nabla w|^2}\, dA\,.$$

Let $\varphi \in \mathscr{C}^1(\overline{\Omega})$, and define

$$\delta\mathcal{A}(u;\varphi) = \lim_{t\to 0} \frac{\mathcal{A}(u + t\varphi) - \mathcal{A}(u)}{t} = \int_\Omega \frac{\nabla u \cdot \nabla\varphi}{\sqrt{1 + |\nabla u|^2}}\, dx\,.$$

If u minimize $\mathcal{A}$, then $\delta\mathcal{A}(u;\varphi) = 0$ for all $\varphi \in \mathscr{C}_c^1(\Omega)$. Assuming that $u \in \mathscr{C}^2(\overline{\Omega})$, we find that u satisfies

$$\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\Big) = 0\,,$$

or expanding the bracket using the Leibnitz rule, we obtain the ***minimal surface equation***

$$(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0 \quad \forall\, (x, y) \in \Omega\,. \tag{1.3}$$

Example 1.14 (System of PDEs - the Euler equations)**.** Let $\Omega \subseteq \mathbb{R}^3$ denote a fluid container, and $\varrho(x, t), \boldsymbol{u}(x, t), p(x, t)$ denotes the fluid density, velocity and pressure at position x and time t. For a given an open subset $\mathcal{O} \subseteq \Omega$ with smooth boundary, the rate of change of the mass in $\mathcal{O}$ is the same as the mass flux through the boundary; thus

$$\frac{d}{dt}\int_{\mathcal{O}} \varrho(x, t)dx = -\int_{\partial\mathcal{O}} (\varrho\boldsymbol{u})(x, t)\cdot \mathbf{N}\, dS\,,$$

where $\mathbf{N}$ is the outward-pointing unit normal of ∂O. The divergence theorem then implies that

$$\frac{d}{dt}\int_{\mathcal{O}} \varrho(x, t)dx = -\int_{\mathcal{O}} \operatorname{div}(\varrho\boldsymbol{u})(x, t)\, dS\,.$$

If ϱ is a smooth function, then $\displaystyle\frac{d}{dt}\int_{\mathcal{O}} \varrho(x, t)dx = \int_{\mathcal{O}} \varrho_t(x, t)dx$; thus

$$\int_{\mathcal{O}} \big[\varrho_t + \operatorname{div}(\varrho\boldsymbol{u})\big](x, t)dx = 0\,.$$

Since $\mathcal{O}$ is chosen arbitrarily, we must have

$$\varrho_t + \operatorname{div}(\varrho\boldsymbol{u}) = 0 \qquad \text{in} \quad \Omega\,. \tag{1.4}$$

Equation (1.4) is called the equation of continuity.

Now we consider that conservation of momentum. Let $\boldsymbol{m} = \varrho\boldsymbol{u}$ be the momentum. The conservation of momentum states that

$$\frac{d}{dt}\int_{\mathcal{O}} \boldsymbol{m}\, dx = -\int_{\partial\mathcal{O}} \boldsymbol{m}(\boldsymbol{u}\cdot\mathbf{N})\, dS - \int_{\partial\mathcal{O}} p\mathbf{N}\, dS + \int_{\mathcal{O}} \varrho\boldsymbol{f}\, dx\,,$$

here we use the fact that the rate of change of momentum of a body is equal to the resultant force acting on the body, and with p denoting the pressure the buoyancy force is given by $\int_{\partial\mathcal{O}} p\mathbf{N}\,dS$. Here we assume that the fluid is *invicid* so that no friction force is presented in the fluid. Therefore, assuming the smoothness of the variables, the divergence theorem implies that

$$\int_{\mathcal{O}} \Big[\boldsymbol{m}_t + \sum_{j=1}^{\mathrm{n}} \frac{\partial(\boldsymbol{m}\boldsymbol{u}^j)}{\partial x_j} + \nabla p - \varrho\boldsymbol{f}\Big]dx = 0\,.$$

Since $\mathcal{O}$ is chosen arbitrarity, we obtain the momentum equation

$$(\varrho\boldsymbol{u})_t + \mathrm{div}(\varrho\boldsymbol{u}\otimes\boldsymbol{u}) = -\nabla p + \varrho\boldsymbol{f}\,. \tag{1.5}$$

Initial conditions: $\varrho(x,0) = \varrho_0(x)$ and $\boldsymbol{u}(x,0) = \boldsymbol{u}_0(x)$ for all $x \in \Omega$.
Boundary condition: $\boldsymbol{u}\cdot\mathbf{N} = 0$ on $\partial\Omega$.

1. If the density is constant (such as water), then (1.4) and (1.5) reduce to

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla p + \boldsymbol{f} \qquad \text{in} \quad \Omega\times(0,T)\,, \tag{1.6a}$$

$$\mathrm{div}\boldsymbol{u} = 0 \qquad \text{in} \quad \Omega\times(0,T)\,. \tag{1.6b}$$

   Equation (1.6) together with the initial and the boundary condition are called the *incompressible Euler equations.*

2. If the pressure p solely depends on the density; that is, $p = p(\varrho)$ (the equation of state), then (1.4) and (1.5) together with are called the *isentropic Euler equations.*

1.2 Solutions and Initial Value Problems

Recall that a general form of an n-th order ODE with t independent, y dependent, can be expressed as

$$F(t,y,y',\cdots,y^{(n)}) = 0\,, \tag{1.7}$$

where F is a function that depends on x, y, and the derivatives of y up to order n. We assume that the equation holds for all t in an open interval $I = (a,b)$. In many cases we can isolate the highest-order term $y^{(n)}$ and write equation (1.7) as

$$y^{(n)} = f(t,y,y',\cdots,y^{(n-1)})\,. \tag{1.8}$$

Definition 1.15. An ***explicit solution*** to a differential equation on I is a function $\varphi(t)$ that, when substituted for y in (1.7) or (1.8), satisfies the differential equations for all $t \in I$.

A relation $G(t,y) = 0$ (which, under certain assumptions, defines an implicit function of t) is said to be an ***implicit solution*** to equation (1.7) on the interval I if it defines one or more explicit solutions on I.

A solution to an ODE is either an explicit solution or an implicit solution of that ODE.

Example 1.16. Show that $\varphi(t) = t^2 - t^{-1}$ is an explicit solution to the linear equation $y'' - \frac{2}{t^2}y = 0$ but $\psi(t) = t^3$ is not.

Example 1.17. Show that for any choice of constants c_1 and c_2, the function $\varphi(t) = c_1 e^{-t} + c_2 e^{2t}$ is an explicit solution to the linear equation $y'' - y' = 2y = 0$.

Example 1.18. Show that the relation $t+y+e^{ty} = 0$ is an implicit solution to the nonlinear equation

$$(1 + te^{ty})y' + 1 + ye^{ty} = 0\,.$$

Example 1.19. Verify that for every constant C the relation $4t^2 - y^2 = C$ is an implicit solution to $yy' - 4t = 0$. Graph the solution curves for $C = 0, \pm 1, \pm 4$. The collection of all such solutions is called a one-parameter family of solutions.

Definition 1.20. By an ***initial value problem*** for an n-th order differential equation

$$F(t, y, y', \cdots, y^{(n)}) = 0$$

we mean: find a solution to the differential equation on an interval I that satisfies at t_0 the n-initial conditions

$$y(t_0) = y_0\,, \qquad y'(t_0) = y_1\,, \qquad \cdots \qquad y^{(n-1)}(t_0) = y_{n-1}\,,$$

where $t_0 \in I$ and $y_0, y_1, \cdots, y_{n-1}$ are given constants.

Example 1.21. Show that $\varphi(t) = \sin t - \cos t$ is a solution to the initial value problem

$$y'' + y = 0\,; \qquad y(0) = -1\,, \qquad y'(0) = 1\,.$$

Example 1.22. As shown in Example 1.17, the function $\varphi(x) = c_1 e^{-t} + c_2 e^{2t}$ is a solution to $y'' - y' - 2y = 0$ for any choice of the constants c_1 and c_2. Determine c_1 and c_2 so that the initial conditions $y(0) = 2$ and $y'(0) = -3$ are satisfied.

Remark 1.23. For an ODE $f\big(x, y, y', y'', \cdots, y^{(2n-1)}, y^{(2n)}\big) = 0$ of even order on a particular interval $[a, b]$, another set of conditions, called the boundary condition for an ODE, can be imposed. The boundary condition of the ODE $f\big(x, y, y', y'', \cdots, y^{(2n-1)}, y^{(2n)}\big) = 0$ is of the form

$$y(a) = c_1, y(b) = d_1, y'(a) = c_2, y'(b) = d_2, \cdots, y^{(n)}(a) = c_{n+1}, y^{(n)}(b) = d_{n+1}\,.$$

Theorem 1.24 (Existence and Uniqueness of Solution/Fundamental theorem of ODE)**.** *Consider the initial value problem*

$$y^{(n)} = f(t, y, y', \cdots, y^{(n-1)})\,, \quad y(t_0) = y_0\,, \quad y'(t_0) = y_1\,, \quad \cdots \quad y^{(n-1)}(t_0) = y_{n-1}\,. \tag{1.9}$$

If f and the first partial derivatives of f with respect to all its variables, possibly except t, are continuous functions in some rectangular domain $R = [a, b] \times [c_0, d_0] \times [c_1, d_1] \times \cdots \times [c_{n-1}, d_{n-1}]$ that contains the point $(t_0, y_0, y_1, \cdots, y_{n-1})$, then the initial value problem has a unique solution $\varphi(t)$ in some interval $I = (t_0 - h, t_0 + h)$ for some positive number h.

Proof. We only establish the case $n = 1$ (this is the version in the textbook), and the proof for the general case is similar since (1.9) is equivalent to $\boldsymbol{z}' = \boldsymbol{f}(t, \boldsymbol{z})$ with initial condition $\boldsymbol{z}(t_0) = \boldsymbol{z}_0$, where

$$\boldsymbol{z} = (y, y', \cdots, y^{(n-1)}), \qquad \boldsymbol{f}(t, \boldsymbol{z}) = (\boldsymbol{z}_2, \boldsymbol{z}_3, \cdots, \boldsymbol{z}_n, f(t, \boldsymbol{z})) \qquad \text{and} \qquad \boldsymbol{z}_0 = (y_0, y_1, \cdots, y_{n-1}).$$

The proof is separated into two parts.

Existence: Choose a constant $k \in (0,1)$ such that $[t_0 - k, t_0 + k] \times [y_0 - k, y_0 + k] \subseteq R$. Since $[t_0 - k, t_0 + k] \times [y_0 - k, y_0 + k]$ is closed and bounded, the continuous functions $|f|$ and $|f_y|$ attain their maximum in $[t_0 - k, t_0 + k] \times [y_0 - k, y_0 + k]$. Assume that for some $M \geqslant 1$, $\big|f(t, y)\big| + \big|f_y(t, y)\big| \leqslant M$ for all $(t, y) \in [t_0 - k, t_0 + k] \times [y_0 - k, y_0 + k]$. Let $h = k/M$ and $I = [t_0 - h, t_0 + h]$. Then for $t \in I$, define the iterative scheme (called ***Picard's iteration***)

$$\varphi_{n+1}(t) = y_0 + \int_{t_0}^{t} f\big(s, \varphi_n(s)\big)\, ds\,, \qquad \varphi_0(t) = y_0\,. \tag{1.10}$$

Note that φ_n is continuous for all $n \in \mathbb{N}$. We show that the sequence of functions $\{\varphi_n\}_{n=1}^{\infty}$ converges to a solution to (1.9).

Claim 1: For all $n \in \mathbb{N} \cup \{0\}$,

$$\big|\varphi_n(t) - y_0\big| \leqslant k \qquad \forall\, t \in I\,. \tag{1.11}$$

Proof of claim 1: We prove claim 1 by induction. Clearly (1.11) holds for $n = 0$. Now suppose that (1.11) holds for $n = N$. Then for $n = N + 1$ and $t \in I$,

$$\big|\varphi_{N+1}(t) - y_0\big| \leqslant \Big|\int_{t_0}^{t} f\big(s, \varphi_N(s)\big)\, ds\Big| \leqslant M|t - t_0| \leqslant k\,.$$

Claim 2: For all $n \in \mathbb{N} \cup \{0\}$,

$$\max_{t \in I} \big|\varphi_{n+1}(t) - \varphi_n(t)\big| \leqslant k^{n+1}\,.$$

Proof of claim 2: Let $e_{n+1}(t) = \varphi_{n+1}(t) - \varphi_n(t)$. Using (1.10) and the mean value theorem, we find that

$$e_{n+1}(t) = \int_{t_0}^{t} \big[f\big(s, \varphi_{n+1}(s)\big) - f\big(s, \varphi_n(s)\big)\big]\, ds = \int_{t_0}^{t} f_y\big(s, \xi_n(s)\big) e_n(s)\, ds$$

for some function ξ_n satisfying $\big|\xi_n(t) - y_0\big| \leqslant k$ in I (by claim 1); thus with ϵ_n denoting $\max\limits_{t \in I} \big|e_n(t)\big|$,

$$\epsilon_{n+1} \leqslant k\epsilon_n \qquad \forall\, n \in \mathbb{N}\,;$$

thus

$$\epsilon_{n+1} \leqslant k\epsilon_n \leqslant k^2\epsilon_{n-1} \leqslant \cdots \leqslant k^n\epsilon_1 = k^n \max_{t \in I} \Big|\int_{t_0}^{t} f(s, y_0)\, ds\Big| \leqslant Mhk^n = k^{n+1}\,.$$

Claim 3: The sequence of functions $\big\{\varphi_n(t)\big\}_{n=1}^{\infty}$ converges for each $t \in I$.

Proof of claim 3: Note that

$$\varphi_{n+1}(t) = y_0 + \sum_{j=0}^{n} \left[\varphi_{j+1}(t) - \varphi_j(t)\right].$$

For each fixed $t \in I$, the series $\sum\limits_{j=0}^{\infty} \left[\varphi_{j+1}(t) - \varphi_j(t)\right]$ converges absolutely (by claim 2 with the comparison test). Therefore, $\{\varphi_n(t)\}_{n=1}^{\infty}$ converges for each $t \in I$.

Claim 4: The limit function φ is continuous in I.

Proof of Claim 4: Let $\varepsilon > 0$ be given. Choose $\delta = \dfrac{\varepsilon}{2M}$. Then if $t_1, t_2 \in I$ satisfying $|t_1 - t_2| < \delta$, we must have

$$\left|\varphi_{n+1}(t_1) - \varphi_{n+1}(t_2)\right| \leqslant \left|\int_{t_1}^{t_2} f\big(s, \varphi_n(s)\big)\, ds\right| \leqslant M|t_1 - t_2| < \frac{\varepsilon}{2}.$$

Passing to the limit as $n \to \infty$, we conclude that

$$\left|\varphi(t_1) - \varphi(t_2)\right| \leqslant \frac{\varepsilon}{2} < \varepsilon \quad \forall\, t_1, t_2 \in I \text{ and } |t_1 - t_2| < \delta$$

which implies that φ is continuous in I.

Claim 5: The limit function φ satisfies $\varphi(t) = y_0 + \displaystyle\int_{t_0}^{t} f\big(s, \varphi(s)\big)\, ds$ for all $t \in I$.

Proof of claim 5: It suffices to show that

$$\lim_{n\to\infty} \int_{t_0}^{t} f\big(s, \varphi_n(s)\big)\, ds = \int_{t_0}^{t} f\big(s, \varphi(s)\big)\, ds \qquad \forall\, t \in I.$$

Let $\varepsilon > 0$ be given. Choose $N > 0$ such that $\dfrac{k^{N+2}}{1-k} < \varepsilon$. Then by claim 2 and the mean value theorem, for $n \geqslant N$,

$$\begin{aligned}\left|\int_{t_0}^{t} f\big(s, \varphi_n(s)\big)\, ds - \int_{t_0}^{t} f\big(s, \varphi(s)\big)\, ds\right| &= \left|\int_{t_0}^{t} f_y\big(s, \xi(s)\big)\left[\varphi_n(s) - \varphi(s)\right] ds\right| \\ \leqslant M\left|\int_{t_0}^{t} \sum_{j=n}^{\infty} \left|\varphi_{j+1}(s) - \varphi_j(s)\right| ds\right| &\leqslant M|t - t_0| \sum_{j=N}^{\infty} k^{j+1} \leqslant \frac{k^{N+2}}{1-k} < \varepsilon.\end{aligned}$$

Claim 6: $y = \varphi(t)$ is a solution to (1.9).

Proof of claim 6: Since φ is continuous, by the fundamental theorem of Calculus,

$$\frac{d}{dt}\left[y_0 + \int_{t_0}^{t} f\big(s, \varphi(s)\big)\, ds\right] = f\big(t, \varphi(t)\big)$$

which implies that $\varphi'(t) = f\big(t, \varphi(t)\big)$. Moreover, $\varphi(0) = y_0$; thus $y = \varphi(t)$ is a solution to (1.9).

Uniqueness: Suppose that $y = \psi(t)$ is another solution to the ODE (1.9) in the time interval I By the continuity of ψ, there must be some interval $J = (t_0 - \delta, t_0 + \delta)$ such that $\left|\psi(t) - y_0\right| \leqslant k$ in J. We first show that $\psi(t) = \varphi(t)$ for all $t \in J$, and then show that $I \subseteq J$.

Let $\vartheta = \varphi - \psi$. Then ϑ solves

$$\vartheta' = f(t,\varphi) - f(t,\psi) = f_y\big(t,\xi(t)\big)\vartheta \qquad \vartheta(t_0) = 0$$

for some ξ in between φ and ψ satisfying $|\xi(t) - y_0| \leqslant k$. Integrating in t over the time interval $[t_0, t]$ we find that

$$\vartheta(t) = \int_{t_0}^{t} f_y(s,\xi(s))\vartheta(s)\,ds\,.$$

(a) If $t > t_0$,

$$|\vartheta(t)| \leqslant \Big|\int_{t_0}^{t} \big|f_y(s,\xi(s))\big|\big|\vartheta(s)\big|\,ds\Big| \leqslant M\int_{t_0}^{t} |\vartheta(s)|\,ds\,;$$

thus the fundamental theorem of Calculus implies that

$$\frac{d}{dt}\Big(e^{-Mt}\int_{t_0}^{t}\big|\vartheta(s)\big|\,ds\Big) = e^{-Mt}\Big(|\vartheta(t)| - M\int_{t_0}^{t}|\vartheta(s)|\Big) \leqslant 0\,.$$

Therefore,

$$e^{-Mt}\int_{t_0}^{t}\big|\vartheta(s)\big|\,ds \leqslant e^{-Mt_0}\int_{t_0}^{t_0}\big|\vartheta(s)\big|\,ds = 0$$

which implies that $\vartheta(t) = 0$ for all $t \in (t_0, t_0 + \delta)$.

(b) If $t < t_0$,

$$|\vartheta(t)| \geqslant -\Big|\int_{t_0}^{t} \big|f_y(s,\xi(s))\big|\big|\vartheta(s)\big|\,ds\Big| \geqslant -M\int_{t}^{t_0} |\vartheta(s)|\,ds = M\int_{t_0}^{t}|\vartheta(s)|\,ds\,;$$

thus the fundamental theorem of Calculus implies that

$$\frac{d}{dt}\Big(e^{-Mt}\int_{t_0}^{t}\big|\vartheta(s)\big|\,ds\Big) = e^{-Mt}\Big(|\vartheta(t)| - M\int_{t_0}^{t}|\vartheta(s)|\Big) \geqslant 0\,.$$

Therefore,

$$e^{-Mt}\int_{t_0}^{t}\big|\vartheta(s)\big|\,ds \geqslant e^{-Mt_0}\int_{t_0}^{t_0}\big|\vartheta(s)\big|\,ds = 0$$

which implies that $\vartheta(t) = 0$ for all $t \in (t_0 - \delta, t_0)$.

Therefore, $\theta(t) \equiv 0$ for all $t \in J$ which implies that the solution φ equals the solution ψ in some open interval J containing t_0.

Finally, we need to argue if it is possible to have a solution $y = \psi(t)$ in the time interval I but $\big|y(t) - y_0\big| > k$ for some $t \in I$. If so, by the continuity of the solution there must be some $t_1 \in I$ such that $\big|\psi(t_1) - y_0\big| = k$. Since ψ satisfies

$$\psi' = f(t,\psi) \qquad \psi(t_1) = \varphi(t_1)\,,$$

the argument above implies that there is an open interval $\widetilde{J} \subseteq I$ in which $\varphi = \psi$. Since $y = \varphi(t)$ is a solution in the time interval J, we must have $\varphi = \psi$ in $I \cap \widetilde{J}$. In other words, $\psi(t)$ stays in $[y_0 - k, y_0 + k]$ as long as $t \in I$. This concludes the uniqueness of the solution to (1.9). $\square$

Remark 1.25. In the proof of the existence and the uniqueness theorem, the condition that f_y is continuous is not essential. This condition can be replaced by that f is (local) Lipschitz in its second variable; that is, there exists $L > 0$ such that

$$\big|f(t, y_1) - f(t, y_2)\big| \leqslant L|y_1 - y_2|\,.$$

Example 1.26. Consider the initial value problem

$$3y' = t^2 - ty^3\,, \qquad y(1) = 6\,.$$

Let $f(t, y) = \frac{1}{3}(t^2 - ty^3)$. Then f and f_y are continuous in $\mathbb{R}^2$; thus the fundamental theorem of ODE provides the existence and uniqueness of the solution in an interval about 1 to the ODE above.

Example 1.27. Consider the initial value problem

$$y' = y^2\,, \qquad y(0) = 1\,. \tag{1.12}$$

Let $f(t, y) = y^2$. Then f and f_y are continuous in $\mathbb{R}^2$; thus the fundamental theorem of ODE provides the existence and uniqueness of the solution in an interval about 0 to (1.12). In fact,

$$y(t) = \frac{1}{1-t} \tag{1.13}$$

satisfies $y' = y^2$ and the initial data $y(0) = 1$; thus the unique solution to (1.12) is given by (1.13) which blows up in finite time. Therefore, even if f and f_y are continuous in any rectangle containing (t_0, y_0), the ODE $y' = f(t, y)$ with initial data $y(t_0) = y_0$ might not have a solution that exists for all time.

Example 1.28. Consider the initial value problem

$$y' = 3y^{\frac{2}{3}}\,, \qquad y(2) = 0\,. \tag{1.14}$$

The fundamental theorem of ODE cannot be applied since the function $f(t, y) \equiv 3y^{\frac{2}{3}}$ does not have continuous partial derivative in any rectangle containing $(2, 0)$. In fact, $\phi_1(t) = 0$ and $\phi_2(t) = (t-2)^3$ are solutions to (1.14). Moreover, for every $a > 2$, the function

$$\phi(t) = \begin{cases} (x-a)^3 & \text{if } x > a\,, \\ 0 & \text{if } x \leqslant a \end{cases}$$

is also a solution to (1.14). Therefore, the initial value problem (1.14) has infinitely many solutions.

Example 1.29. Solve the initial value problem $y' = 2t(1 + y)$ with initial data $y(0) = 0$ using the Picard iteration.

Recall the Picard iteration

$$\varphi_{k+1}(t) = \int_0^t 2s\big(1 + \varphi_k(s)\big)\,ds \qquad \text{with } \ \varphi_0(t) = 0. \tag{1.15}$$

Then $\varphi_1(t) = \int_0^t 2s\,ds = t^2$, and $\varphi_2(t) = \int_0^t 2s(1+s^2)\,ds = t^2 + \frac{t^4}{2}$, and then $\varphi_3(t) = \int_0^t 2s\big(1+s^2+\frac{s^4}{2}\big)\,ds = t^2 + \frac{t^4}{2} + \frac{t^6}{6}$. To see a general rule, we observe that $\varphi_k(t)$ must be a polynomial of the form

$$\varphi_k(t) = \sum_{j=1}^{k} a_j t^{2j}\,,$$

and $\varphi_{k+1}(t) = \varphi_k(t) + a_{k+1}t^{2(k+1)}$. Therefore, we only need to determine the coefficients a_k in order to find the solution. Note that using (1.15) we have

$$\sum_{j=1}^{k+1} a_j t^{2j} = \int_0^t 2s\big(1+\sum_{j=1}^{k} a_j t^{2j}\big)\,ds = t^2 + \sum_{j=1}^{k} \frac{2a_j}{2j+2} t^{2j+2} = t^2 + \sum_{j=2}^{k+1} \frac{a_{j-1}}{j} t^{2j}\,;$$

thus the comparison of coefficients implies that $a_1 = 1$, $a_j = \frac{a_{j-1}}{j}$. Therefore,

$$a_k = \frac{a_{k-1}}{k} = \frac{a_{k-2}}{k(k-1)} = \cdots = \frac{a_1}{k(k-1)\cdots 2} = \frac{1}{k!}$$

which implies that $\varphi_k(t) = \sum\limits_{j=1}^{k} \frac{t^{2j}}{j!} = \sum\limits_{j=0}^{k} \frac{t^{2j}}{j!} - 1$. Using the Maclaurin series of the exponential function, we find that $\varphi_k(t)$ converges to $e^{t^2} - 1$. The function $\varphi(t) = e^{t^2} - 1$ is indeed a solution of the ODE under consideration.

1.3 Direction Fields

A direction field is in particular very useful in the study of first order differential equations of the type:

$$\frac{dy}{dt} = f(t,y)\,,$$

where f is a scalar function. A plot of short line segments (with equal length) drawn at various points in the ty-plane showing the slope of the solution curve there is called a ***direction field*** for the differential equation.

Example 1.30. Plot the direction field for the ODE $\frac{dy}{dt} = -\frac{y}{t}$. Note that for every constant c, the relation $ty = c$ is an implicit solution to the ODE.

Example 1.31. Plot the direction field for the ODE $\frac{dy}{dt} = 3y^{\frac{2}{3}}$.

1.3.1 The method of isoclines

The method of isocline can be used to plot the direction field for differential equations. An ***isocline*** for the differential equation $y' = f(t,y)$ is a set of points in ty-plane where all the solutions have the same slope $\frac{dy}{dt}$; thus it is a level curve for the function $f(t,y)$.

To implement the method of isoclines for sketching direction fields, we draw hash marks with slope c along the isocline $f(t,y) = c$ for a few selected value of c.

Example 1.32. Plot the direction field for the ODE $\frac{dy}{dt} = t^2 - y$. Show that for each constant c, the function $\phi(t) = ce^{-t} + t^2 - 2t + 2$ is an explicit solution to the ODE.

Example 1.33. Consider a falling object whose velocity satisfies the ODE

$$m\frac{dv}{dt} = mg - bv\,.$$

Plot the direction field for the ODE above.

1.3.2 Integral Curves

The so-called integral curves of an ODE is related to the direction field in the sense that at each point of each integral curve, the direction field (at that point) is tangent to the curve.

Definition 1.34. A curve C is said to be an ***integral curve*** of the ODE $\frac{dy}{dx} = f(x, y)$ if there exists a parametrization $(x(t), y(t))$ of C, where where t belongs to some parameter domain I, such that

$$y'(t) = f(x(t), y(t))x'(t) \qquad \forall\, t \in I\,.$$

2 First Order Differential Equations

In general, a first order ODE can be written as

$$\frac{dy}{dt} = f(t, y)$$

for some function f. In this chapter, we are going to solve the linear equation above explicitly with

1. $f(t, y) = g(y)h(t)$;
2. $f(t, y) = p(t)y + q(t)$;
3. $f(t, y) = -F_x(x, y)/F_y(x, y)$ for some function F;
4. $f(t, y) = g(y/t)$ for some function g;

and more.

2.1 Introduction: Motion of a Falling Body

The equation of falling body is a first order ODE

$$m\frac{dv}{dt} = mg - bv\,. \tag{2.1}$$

The technique of ***separation of variables*** (which will be detailed in the next section) implies that

$$\frac{dv}{mg - bv} = \frac{dt}{m}\,.$$

Integrating both sides, we obtain that

$$-\frac{1}{b}\log|mg - bv| = \frac{t}{m} + C$$

for some constant C. Therefore,

$$|mg - bv| = e^{-bC}e^{-\frac{bt}{m}}$$

which implies that there exists A such that $mg - bv = Ae^{-\frac{bt}{m}}$. Therefore,

$$v = \frac{mg}{b} - \frac{A}{b}e^{-\frac{bt}{m}}\,.$$

To determine A, suppose one initial condition

$$v(0) = v_0 \tag{2.2}$$

is imposed. Then

$$v_0 = \frac{mg}{b} - \frac{A}{b} \qquad \text{or equivalently,} \qquad -\frac{A}{b} = v_0 - \frac{mg}{b}\,;$$

thus we conclude that the IVP (2.1 + (2.2) has a solution

$$v = v(t) = \frac{mg}{b} + \Big(v_0 - \frac{mg}{b}\Big)e^{-\frac{bt}{m}}\,.$$

Some information that we obtain from the form of the solution above:

• As $t \to \infty$, the velocity reaches a so-called ***terminal velocity*** $\frac{mg}{b}$. Since the decay is exponential, the falling object reaches the terminal velocity very quickly. The heavier the object, the faster the terminal velocity.

2.2 Separable Equations

Definition 2.1. The ODE $y' = f(t, y)$ is said to be ***separable*** if $f(t, y) = g(y)h(t)$ for some functions g and h.

Suppose that we are given a first order linear equation

$$\frac{dy}{dt} = g(y)h(t) \quad \text{with initial condition } y(t_0) = y_0\,,$$

where $1/g$ is assume to be integrable. Let G be an anti-derivative of $1/g$. Then

$$\begin{aligned}
&\frac{dy}{dt} = g(y)h(t) \Rightarrow \frac{1}{g(y)}\frac{dy}{dt} = h(t) \Rightarrow G'(y)\frac{dy}{dt} = h(t)\\
&\Rightarrow \frac{d}{dt}G(y(t)) = h(t) \Rightarrow \int_a^t \frac{d}{ds}G(y(s))ds = h(t) \Rightarrow G(y(t)) - G(y(t_0)) = \int_{t_0}^t h(s)ds\\
&\Rightarrow G(y(t)) = G(y_0) + \int_{t_0}^t h(s)ds\,,
\end{aligned}$$

and y can be solved if the inverse function of G is known.

Computations in shorthand: To solve the equation $\frac{dy}{dt} = g(y)h(t)$, we formally write

$$\frac{dy}{g(y)} = h(t)dt\,.$$

If G is an anti-derivative of $1/g$ and H is an anti-derivative of h; that is, $\frac{dG}{dy} = \frac{1}{g(y)}$ and $\frac{dH}{dt} = h(t)$, then the equality above implies that

$$dG = dH\,.$$

Therefore, $G(y) = H(t) + C$ for some constant C which can be determined by the initial condition (if there is one).

Example 2.2. Let y be a solution to the ODE $\frac{dy}{dx} = \frac{x^2}{1-y^2}$. Then x, y satisfies $x^3 + y^3 - 3y = C$ for some constant C.

Example 2.3. Let y be a solution to the ODE $\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}$ with initial data $y(0) = -1$. Then $y = 1 - \sqrt{x^3+2x^2+2x+4}$.

Definition 2.4 (Integral Curves)**.** Let $\mathbf{F} = (F_1, \cdots, F_n)$ be a vector field. A parametric curve $\boldsymbol{x}(t) = \big(x_1(t), \cdots, x_n(t)\big)$ is said to be an ***integral curve*** of $\mathbf{F}$ if it is a solution of the following autonomous system of ODEs:

$$\begin{aligned} \frac{dx_1}{dt} &= F_1(x_1, \cdots, x_n)\,, \\ &\vdots \\ \frac{dx_n}{dt} &= F_n(x_1, \cdots, x_n)\,. \end{aligned}$$

In particular, when $n = 2$, the autonomous system above is reduced to

$$\frac{dx}{dt} = F(x,y)\,, \qquad \frac{dy}{dt} = G(x,y) \tag{2.3}$$

for some function F, G. Since at each point $(x_0, y_0) = \big(x(t_0), y(t_0)\big)$ on the integral curve,

$$\frac{dy}{dx}\Big|_{(x,y)=(x_0,y_0)} = \frac{dy/dt}{dx/dt}\Big|_{t=t_0}$$

if $\frac{dx}{dt}\Big|_{t=t_0} \neq 0$, instead of finding solutions to (2.3) we often solve

$$\frac{dy}{dx} = \frac{G(x,y)}{F(x,y)}\,.$$

Example 2.5. Let $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field defined by $\boldsymbol{F}(x,y) = \big(F_1(x,y), F_2(x,y)\big) = (-y, x)$. Then the parametric curve $(\cos t, \sin t)$ is an integral curve of $\boldsymbol{F}$ since $\big(x_1(t), x_2(t)\big) = (\cos t, \sin t)$ satisfies

$$\begin{aligned} x_1'(t) &= -\sin t = -x_2(t) = F_1\big(x_1(t), x_2(t)\big)\,, \\ x_2'(t) &= \cos t = x_1(t) = F_2\big(x_1(t), x_2(t)\big)\,. \end{aligned}$$

Example 2.6. Find the integral curve of the vector field $\mathbf{F}(x,y) = (4+y^3, 4x-x^3)$ passing through $(0,1)$. Answer: $y^4 + 16y + x^4 - 8x^2 = 17$.

2.3 Linear Equations; Method of Integrating Factors

Suppose that we are given a first order linear equation

$$\frac{dy}{dt} + p(t)y = q(t) \quad \text{with initial condition } y(t_0) = y_0 \,.$$

Let $P(t)$ be an anti-derivative of $p(t)$; that is, $P'(t) = p(t)$. Then

$$\begin{aligned} & e^{P(t)}\Big(\frac{dy}{dt} + P'(t)y\Big) = e^{P(t)}q(t) \Rightarrow \frac{d}{dt}\Big(e^{P(t)}y(t)\Big) = e^{P(t)}q(t) \\ \Rightarrow & \int_{t_0}^{t} \frac{d}{ds}\Big(e^{P(s)}y(s)\Big)ds = \int_{t_0}^{t} e^{P(s)}Q(s)ds \Rightarrow e^{P(t)}y(t) - e^{P(t_0)}y(t_0) = \int_{t_0}^{t} e^{P(s)}Q(s)ds \\ \Rightarrow & \, y(t) = e^{P(a)-P(t)}y_0 + \int_{t_0}^{t} e^{P(s)-P(t)}Q(s)ds \,. \end{aligned}$$

How about if we do not know what the initial data is? Then

$$e^{P(t)}\Big(\frac{dy}{dt} + P'(t)y\Big) = e^{P(t)}q(t) \Rightarrow \frac{d}{dt}\Big(e^{P(t)}y(t)\Big) = e^{P(t)}q(t) \Rightarrow e^{P(t)}y(t) = C + \int e^{P(t)}q(t)dt \,,$$

where $\int e^{P(t)}q(t)dt$ denotes an anti-derivative of $e^P Q$. Therefore,

$$y(t) = Ce^{-P(t)} + e^{-P(t)} \int e^{P(t)}q(t)dt \,.$$

Example 2.7. Solve $\frac{dy}{dt} + \frac{1}{2}y = \frac{1}{2}e^{t/3}$. Answer: $y(t) = \frac{3}{5}e^{t/3} + Ce^{-t/2}$.

Example 2.8. Solve $\frac{dy}{dt} - 2y = 4 - t$. Answer: $y(t) = -\frac{7}{4} + \frac{1}{2}t + Ce^{2t}$.

Example 2.9. Solve $\frac{1}{t}\frac{dy}{dt} - \frac{y}{t^2} = t\cos t$, where $t > 0$. Answer: $y(t) = t^2 \sin t + Ct^2$.

Example 2.10. Solve $ty' + 2y = 4t^2$ with $y(1) = 2$. Answer: $y(t) = t^2 + \frac{1}{t^2}$.

2.4 Exact Equations and Integrating Factors

In this section, we focus on solving $\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}$ for M, N satisfying some special relation.

Recall vector calculus:

Definition 2.11 (Vector fields)**.** A ***vector field*** is a vector-valued function whose domain and range are subsets of Euclidean space $\mathbb{R}^n$.

Definition 2.12 (Conservative vector fields)**.** A vector field $\mathbf{F} : \mathcal{D} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is said to be ***conservative*** if $\mathbf{F} = \nabla\varphi$ for some scalar function φ. Such a φ is called a (scalar) potential for $\mathbf{F}$ on $\mathcal{D}$.

Theorem 2.13. *If* $\mathbf{F} = (M, N)$ *is a conservative vector field in a domain* $\mathcal{D}$*, then* $N_x = M_y$ *in* $\mathcal{D}$*.*

Theorem 2.14. *Let $\mathcal{D}$ be an open, connected domain, and let $\mathbf{F}$ be a smooth vector field defined on $\mathcal{D}$. Then the following three statements are equivalent:*

1. $\mathbf{F}$ *is conservative in* $\mathcal{D}$.
2. $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ *for every piecewise smooth, closed curve* C *in* $\mathcal{D}$.
3. *Given and two point* $P_0, P_1 \in \mathcal{D}$, $\int_C \mathbf{F} \cdot d\mathbf{r}$ *has the same value for all piecewise smooth curves in* D *starting at* P_0 *and ending at* P_1.

Definition 2.15. A connected domain $\mathcal{D}$ is said to be ***simply connected*** if every simple closed curve can be continuously shrunk to a point in $\mathcal{D}$ without any part ever passing out of $\mathcal{D}$.

Theorem 2.16. *Let $\mathcal{D}$ be a simply connected domain, and M, N, M_y, N_x be continuous in $\mathcal{D}$. If $M_y = N_x$, then $\mathbf{F} = (M, N)$ is conservative.*

Sketch of the proof. Since $N_x = M_y$,

$$\begin{aligned} N(x,y) &= N(x_0,y) + \int_{x_0}^{x} M_y(z,y)\, dz = N(x_0,y) + \frac{\partial}{\partial y} \int_{x_0}^{x} M(z,y)\, dz \\ &= \frac{\partial}{\partial y}\Big[\Psi(y) + \int_{x_0}^{x} M(z,y)\, dz\Big], \end{aligned}$$

where $\Psi(y)$ is an anti-derivative of $N(x_0, y)$. Let $\varphi(x,y) = \Psi(y) + \int_{x_0}^{x} M(z,y)\, dz$. Then clearly $(M, N) = \nabla\varphi$ which implies that $\mathbf{F} = (M, N)$ is conservative. □

Combining Theorem 2.13 and 2.16, in a simply connected domain a vector field $\mathbf{F} = (M, M)$ is conservative if and only if $M_y = N_x$.

Example 2.17. Let $\mathcal{D} = \mathbb{R}^2 \backslash \{(0,0)\}$, and $M(x,y) = \frac{-y}{x^2+y^2}$, $N(x,y) = \frac{x}{x^2+y^2}$. Then $M_y = N_x = \frac{y^2-x^2}{(x^2+y^2)^2}$ in $\mathcal{D}$; however, $F \neq \nabla\varphi$ for some scalar function φ for if there exists such a φ, φ, up to adding a constant, must be identical to the polar angle $\theta(x,y) \in [0, 2\pi)$.

Now suppose that we are given a differential equation of the form

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)},$$

in which separation of variables is not possible. We note that this is equivalent of finding integral curves of the vector field $\mathbf{F} = (-N, M)$.

Definition 2.18. Let $\mathcal{D} \subseteq \mathbb{R}^2$ be open, and $M, N : \mathcal{D} \to \mathbb{R}$ be continuous. An ODE of the form $\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}$ (or the *differential form* $M(x,y)dx + N(x,y)dy$) is called ***exact*** in $\mathcal{D}$ if there exists a continuously differentiable function $\varphi : \mathcal{D} \to \mathbb{R}$, called the ***potential function***, such that $\varphi_x = M$ and $\varphi_y = N$.

Definition 2.19. A function $\mu = \mu(x, y)$ is said to be an ***integrating factor*** of the differential form $M(x, y)dx + N(x, y)dy$ if $(\mu M)(x, y)dx + (\mu N)(x, y)dy$ is exact (in the domain of interest).

To solve the ODE

$$\frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)}, \tag{2.4}$$

the following two possibilities are most possible situations:

1. If $M_y = N_x$ in a simply connected domain $\mathcal{D}$, then Theorem 2.16 implies that the ODE (2.4) is exact in a simply connected domain $\mathcal{D} \subseteq \mathbb{R}^2$; that is, there exists a potential function φ such that $\nabla\varphi = (M, N)$. Then (2.4) can be rewritten as

$$\varphi_x(x, y) + \varphi_y(x, y)\frac{dy}{dx} = 0\,;$$

   and if $(x(t), y(t))$ is an integral curve, we must have

$$\varphi_x(x(t), y(t))\frac{dx}{dt} + \varphi_y(x(t), y(t))\frac{dy}{dt} = 0 \quad \text{or equivalently,} \quad \frac{d}{dt}\varphi(x(t), y(t)) = 0\,.$$

   Therefore, integral curve satisfies $\varphi(x, y) = C$.

2. If $M_y \neq N_x$, we look for a function μ such that $(\mu M)_y = (\mu N)_x$ in a simply connected domain $\mathcal{D} \subseteq \mathbb{R}^2$. Such a μ always exists (in theory, but may be hard to find the explicit expression), and such a μ μ satisfies the PDE

$$M\mu_y - N\mu_x + (M_y - N_x)\mu = 0\,. \tag{2.5}$$

   Usually solving a PDE as above is as difficult as solving the original ODE.

Example 2.20. Solve $\dfrac{dy}{dx} = -\dfrac{y\cos x + 2xe^y}{\sin x + x^2e^y - 1}$.

Let $M(x, y) = y\cos x + 2xe^y$ and $N(x, y) = \sin x + x^2e^y - 1$. Then $M_y(x, y) = \cos x + 2xe^y = N_x(x, y)$; thus the ODE above is exact in $\mathbb{R}^2$. To find the potential function φ, due to the fact that $\varphi_x = M$ we find that

$$\varphi(x, y) = \Psi(y) + \int M(x, y)dx = \Psi(y) + y\sin x + x^2e^y$$

for some function Ψ. By $\varphi_y = N$, we must have $\Psi'(y) = -1$. Therefore, $\Psi(y) = -y + C$; thus the potential function φ is

$$\varphi(x, y) = y\sin x + x^2e^y - y + C\,.$$

Example 2.21. Show that $\mu(x, y) = xy^2$ is an integrating factor for the differential form

$$(2y - 6x)dx + (3x - 4x^2y^{-1})dy\,.$$

Let $M(x, y) = 2y - 6x$ and $N(x, y) = 3x - 4x^2y^{-1}$. Then

$$(\mu M)_y = \frac{\partial}{\partial y}\big[xy^2(2y - 6x)\big] = 6xy^2 - 12x^2y$$

and

$$(\mu N)_x = \frac{\partial}{\partial x}\big[xy^2(3x - 4x^2y^{-1})\big] = 6xy^2 - 12x^2y\,.$$

Therefore, μ is an integrating factor since $(\mu M)_y = (\mu N)_x$ on a simply connected domain $\mathbb{R}^2$.

Now we find a potential function φ; that is, we look for a continuously differentiable function φ such that $\varphi_x = \mu M$ and $\varphi_y = \mu N$. Since $\mu M = xy^2(2y - 6x) = 2xy^3 - 6x^2y^2$,

$$\varphi(x, y) = \psi(y) + x^2y^3 - 2x^3y^2$$

for some function ψ. Using the identity $\varphi_y = \mu N$, we find that ψ satisfies

$$\psi'(y) + 3x^2y^2 - 4x^3y = \varphi_y = \mu N = 3x^2y^2 - 4x^3y\,.$$

Therefore, $\psi = C$ and the potential function has the form $\varphi(x, y) = x^2y^3 - 2x^3y^2 + C$.

2.5 Special Integrating Factors

There are two special cases in which it is easy to find integrating factors.

1. $\dfrac{M_y - N_x}{N}$ is a continuous function of x but independent of y: then we can assume that μ is a function of x, and (2.5) implies that μ satisfies

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu\,. \tag{2.6}$$

2. $\dfrac{M_y - N_x}{M}$ is a continuous function of y but independent of x: then we can assume that μ is a function of y, and (2.5) implies that μ satisfies

$$\frac{d\mu}{dy} = \frac{N_x - M_y}{M}\mu\,. \tag{2.7}$$

Example 2.22. Solve $\dfrac{dy}{dx} = -\dfrac{3xy + y^2}{x^2 + xy}$.

Let $M(x, y) = 3xy + y^2$ and $N(x, y) = x^2 + xy$. Then $M_y - N_x = x + y$ so $Mdx + Ndy$ is not exact. We observe that $\dfrac{M_y - N_x}{N} = \dfrac{1}{x}$ which is a function of x, we can assume that the integrating factor μ is only a function of x, and (2.6) implies that μ satisfies

$$\frac{d\mu}{dx} = \frac{1}{x}\mu\,;$$

thus $\mu(x) = x$. This choice of μ validates that $(\mu M)_y = (\mu N)_x$ in $\mathbb{R}^2$; thus $\mu Mdx + \mu Ndy$ is exact. We note that the potential function φ has the form

$$\varphi(x, y) = x^3y + \frac{x^2y^2}{2} + C\,.$$

Therefore, we find an implicit solution

$$x^3y + \frac{x^2y^2}{2} = C\,.$$

One can also verify that the function $\mu(x, y) = \dfrac{1}{xy(2x + y)}$ is also a valid integrating factor (in some domain $\mathcal{D} \subseteq \mathbb{R}^2$).

Example 2.23. Solve $\frac{dy}{dx} = -\frac{2x^2 + y}{x^2 y - x}$.

Let $M(x, y) = 2x^2 + y$ and $N(x, y) = x^2 y - x$. Then $M_y - N_x = 2 - 2xy$ which implies that $\frac{M_y - N_x}{N} = -\frac{2}{x}$. Therefore, we assume that $\mu = \mu(x)$ satisfies

$$\frac{d\mu}{dx} = -\frac{2}{x}\mu \,.$$

The separation of variables then provides that $\mu(x) = x^{-2}$, and the potential function for the differential form $\mu M dx + \mu N dy$ has the form

$$\varphi(x, y) = 2x - \frac{y}{x} + \frac{y^2}{2} + C \,.$$

thus we obtain an implicit solution $2x - \frac{y}{x} + \frac{y^2}{2} = C$.

Remark 2.24. It is possible to lose or gain solutions when multiplying and dividing by $\mu(x, y)$.

2.6 Substitution and Transformations

2.6.1 Homogeneous equations

Definition 2.25. The ODE $\frac{dy}{dx} = f(x, y)$ is said to be ***homogeneous*** if $f(x, y) = g\big(\frac{y}{x}\big)$ for some function g.

Example 2.26. The equation $(x - y)dx + xdy = 0$ is homogeneous since it can be rewritten as

$$\frac{dy}{dx} = -\frac{x - y}{x} = -1 + \frac{y}{x}$$

and the right-hand side is a function of $\frac{y}{x}$.

Test for homogeneity: The ODE $\frac{dy}{dx} = f(x, y)$ is homogeneous if and only if $f(tx, ty) = f(x, y)$ for all $t \neq 0$.

To solve a homogeneous equation $\frac{dy}{dx} = g\big(\frac{y}{x}\big)$, we let $v = \frac{y}{x}$ and substitute in $\frac{dy}{dx} = v + x\frac{dv}{dx}$ to obtain that

$$x\frac{dv}{dx} + v = g(v) \,.$$

The new equation is separable so that we can apply the separation of variables to obtain a solution.

Example 2.27. Solve $(xy + y^2 + x^2)dx - x^2 dy = 0$.

If $x \neq 0$, we can rewrite the ODE above as $\frac{dy}{dx} = \frac{xy + y^2 + x^2}{x^2} = \frac{y}{x} + \frac{y^2}{x^2} + 1$; thus the ODE is homogeneous. Therefore, letting $v = \frac{y}{x}$, we find that

$$x\frac{dv}{dx} = v^2 + 1 \,;$$

thus the separation of variables implies that v and x satisfies

$$\frac{dv}{v^2 + 1} = \frac{dx}{x} \,.$$

Therefore, $\tan^{-1} v = \log|x| + C$ or $v = \tan(\log|x| + C)$. We also note that $x = 0$ is a solution.

2.6.2 Equations of the form $\dfrac{dy}{dx} = G(ax+by)$

To solve $\dfrac{dy}{dx} = G(ax+by)$, we let $z = ax+by$ and find that

$$\frac{dz}{dx} = a + b\frac{dy}{dx} = a + bG(z)\,.$$

The separation of variable can be used to solve the ODE above.

Example 2.28. Solve $\dfrac{dy}{dx} = y - x - 1 + \dfrac{1}{x-y+2}$. Letting $z = x - y$, we find that

$$\frac{dz}{dx} = 1 - \frac{dy}{dx} = 1 - \Big(y - x - 1 + \frac{1}{x-y+2}\Big) = 2 + z - \frac{1}{z+2} = \frac{z^2+4z+3}{z+2}\,.$$

Therefore, the separation of variables implies that z and x satisfy

$$\frac{z+2}{z^2+4z+3}dz = dx\,.$$

Using the partial fraction,

$$\frac{z+2}{z^2+4z+3} = \frac{z+2}{z^2+4z+3} = \frac{1}{2}\Big(\frac{1}{z+3} + \frac{1}{z+1}\Big)\,;$$

thus

$$\frac{1}{2}\log|z^2+4z+3| = x + C\,.$$

This implies that

$$(z+2)^2 - 1 = \pm Ce^{2x} = Ce^{2x}$$

which shows that an implicit solution to the given ODE is

$$(x-y+2)^2 = Ce^{2x} + 1\,.$$

2.6.3 Bernoulli equations

Definition 2.29. A first order ODE that can be written in the form

$$\frac{dy}{dx} + p(x)y = q(x)y^r\,,$$

where $p(x)$ and $q(x)$ are continuous on an interval (a,b) and $r \in \mathbb{R}$, is called a ***Bernoulli equation***.

To solve a Bernoulli equation, we focus only on the case that $r \neq 0, 1$ (for otherwise we can solve using the method of integrating factor for $r = 0$ or separation of variable for $r = 1$). Let $v = y^{1-r}$. Then

$$\begin{aligned}\frac{dv}{dx} &= (1-r)y^{-r}\frac{dy}{dx} = (1-r)y^{-r}\big[q(x)y^r - p(x)y\big] = (1-r)\big[q(x) - p(x)y^{1-r}\big]\\ &= (1-r)\big[q(x) - p(x)v\big]\end{aligned}$$

which can be solved using the method of integrating factor.

Remark 2.30. We note that when $r > 0$, $y \equiv 0$ is also a solution to the Bernoulli equation.

Example 2.31. Solve $\dfrac{dy}{dx} - 5y = -\dfrac{5}{2}xy^3$.

Let $v = y^{-2}$. Then

$$\frac{dv}{dx} = -2y^{-3}\frac{dy}{dx} = -2y^{-3}\big(5y - \frac{5}{2}xy^3\big) = -10v + 5x$$

or equivalently,

$$\frac{dv}{dx} + 10v = 5x\,.$$

Using the method of integrating factor,

$$\frac{d}{dx}\big(e^{10x}v\big) = 5xe^{10x}\,;$$

thus

$$e^{10x}v = 5\int xe^{10x}\,dx = 5\Big[\frac{xe^{10x}}{10} - \frac{1}{10}\int e^{10x}\,dx\Big] = \frac{xe^{10x}}{2} - \frac{e^{10x}}{20} + C\,.$$

Therefore, using $v = y^{-2}$ we obtain that

$$y^{-2} = \frac{x}{2} - \frac{1}{20} + Ce^{-10x}\,.$$

2.6.4 Equations with linear coefficients

Consider the ODE

$$(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0\,.$$

1. If $b_1 = a_2$, then there exists a scalar potential such that

$$\nabla\varphi(x,y) = (a_1x + b_1y + c_1, a_2x + b_2y + c_2)\,,$$

   and an implicit solution is given by $\varphi(x,y) = c$.

2. If $a_1b_2 = a_2b_1$, then

$$\frac{dy}{dx} = -\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} = G(ax + by)$$

   for some $a, b \in \mathbb{R}$.

3. If $a_1b_2 \neq a_2b_1$, then there exists h, k such that

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

   Let $u = x + h$ and $v = y + k$. Then

$$\frac{dv}{du} = -\frac{a_1u + b_1v}{a_2u + b_2v} = -\frac{a_1 + b_1\frac{v}{u}}{a_2 + b_2\frac{v}{u}} = g\big(\frac{v}{u}\big)\,.$$

   The equation above is homogeneous and we can solve by the change of variable $w = \dfrac{v}{u}$.

Example 2.32. Solve $(-3x + y + 6)dx + (x + y + 2)dy = 0$. Answer: An implicit solution is given by $(y+3)^2 + 2(x-1)(y+3) - 3(x-1)^2 = c$.

3 Numerical Methods Involving First-Order Equations

3.1 Numerical Approximations: Euler's Method

The goal in this section is to solve the ODE

$$\frac{dy}{dt} = f(t,y) \qquad y(t_0) = y_0 \tag{3.1}$$

numerically (meaning, programming in computer to produce an approximation of the solution) in the time interval $[t_0, t_0 + T]$.

Let Δt denote the time step size (which mean we only care what the approximated solution is at time $t_k = t_0 + k\Delta t$ for all $k \in \mathbb{N}$). Since $\frac{dy}{dt}(t_k) \approx \frac{y(t_{k+1}) - y(t_k)}{\Delta t}$ when $\Delta t \approx 0$, we substitute $\frac{y(t_{k+1}) - y(t_k)}{\Delta t}$ for $\frac{dy}{dt}(t_k)$ and obtain

$$y(t_{k+1}) \approx y(t_k) + f\big(t_k, y(t_k)\big)\Delta t \qquad \forall\, k \in \mathbb{N}\,.$$

The ***forward/explicit Euler method*** is the iterative scheme

$$y_{k+1} = y_k + f(t_k, y_k)\Delta t \qquad \forall\, k \in \Big\{1, 2, \cdots, \big[\frac{T}{\Delta t}\big] - 1\Big\}\,, \quad y_0 \text{ ``}=\text{'' } y_0 \text{ (in theory)}\,. \tag{3.2}$$

Example 3.1. Use the Euler method with step size $\Delta t = 0.1$ to approximate the solution to the initial value problem

$$y' = t\sqrt{y}\,, \qquad y(1) = 4\,.$$

Let $f(t,y) = t\sqrt{y}$ and $(t_0, y_0) = (1, 4)$. The Euler method provides an approximating sequence $\{y_k\}_{k\in\mathbb{N}}$ satisfying $y_0 = 4$ and

$$y_{k+1} = y_k + \Delta t f(1 + k\Delta t, y_k) = y_k + 0.1(1 + k\Delta t)\sqrt{y_k}\,.$$

Then $y_1 = 4 + 0.2 = 4.2$ while $y_2 = 4.2 + 0.1(1.1)\sqrt{4.2} \approx 4.42543$, and etc.

k	t_k	y_k Determined by Euler's Method	Exact Value of $y(t_k)$
0	1	4	4
1	1.1	4.2	4.21276
2	1.2	4.42543	4.45210
3	1.3	4.67787	4.71976
4	1.4	4.95904	5.01760
5	1.5	5.27081	5.34766

Table 1: Computations for $y' = t\sqrt{y}$, $y(1) = 4$

In order to study the "convergence" of a numerical method, we define the global truncation error (associated with a numerical scheme) as follows.

Definition 3.2. Let $\{y_k\}_{k=1}^{\infty}$ be a sequence of numerical solution obtained by a specific numerical method (with step size $\Delta t > 0$ fixed) of solving ODE $y' = f(t, y)$ with initial data $y(t_0) = y_0$. At each time step $t_k = t_0 + k\Delta t$, the ***global truncation error*** (associated with this numerical method) is the number $e_k(\Delta t) = y(t_k) - y_k$.

Therefore, to see if a numerical method produces good approximation of the exact solution, we check if the global truncation error converges to 0 for all $k\big(\leqslant T/\Delta t\big)$ as $\Delta t \to 0$.

Assume that f is bounded and has bounded continuous partial derivatives f_t and f_y; that is, f_t and f_y are continuous and for some constant $M > 0$ $|f(t, y)| + |f_t(t, y)| + |f_y(t, y)| \leqslant M$ for all t, y. Then the fundamental theorem of ODE provides a unique continuously differentiable solution $y = y(t)$ to (3.1). Since f_t and f_y are continuous, we must have that y is twice continuously differentiable and we have

$$y'' = f_t(t, y) + f_y(t, y)y' .$$

By Taylor's theorem, for some $\theta_k \in (0, 1)$ we have

$$\begin{aligned} y(t_{k+1}) &= y(t_k) + y'(t_k)\Delta t + \frac{1}{2}(\Delta t)^2 y''(t_k + \theta_k\Delta t) \\ &= y(t_k) + f\big(t_k, y(t_k)\big)\Delta t + \frac{(\Delta t)^2}{2}\big[f_t + f_y f\big](t_k + \theta_k\Delta t, y(t_k + \theta_k\Delta t)) ; \end{aligned}$$

thus we conclude that

$$y(t_{k+1}) = y(t_k) + f\big(t_k, y(t_k)\big)\Delta t + \tau_k\Delta t$$

for some τ_k satisfying $|\tau_k| \leqslant L\Delta t$ for some constant L.

With e_k denoting $y(t_k) - y_k$, we have

$$e_{k+1} = e_k + \big[f\big(t_k, y(t_k)\big) - f(t_k, y_k)\big]\Delta t + \tau_k\Delta t .$$

The mean value theorem then implies that

$$|e_{k+1}| \leqslant |e_k| + (M\Delta t)|e_k| + L(\Delta t)^2 = (1 + M\Delta t)|e_k| + L(\Delta t)^2 ;$$

thus by iteration we have

$$\begin{aligned} |e_{k+1}| &\leqslant (1 + M\Delta t)|e_k| + L(\Delta t)^2 \leqslant (1 + M\Delta t)\big[(1 + M\Delta t)|e_{k-1}| + L(\Delta t)^2\big] + L(\Delta t)^2 \\ &= (1 + M\Delta t)^2|e_{k-1}| + L(\Delta t)^2\big[1 + (1 + M\Delta t)\big] \\ &\leqslant \cdots\cdots\cdots \\ &\leqslant (1 + M\Delta t)^{k+1}|e_0| + L(\Delta t)^2\big[1 + (1 + M\Delta t) + (1 + M\Delta t)^2 + \cdots + (1 + M\Delta t)^k\big] \\ &= (1 + M\Delta t)^{k+1}|e_0| + \frac{L}{M}\Delta t\big[(1 + M\Delta t)^{k+1} - 1\big] \\ &\leqslant (1 + M\Delta t)^{k+1}\Big(|e_0| + \frac{L}{M}\Delta t\Big) \end{aligned}$$

for all $k \in \big\{1, 2, \cdots, \big[\frac{T}{\Delta t}\big] - 1\big\}$. Since $(1 + M\Delta t) \leqslant e^{M\Delta t}$, we conclude that

$$|e_{k+1}| \leqslant e^{M(k+1)\Delta t}\Big(|e_0| + \frac{L}{M}\Delta t\Big) \leqslant e^{MT}\Big(|e_0| + \frac{L}{M}\Delta t\Big)$$

which further implies that

$$\max_{k\in\{1,\cdots,[\frac{T}{\Delta}]\}} |e_k| \leqslant e^{MT}\Big(|e_0| + \frac{L}{M}\Delta t\Big)\,. \tag{3.3}$$

Therefore, the difference between $y(t_k)$ and y_k approaches zero as $\Delta t \to 0$.

Remark 3.3. The Euler method can also be "derived" in the following way: the solution y at each t_k satisfies

$$y(t_{k+1}) - y(t_k) = \int_{t_k}^{t_{k+1}} y'(t)\,dt = \int_{t_k}^{t_{k+1}} f\big(t, y(t)\big)\,dt\,, \tag{3.4}$$

and the integral on the right-hand side is approximated by the value $f(t_k, y(t_k))(t_{k+1} - t_k)$.

3.2 Improved Euler's Method

Now suppose that we use the trapezoidal rule to approximate the value of the integral on the right-hand side of (3.4); that is,

$$\int_{t_k}^{t_{k+1}} f\big(t, y(t)\big)\,dt \approx \frac{f\big(t_k, y(t_k)\big) + f\big(t_{k+1}, y(t_{k+1})\big)}{2}(t_{k+1} - t_k)\,,$$

then another numberical scheme, called the ***trapezoid scheme***, can be developed: given y_0 and Δt, find y_{k+1} satisfying

$$y_{k+1} = y_k + \frac{f(t_k, y_k) + f(t_{k+1}, y_{k+1})}{2}\Delta t\,. \tag{3.5}$$

We note that the numerical scheme involves solving for y_{k+1} from a non-linear equation which is often very expensive (meaning that it takes a lot of computation time to find y_{k+1}). Since y_{k+1} depends on y_k (and other variables) implicitly, the trapezoid scheme is an ***implicit numerical scheme***.

To develop an explicit scheme (which means y_{k+1} can be explicitly expressed as a function of y_k), we approximate y_{k+1} on the right-hand side of (3.5) using the Euler method; that is, we approximate y_{k+1} by $y_k + \Delta f(t_k, y_k)$ and the trapezoid scheme is replaced by

$$y_{k+1} = y_k + \frac{f(t_k, y_k) + f\big(t_{k+1}, y_k + \Delta t f(t_k, y_k)\big)}{2}\Delta t\,. \tag{3.6}$$

The scheme above is called the ***improved Euler's method***.

Example 3.4. Compute the improved Euler's method approximation to the solution of the ODE $y' = y$, $y(0) = 1$.

Let Δt be the time step size. Then the improved Euler's method provides

$$y_{k+1} = y_k + \frac{y_k + y_k + \Delta t y_k}{2}\Delta t = y_k\Big(1 + \Delta t + \frac{\Delta t^2}{2}\Big)\,.$$

Using the initial condition,

$$y_k = \Big(1 + \Delta t + \frac{\Delta t^2}{2}\Big)^k\,.$$

We note that for each $t > 0$, $k\Delta t \leqslant t < (k+1)\Delta t$ for a unique $k = \Big[\frac{t}{\Delta t}\Big] \in \mathbb{N} \cup \{0\}$, and $k\Delta t \to t$ as $\Delta t \to 0$. Therefore, by the face that

$$\lim_{\Delta t\to 0}\Big(1 + \Delta t + \frac{\Delta t^2}{2}\Big)^{\frac{t}{\Delta t}} = e^t\,,$$

we find that y_k converges to the solution $y(t)$ of the ODE given above.

In general, the so-called **one-step explicit method** is often given in the form

$$y_{k+1} = y_k + \Delta t \Phi(\Delta t, t_k, y_k)\,.$$

For example, in Euler's method the function $\Phi(\Delta t, t_k, y_k) = f(t_k, y_k)$, while in the improved Euler's method,

$$\Phi(\Delta t, t_k, y_k) = \frac{f(t_k, y_k) + f(t_k + \Delta t, y_k + \Delta t f(t_k, y_k))}{2}\,.$$

Definition 3.5. A numerical method is said to be ***consistent*** if

$$\lim_{\Delta t \to 0} \Phi(\Delta t, t, y) = y'(t)\,.$$

3.2.1 Rate of convergence and the local truncation errors

The rate of convergence is used to understand how fast an approximated solution provided by a numerical scheme converges to the solution of an IVP. For a numerical method, we would like to determined the order n such that for a fixed $T > 0$,

$$\frac{|e_k(\Delta t)|}{\Delta t^n} \text{ is bounded for all } k \leqslant \frac{T}{\Delta t} \text{ as } \Delta t \to 0\,,$$

provided that y_0 is the exact initial data. We note that the Euler method is of order 1 due to (3.3).

Definition 3.6 (Big $\mathcal{O}$ and little $\mathcal{o}$)**.** We use the notation

$$f(x) = g(x) + \mathcal{O}(h(x)) \qquad \text{as} \quad x \to a$$

to express the idea that $\left|\frac{f(x) - g(x)}{h(x)}\right|$ is bounded when x is closed to a, and use the notation

$$f(x) = g(x) + \mathcal{o}(h(x)) \qquad \text{as} \quad x \to a$$

to express the idea that $\lim\limits_{x \to a} \left|\frac{f(x) - g(x)}{h(x)}\right| = 0$.

Now assume that f is twice continuously differentiable; that is, f_{tt}, f_{ty} and f_{yy} are continuous. Then y is three times continuously differentiable and Taylor's theorem implies that

$$\begin{aligned} y(t_{k+1}) &= y(t_k) + y'(t_k)\Delta t + \frac{1}{2}(\Delta t)^2 y''(t_k) + \mathcal{O}(\Delta t^3) \\ &= y(t_k) + f\big(t_k, y(t_k)\big)\Delta t + \frac{(\Delta t)^2}{2}\big[f_t + f_y f\big](t_k, y(t_k)) + \mathcal{O}(\Delta t^3)\,. \end{aligned} \tag{3.7}$$

On the other hand, the improved Euler's method produces that

$$\begin{aligned} y_{k+1} &= y_k + \frac{f(t_k, y_k) + f\big(t_{k+1}, y_k + \Delta t f(t_k, y_k)\big)}{2}\Delta t \\ &= y_k + \frac{f(t_k, y_k) + f(t_k, y_k) + f_t(t_k, y_k)\Delta t + f_y(t_k, y_k)\Delta t f(t_k, y_k)}{2}\Delta t + \mathcal{O}(\Delta t^3) \\ &= y_k + f(t_k, y_k)\Delta t + \frac{\Delta t^2}{2}\big[f_t + f_y f\big](t_k, y_k) + \mathcal{O}(\Delta t^3)\,. \end{aligned}$$

Therefore, if we solve the ODE using the initial condition $y(t_k) = y_k$, then the difference between the exact value $y(t_{k+1})$ and the approximated value y_{k+1} is of order Δt^3. This induces the following

Definition 3.7. Let y be the solution to the IVP $y' = f(t, y)$ with initial data $y(t_0) = y_0$. At each time step $t_k = t_0 + k\Delta t$, the ***local truncation error*** associated with the one-step numerical method

$$y_{k+1} = y_k + \Delta t \Phi(\Delta t, t_k, y_k)$$

is the number $\tau_k(\Delta t) = \dfrac{y(t_k) - z_k}{\Delta t}$, where $z_k = y(t_{k-1}) + \Delta t \Phi\big(\Delta t, t_{k-1}, y(t_{k-1})\big)$ is obtained according to the numerical scheme with $y_{k-1} = y(t_{k-1})$.（數值方法在前一步是正確值時走一步的誤差）

By the mean value theorem (for functions of several variables), with $h = \Delta t$ we have

$$\begin{aligned} e_k(h) &= y(t_k) - y_k = y(t_k) - y_{k-1} - h\Phi(h, t_{k-1}, y_{k-1}) \\ &= y(t_k) - y(t_{k-1}) - h\Phi(h, t_{k-1}, y(t_{k-1})) + y(t_{k-1}) - y_{k-1} \\ &\quad + h\big[\Phi(h, t_{k-1}, y(t_{k-1})) - \Phi(h, t_{k-1}, y_{k-1})\big] \\ &= h\tau_k(h) + e_{k-1}(h) + h\Phi_y(h, t_{k-1}, \xi_{k-1})\big[y(t_{k-1}) - y_{k-1}\big] \\ &= h\tau_k(h) + e_{k-1}(h) + h\Phi_y(h, t_{k-1}, \xi_{k-1}) e_{k-1}(h) \end{aligned}$$

for some ξ_{k-1} on the line segment joining $y(t_{k-1})$ and y_{k-1}. If we assume that $|\Phi_y|$ is bounded by M, then the equality above implies that

$$|e_k(h)| \leqslant h|\tau_k(h)| + (1 + hM)|e_{k-1}(h)|\,.$$

Therefore,

$$\begin{aligned} |e_k(h)| &\leqslant h|\tau_k(h)| + (1 + hM)|e_{k-1}(h)|\,, \\ (1 + hM)|e_{k-1}(h)| &\leqslant h(1 + hM)|\tau_{k-1}(h)| + (1 + hM)^2|e_{k-2}(h)|\,, \\ \vdots \quad &\leqslant \quad \vdots \\ (1 + hM)^{k-1}|e_1(h)| &\leqslant h(1 + hM)^{k-1}|\tau_1(h)| + (1 + hM)^k|e_0(h)|\,. \end{aligned}$$

Summing all the inequalities above, we find that

$$|e_k(h)| \leqslant h \sum_{\ell=0}^{k-1} (1 + hM)^\ell \big|\tau_{k-\ell}(h)\big| + (1 + hM)^k |e_0(h)|\,.$$

Suppose that the local truncation error is of order n; that is, τ_k satisfies

$$|\tau_k(h)| \leqslant Ah^n \qquad \forall\, k \in \Big\{0, 1, \cdots, \Big[\frac{T}{h}\Big] - 1\Big\}$$

for some constant A and $n > 0$. Then by the fact $e_0(h) = 0$, we conclude that

$$\begin{aligned} |e_k(h)| &\leqslant h \sum_{\ell=0}^{k-1} (1 + hM)^\ell Ah^n + (1 + hM)^k |e_0(h)| \leqslant h \frac{(1 + hM)^k - 1}{hM} Ah^n \\ &\leqslant \frac{1}{M}\big[(1 + hM)^{\frac{T}{h}} - 1\big] Ah^n \leqslant \frac{1}{M}(e^{MT} - 1)Ah^n\,. \end{aligned}$$

Therefore, we establish the following

Theorem 3.8. *If Φ_y is bounded by M and the local truncation error $\tau_k(h)$ associated with the one step numerical scheme*

$$y_{k+1} = y_k + h\Phi(h, t_k, y_k)$$

satisfies

$$\big|\tau_k(h)\big| \leqslant Ah^n \qquad \forall\, k \in \big\{0, 1, \cdots, \big[\tfrac{T}{h}\big]\big\} \text{ and } h > 0\,,$$

then the global truncation error $e_k(h)$ satisfies

$$|e_k(h)| \leqslant \frac{A}{M}(e^{MT} - 1)h^n \qquad \forall\, h > 0\,.$$

In shorthand, if $\tau_k(h) = \mathcal{O}(h^n)$, then $e_k(h) = \mathcal{O}(h^n)$.

Example 3.9. The improved Euler's method is a second order numerical scheme.

Example 3.10. Consider solving the initial value problem

$$y' = \sin(t^2 + y)\,, \qquad y(0) = 0$$

numerically (in the time interval $[0, 1]$) using the improved Euler method. First we compute the derivative of y:

$$\begin{aligned} y'' &= \cos(t^2 + y)(2t + y') = \cos(t^2 + y)\big(2t + \sin(t^2 + y)\big)\,,\\ y''' &= \sin(t^2 + y)\big(2t + \sin(t^2 + y)\big)^2 + \cos(t^2 + y)\big[2 + \cos(t^2 + y)\big(2t + \sin(t^2 + y)\big)\big]\,. \end{aligned}$$

Therefore, writing the improved Euler in the format $y_{k+1} = y_k + h\Phi(h, t_k, y_k)$, we have

$$\begin{aligned} \Phi(h, t, y) &= \frac{1}{2}\big[\sin(t^2 + y) + \sin\big((t + h)^2 + y + h\sin(t^2 + y)\big)\big]\\ &= \frac{1}{2}\big[\sin(t^2 + y) + \sin(t^2 + y + 2th + h^2 + h\sin(t^2 + y))\big]\,. \end{aligned}$$

A direct computation shows that

$$\Phi_y(h, t, y) = \frac{1}{2}\big[\cos(t^2 + y) + \cos((t + h)^2 + y + h\sin(t^2 + y))\big(1 + h\cos(t^2 + y)\big)\big]\,;$$

thus if $t \in [0, 1]$, we must have $h \in [0, 1]$ which implies that

$$\big|\Phi_y(h, t, y)\big| \leqslant \frac{1}{2}\big(1 + 1 + h\big) = \frac{2 + h}{2} \leqslant \frac{3}{2} \qquad \text{if} \quad t \in [0, 1]\,.$$

By the Taylor theorem,

$$\begin{aligned} y(t_k) &= y(t_{k-1}) + hy'(t_{k-1}) + \frac{h^2}{2}y''(t_{k-1}) + \frac{h^3}{6}y'''(\xi)\\ &= y(t_{k-1}) + h\sin\big(t_{k-1}^2 + y(t_{k-1})\big) + \frac{h^2}{2}\Big[2t_{k-1}\cos(t_{k-1}^2 + y(t_{k-1}))\\ &\qquad + \sin(t_{k-1}^2 + y(t_{k-1}))\cos(t_{k-1}^2 + y(t_{k-1}))\Big] + \frac{h^3}{6}y'''(\xi_{k-1}) \end{aligned}$$

for some ξ_{k-1} in between t_{k-1} and t_k. Moreover,

$$\begin{aligned} y_k &= y_{k-1} + h\Phi(h, t_{k-1}, y_{k-1}) \\ &= y_{k-1} + \frac{h}{2}\Big[2\sin(t_{k-1}^2 + y_{k-1}) + \cos(t_{k-1}^2 + y_{k-1})\big(2t_{k-1}h + h^2 + h\sin(t_{k-1}^2 + y_{k-1})\big) \\ &\qquad\qquad - \frac{1}{2}\sin\eta_{k-1}\big(2t_{k-1}h + h^2 + h\sin(t_{k-1}^2 + y_{k-1})\big)^2\Big] \end{aligned}$$

for some η_{k-1} in between $t_{k-1}^2 + y_{k-1}$ and $(t_{k-1} + h)^2 + y_{k-1} + h\sin(t_{k-1}^2 + y_{k-1})$. Therefore, in the time interval $[0, 1]$ the local truncation error $\tau_k(h)$ satisfies

$$\begin{aligned} \big|\tau_k(h)\big| &\leqslant \frac{h^2}{2}\big|\cos(t_{k-1}^2 + y(t_{k-1}))\big| + \frac{1}{4}\big|\sin\eta_{k-1}\big|\big(2t_{k-1}h + h^2 + h\sin(t_{k-1}^2 + y_{k-1})\big)^2 + \frac{h^2}{6}\big|y'''(\xi_{k-1})\big| \\ &\leqslant \frac{h^2}{2} + \frac{h^2}{4}(3+h)^2 + \frac{h^2}{6}\cdot\big((2+1)^2 + 2 + 3\big) \leqslant \Big(\frac{1}{2} + 4 + \frac{7}{3}\Big)h^2 \leqslant 7h^2\,. \end{aligned}$$

To obtain numerical solution which is accurate to the six decimals, by Theorem 3.8 we need to choose $h > 0$ such that

$$\frac{7}{3/2}(e^{\frac{3}{2}} - 1)h^2 \leqslant 10^{-7}\,.$$

Solving for h, we find that as long as $0 < h < 7.8452 \times 10^{-5}$, the global truncation error $e_k(h)$ is bounded above by 10^{-7} for all k (such that $t_k \leqslant 1$).

3.3 The Taylor Method and the Runge-Kutta Method

Motivated by the definition of the local truncation error and Theorem 3.8, if f is smooth enough, we can design one step numerical method as follows: by Taylor's theorem,

$$y(t_{k+1}) = y(t_k) + y'(t_k)h + \frac{y''(t_k)}{2}h^2 + \cdots + \frac{y^{(n)}(t_k)}{n!}h^n + \mathcal{O}(h^{n+1}) \qquad \text{as} \quad h \to 0\,.$$

Since $y'(t) = f(t, y)$, each derivative $y^{(j)}(t_k)$ can be expressed in terms of f and its partial derivatives. For example, we have used

$$y''(t) = (f_t + ff_y)(t, y)$$

to derive that the improved Euler's method is of order 2. The Taylor method of order 3 would require we compute $y'''(t)$ in terms of f and its partial derivatives. Since

$$\begin{aligned} y'''(t) &= \frac{d}{dt}(f_t + ff_y)(t, y) = (f_t + ff_y)_t(t, y) + (f_t + ff_y)_y(t, y)f(t, y) \\ &= (f_{tt} + f_tf_y + 2ff_{ty} + ff_y^2 + f^2f_{yy})(t, y)\,, \end{aligned}$$

the Taylor method of order 3 is given by

$$y_{k+1} = y_k + hf(t_k, y_k) + \frac{h^2}{2}(f_t + ff_y)(t_k, y_k) + \frac{h^3}{6}(f_{tt} + f_tf_y + 2ff_{ty} + ff_y^2 + f^2f_{yy})(t_k, y_k)\,. \quad (3.8)$$

Similarly, since

$$\begin{aligned} y^{(4)}(t) &= \frac{d}{dt}(f_{tt} + f_tf_y + 2ff_{ty} + ff_y^2 + f^2f_{yy})(t, y) \\ &= (f_{ttt} + f_{tt}f_y + 3f_tf_{ty} + 2ff_{tty} + f_tf_y^2 + 2ff_yf_{ty} + 2ff_tf_{yy} + f^2f_{tyy})(t, y) \\ &\quad + (f_{tty} + f_{ty}f_y + f_tf_{yy} + 2f_yf_{ty} + 2ff_{tyy} + f_y^3 + 4ff_yf_{yy} + f^2f_{yyy})(t, y)f(t, y)\,, \end{aligned}$$

the Taylor method of order 4 is given by

$$
\begin{aligned}
y_{k+1} = {} & y_k + hf(t_k,y_k) + \frac{h^2}{2}(f_t + ff_y)(t_k,y_k) + \frac{h^3}{6}(f_{tt} + f_tf_y + 2ff_{ty} + ff_{ty} + ff_y^2 + f^2f_{yy})(t_k,y_k) \\
& + \frac{h^4}{24}(f_{ttt} + f_{tt}f_y + 3f_tf_{ty} + 2ff_{tty} + f_tf_y^2 + 2ff_yf_{ty} + 2ff_tf_{yy} + f^2f_{tyy})(t_k,y_k) \\
& + \frac{h^4}{24}(f_{tty} + f_{ty}f_y + f_tf_{yy} + 2f_yf_{ty} + 2ff_{tyy} + f_y^3 + 4ff_yf_{yy} + f^2f_{yyy})(t_k,y_k)f(t_k,y_k)\,.
\end{aligned} \tag{3.9}
$$

Example 3.11. Find the third order Taylor's method for solving the IVP

$$y' = ty^2\,, \qquad y(0) = 1\,.$$

Let $f(t,y) = ty^2$. Then $f_t = y^2$, $f_y = 2ty$, $f_{tt} = 0$, $f_{ty} = 2y$ and $f_{yy} = 2t$. Therefore, using (3.8) the 3rd order Taylor's method is

$$
\begin{aligned}
y_{k+1} &= y_k + ht_ky_k^2 + \frac{h^2}{2}(y_k^2 + 2t_k^2y_k^3) + \frac{h^3}{6}(2t_ky_k^3 + 4t_ky_k^3 + 4t_k^3y_k^4 + 2t_k^3y_k^4) \\
&= y_k + ht_ky_k^2 + \frac{h^2}{2}(y_k^2 + 2t_k^2y_k^3) + h^3(t_ky_k^3 + t_k^3y_k^4)
\end{aligned}
$$

which starts at $y_0 = 1$.

To implement the Taylor method, it requires that we compute the derivatives of $y^{(j)}$ in terms of f and its partial derivative by hand. Moreover, for Taylor's method of higher order, the iterative relation becomes very lengthy so it becomes even harder for coding purposes. There are higher order one step explicit method for solving the IVP which does not require that we differentiate f by hand, and it is easy to implement. One of such one step explicit method is the Runge-Kutta method.

Let us start with a second order Runge-Kutta method to illustrate the idea. The idea of the second order Runge-Kutta method is to find a, b, α, β such that the one-step numerical scheme

$$r_1 = hf(t_k, y_k)\,, \tag{3.10a}$$

$$r_2 = hf(t_k + \alpha h, y_k + \beta r_1)\,, \tag{3.10b}$$

$$y_{k+1} = y_k + ar_1 + br_2 \tag{3.10c}$$

which produces a second order method. In order to make sure that (3.10) is of order 2, we compute the local truncation error (by assuming that f is smooth enough). By Taylor's theorem, we find that

$$
\begin{aligned}
y_{k+1} &= y_k + ahf(t_k,y_k) + bh\Big[f(t_k,y_k) + f_t(t_k,y_k)\alpha h + f_y(t_k,y_k)\beta r_1\Big] + \mathcal{O}(h^3) \\
&= y_k + (a+b)hf(t_k,y_k) + \big[b\alpha f_t(t_k,y_k) + b\beta f(t_k,y_k)f_t(t_k,y_k)\big]h^2 + \mathcal{O}(h^3)\,;
\end{aligned}
$$

thus comparing with (3.7) (with h replacing Δt) and applying Theorem 3.8 we conclude that (3.10) is of order 2 if

$$a + b = 1\,, \quad b\alpha = \frac{1}{2} \quad \text{and} \quad b\beta = \frac{1}{2}\,.$$

This is a system of three equations with four unknowns and has infinitely many solutions. In particular, $a = b = \frac{1}{2}$ and $\alpha = \beta = 1$ provides the improved Euler's method.

Similarly, a fourth order Runge-Kutta method in general is given by

$$
\begin{aligned}
r_1 &= hf(t_k, y_k)\,,\\
r_2 &= hf(t_k + \alpha_1 h, y_k + \beta_1 r_1)\,,\\
r_3 &= hf(t_k + \alpha_2 h, y_k + \beta_2 r_1 + \beta_3 r_2)\,,\\
r_4 &= hf(t_k + \alpha_3 h, y_k + \beta_4 r_1 + \beta_5 r_2 + \beta_6 r_3)\,,\\
y_{k+1} &= y_k + ar_1 + br_2 + cr_3 + dr_4
\end{aligned}
$$

such that it agrees with (3.9) up to the fourth order. One of the most popular choices of parameters in the fourth order Runge-Kutta is given by

$$r_1 = hf(t_k, y_k)\,, \tag{3.11a}$$

$$r_2 = hf(t_k + \frac{1}{2}h, y_k + \frac{1}{2}r_1)\,, \tag{3.11b}$$

$$r_3 = hf(t_k + \frac{1}{2}h, y_k + \frac{1}{2}r_2)\,, \tag{3.11c}$$

$$r_4 = hf(t_k + h, y_k + r_3)\,, \tag{3.11d}$$

$$y_{k+1} = y_k + \frac{r_1 + 2r_2 + 2r_3 + r_4}{6}\,. \tag{3.11e}$$

4 Second Order Linear Equations

Recall that a second order ordinary differential equation has the form

$$f\Big(t, y, \frac{dy}{dt}, \frac{d^2y}{dt^2}\Big) = 0 \tag{4.1}$$

for some given function f. The ODE (4.1) is called ***linear*** if the function f takes the form

$$f\Big(t, y, \frac{dy}{dt}, \frac{d^2y}{dt^2}\Big) = P(t)\frac{d^2y}{dt^2} + Q(t)\frac{dy}{dt} + R(t)y - G(t)\,,$$

where P is a function which never vanishes for all $t > 0$. The ODE (4.1) is called ***nonlinear*** if it is not linear. The functions P, Q, R are called the ***coefficients*** of the ODE, and G is called the ***forcing*** of the ODE. The ***initial condition*** for (4.1) is $\big(y(t_0), y'(t_0)\big) = (y_0, y_1)$.

In this chapter, the main focus is on solving linear second order ODE

$$P(t)\frac{d^2y}{dt^2} + Q(t)\frac{dy}{dt} + R(t)y = G(t)\,. \tag{4.2}$$

The prototype model of such kind of ODEs is the ODE

$$my'' = -ky - by' + f(t)$$

which is used to model the mass-spring oscillator, where m is the mass, k is the Hooke constant, b is the damping coefficient, and f is the external forcing acting on the mass.

4.1 Basic Theory for Second Order Linear Equations

Let $I \subseteq \mathbb{R}$ be an interval containing t_0 as an interior point. Suppose that $P, Q, R, F : I \to \mathbb{R}$ are continuous and $P(t) \neq 0$ for all $t \in I$. By the fundamental theorem of ODE, the initial value problem (4.2) with initial condition $y(t_0) = y_0$, $y'(t_0) = y_1$ has a unique solution in some time interval containing t_0 as an interior point.

Since $P \neq 0$ on $\mathbb{R}$, the functions $p \equiv \dfrac{Q}{P}$, $q \equiv \dfrac{R}{P}$ and $g \equiv \dfrac{G}{P}$ are also continuous on I, and (4.2) is equivalent to

$$y'' + p(t)y' + q(t)y = g(t)\,. \tag{4.3}$$

Theorem 4.1. *Let $I \subseteq \mathbb{R}$ be an interval, and $p, q, g : I \to \mathbb{R}$ be continuous. Then the initial value problem*

$$y'' + p(t)y' + q(t)y = g(t)\,, \qquad y(t_0) = y_0, \quad y'(t_0) = y_1 \tag{4.4}$$

has a unique solution $y : I \to \mathbb{R}$.

Proof. By the fundamental theorem of ODE, it suffices to show that $y(t)$ exists for all $t \in I$.

Suppose that the maximal interval for the existence of y is $(a, b) \subsetneq I$ (which means $\lim\limits_{t\to b^-} y(t)$ and $\lim\limits_{t\to a^+} y(t)$ do not exist). For $t \in (a, b)$, multiplying (4.4) by $y'(t)$ we find that

$$\frac{1}{2}\frac{d}{dt}\big|y'(t)\big|^2 + q(t)\big|y'(t)\big|^2 + r(t)y(t)y'(t) = g(t)y'(t)\,. \tag{4.5}$$

By the fundamental theorem of calculus,

$$y(t) = y(t_0) + \int_{t_0}^{t} y'(s)\,ds = y_0 + \int_{t_0}^{t} y'(s)\,ds\,;$$

thus the Cauchy-Schwarz inequality implies that for $a < t < b$

$$\big|y(t)\big|^2 \leqslant 2\Big[|y_0|^2 + (t - t_0)\int_{t_0}^{t} \big|y'(s)\big|^2\,ds\Big]\,.$$

Therefore, letting $M = \sup\limits_{t\in[a,b]} \Big[\big|p(t)\big| + \big|q(t)\big|\Big]$, for $t \in (a, b)$ (4.5) implies that

$$\begin{aligned}\frac{d}{dt}\big|y'(t)\big|^2 &\leqslant 2\big|q(t)\big|\big|y'(t)\big|^2 + \big|r(t)\big|\big|y(t)\big|^2 + \big|r(t)\big|\big|y'(t)\big|^2 + \big|g(t)\big|^2 + \big|y'(t)\big|^2 \\ &\leqslant (3M + 1)\big|y'(t)\big|^2 + 2M\Big[|y_0|^2 + (t - t_0)\int_{t_0}^{t} \big|y'(s)\big|^2\,ds\Big] + \big|g(t)\big|^2\,.\end{aligned}$$

Let $X(t) = \displaystyle\int_{t_0}^{t} \big|y'(s)\big|^2\,ds$ and $N = 2M(b - a)|y_0|^2 + |y_1|^2 + \displaystyle\int_a^b \big|g(s)\big|^2\,ds$. Integrating the inequality above in t, by the fact that $X'(t) = \big|y'(t)\big|^2$ we find that

$$X'(t) \leqslant N + \big[3M + 1 + (b - a)^2\big]X(t) \qquad \forall\, t_0 < t < b$$

and

$$-X'(t) \leqslant N - \big[3M + 1 + (b - a)^2\big]X(t) \qquad \forall\, a < t < t_0\,.$$

Therefore, using the method of integrating factor, we obtain that

$$\big|X(t)\big| \leqslant N e^{[3M+1+(b-a)^2](t-t_0)} \qquad \forall\, a < t < b$$

which in turn implies that $\big|y'(t)\big|$ is bounded above by a fixed constant C for all $a < t < b$.

Let $\{t_n\}_{n=1}^{\infty} \subseteq (a,b)$ be a convergent sequence with limit a (or b). Then the mean value theorem implies that

$$\big|y(t_n) - y(t_m)\big| \leqslant C|t_n - t_m|$$

which implies that $\big\{y(t_n)\big\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$. Therefore, $\big\{y(t_n)\big\}_{n=1}^{\infty}$ converges as long as $\{t_n\}_{n=1}^{\infty}$ converges (to b or a). This shows that $\lim\limits_{t\to b^-} y(t)$ and $\lim\limits_{t\to a^+} y(t)$ exist, a contradiction. □

Definition 4.2. The ODE (4.3) is called ***homogeneous*** if $g \equiv 0$, otherwise it is called ***non-homogeneous***. When $g \not\equiv 0$, the term $g(t)$ in (4.1) is called the non-homogeneous term.

Let $I \subseteq \mathbb{R}$ be an interval, and $p, q, g : I \to \mathbb{R}$ be given continuous functions. For a twice differentiable function $y : \mathbb{R} \to \mathbb{R}$, let $L[y]$ denote the function

$$(L[y])(t) \equiv y''(t) + p(t)y'(t) + q(t)y(t)\,.$$

The kernel of L, denoted by $\mathrm{Ker}(L)$, consists of solutions to the homogeneous equation

$$y'' + p(t)y' + q(t)y(t) = 0\,.$$

The kernel of L is called the solution space of the homogeneous equation above. We note that $\mathrm{Ker}(L)$ is a vector space.

Theorem 4.3 (Principle of Superposition)**.** *If $y = \varphi_1$ and $y = \varphi_2$ are two solutions of the differential equation*

$$L[y] = y'' + py' + qy = 0\,, \tag{4.6}$$

then the linear combination $c_1\varphi_1 + c_2\varphi_2$ is also a solution for any values of the constants c_1 and c_2. In other words, the collection of solutions to (4.6) is a vector spaces.

Let $Y_i(t)$, $i = 1, 2$, be the solution to the IVP

$$y'' + p(t)y' + q(t)y = 0\,, \qquad \big(y(t_0), y'(0)\big) = \mathbf{e}_i$$

respectively, where $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$. Then the solution to the IVP

$$y'' + p(t)y' + q(t)y = 0\,, \qquad y(t_0) = y_0, \quad y'(t_0) = y_1$$

is $y(t) = y_0 Y_1(t) + y_1 Y_1(t)$. Therefore, the solution to the ODE

$$y'' + p(t)y' + q(t)y = 0$$

must be of the form $y(t) = c_1Y_1(t) + c_2Y_2(t)$. On the other hand, there is no non-zero vector (c_1, c_2) such that $c_1Y_1(t) + c_2Y_2(t) = 0$ for all $t \in \mathbb{R}$, the set $\{Y_1, Y_2\}$ is linearly independent. Therefore,

$$\dim \operatorname{Ker}(L) = 2 \text{ and } \{Y_1, Y_2\} \text{ is a basis for } \operatorname{Ker}(L).$$

It is natural to ask "are two given functions φ_1, φ_2 in $\operatorname{Ker}(L)$ linearly independent?" Suppose that for given initial data y_0, y_1 there exist constants c_1, c_2 such that $y(t) = c_1\varphi_1(t) + c_2\varphi_2(t)$ is a solution to (4.4). Then

$$\begin{bmatrix} \varphi_1(t_0) & \varphi_2(t_0) \\ \varphi_1'(t_0) & \varphi_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

So for any given initial data (y_0, y_1) the solution to (4.4) can be written as a linear combination of φ_1 and φ_2 if the matrix $\begin{bmatrix} \varphi_1(t_0) & \varphi_2(t_0) \\ \varphi_1'(t_0) & \varphi_2'(t_0) \end{bmatrix}$ is non-singular. This induces the following

Definition 4.4. Let φ_1 and φ_2 be two differentiable functions. The ***Wronskian*** or ***Wronskian determinant*** of φ_1 and φ_2 at point t_0 is the number

$$W[\varphi_1, \varphi_2](t_0) = \det\Big(\begin{bmatrix} \varphi_1(t_0) & \varphi_2(t_0) \\ \varphi_1'(t_0) & \varphi_2'(t_0) \end{bmatrix}\Big) = \varphi_1(t_0)\varphi_2'(t_0) - \varphi_2(t_0)\varphi_1'(t_0).$$

The collection of functions $\{\varphi_1, \varphi_2\}$ is called a ***fundamental set*** of the initial value problem (4.6) if $W[\varphi_1, \varphi_2](t) \neq 0$ for some t in the interval of interest.

Moreover, we also establish the following

Theorem 4.5. *Suppose that $y = \varphi_1$ and $y = \varphi_2$ are two solutions of the initial value problem (4.6). Then for any arbitrarily given (y_0, y_1), the solution to the ODE*

$$L[y] = y'' + py' + qy = 0 \quad \textit{with initial condition } y(t_0) = y_0 \textit{ and } y'(t_0) = y_1,$$

can be written as a linear combination of φ_1 and φ_2 if and only if the Wronskian of φ_1 and φ_2 at t_0 does not vanish.

Theorem 4.6. *Let φ_1 and φ_2 be solutions to the differential equation (4.6) satisfying the initial conditions $\big(\varphi_1(t_0), \varphi_1'(t_0)\big) = (1, 0)$ and $\big(\varphi_2(t_0), \varphi_2'(t_0)\big) = (0, 1)$. Then $\{\varphi_1, \varphi_2\}$ is a fundamental set of equation (4.6), and for any (y_0, y_1), the solution to (4.4) can be written as $y = y_0\varphi_1 + y_1\varphi_2$.*

Next, suppose that φ_1, φ_2 are solutions to (4.6) and $W[\varphi_1, \varphi_2](t_0) \neq 0$. We would like to know if $\{\varphi_1, \varphi_2\}$ can be used to construct solutions to the differential equation

$$L[y] = y'' + py' + qy = 0 \quad \text{with initial condition } y(t_1) = y_0 \text{ and } y'(t_1) = y_1 \tag{4.7}$$

for some $t_1 \neq t_0$. In other words, we would like to know if $W[\varphi_1, \varphi_2](t_1)$ vanishes or not. This question is answered by the following

Theorem 4.7 (Abel)**.** *Let φ_1 and φ_2 be two solutions of* (4.6) *in which p, q are continuous in an open interval I, and the Wronskian $W[\varphi_1, \varphi_2]$ does not vanish at $t_0 \in I$. Then*

$$W[\varphi_1, \varphi_2](t) = W[\varphi_1, \varphi_2](t_0) \exp\Big(-\int_{t_0}^{t} p(s)ds \Big) .$$

In particular, $W[\varphi_1, \varphi_2](t)$ is never zero for all $t \in I$.

Proof. Since φ_1 and φ_2 are solutions to (4.6), we have

$$\varphi_1''(t) + p(t)\varphi_1'(t) + q(t)\varphi_1(t) = 0 , \tag{4.8a}$$
$$\varphi_2''(t) + p(t)\varphi_2'(t) + q(t)\varphi_2(t) = 0 . \tag{4.8b}$$

Computing $(4.8b) \times \varphi_1 - (4.8a) \times \varphi_2$, we obtain that

$$(\varphi_2\varphi_1'' - \varphi_1\varphi_2'') + p(\varphi_2\varphi_1' - \varphi_1\varphi_2') = 0$$

Therefore, letting $W = \varphi_2\varphi_1' - \varphi_1\varphi_2'$ be the Wronskian of φ_1 and φ_2. Then $W' + pW = 0$; thus

$$W(t) = W(t_0) \exp\Big(-\int_{t_0}^{t} p(s)ds \Big) .$$

Since p is continuous on $[t_0, t]$ (or $[t, t_0]$), the integral $\displaystyle\int_{t_0}^{t} p(s)ds$ is finite; thus $W(t) \neq 0$. □

4.2 Homogeneous Equations with Constant Coefficients: The General Solution

In this section, we consider homogeneous second order linear ODE with constant coefficients

$$Py'' + Qy' + Ry = 0 ,$$

where P, Q, R are independent of t. Since $P \neq 0$, the ODE reduces to

$$y'' + by' + cy = 0 . \tag{4.9}$$

Consider the equation $\lambda^2 + b\lambda + c = 0$.

1. Suppose that there are two distinct real roots λ_1 and λ_2. Then

$$\Big(\frac{d}{dt} - \lambda_1\Big)\Big(\frac{d}{dt} - \lambda_2\Big)y = 0 .$$

Therefore, if $z = \big(\frac{d}{dt} - \lambda_2\big)y$, then $\big(\frac{d}{dt} - \lambda_1\big)z = 0$ which further implies that $z = c_1 e^{\lambda_1 t}$ for some constant c_1. Then

$$y' - \lambda_2 y = c_1 e^{\lambda_1 t} \Rightarrow \big(e^{-\lambda_2 t}y\big)' = c_1 e^{(\lambda_1-\lambda_2)t} \Rightarrow e^{-\lambda_2 t}y = \frac{c_1}{\lambda_1 - \lambda_2} e^{(\lambda_1-\lambda_2)t} + c_2$$
$$\Rightarrow y = \frac{c_1}{\lambda_1 - \lambda_2} e^{\lambda_1 t} + c_2 e^{\lambda_2 t} .$$

In other words, a solution to the ODE (4.9) is a linear combination of $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ if λ_1 and λ_2 are distinct real roots of $\lambda^2 + b\lambda + c = 0$, and is called the ***general solution*** to (4.9).

2. Suppose that there is a real double root λ. Then the argument show that y satisfies

$$y' - \lambda y = c_1 e^{\lambda t} \Rightarrow (e^{-\lambda t} y)' = c_1 \Rightarrow e^{-\lambda t} y = c_1 t + c_2 \Rightarrow y = c_1 t e^{\lambda t} + c_2 e^{\lambda t}\,.$$

In other words, a solution to the ODE (4.9) is a linear combination of $te^{\lambda t}$ and $e^{\lambda t}$ if λ is the real double root of $\lambda^2 + b\lambda + c = 0$, and is called the general solution to (4.9).

Example 4.8. Find the general solution to $y'' + 5y' - 6y = 0$. Answer: $y(t) = C_1 e^t + C_2 e^{-6t}$.

Example 4.9. Solve the initial value problem

$$y'' + 2y' - y = 0\,, \qquad y(0) = 0, \quad y'(0) = -1\,.$$

Answer: $y(t) = -\frac{\sqrt{2}}{4} e^{(-1+\sqrt{2})t} + \frac{\sqrt{2}}{4} e^{(-1-\sqrt{2})t}$.

Example 4.10. Find the solution to the initial value problem

$$y'' + 4y' + 4y = 0\,, \qquad y(0) = 1, \quad y'(0) = 3\,.$$

Answer: $y(t) = 3te^{-2t}$.

Question: What happened if there are complex roots for $\lambda^2 + b\lambda + c = 0$?

Definition 4.11. The equation $\lambda^2 + b\lambda + c = 0$ is called the characteristic equation associated with the ODE (4.9).

Another way to derive the characteristic equations: Consider $y'' + by' + cy = 0$. Let $y' = z$. Then

$$\frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}.$$

Write $\boldsymbol{x} = [y, z]^{\mathrm{T}}$ and $A = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix}$. Then $\boldsymbol{x}' = A\boldsymbol{x}$.

Suppose that $A = P\Lambda P^{-1}$ for some diagonal matrix Λ; that is, A is diagonalizable (with eigenvectors of A form the columns of P and eigenvalues forms the diagonal entry of Λ), then $P^{-1}\boldsymbol{x}' = \Lambda P^{-1}\boldsymbol{x}$. Letting $\boldsymbol{u} = P^{-1}\boldsymbol{x}$, then $\boldsymbol{u}' = \Lambda\boldsymbol{u}$ or equivalently,

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Therefore, $u_1' = \lambda_1 u_1$ and $u_2' = \lambda_2 u_2$ that further imply that $u_1 = c_1 e^{\lambda_1 t}$ and $u_2 = c_2 e^{\lambda_2 t}$. Since $\boldsymbol{x} = P\boldsymbol{u}$, we conclude that y is a linear combination of $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$.
What are eigenvalues of A? Let λ be an eigenvalue of A. Then

$$\begin{vmatrix} -\lambda & 1 \\ -c & -b-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^2 + b\lambda + c = 0$$

which is the characteristic equation. Therefore, eigenvalues of A are the roots of the characteristic equation for the ODE (4.9).

4.3 Characteristic Equations with Complex Roots

Consider again the 2nd order linear homogeneous ordinary differential equation

$$y'' + by' + cy = 0 \tag{4.9}$$

where b and c are both constants. Suppose that the characteristic equation $r^2 + br + c = 0$ has two complex roots $\lambda \pm i\mu$. We expect that the solution to (4.9) can be written as a linear combination of $e^{(\lambda+i\mu)t}$ and $e^{(\lambda-i\mu)t}$.

What is $e^{i\mu t}$? The Euler identity says that $e^{i\theta} = \cos\theta + i\sin\theta$; thus

$$e^{(\lambda\pm i\mu)t} = e^{\lambda t}\big[\cos(\mu t) \pm i\sin(\mu t)\big] .$$

Considering the real and imaginary parts of $e^{\lambda t \pm \mu t}$, we expect that $\varphi_1(t) = e^{\lambda t}\cos(\mu t)$ and $e^{\lambda t}\sin(\mu t)$ are solutions to (4.9).

φ_1 and φ_2 are solutions: left as an exercise.

Linear independence of φ_1 and φ_2: Computing the Wronskian of φ_1 and φ_2, we find that

$$W[\varphi_1, \varphi_2](t) = \begin{vmatrix} e^{\lambda t}\cos(\mu t) & e^{\lambda t}\sin(\mu t) \\ e^{\lambda t}\big(\lambda\cos(\mu t) - \mu\sin(\mu t)\big) & e^{\lambda t}\big(\lambda\sin(\mu t) + \mu\cos(\mu t)\big) \end{vmatrix} = \mu e^{\lambda t}$$

which is non-zero if $\mu \neq 0$. Therefore, Theorem 4.6 implies that any solution to (4.9) can be written as a linear combination of φ_1 and φ_2 if $b^2 - 4c < 0$ and is called the general solution to (4.9).

Example 4.12. Find the general solution of $y'' + 2y' + 4y = 0$. Answer: $y(t) = C_1 e^{-t}\cos\sqrt{3}t + C_2 e^{-t}\sin\sqrt{3}t$.

Example 4.13. Consider the motion of an object attached to a spring. The dynamics is described by the 2nd order ODE:

$$m\ddot{x} = -kx - r\dot{x} \,, \tag{4.10}$$

where m is the mass of the object, k is the Hooke constant of the spring, and r is the friction/damping coefficient.

1. If $r^2 - 4mk > 0$: There are two distinct negative roots $\dfrac{-r \pm \sqrt{r^2 - 4mk}}{2m}$ to the characteristic equation, and the solution of (4.10) can be written as

$$x(t) = C_1 \exp\left(\frac{-r + \sqrt{r^2 - 4mk}}{2m}t\right) + C_2 \exp\left(\frac{-r - \sqrt{r^2 - 4mk}}{2m}t\right).$$

   The solution $x(t)$ approaches zero as $t \to \infty$.

2. If $r^2 - 4mk = 0$: There is one negative double root $\dfrac{-r}{2m}$ to the characteristic equation, and the solution of (4.10) can be written as

$$x(t) = C_1 \exp\left(\frac{-rt}{2m}\right) + C_2 t \exp\left(\frac{-rt}{2m}\right).$$

   The solution $x(t)$ approaches zero as $t \to \infty$.

3. If $r^2 - 4mk < 0$: There are two complex roots $\dfrac{-r \pm i\sqrt{4mk - r^2}}{2m}$ to the characteristic equation, and the solution of (4.10) can be written as

$$x(t) = C_1 e^{-\frac{rt}{2m}} \cos\Big(\frac{\sqrt{4mk - r^2}}{2m} t\Big) + C_2 e^{-\frac{rt}{2m}} \sin\Big(\frac{\sqrt{4mk - r^2}}{2m} t\Big).$$

   (a) If $r = 0$, the motion of the object is periodic with period $\dfrac{4m\pi}{\sqrt{4mk - r^2}}$, and is called **simple harmonic motion**.

   (b) If $r > 0$, the object oscillates about the equilibrium point ($x = 0$) but approaches to zero exponentially.

4.4 Nonhomogeneous Equations

In this section, we focus on the second order nonhomogeneous ODE

$$y'' + p(t)y' + q(t)y = g(t). \tag{4.11}$$

Definition 4.14. A ***particular solution*** to (4.11) is a twice differentiable function validating (4.11). In other words, a particular solution is a solution of (4.11). The space of ***complementary solutions*** to (4.11) is the collection of solutions to the corresponding homogeneous equation

$$y'' + p(t)y' + q(t)y = 0. \tag{4.12}$$

Let $y = Y(t)$ be a particular solution to (4.11). If $y = \varphi(t)$ is another solution to (4.11), then $y = \varphi(t) - Y(t)$ is function in the space of complementary solutions to (4.11). By Theorem 4.6, there exist two function φ_1 and φ_2 such that $y = \varphi_j(t)$, $j = 1, 2$, are linearly independent solutions to (4.12), and $\varphi(t) - Y(t) = C_1\varphi_1(t) + C_2\varphi_2(t)$ for some constants C_1 and C_2. This observation shows the following

Theorem 4.15. *The general solution of the nonhomogeneous equation* (4.11) *can be written in the form*

$$y = \varphi(t) = C_1\varphi_1(t) + C_2\varphi_2(t) + Y(t),$$

where $\{\varphi_1, \varphi_2\}$ *is a fundamental set of* (4.12)*,* C_1 *and* C_2 *are arbitrary constants, and* $y = Y(t)$ *is a particular solution of the nonhomogeneous equation* (4.11)*.*

General strategy of solving nonhomogeneous equation (4.11):

1. Find the space of complementary solution to (4.11); that is, find the general solution $y = C_1\varphi_1(t) + C_2\varphi_2(t)$ of the homogeneous equation (4.12).

2. Find a particular solution $y = Y(t)$ of the nonhomogeneous equation (4.11).

3. Apply Theorem 4.15.

4.5 The Method of Undetermined Coefficients

In this sub-section, we focus on solving

$$y'' + by' + cy = g(t)\,. \tag{4.13}$$

Suppose that λ_1 and λ_2 are two roots of $r^2 + br + c = 0$ (λ_1 and λ_2 could be identical or complex-valued). Then (4.13) can be written as

$$\Big(\frac{d}{dt} - \lambda_1\Big)\Big(\frac{d}{dt} - \lambda_2\Big)y(t) = g(t)\,.$$

Letting $y' - \lambda_2 y = z$, we have $z' - \lambda_1 z = g(t)$; thus

$$z(t) = e^{\lambda_1 t}\int e^{-\lambda_1 t} g(t)\,dt\,.$$

Solving for y we obtain that

$$y(t) = e^{\lambda_2 t}\int \Big(e^{(\lambda_1-\lambda_2)t}\int e^{-\lambda_1 t} g(t)\,dt\Big)\,dt\,. \tag{4.14}$$

Consider the following three types of forcing function g:

1. $g(t) = p_n(t)e^{\alpha t}$ for some polynomial $p_n(t) = a_n t^n + \cdots + a_1 t + a_0$ of degree n: note that

$$\int e^{\gamma t} t^k\,dt = \begin{cases} \dfrac{1}{\gamma}e^{\gamma t}t^k - \dfrac{k}{\gamma}\displaystyle\int e^{\gamma t}t^{k-1}dt & \text{if } \gamma \neq 0\,, \\ \dfrac{1}{k+1}t^{k+1} + C & \text{if } \gamma = 0\,. \end{cases} \tag{4.15}$$

   Therefore, in this case a particular solution is of the form

$$Y(t) = t^s(A_n t^n + \cdots + A_1 t + A_0)e^{\alpha t}$$

   for some unknown s and coefficients $A_i's$, and we need to determine the values of these unknowns.

   (a) If $\lambda_1 \neq \alpha$ and $\lambda_2 \neq \alpha$, then $s = 0$.

   (b) If $\lambda_1 = \alpha$ but $\lambda_2 \neq \alpha$, then $s = 1$.

   (c) If $\lambda_1 = \lambda_2 = \alpha$, then $s = 2$.

2. $g(t) = p_n(t)e^{\alpha t}\cos(\beta t)$ or $g(t) = p_n(t)e^{\alpha t}\sin(\beta t)$ for some polynomial p_n of degree n and $\beta \neq 0$: note that (4.15) also holds for $\gamma \in \mathbb{C}$. Therefore, in this case we assume that a particular solution is of the form

$$Y(t) = t^s\Big[(A_n t^n + \cdots + A_1 t + A_0)e^{\alpha t}\cos(\beta t) + (B_n t^n + \cdots + B_1 t + B_0)e^{\alpha t}\sin(\beta t)\Big]$$

   for some unknown s and coefficients $A_i's$, $B_i's$, and we need to determine the values of these unknowns.

(a) If $\lambda_1, \lambda_2 \in \mathbb{R}$, then $s = 0$.

(b) If $\lambda_1, \lambda_2 \notin \mathbb{R}$; that is, $\lambda_1 = \gamma + i\delta$ and $\lambda_2 = \gamma - i\delta$ for some $\delta \neq 0$:

(1) If $\lambda_1 = \gamma + i\delta$ and $\lambda_2 = \gamma - i\delta$ for some $\gamma \neq \alpha$ or $\delta \neq \pm\beta$, then $s = 0$.

(2) If $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$, then $s = 1$.

Example 4.16. Find a particular solution of $y'' - 3y' - 4y = 3e^{2t}$.

Since the roots of the characteristic equation $r^2 - 3r - 4$ are different from -1, we expect that a particular solution to the ODE above is of the form Ae^{2t}. Solving for A, we find that $A = -\frac{1}{2}$; thus a particular solution is $Y(t) = -\frac{1}{2}e^{2t}$.

Example 4.17. Find a particular solution of $y'' - 3y' - 4y = 2\sin t$.

Since the roots of $r^2 - 3r - 4 = 0$ are real, we expect that a particular solution is of the form $Y(t) = A\cos t + B\sin t$ for some constants A, B to be determined. In other words, we look for A, B such that

$$(A\cos t + B\sin t)'' - 3(A\cos t + B\sin t)' - 4(A\cos t + B\sin t) = 2\sin t\,.$$

By expanding the derivatives and comparing the coefficients, we find that (A, B) satisfies

$$\begin{cases} 3A - 5B = 2\,, \\ 5A + 3B = 0\,, \end{cases}$$

and the solution to the equation above is $(A, B) = (\frac{3}{17}, \frac{-5}{17})$. Therefore, a particular solution is

$$Y(t) = \frac{3}{17}\cos t - \frac{5}{17}\sin t\,.$$

Example 4.18. Find a particular solution of $y'' - 3y' - 4y = -8e^t\cos 2t$.

Since the roots of $r^2 - 3r - 4 = 0$ are real, we expect that a particular solution is of the form $Y(t) = Ae^t\cos 2t + Be^t\sin 2t$ for some constants A, B to be determined. In other words, we look for A, B such that

$$(Ae^t\cos 2t + Be^t\sin 2t)'' - 3(Ae^t\cos 2t + Be^t\sin 2t)' - 4(Ae^t\cos 2t + Be^t\sin 2t) = -8e^t\cos 2t\,.$$

By expanding the derivatives,

	$(e^t\cos 2t)''$	$(e^t\sin 2t)''$	$(e^t\cos 2t)'$	$(e^t\sin 2t)'$	$e^t\cos 2t$	$e^t\sin t$
$e^t\cos 2t$	-3	4	1	2	1	0
$e^t\sin 2t$	-4	-3	-2	1	0	1

thus

$$-3A + 4B - 3A - 6B - 4A = -8\,,$$
$$-4A - 3B + 6A - 3B - 4B = 0\,.$$

Therefore, $(A, B) = (\frac{10}{13}, \frac{2}{13})$; thus a particular solution is

$$Y(t) = \frac{10}{13}e^t\cos 2t + \frac{2}{13}e^t\sin 2t\,.$$

Example 4.19. Find a particular solution of $y'' - 3y' - 4y = 2e^{-t}$.

Since one of the roots of the characteristic equation $r^2 - 3r - 4$ is -1, we expect that a particular solution to the ODE above is of the form Ate^{-t} for some constant A to be determined. In other words, we look for A such that

$$(Ate^{-t})'' - 3(Ate^{-t})' - 4Ate^{-t} = 2e^{-t}\,.$$

By expanding the derivatives, we find that $-5A = 2$ which implies that $A = -\frac{2}{5}$. Therefore, a particular solution is given by $Y(t) = -\frac{2}{5}te^{-t}$.

How about if we forget what s is? - By trial and error! Starting from $s = 0$. If a particular of the form with $s = 0$ cannot be found, then try $s = 1$, and so on.

Example 4.20. Find a particular solution of $y'' - 4y' + 5y = -2e^{2t}\sin t$.

We first look for a particular solution of the form $Y(t) = Ae^{2t}\cos t + Be^{2t}\sin t$, and find that this leads to that $0 = e^{2t}\sin t$ which is impossible. Therefore, we look for a particular solution of the form $Y(t) = t(Ae^{2t}\cos t + Be^{2t}\sin t)$. Note that

	$(te^{2t}\cos t)''$	$(te^{2t}\sin t)''$	$(te^{2t}\cos t)'$	$(te^{2t}\sin t)'$	$te^{2t}\cos t$	$te^{2t}\sin t$
$te^{2t}\cos t$	3	4	2	1	1	0
$te^{2t}\sin t$	-4	3	-1	2	0	1
$e^{2t}\cos t$	4	2	1	0	0	0
$e^{2t}\sin t$	-2	4	0	1	0	0

thus by assuming this form of particular solutions we find that

$$\begin{aligned} 3A + 4B - 8A - 4B + 5A &= 0\,, \\ -4A + 3B + 4A - 8B + 5B &= 0\,, \\ 4A + 2B - 4A &= 0\,, \\ -2A + 4B - 4B &= -2\,. \end{aligned}$$

Therefore, $(A, B) = (1, 0)$, and a particular solution is $Y(t) = te^t \cos t$.

If the forcing g is the sum of functions of different types, the construction of a particular solution relies on the following

Theorem 4.21. *If $y = \varphi_j(t)$ is a particular solution to the ODE*

$$y'' + p(t)y' + q(t)y = g_j(t)$$

for all $j = 1, \cdots n$, then the function $y = \sum_{j=1}^{n} \varphi_j(t)$ is a particular to the ODE

$$y'' + p(t)y' + q(t)y = g(t) \equiv \sum_{j=1}^{n} g_j(t)\,.$$

Example 4.22. Find a particular solution of $y'' - 3y' - 4y = 3e^{2t} - 8e^t\cos 2t + 2e^{-t}$.

By Example 4.16, 4.18 and 4.19, a particular solution to the ODE above is

$$Y(t) = -\frac{1}{2}e^{2t} + \frac{10}{13}e^t\cos 2t + \frac{2}{13}e^t\sin 2t - \frac{2}{5}te^{-t}\,.$$

4.6 Repeated Roots; Reduction of Order

In Section 4.2 we have discussed the case that the characteristic equation of the homogeneous equation with constant coefficients

$$y'' + by' + cy = 0 \tag{4.9}$$

has one double root. We recall that in such case $b^2 = 4c$, and $\varphi_1(t) = \exp\left(\frac{-bt}{2}\right)$, $\varphi_2(t) = t\exp\left(\frac{-bt}{2}\right)$ together form a fundamental set of (4.9).

Suppose that we are given a solution $\varphi_1(t)$. Since (4.9) is a second order equation, there should be two linearly independent solutions. One way of finding another solution, using information that φ_1 provides, is **the variation of constant**: suppose another solution is given by $\varphi_2(t) = v(t)\varphi_1(t)$. Then

$$v''\varphi_1 + 2v'\varphi_1' + v\varphi_1'' + b\big(v'\varphi_1 + v\varphi_1'\big) + cv\varphi_1 = 0\,.$$

Since $y = \varphi_1(t)$ verifies (4.9), we find that

$$v''\varphi_1 + 2v'\varphi_1' + bv'\varphi_1 = 0\,;$$

thus using $\varphi_1(t) = \exp\left(\frac{-bt}{2}\right)$ we obtain $v''\varphi_1 = 0$. Since φ_1 never vanishes, $v''(t) = 0$ for all t. Therefore, $v(t) = C_1 t + C_2$ for some constant C_1 and C_2. Therefore, another solution to (4.9), when $b^2 = 4c$, is $\varphi_2(t) = t\exp\left(\frac{-bt}{2}\right)$.

The idea of the variation of constant can be generalize to homogeneous equations with variable coefficients. Suppose that we have found a solution $y = \varphi_1(t)$ to the second order homogeneous equation

$$y'' + p(t)y' + q(t)y = 0\,. \tag{4.16}$$

Assume that another solution is given by $y = v(t)\varphi_1(t)$. Then v satisfies

$$v''\varphi_1 + 2v'\varphi_1' + v\varphi_1'' + p(v'\varphi_1 + v\varphi_1') + qv\varphi_1 = 0\,.$$

By the fact that φ_1 solves (4.16), we find that v satisfies

$$v''\varphi_1 + 2v'\varphi_1' + pv'\varphi_1 = 0 \quad \text{or equivalently,} \quad v''\varphi_1 + v'(2\varphi_1' + p\varphi_1) = 0\,. \tag{4.17}$$

The equation above can be solved (for v') using the method of integrating factor, and is essentially a first order equation.

Let P be an anti-derivative of p. If $\varphi_1(t) \neq 0$ for all $t \in I$, then (4.17) implies that

$$\big(\varphi_1^2(t)e^{P(t)}v'(t)\big)' = 0 \quad \Rightarrow \quad \varphi_1^2(t)e^{P(t)}v'(t) = C \quad \Rightarrow \quad \varphi_1^2(t)v'(t) = Ce^{-P(t)} \qquad \forall\, t \in I\,.$$

As a consequence,

$$W[\varphi_1, \varphi_2](t) = \begin{vmatrix} \varphi_1(t) & v(t)\varphi_1(t) \\ \varphi_1'(t) & v'(t)\varphi_1(t) + v(t)\varphi_1'(t) \end{vmatrix} = \begin{vmatrix} \varphi_1(t) & 0 \\ \varphi_1'(t) & v'(t)\varphi_1(t) \end{vmatrix} = \varphi_1^2(t)v'(t) = Ce^{-P(t)} \neq 0$$

which implies that $\{\varphi_1, v\varphi_1\}$ is indeed a fundamental set of (4.16).

Example 4.23. Given that $y = \varphi_1(t) = \dfrac{1}{t}$ is a solution of

$$2t^2y'' + 3ty' - y = 0 \qquad \text{for } t > 0\,, \tag{4.18}$$

find a fundamental set of the equation.

Suppose another solution is given by $y = v(t)\varphi_1(t) = v(t)/t$. Then (4.17) implies that v satisfies

$$v''(t)\frac{1}{t} + v'(-\frac{2}{t^2} + \frac{3}{2t}\frac{1}{t}) = 0\,.$$

Therefore, $v'' = \dfrac{v'}{2t}$; thus $v'(t) = C_1\sqrt{t}$ which further implies that $v(t) = \dfrac{2}{3}C_1 t^{\frac{3}{2}} + C_2$. Therefore, one solution to (4.18) is

$$y = \frac{2}{3}C_1\sqrt{t} + C_2\frac{1}{t}$$

which also implies that $y = \varphi_2(t) = \sqrt{t}$ is a solution to (4.18). Note that the Wronskian

$$W[\varphi_1,\varphi_2](t) = \begin{vmatrix} \frac{1}{t} & \sqrt{t} \\ -\frac{1}{t^2} & \frac{1}{2\sqrt{t}} \end{vmatrix} = \frac{3}{2}t^{-\frac{3}{2}} \neq 0 \quad \text{for } t > 0\,; \tag{4.19}$$

thus $\{\varphi_1, \varphi_2\}$ is indeed a fundamental set of (4.18).

4.6.1 Method of Variation of Parameters

This method can be used to solve a nonhomogeneous ODE when one solution to the corresponding homogeneous equation is known.

Consider

$$y'' + p(t)y' + q(t)y = g(t)\,. \tag{4.11}$$

Suppose that we are given one solution $y = \varphi_1(t)$ to the corresponding homogeneous euqation

$$y'' + p(t)y' + q(t)y = 0\,. \tag{4.12}$$

Using the procedure in Section 4.6, we can find another solution $y = \varphi_2(t)$ to (4.12) so that $\{\varphi_1, \varphi_2\}$ forms a fundamental set of (4.12). Our goal next is to obtain a particular solution to (4.11).

Suppose a particular solution $y = Y(t)$ can be written as the product of two functions u and φ_1; that is, $Y(t) = u(t)\varphi_1(t)$. Then similar computations as in Section 4.6 show that

$$u''\varphi_1 + u'(2\varphi_1' + p\varphi_1) = g \quad \Rightarrow \quad (\varphi_1^2 e^P u')' = \varphi_1 e^P g\,,$$

where P is an anti-derivative of p. Therefore,

$$\varphi_1^2(t)e^{P(t)}u'(t) = \int \varphi_1(t)e^{P(t)}g(t)\,dt\,,$$

and further computations yield that

$$u(t) = \int \frac{\int \varphi_1(t)e^{P(t)}g(t)\,dt}{\varphi_1^2(t)e^{P(t)}}\,dt\,.$$

Therefore, a particular solution is of the form

$$Y(t) = \varphi_1(t) \int \frac{\int \varphi_1(t) e^{P(t)} g(t)\, dt}{\varphi_1^2(t) e^{P(t)}}\, dt\,. \tag{4.20}$$

Example 4.24. As in Example 4.23, let $y = \varphi_1(t) = \dfrac{1}{t}$ be a given solution to

$$2t^2 y'' + 3ty' - y = 0 \qquad \text{for } t > 0\,, \tag{4.18}$$

Suppose that we are looking for solutions to

$$2t^2 y'' + 3ty' - y = 2t^2 \qquad \text{for } t > 0\,. \tag{4.21}$$

Using (4.20) (noting that in this case $g(t) = 1$), we know that a particular solution is given by

$$Y(t) = \frac{1}{t} \int \frac{\int t^{-1} e^{3/2 \log t} dt}{t^{-2} e^{3/2 \log t}}\, dt = \frac{1}{t} \int \Big(t^{\frac{1}{2}} \int t^{\frac{1}{2}} dt \Big) dt = \frac{2}{9} t^2\,.$$

Therefore, combining with the fact that $\varphi_2(t) = \sqrt{t}$ is a solution to (4.18), we find that a general solution to (4.21) is given by

$$y = \frac{C_1}{t} + C_2 \sqrt{t} + \frac{2}{9} t^2\,.$$

Let $\{\varphi_1, \varphi_2\}$ be a fundamental set of (4.12) (here φ_2 is either given or obtained using the procedure in previous section), we can also look for a particular solution to (4.11) of the form

$$Y(t) = c_1(t)\varphi_1(t) + c_2(t)\varphi_2(t)\,.$$

Plugging such Y in (4.11)), we find that

$$c_1''\varphi_1 + c_1'(2\varphi_1' + p\varphi_1) + c_2''\varphi_2 + c_2'(2\varphi_2' + p\varphi_2) = g\,. \tag{4.22}$$

Since we increase the degree of freedom (by adding another function c_2), we can impose an additional constraint. Assume that the additional constraint is

$$c_1'\varphi_1 + c_2'\varphi_2 = 0\,. \tag{4.23}$$

Then $c_1''\varphi_1 + c_2''\varphi_2 = -c_1'\varphi_1' - c_2'\varphi_2'$; thus (4.22) reduces to

$$c_1'\varphi_1' + c_2'\varphi_2' = g\,. \tag{4.24}$$

Solving (4.23) and (4.24), we find that

$$c_1' = \frac{-g\varphi_2}{W[\varphi_1, \varphi_2]} \qquad \text{and} \qquad c_2' = \frac{g\varphi_1}{W[\varphi_1, \varphi_2]}\,.$$

The discussion above establishes the following

Theorem 4.25. *If the function p, q and g are continuous in an open interval I, and $\{\varphi_1, \varphi_2\}$ be a fundamental set of the ODE* (4.12)*. Then a particular solution to* (4.11) *is*

$$Y(t) = -\varphi_1(t) \int_{t_0}^{t} \frac{g(s)\varphi_2(s)}{W[\varphi_1, \varphi_2](s)}\, ds + \varphi_2(t) \int_{t_0}^{t} \frac{g(s)\varphi_1(s)}{W[\varphi_1, \varphi_2](s)}\, ds\,, \tag{4.25}$$

where $t_0 \in I$ can be arbitrarily chosen, and the general solution to (4.11) *is*

$$y = C_1\varphi_1(t) + C_2\varphi_2(t) + Y(t)\,.$$

Example 4.26. Given two solutions $\varphi_1(t) = \dfrac{1}{t}$ and $\varphi_2(t) = \sqrt{t}$ to the ODE

$$2t^2 y'' + 3ty' - y = 0 \qquad \text{for } t > 0\,. \tag{4.18}$$

To solve

$$2t^2 y'' + 3ty' - y = 2t^2 \qquad \text{for } t > 0\,, \tag{4.21}$$

we use (4.25) and (4.19) to obtain that a particular solution to (4.21) is given by

$$Y(t) = -\frac{1}{t} \int \frac{\sqrt{t}}{\frac{3}{2}t^{-3/2}}\, dt + \sqrt{t} \int \frac{t^{-1}}{\frac{3}{2}t^{-3/2}}\, dt = \frac{2}{9}t^2\,.$$

Therefore, a general solution to (4.21) is given by

$$y = \frac{C_1}{t} + C_2\sqrt{t} + \frac{2}{9}t^2\,.$$

4.7 Mechanical Vibrations

We have been discussing the motion of an object attached to a spring without external force in Example 4.13. Now we explain what if there are presence of external forcings. We consider

$$m\ddot{x} = -kx - r\dot{x} + g(t)\,, \tag{4.26}$$

where m, k, r are positive constants. We remark that the term $-r\dot{x}$ is sometimes called a ***damping*** or ***resistive*** force, and r is called the ***damping coefficient***.

1. **Undamped Free Vibrations**: This case refers to that $g \equiv 0$ and $r = 0$. The solution to (4.26) is then

$$x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega t = R\cos(\omega_0 t - \phi)\,,$$

where $R = \sqrt{C_1^2 + C_2^2}$ is called the ***amplitude***, $\omega_0 = \sqrt{\dfrac{k}{m}}$ is called the ***natural frequency*** and $\phi = \arctan \dfrac{C_2}{C_1}$ is called the ***phase angle***. The period of this vibration is $T = \dfrac{2\pi}{\omega_0}$.

2. **Dampled Free Vibrations**: This case refers to that $g \equiv 0$ and $r > 0$. The solution to (4.26) is then

$$x(t) = C_1 e^{-\frac{rt}{2m}} \cos \mu t + C_2 e^{-\frac{rt}{2m}} \sin \mu t = Re^{-\frac{rt}{2m}} \cos(\mu t - \phi)\,,$$

where $R = \sqrt{C_1^2 + C_2^2}$, $\mu = \dfrac{\sqrt{4km - r^2}}{2m}$, and $\phi = \arctan \dfrac{C_2}{C_1}$. Here μ is called the ***quasi frequency***, and we note that

$$\frac{\mu}{\omega_0} = \Big(1 - \frac{r^2}{4km}\Big)^{\frac{1}{2}} \approx 1 - \frac{r^2}{8km} \,,$$

where the last approximation holds if $\dfrac{r^2}{4km} \ll 1$. The period of this vibration $\dfrac{2\pi}{\mu}$ is called the ***quasi period***.

(a) **Critical damped**: In this case, $r^2 = 4km$.

(b) **Overdamped**: This case refers to that $r^2 > 4km$, and in this case the attached object pass the equilibrium at most once and does not oscillate about equilibrium.

3. **Forced Vibrations with Damping**: We only consider

$$m\ddot{x} + r\dot{x} + kx = F_0 \cos \omega t \tag{4.27}$$

for some non-zero r, F_0 and ω. Let $\{\varphi_1, \varphi_2\}$ be a fundamental set of the corresponding homogeneous equation of (4.27). From the discussion above, φ_1 and φ_2 both decay to zero (die out) as $t \to \infty$. Using what we learn from the method of undetermined coefficients, the general solution to (4.27) is

$$x = \underbrace{C_1\varphi_1(t) + C_2\varphi_2(t)}_{\equiv x_c(t)} + \underbrace{A\cos\omega t + B\sin\omega t}_{\equiv X(t)} \,,$$

where C_1 and C_2 are chosen to satisfy the initial condition, and A and B are some constants so that $X(t) = A\cos\omega t + B\sin\omega t$ is a particular solution to (4.27). The part $x_c(t)$ is called the ***transient solution*** and it decays to zero (die out) as $t \to \infty$; thus as $t \to \infty$, one sees that only a steady oscillation with the same frequency as the external force remains in the motion. $x = X(t)$ is called the ***steady state solution*** or the ***forced response***.

Since $x = X(t)$ is a particular solution to (4.27), (A, B) satisfies

$$\begin{aligned} (k - \omega^2 m)A + r\omega B &= F_0 \,, \\ -r\omega A + (k - \omega^2 m)B &= 0 \,; \end{aligned}$$

thus with ω_0 denoting the natural frequency; that is, $\omega_0 = \dfrac{k}{m}$, we have

$$(A, B) = \Big(\frac{F_0 m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + r^2\omega^2}, \frac{F_0 r\omega}{m^2(\omega_0^2 - \omega^2)^2 + r^2\omega^2}\Big) \,.$$

Let $\alpha = \dfrac{\omega}{\omega_0}$, and $\Gamma = \dfrac{r^2}{mk}$. Then

$$(A, B) = \frac{F_0}{k}\Big(\frac{1 - \alpha^2}{(1 - \alpha^2)^2 + \Gamma\alpha^2}, \frac{\sqrt{\Gamma}\alpha}{(1 - \alpha^2)^2 + \Gamma\alpha^2}\Big) \,;$$

thus

$$X(t) = R\cos(\omega t - \phi)\,,$$

where with Δ denoting the number $\sqrt{(1-\alpha^2)^2 + \Gamma\alpha^2}$, we have

$$R = \sqrt{A^2+B^2} = \frac{F_0}{k\Delta} \quad \text{and} \quad \phi = \arccos\frac{1-\alpha^2}{\Delta}\,.$$

Then if $\alpha \ll 1$, $R \approx \dfrac{F_0}{k}$ and $\phi \approx 0$, while if $\alpha \gg 1$, $R \ll 1$ and $\phi \approx \pi$.

In the intermediate region, some α, called $\alpha_{\rm max}$, maximize the amplitude R. Then $\alpha_{\rm max}$ minimize $(1-\alpha^2)^2 + \Gamma\alpha^2$ which implies that $\alpha_{\rm max}$ satisfies

$$\alpha_{\rm max}^2 = 1 - \frac{\Gamma}{2}$$

and, when $\Gamma < 1$, the corresponding maximum amplitude $R_{\rm max}$ is

$$R_{\rm max} = \frac{F_0}{k}\frac{1}{\sqrt{\Gamma}\sqrt{1-\Gamma/4}} \approx \frac{F_0}{k\sqrt{\Gamma}}\Big(1+\frac{\Gamma}{8}\Big)\,,$$

where the last approximation holds if $\Gamma \ll 1$. If $\Gamma > 2$, the maximum of R occurs at $\alpha = 0$ (and the maximum amplitude is $R_{\rm max} = \dfrac{F_0}{k}$).

For lightly damped system; that is, $r \ll 1$ (which implies that $\Gamma \ll 1$), the maximum amplitude $R_{\rm max}$ is closed to a very large number $\dfrac{F_0}{k\sqrt{\Gamma}}$. In this case $\alpha_{\rm max} \approx 1$, and this implies that the frequency $\omega_{\rm max}$, where the maximum of R occurs, is very close to ω_0. We call such a phenomena (that $R_{\rm max} \gg 1$ when $\omega \approx \omega_0$) ***resonance***. In such a case, $\alpha_{\rm max} \approx 1$; thus $\phi = \dfrac{\pi}{2}$ which means the response occur $\dfrac{\pi}{2}$ later than the peaks of the excitation.

4. **Forced Vibrations without Damping**:

   (a) When $r = 0$, if $\omega \neq \omega_0$, then general solution to (4.27) is

   $$x = C_1\cos\omega_0 t + C_2\sin\omega_0 t + \frac{F_0}{m(\omega_0^2-\omega^2)}\cos\omega t\,,$$

   where C_1 and C_2 depends on the initial data. We are interested in the case that $x(0) = x'(0) = 0$. In this case,

   $$C_1 = -\frac{F_0}{m(\omega_0^2-\omega^2)} \quad \text{and} \quad C_2 = 0\,,$$

   so the solution to (4.27) (with initial condition $x(0) = x'(0) = 0$) is

   $$x = \frac{F_0}{m(\omega_0^2-\omega^2)}\big(\cos\omega t - \cos\omega_0 t\big) = \frac{2F_0}{m(\omega_0^2-\omega^2)}\sin\frac{\omega_0-\omega}{2}t\sin\frac{\omega_0+\omega}{2}t\,.$$

   When $\omega \approx \omega_0$, $R = \dfrac{2F_0}{m(\omega_0^2-\omega^2)}\sin\dfrac{\omega_0-\omega}{2}t$ presents a slowly varying sinusoidal amplitude. This type of motion, possessing a periodic variation of amplitude, is called a ***beat***.

   (b) When $r = 0$ and $\omega = \omega_0$, the general solution to (4.27) is

   $$x = C_1\cos\omega_0 t + C_2\sin\omega_0 t + \frac{F_0}{2m\omega_0}t\sin\omega_0 t\,.$$

5 Theory of Higher Order Linear Differential Equations

5.1 Basic Theory of Linear Differential Equations

An n-th order linear ordinary differential equations is an equation of the form

$$P_n(t)\frac{d^n y}{dt^n} + P_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + P_1\frac{dy}{dt} + P_0(t)y = G(t)\,,$$

where P_n is never zero in the time interval of interest. Divide both sides by $P_n(t)$, we obtain

$$L[y] = \frac{d^n y}{dt} + p_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_1(t)\frac{dy}{dt} + p_0(t)y = g(t)\,. \tag{5.1}$$

Suppose that $p_j \equiv 0$ for all $0 \leqslant j \leqslant n-1$. Then to determine y, it requires n times integration and each integration produce an arbitrary constant. Therefore, we expect that to determine the solution y to (5.1) uniquely, it requires n initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_1, \quad \cdots, \quad y^{(n-1)}(t_0) = y_{n-1}\,, \tag{5.2}$$

where t_0 is some point in an open interval I, and $y_0, y_1, \cdots, y_{n-1}$ are some given constants.

Definition 5.1. Equation (5.1) is called ***homogeneous*** if $g \equiv 0$.

Similar to Theorem 4.1, we have the following

Theorem 5.2. *If the functions $p_0, \cdots, p_{n-1}$ and g are continuous on an open interval I, then there exists exactly one solution $y = \varphi(t)$ of the differential equation* (5.1) *with initial condition* (5.2), *where t_0 is any point in I. This solution exists throughout the interval I.*

Definition 5.3. Let $\{\varphi_1, \cdots, \varphi_n\}$ be a collection of n differentiable functions defined on an open interval I. The Wronskian of $\varphi_1, \varphi_2, \cdots, \varphi_n$ at $t_0 \in I$, denoted by $W[\varphi_1, \cdots, \varphi_n](t_0)$, is the number

$$W[\varphi_1, \cdots, \varphi_n](t_0) = \begin{vmatrix} \varphi_1(t_0) & \varphi_2(t_0) & \cdots & \varphi_n(t_0) \\ \varphi_1'(t_0) & \varphi_2'(t_0) & \cdots & \varphi_n'(t_0) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1^{(n-1)}(t_0) & \varphi_2^{(n-1)}(t_0) & \cdots & \varphi_n^{(n-1)}(t_0) \end{vmatrix}.$$

The following theorem can be viewed as a generalization of Theorem 4.7.

Theorem 5.4. *Let $y = \varphi_1(t)$, $y = \varphi_2(t)$, $\cdots$, $y = \varphi_n(t)$ be solutions to the homogeneous equation*

$$L[y] = \frac{d^n y}{dt} + p_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_1(t)\frac{dy}{dt} + p_0(t)y = 0\,. \tag{5.3}$$

Then the Wronskian of $\varphi_1, \varphi_2, \cdots, \varphi_n$ satisfies

$$\frac{d}{dt}W[\varphi_1, \cdots, \varphi_n](t) + p_{n-1}(t)W[\varphi_1, \cdots, \varphi_n](t) = 0\,.$$

Proof. By the differentiation of the determinant, we find that

$$\frac{d}{dt}W[\varphi_1,\cdots,\varphi_n] = \begin{vmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1' & \varphi_2'(t_0) & \cdots & \varphi_n' \\ \vdots & \vdots & & \vdots \\ \varphi_1^{(n-2)} & \varphi_2^{(n-2)} & \cdots & \varphi_n^{(n-2)} \\ \varphi_1^{(n)} & \varphi_2^{(n)} & \cdots & \varphi_n^{(n)} \end{vmatrix}$$

$$= \begin{vmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1' & \varphi_2'(t_0) & \cdots & \varphi_n' \\ \vdots & \vdots & & \vdots \\ \varphi_1^{(n-2)} & \varphi_2^{(n-2)} & \cdots & \varphi_n^{(n-2)} \\ -p_{n-1}\varphi_1^{(n-1)} - \cdots - p_0\varphi_1 & -p_{n-1}\varphi_2^{(n-1)} - \cdots - p_0\varphi_2 & \cdots & -p_{n-1}\varphi_n^{(n-1)} - \cdots - p_0\varphi_n \end{vmatrix}$$

$$= -p_{n-1}W[\varphi_1,\cdots,\varphi_n]\,. \qquad \square$$

Theorem 5.5. *Suppose that the functions $p_0,\cdots,p_{n-1}$ are continuous on an open interval I. If $y=\varphi_1(t)$, $y=\varphi_2(t)$, $\cdots$, $y=\varphi_n(t)$ are solutions to the homogeneous equation* (5.3) *and the Wronskian $W[\varphi_1,\cdots,\varphi_n](t)\neq 0$ for at least one point $t\in I$, then every solution of* (5.3) *can be expressed as a linear combination of $\varphi_1,\cdots,\varphi_n$.*

Proof. Let $y=\varphi(t)$ be a solution to (5.3), and suppose that $W[\varphi_1,\cdots,\varphi_n](t_0)\neq 0$. Define $(y_0,y_1,\cdots,y_{n-1}) = \big(\varphi(t_0),\varphi'(t_0),\cdots,\varphi^{(n-1)}(t_0)\big)$, and let $C_1,\cdots,C_n\in\mathbb{R}$ be the solution to

$$\begin{bmatrix} \varphi_1(t_0) & \varphi_2(t_0) & \cdots & \varphi_n(t_0) \\ \varphi_1'(t_0) & \varphi_2'(t_0) & \cdots & \varphi_n'(t_0) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1^{(n-1)}(t_0) & \varphi_2^{(n-1)}(t_0) & \cdots & \varphi_n^{(n-1)}(t_0) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix}.$$

We note that the system above has a unique solution since $W[\varphi_1,\cdots,\varphi_n](t_0)\neq 0$.
Claim: $\varphi(t) = C_1\varphi_1(t)+\cdots+C_n\varphi_n(t)$.
Proof of Claim: Note that $y=\varphi(t)$ and $y=C_1\varphi_1(t)+\cdots+C_n\varphi_n(t)$ are both solutions to (5.3) satisfying the same initial condition. Therefore, by Theorem 5.2 the solution is unique, so the claim is concluded. $\square$

Definition 5.6. A collection of solutions $\{\varphi_1,\cdots,\varphi_n\}$ to (5.3) is called a fundamental set of equation (5.3) if $W[\varphi_1,\cdots,\varphi_n](t)\neq 0$ for some t in the interval of interest.

5.1.1 Linear Independence of Functions

Recall that in a vector space $(\mathcal{V},+,\cdot)$ over scalar field $\mathbb{F}$, a collection of vectors $\{\boldsymbol{v}_1,\cdots,\boldsymbol{v}_n\}$ is called linearly dependent if there exist constants c_1, $\cdots$, c_n in $\mathbb{F}$ such that $\prod_{i=1}^{n} c_i \equiv c_1\cdot c_2\cdot\cdots\cdot c_{n-1}\cdot c_n \neq 0$ and

$$c_1\cdot\boldsymbol{v}_1+\cdots+c_n\cdot\boldsymbol{v}_n = \mathbf{0}\,.$$

If no such $c_1, \cdots, c_n$ exists, $\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n\}$ is called linearly independent. In other words, $\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n\} \subseteq \mathcal{V}$ is linearly independent if and only if

$$c_1 \cdot \boldsymbol{v}_1 + \cdots + c_n \cdot \boldsymbol{v}_n = \mathbf{0} \quad \Leftrightarrow \quad c_1 = c_2 = \cdots = c_n = 0\,.$$

Now let $\mathcal{V}$ denote the collection of all $(n-1)$-times differentiable functions defined on an open interval I. Then $(\mathcal{V}, +, \cdot)$ clearly is a vector space over $\mathbb{R}$. Given $\{f_1, \cdots, f_n\} \subseteq \mathcal{V}$, we would like to determine the linear dependence or independence of the n-functions $\{f_1, \cdots, f_n\}$. Suppose that

$$c_1 f_1(t) + \cdots + c_n f_n(t) = 0 \qquad \forall\, t \in I\,.$$

Since each f_j are $(n-1)$-times differentiable, we have for $1 \leqslant k \leqslant n-1$,

$$c_1 f_1^{(k)}(t) + \cdots + c_n f_n^{(k)}(t) = 0 \qquad \forall\, t \in I\,.$$

In other words, $c_1, \cdots, c_n$ satisfy

$$\begin{bmatrix} f_1(t) & f_2(t) & \cdots & f_n(t) \\ f_1'(t) & f_2'(t) & \cdots & f_n'(t) \\ \vdots & & \vdots & \\ f_1^{(n-1)}(t) & f_2^{(n-1)}(t) & \cdots & f_n^{(n-1)}(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \forall\, t \in I\,.$$

If there exists $t_0 \in I$ such that the matrix $\begin{bmatrix} f_1(t_0) & f_2(t_0) & \cdots & f_n(t_0) \\ f_1'(t_0) & f_2'(t_0) & \cdots & f_n'(t_0) \\ \vdots & & \vdots & \\ f_1^{(n-1)}(t_0) & f_2^{(n-1)}(t_0) & \cdots & f_n^{(n-1)}(t_0) \end{bmatrix}$ is non-singular, then $c_1 = c_2 = \cdots = c_n = 0$. Therefore, a collection of solutions $\{\varphi_1, \cdots, \varphi_n\}$ is a fundamental set of equation (5.3) if and only if $\{\varphi_1, \cdots, \varphi_n\}$ is linearly independent.

5.1.2 The Homogeneous Equations - Reduction of Orders

Suppose that $y = \varphi_1(t)$ is a solution to (5.3). Now we look for a function v such that $y = v(t)\varphi_1(t)$ is also a solution to (5.3). The derivative of this v satisfies an $(n-1)$-th order homogeneous ordinary differential equation.

Example 5.7. Suppose that we are given $y = \varphi_1(t) = e^t$ as a solution to

$$(2-t)y''' + (2t-3)y'' - ty' + y = 0 \quad \text{for} \quad t < 2\,. \tag{5.4}$$

Suppose that $y = v(t)e^t$ is also a solution to (5.4). Then

$$(2-t)(v'''e^t + 3v''e^t + 3v'e^t + ve^t) + (2t-3)(v''e^t + 2v'e^t + ve^t) - t(v'e^t + ve^t) + ve^t = 0$$

which implies that v satisfies

$$(2-t)v''' + \big[3(2-t) + (2t-3)\big]v'' + \big[3(2-t) + 2(2t-3) - t\big]v' = 0$$

or equivalently, with u denoting v'',

$$(2-t)u' + (3-t)u = 0\,.$$

Solving the ODE above, we find that $u(t) = C_1(2-t)e^{-t}$ for some constant C_1; thus

$$v(t) = C_3 + C_2 t + C_1 e^{-t} - C_1(t+1)e^{-t} = C_3 + C_2 t - C_1 t e^{-t}\,.$$

Therefore, a fundamental set of (5.4) is $\{e^t, te^t, t\}$.

Example 5.8. Suppose that we are given $y = \varphi_1(t) = t^2$ as a solution to

$$t^2(t+3)y''' - 3t(t+2)y'' + 6(1+t)y' - 6y = 0 \qquad \text{for} \quad t > 0\,. \tag{5.5}$$

Suppose that $y = v(t)t^2$ is also a solution to the ODE above. Then

$$t^2(t+3)(v'''t^2 + 6v''t + 6v') - 3t(t+2)(v''t^2 + 4v't + 2v) + 6(1+t)(v't^2 + 2vt) - 6vt^2 = 0$$

which implies that v satisfies

$$t^4(t+3)v''' + \big[6t^3(t+3) - 3t^3(t+2)\big]v'' + \big[6t^2(t+3) - 12t^2(t+2) + 6t^2(1+t)\big]v' = 0$$

or equivalently, with u denoting v'',

$$t(t+3)u' + 3(t+4)u = 0\,.$$

Solving the ODE above, we find that $u(t) = C_1 t^{-4}(t+3)$ for some constant C_1; thus

$$v(t) = \frac{C_1}{2}(t^{-2} + t^{-1}) + C_2 t + C_3$$

for some constants C_2 and C_2. Therefore, the general solution to (5.5) is given by $y(t) = C_1(1+t) + C_2 t^3 + C_3 t^2$ which implies that $\{t^2, t^3, 1+t\}$ is a fundamental set of the ODE.

5.1.3 The Nonhomogeneous Equations

Let $y = Y_1(t)$ and $y = Y_2(t)$ be solutions to (5.1). Then $y = Y_1(t) - Y_2(t)$ is a solution to the homogeneous equation (5.3); thus if $\{\varphi_1, \cdots, \varphi_n\}$ is a fundamental set of (5.3), then

$$Y_1(t) - Y_2(t) = C_1\varphi_1(t) + \cdots + C_n\varphi_n(t)\,.$$

Therefore, we establish the following theorem which is similar to Theorem 4.15.

Theorem 5.9. *The general solution of the nonhomogeneous equation* (5.1) *can be written in the form*

$$y = \varphi(t) = C_1\varphi_1(t) + C_2\varphi_2(t) + \cdots + C_n\varphi_n(t) + Y(t)\,,$$

where $\{\varphi_1, \cdots, \varphi_n\}$ *is a fundamental set of* (5.3), $C_1, \cdots, C_n$ *are arbitrary constants, and* $y = Y(t)$ *is a particular solution of the nonhomogeneous equation* (5.1).

In general, in order to solve (5.1), we follow the procedure listed below:

1. Find the space of complementary solution to (5.3); that is, find the general solution $y = C_1\varphi_1(t) + C_2\varphi_2(t) + \cdots + C_n\varphi_n$ of the homogeneous equation (5.3).

2. Find a particular solution $y = Y(t)$ of the nonhomogeneous equation (5.1).

3. Apply Theorem 5.9.

5.2 Homogeneous Linear Equations with Constant Coefficients

We now consider the n-th order linear homogeneous ODE with constant coefficients

$$L[y] = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0\,, \tag{5.6}$$

where a_j's are constants for all $j \in \{0, 1, \cdots, n-1\}$. Suppose that $r_1, r_2, \cdots, r_n$ are solutions to the characteristic equation of (5.6)

$$r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = 0\,.$$

Then (5.6) can be written as

$$\Big(\frac{d}{dt} - r_1\Big)\Big(\frac{d}{dt} - r_2\Big)\cdots\Big(\frac{d}{dt} - r_n\Big)y = 0$$

1. If the characteristic equation of (5.6) has distinct roots, then

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + \cdots + C_n e^{r_n t}\,. \tag{5.7}$$

   Reason: Let $z_1 = \big(\frac{d}{dt} - r_2\big)\cdots\big(\frac{d}{dt} - r_n\big)y$. Then $z_1' - r_1 z_1 = 0$; thus $z_1(t) = c_1 e^{r_1 t}$.

   Let $z_2 = \big(\frac{d}{dt} - r_3\big)\cdots\big(\frac{d}{dt} - r_n\big)y$. Then $z_2' - r_2 z_2 = c_1 z_1$; thus using the method of integrating factors, we find that

$$\frac{d}{dt}\big(e^{-r_2 t} z_2\big) = c_1 e^{(r_1 - r_2)t} \quad \Rightarrow \quad z_2(t) = \frac{c_1}{r_1 - r_2} e^{r_1 t} + c_2 e^{r_2 t}\,. \tag{5.8}$$

   Repeating the process, we conclude (5.7).

   How about if there are complex roots? Suppose that $r_1 = a + bi$ and $r_2 = a - bi$, then the Euler identity implies that, by choosing complex c_1 and c_2 in (5.8), we find that

$$z_2(t) = c_1 e^{at}\cos bt + c_2 e^{at}\sin bt$$

   for some constants c_1 and c_2. Therefore, suppose that we have complex roots $a_k \pm b_k i$ for $k = 1, \cdots, \ell$ and real roots $r_{2\ell+1}, \cdots, r_n$. Then the general solution to (5.7) is

$$\begin{aligned} y(t) = {} & C_1 e^{a_1 t}\cos b_1 t + C_2 e^{a_1 t}\sin b_1 t + \cdots + C_{2\ell-1} e^{a_\ell t}\cos b_\ell t + C_{2\ell} e^{a_\ell t}\sin b_\ell t \\ & + C_{2\ell+1} e^{r_{2\ell+1} t} + \cdots + C_n e^{r_n t}\,. \end{aligned}$$

2. If the characteristic equation of (5.6) has repeated roots, we group the roots in such a way that $r_1 = r_2 = \cdots = r_\ell$ and so on; that is, repeated roots appear in a successive order. Then the implication in (5.8) is modified to

$$\frac{d}{dt}\big(e^{-r_2 t} z_2\big) = c_1 e^{(r_1 - r_2)t} = c_1 \quad \Rightarrow \quad z_2(t) = (c_1 t + c_2) e^{r_1 t}\,.$$

(a) Suppose that $r_3 = r_2 = r_1 = r$. Letting $z_3 = \big(\frac{d}{dt} - r_4\big) \cdots \big(\frac{d}{dt} - r_n\big)y$, we find that

$$z_3' - rz_3 = (c_1 t + c_2)e^{rt}\,;$$

thus the method of integrating factor implies that

$$\frac{d}{dt}\big(e^{-rt}z_3\big) = c_1 t + c_2 \qquad \Rightarrow \qquad z_3(t) = \big(\frac{c_1}{2}t^2 + c_2 t + c_3\big)e^{rt}\,.$$

(b) Suppose that $r_1 = r_2 = r$ and $r_3 \neq r_2$. Letting $z_3 = \big(\frac{d}{dt} - r_4\big) \cdots \big(\frac{d}{dt} - r_n\big)y$, we find that

$$z_3' - r_3 z_3 = (c_1 t + c_2)e^{rt}\,;$$

thus the method of integrating factor implies that

$$\frac{d}{dt}\big(e^{-r_3 t}z_3\big) = (c_1 t + c_2)e^{(r-r_3)t} \qquad \Rightarrow \qquad z_3(t) = \big(\widetilde{c_1}t + \widetilde{c_2}\big)e^{rt} + c_3 e^{r_3 t}\,.$$

From the discussion above, we "conjecture" that if r_j's are roots of the characteristic equation of (5.6) with multiplicity n_j (so that $n_1 + \cdots + n_k = n$), then the general solution to (5.6) is

$$y(t) = \sum_{j=1}^{k} p_j(t)e^{r_j t}\,,$$

where $p_j(t)$'s are some polynomials of degree $n_j - 1$. Note that in each p_j there are n_j constants to be determined by the initial conditions.

If there are repeated complex roots, say $r_1 = a + bi$ and $r_2 = a - bi$ with $n_1 = n_2$. Then p_1 and p_2 are polynomials of degree n_1; thus by adjusting constants in the polynomials properly, we find that

$$p_1(t)e^{r_1 t} + p_2(t)e^{r_2 t} = \widetilde{p}_1(t)e^{at}\cos bt + \widetilde{p}_2(t)e^{at}\sin bt\,.$$

In other words, if r_j are real roots of the characteristic equation of (5.6) with multiplicity n_j and $a_k \pm ib_k$ are complex roots of the characteristic equation of (5.6) with multiplicity m_k (so that $\sum\limits_j n_j + \sum\limits_k 2m_k = n$), then the general solution to (5.6) is

$$y(t) = \sum_j p_j(t)e^{r_j t} + \sum_k e^{a_k t}\big(q_k^1(t)\cos b_k t + q_k^2(t)\sin b_k t\big)\,,$$

where $p_j(t)$'s are some polynomials of degree $n_j - 1$ and q_k^1, q_k^2's are some polynomials of degree $m_k - 1$.

Example 5.10. Find the general solution of

$$y^{(4)} + y''' - 7y'' - y' + 6y = 0\,.$$

The roots of the characteristic equation is $r = \pm 1$, $r = 2$ and $r = -3$; thus the general solution to the ODE above is

$$y = C_1 e^t + C_2 e^{-t} + C_3 e^{2t} + C_4 e^{-3t}\,.$$

If we are looking for a solution to the ODE above satisfying the initial conditions $y(0) = 1$, $y'(0) = 0$, $y''(0) = -1$ and $y'''(0) = -1$, then C_1, C_2, C_3, C_4 have to satisfy

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -3 \\ 1 & 1 & 4 & 9 \\ 1 & -1 & 8 & -27 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}.$$

Solving the linear system above, we find that the solution solving the ODE with the given initial data is

$$y = \frac{11}{8}e^t + \frac{5}{12}e^{-t} - \frac{2}{3}e^{2t} - \frac{1}{8}e^{-3t}.$$

Example 5.11. Find the general solution of

$$y^{(4)} - y = 0.$$

Also find the solution that satisfies the initial condition

$$y(0) = \frac{7}{2}, \quad y'(0) = -4, \quad y''(0) = \frac{5}{2}, \quad y'''(0) = -2.$$

The roots of the characteristic equation are $r = \pm 1$ and $r = \pm i$. Therefore, the general solution to the ODE above is

$$y = C_1 e^t + C_2 e^{-t} + C_3 \cos t + C_4 \sin t.$$

To satisfy the initial condition, $C_1, \cdots, C_4$ has to satisfy

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} \frac{7}{2} \\ -4 \\ \frac{5}{2} \\ -2 \end{bmatrix}.$$

Solving the linear system above, we find that the solution solving the ODE with the given initial data is

$$y = 3e^{-t} + \frac{1}{2}\cos t - \sin t.$$

Example 5.12. Find the general solution of $y^{(4)} + y = 0$.

The roots of the characteristic equation are $r = \pm\big(\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}i\big)$. Therefore, the general solution to the ODE above is

$$y = \exp\big(\frac{\sqrt{2}}{2}t\big)\big(C_1 \cos\frac{\sqrt{2}}{2}t + C_2 \sin\frac{\sqrt{2}}{2}t\big) + \exp\big(-\frac{\sqrt{2}}{2}t\big)\big(C_2 \cos\frac{\sqrt{2}}{2}t + C_4 \sin\frac{\sqrt{2}}{2}t\big).$$

5.3 Undetermined coefficients and the Annihilator Method

Definition 5.13. A linear differential operator L is a linear map sending smooth (meaning infinitely many times differentiable) function y to a function $L[y]$ given by

$$L[y](t) = p_n(t)\frac{d^n y}{dt^n}(t) + p_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}}(t) + \cdots + p_1(t)\frac{dy}{dt}(t) + p_0(t)y(t)$$

for some coefficient functions $p_0, p_1, \cdots, p_n$, where n is called the order of L if $p_n \neq 0$. In this case, L is usually expressed as

$$L = p_n(t)\frac{d^n}{dt^n} + p_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + p_1(t)\frac{d}{dt} + p_0(t)\,.$$

A linear differential operator L is said to annihilate a function y if $L[y] = 0$.

Theorem 5.14. *Let L_1 and L_2 be two differential operators with constant coefficients. Then L_1, L_2 commute; that is $L_1L_2 = L_2L_1$ or equivalently, for any smooth function y,*

$$L_1\big[L_2[y]\big] = L_2\big[L_1[y]\big]\,.$$

Example 5.15. Find a differentiable operator that annihilates $y(t) = 6te^{-4t} + 5e^t \sin 2t$.

Note that $L_1 = \dfrac{d^2}{dt^2} + 8\dfrac{d}{dt} + 16$ annihilates the function $\phi_1(t) = te^{-4t}$ and $L_2 = \dfrac{d^2}{dt^2} - 2\dfrac{d}{dt} + 5$ annihilates the function $\phi_2(t) = e^t \sin 2t$. Let $L = L_1L_2$; that is, for a given function ϕ, $L[\phi] = L_1\big[L_2[\phi]\big]$. Then $L = L_2L_1$ and

$$L[\phi_1] = L_2\big[L_1[\phi_1]\big] = L_2[0] = 0 \quad \text{and} \quad L[\phi_2] = L_1\big[L_2[\phi_2]\big] = L_1[0] = 0\,.$$

Therefore, the differential operator

$$L = L_1L_2 = \Big(\frac{d^2}{dt^2} + 8\frac{d}{dt} + 16\Big)\Big(\frac{d^2}{dt^2} - 2\frac{d}{dt} + 5\Big) = \frac{d^4}{dt^4} + 6\frac{d^3}{dt^3} + 5\frac{d^2}{dt^2} + 8\frac{d}{dt} + 80$$

annihilates y.

5.3.1 Method of annihilator

Example 5.16. Find a general solution to

$$y'' - y = te^t + \sin t\,. \tag{5.9}$$

As in the previous example, we find that

$$\Big(\frac{d^2}{dt^2} - 2\frac{d}{dt} + 1\Big)\Big(\frac{d^2}{dt^2} + 1\Big) = \frac{d^4}{dt^4} - 2\frac{d^3}{dt^3} + 2\frac{d^2}{dt^2} - 2\frac{d}{dt} + 1$$

is an annihilator of the function $\phi(t) = te^t + \sin t$. Therefore,

$$\Big(\frac{d^4}{dt^4} - 2\frac{d^3}{dt^3} + 2\frac{d^2}{dt^2} - 2\frac{d}{dt} + 1\Big)\Big(\frac{d^2}{dt^2} - 1\Big)y = 0$$

which implies that y is of the form

$$y(t) = \big(C_1t^2 + C_2t + C_3\big)e^t + C_4e^{-t} + C_5\cos t + C_6\sin t \tag{5.10}$$

since the characteristic equation has roots

$$\lambda = 1 \text{ (triple roots)}, -1, \pm i\,.$$

Substituting (5.10) into (5.9), we find that

$$y'' - y = 2C_1 e^t + 2(2C_1 t + C_2)e^t - 2C_5 \cos t - 2C_6 \sin t = te^t + \sin t\,;$$

thus $C_1 = \frac{1}{4}$, $C_2 = -\frac{1}{4}$, $C_5 = 0$ and $C_6 = -\frac{1}{2}$. Therefore, the general solution to (5.9) is

$$y(t) = \frac{1}{4}(t^2 - t)e^t - \frac{1}{2}\sin t + Ae^t + Be^{-t}\,.$$

Example 5.17. Find a general solution, using the annihilator method, to

$$y''' - 3y'' + 4y = te^{2t}\,.$$

Since $\Big(\frac{d^2}{dt^2} - 4\frac{d}{dt} + 4\Big)$ annihilates the function $\phi(t) = te^{2t}$, we find that the general solution y to the ODE above satisfies

$$\Big(\frac{d^2}{dt^2} - 4\frac{d}{dt} + 4\Big)\Big(\frac{d^3}{dt^3} - 3\frac{d^2}{dt^2} + 4\Big)y = 0\,. \tag{5.11}$$

Since the characteristic equation of (5.11) has zeros 2 (with multiplicity 4) and the general solution y to (5.11) can be written as

$$y(t) = (C_1 t^3 + C_2 t^2 + C_3 t + C_4)e^{2t} + C_5 e^{-t}\,. \tag{5.12}$$

Substituting (5.12) into (5.11), we find that

$$y(t) = \frac{1}{18}(t^3 - t^2)e^{2t} + (At + B)e^{2t} + Ce^{-t}\,.$$

5.3.2 Method of undetermined coefficients

A particular solution to the constant-coefficient differential equation $L[y] = Ct^m e^{rt}$, where m is a non-negative integer, has the form

$$y_p(t) = t^s\big(A_m t^m + \cdots + A_1 t + A_0\big)e^{rt}\,,$$

where $s = 0$ if r is not a root of the associated characteristic equation or s equals the multiplicity of this root.

A particular solution to the constant-coefficient differential equation $L[y] = Ct^m e^{\alpha t}\cos\beta t$ or $L[y] = Ct^m e^{\alpha t}\sin\beta t$, where $\beta \neq 0$, has the form

$$y_p(t) = t^s\big(A_m t^m + \cdots + A_1 t + A_0\big)e^{\alpha t}\cos\beta t + t^s\big(B_m t^m + \cdots + B_1 t + B_0\big)e^{\alpha t}\sin\beta t\,,$$

where $s = 0$ if $\alpha + i\beta$ is not a root of the associated characteristic equation or s equals the multiplicity of this root.

5.4 Method of Variation of Parameters

To solve a non-homogeneous ODE

$$L[y] = \frac{d^n y}{dt} + p_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_1(t)\frac{dy}{dt} + p_0(t)y = g(t)\,, \tag{5.1}$$

often times we apply the method of variation of parameters to find a particular solution. Suppose that $\{\varphi_1, \cdots, \varphi_n\}$ is a fundamental set of the homogeneous equation (5.3), we assume that a particular solution can be written as

$$y = Y(t) = u_1(t)\varphi_1(t) + \cdots + u_n(t)\varphi_n(t)\,.$$

Assume that $u_1, \cdots, u_n$ satisfy

$$u_1'\varphi_1^{(j)} + \cdots + u_n'\varphi_n^{(j)} = 0$$

for $j = 0, \cdots, 1, n-2$. Then

$$\begin{aligned} Y' &= u_1\varphi_1' + \cdots + u_n\varphi_n'\,, \\ Y'' &= u_1\varphi_1'' + \cdots + u_n\varphi_n''\,, \\ &\vdots \\ Y^{(n-1)} &= u_1\varphi_1^{(n-1)} + \cdots + u_n\varphi_n^{(n-1)}\,, \end{aligned}$$

and

$$Y^{(n)} = u_1'\varphi_1^{(n-1)} + \cdots + u_n'\varphi_n^{(n-1)} + u_1\varphi_1^{(n)} + \cdots + u_n\varphi_n^{(n)}\,.$$

Since $y = Y(t)$ is assumed to be a particular solution of (5.1), we have

$$u_1'\varphi_1^{(n-1)} + \cdots + u_n'\varphi_n^{(n-1)} = g(t)\,.$$

Therefore, $u_1, \cdots, u_n$ satisfy

$$\begin{bmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1' & \varphi_2' & \cdots & \varphi_n' \\ \vdots & & \ddots & \vdots \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} & \cdots & \varphi_n^{(n-1)} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ g \end{bmatrix}.$$

Let W_m denote the Wronskian of $\{\varphi_1, \cdots, \varphi_{m-1}, \varphi_{m+1}, \cdots, \varphi_n\}$; that is,

$$W_m = \begin{vmatrix} \varphi_1 & \cdots & \varphi_{m-1} & \varphi_{m+1} & \cdots & \varphi_n \\ \varphi_1' & \cdots & \varphi_{m-1}' & \varphi_{m+1}' & \cdots & \varphi_n' \\ \vdots & & \vdots & \vdots & & \vdots \\ \varphi_i^{(n-2)} & \cdots & \varphi_{m-1}^{(n-2)} & \varphi_{m+1}^{(n-2)} & \cdots & \varphi_n^{(n-2)} \end{vmatrix}.$$

Then $u_i' = (-1)^{n+i}\dfrac{W_i}{W[\varphi_1, \cdots, \varphi_n]}$ which implies that

$$Y(t) = \sum_{i=1}^{n} (-1)^{n+i}\varphi_i(t)\int_{t_0}^{t} \frac{W_i(s)g(s)}{W[\varphi_1, \cdots, \varphi_n](s)}\,ds\,. \tag{5.13}$$

Example 5.18. Find the general solution to

$$y''' - y'' - y' + y = g(t)\,. \tag{5.14}$$

Note the the roots of the characteristic equation $r^3 - r^2 - r + 1 = 0$ of the homogeneous equation

$$y''' - y'' - y' + y = 0 \tag{5.15}$$

are $r = 1$ (double) and $r = -1$; thus we have a fundamental set $\{e^t, te^t, e^{-t}\}$ of equation (5.15). Let $\varphi_1(t) = e^t$, $\varphi_2(t) = te^t$ and $\varphi_3(t) = e^{-t}$. Then

$$W[\varphi_1, \varphi_2, \varphi_3](t) = \begin{vmatrix} e^t & te^t & e^{-t} \\ e^t & (t+1)e^t & -e^{-t} \\ e^t & (t+2)e^t & e^{-t} \end{vmatrix} = \big[(t+1) + (t+2) - t - (t+1) - t + (t+2)\big]e^t = 4e^t\,,$$

and $W_1(t) = -2t - 1$, $W_2(t) = -2$ and $W_3(t) = e^{2t}$. Therefore, a particular solution is

$$\begin{aligned} Y(t) &= e^t \int_0^t \frac{(-2s-1)}{4e^s} g(s)ds - te^t \int_0^s \frac{-2}{4e^s} g(s)ds + e^{-t} \int_0^t \frac{e^{2s}}{4e^s} g(s)ds \\ &= \frac{1}{4}\int_0^t \big[2(t-s) - 1)e^{t-s} + e^{s-t}\big] g(s)ds\,, \end{aligned}$$

and the general solution to (5.14) is

$$y = C_1 e^t + C_2 te^t + C_3 e^{-t} + Y(t)\,.$$

Example 5.19. Recall that in Example 5.8 we have found a fundamental set $\{t^2, t^3, t+1\}$ to the ODE

$$t^2(t+3)y''' - 3t(t+2)y'' + 6(1+t)y' - 6y = 0 \qquad \text{for} \quad t > 0\,.$$

Now we consider the inhomogeneous equation

$$t^2(t+3)y''' - 3t(t+2)y'' + 6(1+t)y' - 6y = t^2(t+3)^2 \qquad \text{for} \quad t > 0\,.$$

Let $\varphi_1(t) = t^2$, $\varphi_2(t) = t^3$ and $\varphi_3(t) = t + 1$. Then

$$W[\varphi_1, \varphi_2, \varphi_3](t) = \begin{vmatrix} t^2 & t^3 & 1+t \\ 2t & 3t^2 & 1 \\ 2 & 6t & 0 \end{vmatrix} = 12t^2(1+t) + 2t^3 - 6t^2(1+t) - 6t^3 = 2t^2(t+3)\,.$$

and

$$W_1(t) = \begin{vmatrix} t^3 & 1+t \\ 3t^2 & 1 \end{vmatrix} = -2t^3 - 3t^2\,,\ W_2(t) = \begin{vmatrix} t^2 & 1+t \\ 2t & 1 \end{vmatrix} = -t^2 - 2t\,,\ W_3(t) = \begin{vmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{vmatrix} = t^4\,.$$

Rewrite the initial value problem as

$$y''' - \frac{3t(t+2)}{t^2(t+3)}y'' + \frac{6(1+t)}{t^2(t+3)}y' - \frac{6}{t^2(t+3)}y = (t+3)\,.$$

Let $g(t) = t+3$. Using formula (5.13), we find that the general solution to the inhomogeneous ODE is given by

$$\begin{aligned} y(t) &= -\varphi_1(t)\int \frac{(2t^3+3t^2)g(t)}{2t^2(t+3)}\,dt + \varphi_2(t)\int \frac{(t^2+2t)g(t)}{2t^2(t+3)}\,dt + \varphi_3(t)\int \frac{t^4 g(t)}{2t^2(t+3)}\,dt \\ &= -\varphi_1(t)\big(\frac{1}{2}t^2+\frac{3}{2}t\big) + \varphi_2(t)\big(\frac{t}{2}+\ln t\big) + \varphi_3(t)\frac{t^3}{6} + C_1\varphi_1(t) + C_2\varphi_2(t) + C_3\varphi_3(t) \\ &= C_1\varphi_1(t) + C_2\varphi_2(t) + C_3\varphi_3(t) + \frac{1}{6}t^4 - \frac{4}{3}t^3 + t^3\ln t\,. \end{aligned}$$

6 The Laplace Transform

6.1 Definition of the Laplace Transform

Definition 6.1 (Integral transform)**.** An ***integral transform*** is a relation between two functions f and F of the form

$$F(s) = \int_\alpha^\beta K(s,t)f(t)\,dt\,, \tag{6.1}$$

where $K(\cdot,\cdot)$ is a given function, called the ***kernel*** of the transformation, and the limits of integration α, β are also given (here α, β could be ∞ and in such cases the integral above is an improper integral). The relation (6.1) transforms function f into another function F called the transformation of f.

Proposition 6.2. *Every integral transform is linear; that is, for all functions f and g (defined on (α,β)) and constant a,*

$$\int_\alpha^\beta K(s,t)\big(af(t)+g(t)\big)\,dt = a\int_\alpha^\beta K(s,t)f(t)\,dt + \int_\alpha^\beta K(s,t)g(t)\,dt\,.$$

Example 6.3. Let $f:\mathbb{R}\to\mathbb{R}$ be a function such that $\displaystyle\int_{-\infty}^\infty |f(x)|\,dx < \infty$. The ***Fourier transform*** of f, denoted by $\mathscr{F}(f)$, is defined by

$$\mathscr{F}(f)(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-ist}f(t)\,dt\ \Big(= \lim_{\alpha,\beta\to\infty}\int_{-\alpha}^\beta e^{-ist}f(t)dt\Big)\,,$$

where the kernel K is a complex function (i.e., the value of K is complex). We will discuss the Fourier transform later.

Definition 6.4 (Laplace transform)**.** Let $f:[0,\infty]\to\mathbb{R}$ be a function. The ***Laplace transform*** of f, denoted by $\mathscr{L}(f)$, is defined by

$$\mathscr{L}(f)(s) = \int_0^\infty e^{-st}f(t)\,dt\ \Big(= \lim_{R\to\infty}\int_0^R e^{-st}f(t)dt\Big)\,,$$

provided that the improper integral exists.

Example 6.5. Let $f:[0,\infty)\to\mathbb{R}$ be defined by $f(t) = e^{at}$, where $a\in\mathbb{R}$ is a constant. Since the improper integral

$$\int_0^\infty e^{(a-s)t}\,dt = \lim_{R\to\infty}\int_0^R e^{(a-s)t}\,dt \overset{(s\neq a)}{=} \lim_{R\to\infty}\Big(-\frac{e^{(a-s)t}}{(s-a)}\Big|_{t=0}^{t=R}\Big) = \lim_{R\to\infty}\frac{1-e^{(a-s)R}}{s-a}$$

exists for $s > a$, we find that

$$\mathscr{L}(f)(s) = \frac{1}{s-a} \qquad \forall\, s > a\,.$$

Example 6.6. Let $f : [0,\infty) \to \mathbb{R}$ be given by $f(t) = \sin(at)$. Note that

$$\begin{aligned}
\int_0^R \underbrace{e^{-st}}_{\equiv u} \underbrace{\sin(at)\,dt}_{\equiv dv} &= -e^{-st}\frac{\cos(at)}{a}\Big|_{t=0}^{t=R} + \int_0^R (-s)e^{-st}\frac{\cos(at)}{a}\,dt \\
&= \frac{1}{a}\Big(1 - e^{-Rs}\cos(aR)\Big) - \frac{s}{a}\int_0^R e^{-st}\cos(at)\,dt \qquad (6.2)\\
&= \frac{1}{a}\Big(1 - e^{-Rs}\cos(aR)\Big) - \frac{s}{a}\Big(e^{-st}\frac{\sin(at)}{a}\Big|_{t=0}^{t=R} + \frac{s}{a}\int_0^R e^{-st}\sin(at)\,dt\Big) \\
&= \frac{1}{a}\Big(1 - e^{-Rs}\cos(aR)\Big) - \frac{s}{a^2}e^{-Rs}\sin(aR) - \frac{s^2}{a^2}\int_0^R e^{-st}\sin(at)\,dt\,;
\end{aligned}$$

thus we obtain that

$$\Big(1 + \frac{s^2}{a^2}\Big)\int_0^R e^{-st}\sin(at)\,dt = \frac{1}{a}\Big(1 - e^{-Rs}\cos(aR)\Big) - \frac{s}{a^2}e^{-Rs}\sin(aR)\,.$$

Therefore, the improper integral

$$\begin{aligned}
\int_0^\infty e^{-st}\sin(at)\,dt &= \lim_{R\to\infty}\int_0^R e^{-st}\sin(at)\,dt \\
&= \lim_{R\to\infty}\Big[\frac{a}{s^2+a^2}\Big(1 - e^{-Rs}\cos(aR)\Big) - \frac{s}{s^2+a^2}e^{-Rs}\sin(aR)\Big]
\end{aligned}$$

exists for all $s > 0$ which implies that

$$\mathscr{L}(f)(s) = \frac{a}{s^2+a^2} \qquad \forall\, s > 0\,.$$

Moreover, (6.2) further implies that

$$\int_0^\infty e^{-st}\cos(at)\,dt = \frac{a}{s}\Big(\frac{1}{a} - \frac{a}{s^2+a^2}\Big) = \frac{s}{s^2+a^2}\,.$$

Example 6.7. Let $f : [0,\infty) \to \mathbb{R}$ be defined by

$$f(t) = \begin{cases} 1 & \text{if } 0 \leqslant t < 1\,, \\ k & \text{if } t = 1\,, \\ 0 & \text{if } t > 1\,, \end{cases}$$

where k is a given constant. Since the improper integral

$$\int_0^\infty e^{-st} f(t)\,dt = \int_0^1 e^{-st}dt = \frac{1-e^{-s}}{s}$$

exists as long as $s \neq 0$, we find that

$$\mathscr{L}(f)(s) = \frac{1-e^{-s}}{s} \qquad \forall\, s \neq 0\,.$$

We note that the Laplace transform in this case is independent of the choice of k; thus the Laplace transform is not one-to-one (in the classical/pointwise sense).

Example 6.8. Let $f : [0, \infty) \to \mathbb{R}$ be defined by

$$f(t) = \begin{cases} 2 & \text{if } 0 < t < t\,, \\ 0 & \text{if } 5 < t < 10\,, \\ e^{4t} & \text{if } t > 10\,. \end{cases}$$

Then for $R > 10$ and $s \neq 0, 4$,

$$\begin{aligned} \int_0^R e^{-st} f(t)\, dt &= \int_0^5 2e^{-st}\, dt + \int_{10}^R e^{-st} e^{4t}\, dt = -\frac{2e^{-st}}{s}\Big|_{t=0}^{t=5} + \frac{e^{(4-s)t}}{4-s}\Big|_{t=10}^{t=R} \\ &= \frac{2(1-e^{-5s})}{s} + \frac{e^{(4-s)R} - e^{10(4-s)}}{4-s}\,. \end{aligned}$$

Therefore, if $s > 4$, passing to the limit as $R \to \infty$,

$$\mathscr{L}(f)(s) = \lim_{R\to\infty} \int_0^R e^{-st} f(t)\, dt = \frac{2(1-e^{-5s})}{s} + \frac{e^{-10(s-4)}}{s-4}\,.$$

One can verify that $\mathscr{L}(f)(s)$ does not exist for $s \leqslant 4$. Therefore,

$$\mathscr{L}(f)(s) = \frac{2}{s} - \frac{e^{-5s}}{s} + \frac{e^{-10(s-4)}}{s-4} \qquad \text{for } s > 4\,.$$

Theorem 6.9 (Linearity of the Laplace transform)**.** *Let $f, g : [0, \infty) \to \mathbb{R}$ be functions whose Laplace transform exist for $s > \alpha$ and c be a constant. Then for $s > \alpha$,*

1. *$\mathscr{L}(f + g)(s) = \mathscr{L}(f)(s) + \mathscr{L}(g)(s)$.*
2. *$\mathscr{L}(cf)(s) = c\mathscr{L}(f)(s)$.*

Example 6.10. Find the Laplace transform of the function $f(t) = 11 + 5e^{4t} - 6 \sin 2t$.

By Example 6.5, 6.6 and the linearity of the Laplace transform,

$$\mathscr{L}(f)(s) = \frac{11}{s} + \frac{5}{s-4} - \frac{12}{s^2+4} \qquad \text{for } s > 4\,.$$

6.1.1 Existence of the Laplace transform

There are functions whose Laplace transform does not exist for any s. For example, the function $f(t) = e^{t^2}$ does not have Laplace transform since it grows too rapidly as $t \to \infty$.

Definition 6.11. A function $f : [a, b] \to \mathbb{R}$ is said to have a ***jump discontinuity*** at $t_0 \in (a, b)$ if f is discontinuous at t_0 but $\lim\limits_{t\to t_0^+} f(t)$ and $\lim\limits_{t\to t_0^-} f(t)$ both exist. A function f is said to be ***piecewise continuous*** on a finite interval $[a, b]$ if f is continuous on $[a, b]$, except possibly for a finite number points at which f has jump discontinuities. A function f is said to be piecewise continuous on $[0, \infty)$ if f is piecewise continuous on $[0, N]$ for all $N > 0$.

Definition 6.12. A function f is said to be of ***exponential order*** α if there exists M such that

$$\big|f(t)\big| \leqslant Me^{\alpha t} \qquad \forall\, t \geqslant 0\,.$$

Proposition 6.13. *Let $f : [0, \infty) \to \mathbb{R}$ be a function. Suppose that*

1. *f is piecewise continuous on $[0,\infty)$, and*

2. *f is of exponential order α.*

Then the Laplace transform of f exists for $s > \alpha$, and $\lim\limits_{s\to\infty} \mathscr{L}(f)(s) = 0$, where $\mathscr{L}(f)$ is the Laplace transform of f.

Proof. Since f is piecewise continuous on $[0,R]$, the integral $\displaystyle\int_0^R e^{-st} f(t)\,dt$ exists. By the fact that $|f(t)| \leqslant Me^{\alpha t}$ for $t \geqslant 0$ for some M and α, we find that for $R_2 > R_1 > 0$,

$$\Big|\int_{R_1}^{R_2} e^{-st} f(t)\,dt\Big| \leqslant \int_{R_1}^{R_2} e^{-st} M e^{\alpha t}\,dt = M\frac{e^{(\alpha-s)R_2} - e^{(\alpha-s)R_1}}{\alpha - s}$$

which converges to 0 as $R_1, R_2 \to \infty$ if $s > \alpha$. Therefore, the improper integral $\displaystyle\int_0^\infty e^{-st} f(t)\,dt$ exists.

Finally,

$$\begin{aligned}\big|\mathscr{L}(f)(s)\big| &= \Big|\int_0^\infty e^{-st} f(t)\,dt\Big| \leqslant \int_0^\infty e^{-st}\big|f(t)\big|\,dt \leqslant \int_0^\infty e^{-st} M e^{\alpha t}\,dt \\ &= M\int_0^\infty e^{(\alpha-s)t}\,dt \leqslant \frac{M}{s-\alpha} \qquad \forall\, s > \alpha\,.\end{aligned}$$

As $s \to \infty$, the Sandwich lemma implies that $\lim\limits_{s\to\infty} \mathscr{L}(f)(s) = 0$. □

Example 6.14. Let $f : [0,\infty) \to \mathbb{R}$ be given by $f(t) = t^p$ for some $p > -1$. Recall that the Gamma function $\Gamma : (0,\infty) \to \mathbb{R}$ is defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\,dt\,.$$

We note that if $-1 < p < 0$, f is not of exponential order a for all $a \in \mathbb{R}$; however, the Laplace transform of f still exists. In fact, for $s > 0$,

$$\mathscr{L}(f)(s) = \lim_{R\to\infty} \int_0^R e^{-st} t^p\,dt = \lim_{R\to\infty} \int_0^{sR} e^{-t} \Big(\frac{t}{s}\Big)^p \frac{dt}{s} = \frac{\Gamma(p+1)}{s^{p+1}}\,.$$

In particular, if $p = n \in \mathbb{N} \cup \{0\}$, then

$$\mathscr{L}(f)(s) = \frac{n!}{s^{n+1}} \qquad \forall\, s > 0\,.$$

6.1.2 The Inverse Laplace Transform

Even though Example 6.7 shows that the Laplace transform is not one-to-one in the classical sense, we are still able to talk about the "inverse" of the Laplace transform because of the following

Theorem 6.15 (Lerch)**.** *Suppose that $f, g : [0,\infty) \to \mathbb{R}$ are continuous and of exponential order a. If $\mathscr{L}(f)(s) = \mathscr{L}(g)(s)$ for all $s > a$, then $f(t) = g(t)$ for all $t \geqslant 0$.*

Remark 6.16. The ***inverse Laplace transform*** of a function F is given by

$$\mathscr{L}^{-1}(F)(t) = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{\gamma - iR}^{\gamma + iR} e^{st} F(s)\,ds\,,$$

where the integration is done along the vertical line $\mathrm{Re}(s) = \gamma$ in the complex plane such that γ is greater than the real part of all singularities of F.

6.2 Properties of the Laplace Transform

Theorem 6.17. *Let $f : [0,\infty) \to \mathbb{R}$ be a function whose Laplace transform exists for $s > \alpha$. If $g(t) = e^{\beta t} f(t)$, then*

$$\mathscr{L}(g)(s) = \mathscr{L}(f)(s-\beta) \qquad \forall\, s > \alpha + \beta\,.$$

Proof. By the definition of the Laplace transform,

$$\mathscr{L}(g)(s) = \int_0^\infty e^{-st} e^{\beta t} f(t)\,dt = \int_0^\infty e^{-(s-\beta)t} f(t)\,dt = \mathscr{L}(f)(s-\beta)\,,$$

where the Laplace transform of g exists for $s - \beta > \alpha$ or equivalently, $s > \alpha + \beta$. □

Example 6.18. By Theorem 6.17 and Example 6.6, the Laplace transform of the function $f(t) = e^{at}\sin(bt)$ is

$$\mathscr{L}(f)(s) = \frac{b}{(s-a)^2 + b^2} \qquad \forall\, s > a\,.$$

Theorem 6.19. *Suppose that $f : [0,\infty) \to \mathbb{R}$ is continuous with piecewise continuous derivative, and f is of exponential order α. Then the Laplace transform of f' exist for $s > \alpha$, and*

$$\mathscr{L}(f')(s) = s\mathscr{L}(f)(s) - f(0)\,.$$

Proof. Since f is of exponential order, the Laplace transform of f exists. Since f is continuous, integrating by parts we find that

$$\int_0^R e^{-st} f'(t)\,dt = e^{-st} f(t)\Big|_{t=0}^{t=R} - \int_0^R (-s)e^{-st} f(t)\,dt = e^{-Rs} f(R) - f(0) + s\int_0^R e^{-st} f(t)\,dt\,.$$

Since f is of exponential order α, $e^{-Rs} f(R) \to 0$ as $s \to \infty$; thus

$$\mathscr{L}(f')(s) = \lim_{R\to\infty} \int_0^R e^{-st} f'(t)\,dt = -f(0) + s \lim_{R\to\infty} \int_0^R e^{-st} f(t)\,dt = s\mathscr{L}(f)(s) - f(0)\,.$$

□

Corollary 6.20. *Suppose that $f : [0,\infty) \to \mathbb{R}$ is a function such that $f, f', f'', \cdots, f^{(n-1)}$ are continuous of exponential order α, and $f^{(n)}$ is piecewise continuous. Then $\mathscr{L}(f^{(n)})(s)$ exists for all $s > \alpha$, and*

$$\mathscr{L}(f^{(n)})(s) = s^n \mathscr{L}(f)(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)\,. \tag{6.3}$$

Proof. Theorem implies that (6.3) holds for $n = 1$. Suppose that (6.3) holds for $n = k$. Then

$$\begin{aligned}
\mathscr{L}(f^{(k+1)})(s) &= \mathscr{L}((f')^{(k)})(s) \\
&= s^k \mathscr{L}(f')(s) - s^{k-1} f'(0) - s^{k-2} (f')'(0) - \cdots - s(f')^{(k-2)}(0) - (f')^{(k-1)}(0) \\
&= s^k \big[s\mathscr{L}(f)(s) - f(0)\big] - s^{k-1} f'(0) - s^{k-2} f''(0) - \cdots - s f^{(k-1)}(0) - f^{(k)}(0) \\
&= s^{k+1} \mathscr{L}(f)(s) - s^k f(0) - s^{k-1} f'(0) - \cdots - s f^{(k-1)}(0) - f^{(k)}(0)
\end{aligned}$$

which implies that (6.3) holds for $n = k+1$. By induction, we conclude that (6.3) holds for all $n \in \mathbb{N}$.

□

Example 6.21. Let $f : [0,\infty) \to \mathbb{R}$ be continuous such that the Laplace transform of f exists. Define $F(t) = \int_0^t f(\tau)\,d\tau$. The fundamental theorem of calculus implies that $F' = f$; thus Theorem 6.19 implies that

$$\mathscr{L}(f)(s) = \mathscr{L}(F')(s) = s\mathscr{L}(F)(s) - F(0) = s\mathscr{L}(F)(s)$$

which shows that $\mathscr{L}(F)(s) = \frac{1}{s}\mathscr{L}(f)(s)$. On the other hand, we can also compute $\mathscr{L}(F)$ directly as follows: by the Fubini theorem,

$$\begin{aligned}\mathscr{L}(F)(s) &= \int_0^\infty e^{-st}\Big(\int_0^t f(\tau)\,d\tau\Big)\,dt = \int_0^\infty f(\tau)\Big(\int_\tau^\infty e^{-st}\,dt\Big)\,d\tau \\ &= \int_0^\infty f(\tau)\frac{-e^{-st}}{s}\Big|_{t=\tau}^{t=\infty}\,d\tau = \frac{1}{s}\int_0^\infty f(\tau)e^{-s\tau}\,d\tau = \frac{1}{s}\mathscr{L}(f)(s)\,.\end{aligned}$$

Theorem 6.22. *Let $f : [0,\infty) \to \mathbb{R}$ be piecewise continuous of exponential order α, and $g_n(t) = (-t)^n f(t)$. Then*

$$\mathscr{L}(g_n)(s) = \frac{d^n}{ds^n}\mathscr{L}(f)(s) \qquad \forall\, s > \alpha\,.$$

The proof of Theorem 6.22 requires the dominated convergence theorem (in which the integrability is equivalent to the existence of the improper integral) stated below

> Let $f_n : [0,\infty) \to \mathbb{R}$ be a sequence of integrable functions such that $\{f_n\}_{n=1}^\infty$ converges pointwise to some integrable function f on $[0,\infty)$. Suppose that there is an integrable function g such that $|f_n(x)| \leqslant g(x)\ \forall\, x \in [0,\infty)$. Then $\lim_{n\to\infty}\int_0^\infty f_n(x)\,dx = \int_0^\infty f(x)\,dx$.

We will not prove the dominated convergence theorem. The proof of the dominated convergence theorem can be found in all real analysis textbook.

Proof of Theorem 6.22. First we note that

$$1 - t \leqslant e^{-t} \leqslant 1 - t + \frac{t^2}{2} \qquad \forall\, t \in \mathbb{R}\,;$$

thus

$$-\frac{h}{|h|}t \leqslant \frac{e^{-ht}-1}{|h|} \leqslant -\frac{h}{|h|}t + \frac{|h|t^2}{2} \qquad \forall\, h \in \mathbb{R} \text{ and } t > 0\,.$$

Therefore,

$$\Big|\frac{e^{-(s+h)t} - e^{-st}}{h}\Big| = e^{-st}\Big|\frac{e^{-ht}-1}{h}\Big| \leqslant \Big(t + \frac{t^2}{2}\Big)e^{-st} \qquad \forall\, |h| \leqslant 1 \text{ and } t > 0\,.$$

Now, since f is piecewise continuous and of exponential order α, there exists $M > 0$ such that $|f(t)| \leqslant Me^{at}$ for all $t > 0$. Let $g(t) = Me^{(\alpha-s)t}\big(t + \frac{t^2}{2}\big)$. Then for $s > \alpha$, g is integrable (that is, $\int_0^\infty g(t)\,dt < \infty$) and $\Big|\frac{e^{-(s+h)t} - e^{-st}}{h}f(t)\Big| \leqslant g(t)$; thus the dominated convergence theorem implies that for $s > \alpha$,

$$F'(s) = \lim_{h \to 0} \int_0^\infty \frac{e^{-(s+h)t} - e^{-st}}{h} f(t)\,dt = \int_0^\infty \lim_{h \to 0} \frac{e^{-(s+h)t} - e^{-st}}{h} f(t)\,dt = \int_0^t \frac{\partial}{\partial s} e^{-st} f(t)\,dt$$
$$= \int_0^t (-t) e^{-st} f(t)\,dt = \mathscr{L}(-tf(t))(s) = \mathscr{L}(g_1)(s)\,.$$

Moreover, g_1 is of exponential order β as long as $\beta > \alpha$; thus for $s > \alpha$, $s > \beta$ for some $\beta > \alpha$ and using what we just established we find that

$$\frac{d^2}{ds^2} F(s) = \frac{d}{ds} \mathscr{L}(g_1)(s) = \mathscr{L}(-tg_1(t))(s) = \mathscr{L}(g_2)(s)\,.$$

By induction, we conclude that $F^{(n)}(s) = \mathscr{L}(g_n)(s)$ for $s > \alpha$. □

Example 6.23. Find the Laplace transform of the function $f(t) = te^t \cos t$.

Instead of computing the Laplace transform directly, we apply Theorem 6.17 and 6.22 to obtain that

$$\mathscr{L}(f)(s) = -\frac{d}{ds} \mathscr{L}(e^t \cos t)(s) = -\frac{d}{ds} \frac{s-1}{(s-1)^2+1} = -\frac{(s-1)^2 + 1 - 2(s-1)(s-1)}{\big[(s-1)^2+1\big]^2}$$
$$= \frac{(s-1)^2 - 1}{\big[(s-1)^2+1\big]^2} = \frac{s^2 - 2s}{\big[(s-1)^2+1\big]^2}\,.$$

Example 6.24. Let $f : \mathbb{R} \to \infty$ be defined by

$$f(t) = \begin{cases} \dfrac{\sin t}{t} & \text{if } t \neq 0\,, \\ 1 & \text{if } t = 1\,. \end{cases}$$

Then $tf(t) = \sin t$; thus $-\frac{d}{ds}\mathscr{L}(f)(s) = \frac{1}{s^2+1}$. This implies that $\mathscr{L}(f)(s) = -\tan^{-1} s + C$ for some constant C. Since $\int_0^\pi \frac{\sin t}{t}\,dt = \frac{\pi}{2}$, we have $\mathscr{L}(f)(0) = \frac{\pi}{2}$. Therefore,

$$\mathscr{L}(f)(s) = \frac{\pi}{2} - \tan^{-1} s = \tan^{-1} \frac{1}{s}\,.$$

Example 6.25. Find the inverse Laplace transform of $F(s) = \log \frac{s+2}{s-5}$.

Suppose that $\mathscr{L}(f) = F$. Since $F'(s) = \frac{1}{s+2} - \frac{1}{s-5}$, by Theorem 6.22 we find that

$$\mathscr{L}\big(-tf(t)\big)(s) = F'(s) = \frac{1}{s+2} - \frac{1}{s-5} = \mathscr{L}(e^{-2t})(s) - \mathscr{L}(e^{5t})(s)\,;$$

thus $f(t) = \dfrac{e^{5t} - e^{-2t}}{t}$.

6.3 Solution of Initial Value Problems

Theorem 6.19 provides a way of solving of an ODE with constant coefficients. Suppose that we are looking for solutions to

$$y'' + by' + cy = f(t)\,.$$

Then taking the Laplace transform of the equation above (here we assume that y and y' are of exponential order a for some $a \in \mathbb{R}$), we find that

$$s^2\mathscr{L}(y)(s) - sy(0) - y'(0) + b\big(s\mathscr{L}(y)(s) - y(0)\big) + c\mathscr{L}(y)(s) = \mathscr{L}(f)(s)$$

which implies that the Laplace transform of the solution y satisfies

$$\mathscr{L}(y)(s) = \frac{(s+b)y(0) + y'(0)}{s^2 + bs + c} + \frac{\mathscr{L}(f)(s)}{s^2 + bs + c}. \tag{6.4}$$

The ODE is then solved provided that we can find the function $y = \varphi(t)$ whose Laplace transform is the right-hand side of (6.4).

Example 6.26. Consider the ODE

$$y'' - y' - 2y = 0.$$

If the solution y and its derivative y' are of exponential order a for some $a \in \mathbb{R}$, then by taking the Laplace transform of the equation above we find that

$$\big[s^2\mathscr{L}(y) - sy(0) - y'(0)\big] - \big[s\mathscr{L}(y) - y(0)\big] - 2\mathscr{L}(y) = 0;$$

thus

$$\begin{aligned}\mathscr{L}(y)(s) &= \frac{sy(0) + y'(0) - y(0)}{s^2 - s - 2} = \frac{sy(0) + y'(0) - y(0)}{(s-2)(s+1)} \\ &= \frac{y(0)}{s+1} + \frac{y'(0) + y(0)}{(s-2)(s+1)} = \frac{y(0)}{s+1} + \frac{y'(0) + y(0)}{3}\Big(\frac{1}{s-2} - \frac{1}{s+1}\Big).\end{aligned}$$

By Example 6.5 and Theorem 6.15, we find that

$$y(t) = y(0)e^{-t} + \frac{y'(0) + y(0)}{3}\big(e^{2t} - e^{-t}\big).$$

Example 6.27. Find the solution of the ODE $y'' + y = \sin 2t$ with initial condition $y(0) = 2$ and $y'(0) = 1$. If y is the solution to the ODE and y, y' are of exponential order a for some $a \in \mathbb{R}$, then (6.4) and Example 6.6 imply that the Laplace transform of y is given by

$$\mathscr{L}(y)(s) = \frac{2s+1}{s^2+1} + \frac{2}{(s^2+1)(s^2+4)}.$$

Using partial fractions, we expect that

$$\frac{2}{(s^2+1)(s^2+4)} = \frac{as+b}{s^2+1} + \frac{cs+d}{s^2+4} = \frac{(a+c)s^3 + (b+d)s^2 + (4a+c)s + (4b+d)}{(s^2+1)(s^2+4)}.$$

Therefore, $a + c = b + d = 4a + c = 0$ and $4b + d = 2$; thus $a = c = 0$ and $b = -d = \frac{2}{3}$. This provides that

$$\mathscr{L}(y)(s) = \frac{2s+1}{s^2+1} + \frac{2}{3}\frac{1}{s^2+1} - \frac{2}{3}\frac{1}{s^2+4} = \frac{2s}{s^2+1} + \frac{5}{3}\frac{1}{s^2+1} - \frac{1}{3}\frac{2}{s^2+4}.$$

By Proposition 6.2 and Example 6.6, we find that

$$y(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t.$$

Example 6.28. Find the solution of the ODE $y^{(4)} - y = 0$ with initial condition $y(0) = y''(0) = y'''(0) = 0$ and $y'(0) = 1$ and y, y' are of exponential order a for some $a \in \mathbb{R}$. If y is the solution to the ODE, then Corollary 6.20 implies that the Laplace transform of y satisfies

$$s^4\mathscr{L}(y)(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) - \mathscr{L}(y)(s) = 0\,;$$

thus

$$\mathscr{L}(y)(s) = \frac{s^2}{s^4-1} = \frac{s^2}{(s-1)(s+1)(s^2+1)}\,.$$

Using partial fractions, we assume that

$$\begin{aligned}\mathscr{L}(y)(s) = \frac{s^2}{s^4-1} &= \frac{a}{s-1} + \frac{b}{s+1} + \frac{cs+d}{s^2+1} = \frac{(a+b)s+(a-b)}{s^2-1} + \frac{cs+d}{s^2+1} \\ &= \frac{(a+b+c)s^3 + (a-b+d)s^2 + (a+b-c)s + (a-b-d)}{s^4-1}\,.\end{aligned}$$

Therefore, $a+b+c = a+b-c = a-b-d = 0$ and $a-b+d = 1$; thus $a = \frac{1}{4}$, $b = -\frac{1}{4}$, $c = 0$ and $d = \frac{1}{2}$. This provides that

$$\mathscr{L}(y)(s) = \frac{1}{4}\frac{1}{s-1} - \frac{1}{4}\frac{1}{s+1} + \frac{1}{2}\frac{1}{s^2+1}\,.$$

By Example 6.5 and 6.6, we conclude that the solution to the ODE is

$$y(t) = \frac{1}{4}e^t - \frac{1}{4}e^{-t} + \frac{1}{2}\sin t\,.$$

• **Advantages of the Laplace transform method**:

1. Converting a problem of solving a differential equation to a problem of solving an algebraic equation.

2. The dependence on the initial data is automatically build in. The task of determining values of arbitrary constants in the general solution is avoided.

3. Non-homogeneous equations can be treated in exactly the same way as the homogeneous ones, and it is not necessary to solving the corresponding homogeneous equation first.

• **Difficulties of the Laplace transform method**: Need to find the function whose Laplace transform is given - the inverse Laplace transform has to be performed in general situations.

It is also possible to consider the ODE with variable coefficient using the Laplace transform. We use to following two examples to illustrate the idea.

Example 6.29. Find the solution to the initial value problem

$$y'' + ty' - y = 0\,, \qquad y(0) = 0\,, \quad y'(0) = 3\,.$$

Assume that y is continuously differentiable of exponential order α for some $\alpha > 0$, and y'' is piecewise continuous on $[0, \infty)$. Let $Y(s) = \mathscr{L}(y)(s)$. By Corollary 6.20 and Theorem 6.22,

$$s^2Y(s) - 3 - \big[sY(s)\big]' - Y(s) = 0 \qquad \forall\, s > \alpha\,;$$

thus

$$Y'(s) + \Big(\frac{2}{s} - s\Big)Y(s) = -\frac{3}{s} \qquad \forall\, s > \alpha\,.$$

Using the integrating factor $s^2 e^{-\frac{s^2}{2}}$, we find that

$$\big[s^2 e^{-\frac{s^2}{2}} Y(s)\big]' = -3se^{-\frac{s^2}{2}}$$

which shows that

$$s^2 e^{-\frac{s^2}{2}} Y(s) = 3e^{-\frac{s^2}{2}} + C\,.$$

Therefore, $Y(s) = \dfrac{3}{s^2} + Ce^{\frac{s^2}{2}}$. By Proposition 6.13, $\lim\limits_{s\to\infty} Y(s) = 0$; thus $C = 0$. This implies that $y(t) = 3t$.

Example 6.30. Find the solution to the initial value problem

$$ty'' - ty' + y = 2\,, \qquad y(0) = 2\,, \quad y'(0) = -1\,.$$

Assume that y is continuously differentiable of exponential order α for some $\alpha > 0$, and y'' is piecewise continuous on $[0, \infty)$. Let $Y(s) = \mathscr{L}(y)(s)$. Then

$$-\big[s^2 Y(s) - 2s + 1\big]' + \big[sY(s) - 2\big]' + Y(s) = \frac{2}{s} \qquad \forall\, s > \alpha\,.$$

Further computations shows that

$$(s^2 - s)Y'(s) + (2s - 2)Y(s) = 2 - \frac{2}{s} \qquad \forall\, s > \alpha$$

which can be reduced to

$$s^2 Y'(s) + 2sY(s) = 2\,.$$

Therefore, $(s^2 Y)' = 2$ which implies that $s^2 Y(s) = 2s + C$; thus we find that

$$Y(s) = \frac{2}{s} + \frac{C}{s^2}\,.$$

Taking the inverse Laplace transform, we obtain that the general solution to the ODE is given by

$$y(t) = 2 + Ct\,.$$

To validate the initial condition, we find that $C = -1$, so the solution to the initial value problem is $y(t) = 2 - t$.

6.4 Transforms of Discontinuous and Periodic Functions

In the following two sections we are concerned with the Laplace transform of discontinuous functions with jump discontinuities.

Definition 6.31. The ***unit step function*** is the function

$$u(t) = \begin{cases} 0 & \text{if } t < 0\,, \\ 1 & \text{if } t > 0\,. \end{cases}$$

Example 6.32.

1. For $c \in \mathbb{R}$, we define $u_c(t) = u(t - c)$. Then the graph of u_c jumps up from 0 to 1 at $t = c$.

2. The graph of $-u_c$ jumps down from 0 to -1 at $t = c$.

3. Let $a < b$. The characteristic/indicator function $\mathbf{1}_{(a,b)}$ can be expressed by

$$\mathbf{1}_{(a,b)}(t) = u_a(t) - u_b(t) .$$

The function $\mathbf{1}_{(a,b)}$ is called the ***rectangular window function*** (and is denoted by $\Pi_{a,b}$ in the textbook).

4. Let $0 = c_0 < c_1 < \cdots < c_n < c_{n+1} = \infty$. The step function

$$f(t) = \sum_{i=0}^{n} f_i \mathbf{1}_{(c_i,c_{i+1})}(t) \tag{6.5}$$

can be expressed by

$$f(t) = f_0 \mathbf{1}_{(0,c_1)}(t) + \sum_{k=0}^{n} (f_{k+1} - f_k) u_{c_k}(t) .$$

Example 6.33. We can write the function $f : (0, \infty) \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} 3 & \text{if } t < 2 \, , \\ 1 & \text{if } 2 < t < 5 \, , \\ t & \text{if } 5 < t < 8 \, , \\ t^2/10 & \text{if } 8 < t \end{cases}$$

in terms of window and step functions as

$$f(t) = 3\mathbf{1}_{(0,2)}(t) + \mathbf{1}_{(2,5)}(t) + t\mathbf{1}_{(5,8)}(t) + \frac{t^2}{10} u_8(t) .$$

• **The Laplace transform of** u_c: Next, we compute the Laplace transform of the step function f given by (**??**). We note that even though f is not defined on $[0, \infty)$ so in principle $\mathscr{L}(f)$ does not exists. However, if $g, h : [0, \infty) \to \mathbb{R}$ are identical to f on the domain of f, then $\mathscr{L}(g) = \mathscr{L}(h)$. This means any "extension" of f has the same Laplace transform, and the Laplace transform of one of such extensions is viewed as the Laplace transform of f.

To compute the Laplace transform of the step function f given by (**??**), by Proposition 6.2 it suffices to find the Laplace transform of u_c.

1. If $c \leqslant 0$, then

$$\mathscr{L}(u_c)(s) = \int_0^\infty e^{-st} \, dt = \frac{1}{s} \qquad \forall \, s > 0 \, .$$

2. If $c > 0$, then

$$\mathscr{L}(u_c)(s) = \int_c^\infty e^{-st} \, dt = \frac{e^{-cs}}{s} \qquad \forall \, s > 0 \, .$$

Therefore,

$$\mathscr{L}(u_c)(s) = \frac{e^{-\max\{c,0\}s}}{s}\,.$$

Theorem 6.34. *Let $f : [0,\infty) \to \mathbb{R}$ be a function such that the Laplace transform $\mathscr{L}(f)(s)$ of f exists for $s > \alpha$. If c is a positive constant and $g(t) = u_c(t)f(t-c)$, then*

$$\mathscr{L}(g)(s) = e^{-cs}\mathscr{L}(f)(s) \qquad \textit{for } s > \alpha\,.$$

Conversely, if $G(s) = e^{-cs}\mathscr{L}(f)(s)$, then $\mathscr{L}^{-1}(G)(t) = u_c(t)f(t-c)$.

Proof. If $c > 0$ and $g(t) = u_c(t)f(t-c)$, then the change of variable formula implies that

$$\begin{aligned}\mathscr{L}(g)(s) &= \lim_{R\to\infty}\int_c^R e^{-st}f(t-c)\,dt = \lim_{R\to\infty}\int_0^{R-c} e^{-s(t+c)}f(t)\,dt \\ &= e^{-cs}\lim_{R\to\infty}\int_0^{R-c} e^{-st}f(t)\,dt = e^{-cs}\mathscr{L}(f)(s)\,.\end{aligned}$$ □

Example 6.35. Let $f : [0,\infty) \to \mathbb{R}$ be defined by

$$f(t) = \begin{cases} \sin t & \text{if } 0 \leqslant t < \frac{\pi}{4}\,, \\ \sin t + \cos\left(t - \frac{\pi}{4}\right) & \text{if } t \geqslant \frac{\pi}{4}\,. \end{cases}$$

Then $f(t) = \sin t + u_{\frac{\pi}{4}}(t)\cos\left(t - \frac{\pi}{4}\right)$; thus by Example 6.6 and Theorem 6.34 we find that

$$\mathscr{L}(f)(s) = \frac{1}{s^2+1} + e^{-\frac{\pi}{4}s}\frac{s}{s^2+1} = \frac{1+se^{-\frac{\pi}{4}s}}{s^2+1}\,.$$

Example 6.36. Let $f : [0,\infty) \to \mathbb{R}$ be defined by $f(t) = t^2u_1(t)$. Then

$$f(t) = (t-1+1)^2u_1(t) = (t-1)^2u_1(t) + 2(t-1)u_1(t) + u_1(t)\,;$$

thus the Laplace transform of f is given by

$$\mathscr{L}(f)(s) = \frac{2e^{-s}}{s^3} + \frac{2e^{-s}}{s^2} + \frac{e^{-s}}{s} = e^{-s}\Big(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\Big)\,.$$

Example 6.37. Find the inverse Laplace transform of $F(s) = \dfrac{1-e^{-2s}}{s^2}$.

By Example 6.14, the inverse Laplace transform of s^{-2} is $\dfrac{t}{\Gamma(1+1)} = t$; thus Theorem 6.34 implies that

$$\mathscr{L}^{-1}(F)(t) = t - u_2(t)(t-2)\,.$$

Definition 6.38. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be ***periodic of period*** $T(\neq 0)$ if

$$f(t+T) = f(t) \qquad \forall\, t \in \mathbb{R}\,.$$

Theorem 6.39. *Let $f : \mathbb{R} \to \mathbb{R}$ be periodic of period T, and $F_T(s) = \displaystyle\int_0^T e^{-st}f(t)\,dt$. Then*

$$\mathscr{L}(f)(s) = \frac{F_T(s)}{1-e^{-sT}}\,.$$

Proof. By the periodicity of f, $f(t) = f(t-T)$ for all $t \in \mathbb{R}$; thus

$$\begin{aligned} F_T(s) &= \int_0^\infty e^{-st} f(t)\mathbf{1}_{(0,T)}(t)\,dt = \int_0^\infty e^{-st} f(t)\big(1-u_T(t)\big)\,dt \\ &= \mathscr{L}(f)(s) - \int_0^\infty e^{-st} f(t-T)u_T(t)\,dt = \mathscr{L}(f)(s) - e^{-sT}\mathscr{L}(f)(s)\,, \end{aligned}$$

where we have used Theorem 6.34 to conclude the last equality. ◻

6.5 Differential Equations with Discontinuous Forcing Functions

In this section, we consider solving the ODE

$$y'' + by' + cy = f(t) \qquad t > 0 \tag{6.6}$$

with constant coefficients b, c and piecewise continuous forcing function f.

Suppose that $f : [0,\infty) \to \mathbb{R}$ is a function defined by

$$f(t) = \begin{cases} f_1(t) & \text{if } 0 < t < c\,, \\ f_2(t) & \text{if } t > c\,, \\ d & \text{if } t = c\,, \end{cases}$$

where f_1, f_2 are continuous functions on $[0,\infty)$, $d \in \mathbb{R}$ is a given number, and $\lim\limits_{t\to c^+} f_2(t) - \lim\limits_{t\to c^-} f_1(t) = A \neq 0$; that is, c is a jump discontinuity of f. Define

$$g(t) = \begin{cases} f_1(t) & \text{if } 0 < t < c\,, \\ \lim_{t\to c^-} f_1(t) & \text{if } t = c\,, \\ f_2(t) - Au_c(t) & \text{if } t > c\,. \end{cases}$$

Then $g : [0,\infty) \to \mathbb{R}$ is continuous, and $f = g + Au_c$. Similarly, if f is a piecewise continuous function has jump discontinuities $\{c_1, c_2, \cdots, c_n\}$, then f is continuous on (c_k, c_{k+1}) for all $k \in \{1, \cdots, n-1\}$ and by introducing $c_0 = 0$, $c_{n+1} = \infty$, and $A_k \equiv \lim\limits_{t\to c_k^+} (f\mathbf{1}_{(c_k,c_{k+1})})(t) - \lim\limits_{t\to c_k^-} (f\mathbf{1}_{(c_{k-1},c_k)})(t)$, the function $g : [0,\infty) \to \mathbb{F}$ defined by

$$g(t) = \begin{cases} f(t) - \sum\limits_{k=1}^n A_k u_{c_k}(t) & \text{if } t \notin \{c_1, \cdots, c_n\} \\ \lim\limits_{t\to c_j} \Big(f(t) - \sum\limits_{k=1}^n A_k u_{c_k}(t)\Big) & \text{if } t \in \{c_1, \cdots, c_n\} \end{cases}$$

is continuous on $\mathbb{R}$, and $f = g + \sum\limits_{k=1}^n A_k u_{c_k}$. Therefore, in order to understand the solution of (6.6) with piecewise continuous function f, it suffices to consider the case $f = Au_d$ for some constants A and d.

Before proceeding, let us consider the ODE

$$y'' = u_c(t) \qquad t > 0 \tag{6.7}$$

for some $c > 0$. Intuitively, a solution of such an ODE can be obtained by integrating the ODE twice directly, and we find that

$$y'(t) = \begin{cases} y'(0) & \text{if } 0 \leqslant t < c\,, \\ y'(0) + t - c & \text{if } t \geqslant c\,, \end{cases}$$

and

$$y(t) = \begin{cases} y(0) + y'(0)t & \text{if } 0 \leqslant t < c\,, \\ y(0) + y'(0)t + \dfrac{t-c}{2}(2y'(0) + t - c) & \text{if } t \geqslant c\,. \end{cases}$$

We note that such y does not possess second derivative at c, and this fact indicates that it seems impossible to find a twice differentiable function y such that (6.7) for all $t > 0$. Therefore, to solve ODE with piecewise discontinuous forcing function, it requires that we modify the concept of solutions. We have the following

Definition 6.40. Let $f : [0, \infty) \to \mathbb{R}$ be a function. A function y is said to be a solution to the initial value problem

$$y'' + by' + cy = f(t) \qquad y(0) = y_0, \quad y'(0) = y_1$$

if y is continuously differentiable and satisfies the initial condition, y'' exists at every continuity of f, and the ODE holds at every continuity of f.

Now suppose that we are looking for a solution to

$$y'' + by' + cy = f(t)\,, \tag{6.8}$$

where f is a piecewise continuous function on $[0, \infty)$ and has jump discontinuities only at $\{c_1, c_2, \cdots, c_n\}$ as described above. We note that the existence theorem (Theorem 1.24) cannot be applied due to the discontinuity of the forcing function, so in general we do not know if a solution exists. However, if there indeed exists a twice differentiable function y validating (6.8), then the solution must be unique since if y_1 and y_2 are two solutions with the same initial condition, then $y = y_1 - y_2$ is a solution to $y'' + by' + cy = 0$ with $y(0) = y'(0) = 0$; thus y must be zero. On the other hand, if (6.8) has a solution y, then y'' must be piecewise continuous. If in addition y and y' are of exponential order α for some $\alpha \in \mathbb{R}$, we can apply Theorem 6.20 to find the Laplace transform of the solution y as introduced in Section 6.3 which in principle provides information of how the solution can be found.

Now we focus on solving the ODE

$$y'' + by' + cy = Fu_\alpha(t)\,, \qquad y(0) = y_0\,, \quad y'(0) = y_1\,, \tag{6.9}$$

where F is a constant and $\alpha > 0$. We only consider the case that $c \neq 0$ for otherwise the ODE can reduced to a first order ODE (by integrating the ODE).

If y is a twice differentiable solution to (6.9), taking the Laplace transform of the ODE we find that

$$s^2\mathscr{L}(y)(s) - sy_0 - y_1 + b\big[s\mathscr{L}(y)(s) - y_0\big] + c\mathscr{L}(y)(s) = F\frac{e^{-\alpha s}}{s}\,;$$

thus

$$\mathscr{L}(y)(s) = \frac{(s+b)y_0 + y_1}{s^2+bs+c} + F\frac{e^{-\alpha s}}{s(s^2+bs+c)}\,.$$

Using partial fractions, we obtain that $\dfrac{1}{s(s^2+bs+c)} = \dfrac{1}{c}\Big[\dfrac{1}{s} - \dfrac{s+b}{s^2+bs+c}\Big]$; thus with z denoting the solution to the ODE

$$z'' + bz' + cz = 0\,, \qquad z(0) = 1\,, \quad z'(0) = 0\,, \tag{6.10}$$

we find that

$$\frac{e^{-\alpha s}}{s(s^2+bs+c)} = \frac{e^{-\alpha s}}{c}\mathscr{L}(1-z)(s)\,.$$

Therefore, Theorem 6.34 implies that

$$y(t) = Y(t) + \frac{F}{c}u_\alpha(t)\big[1 - z(t-\alpha)\big]\,, \tag{6.11}$$

here Y is the solution to (6.9) with $F = 0$. We note that even though u_α is not defined at $t = \alpha$, the function y given in (6.11) is continuous for all t since $z(0) = 1$. Moreover, the function y clearly satisfies the initial condition $y(0) = y_0$.

• **The first derivative of** y: It is clear that $y'(t)$ exists for $t \neq \alpha$ and can be computed by

$$y'(t) = Y'(t) - \frac{F}{c}u_\alpha(t)z'(t-\alpha) \qquad \forall\, t > 0, t \neq \alpha\,. \tag{6.12}$$

Therefore, $y'(0) = Y'(0) = y_1$. Now we check the differentiability of y at $t = \alpha$ by looking at the limits

$$\lim_{h\to 0^-}\frac{y(\alpha+h) - y(\alpha)}{h} \quad \text{and} \quad \lim_{h\to 0^+}\frac{y(\alpha+h) - y(\alpha)}{h}\,.$$

By the differentiability of Y,

$$\lim_{h\to 0^-}\frac{y(\alpha+h) - y(\alpha)}{h} = Y'(\alpha) + \frac{F}{c}\lim_{h\to 0^-}\frac{u_\alpha(\alpha+h)\big(1 - z(h)\big)}{h} = Y'(\alpha)$$

and

$$\begin{aligned}
\lim_{h\to 0^+}\frac{y(\alpha+h) - y(\alpha)}{h} &= Y'(\alpha) + \frac{F}{c}\lim_{h\to 0^+}\frac{u_\alpha(\alpha+h)\big(1 - z(h)\big)}{h} \\
&= Y'(\alpha) + \frac{F}{c}\lim_{h\to 0^+}\frac{1 - z(h)}{h} = Y'(\alpha) - \frac{F}{c}\frac{z(h) - z(0)}{h} \\
&= Y'(\alpha) - \frac{F}{c}z'(0) = Y'(\alpha)\,.
\end{aligned}$$

Therefore, y' exists at $t = \alpha$ and $y'(\alpha) = Y'(\alpha)$ which also validates (6.12) for $t = \alpha$; thus (6.12) holds for all $t > 0$. We note that y' given by (6.12) is continuous since

$$\lim_{y\to\alpha} y'(t) = Y'(\alpha) = y'(\alpha)\,.$$

• **The second derivative of** y: Now we turn our attention to the second derivative of y. It is clear that

$$y''(t) = Y''(t) - \frac{F}{c}\big[u_\alpha(t)z''(t-\alpha)\big] \qquad \forall\, t > 0\,, t \neq \alpha\,. \tag{6.13}$$

Therefore, the second derivative of y exists at every continuity of the forcing function Fu_α.

• **The validity of the ODE**: Using (6.11), (6.12) and (6.13), we find that for $t > 0$ and $t \neq \alpha$,

$$(y'' + by' + cy)(t) = (Y'' + bY' + cY)(t) - \frac{F}{c} u_\alpha(t)(z'' + bz' + cz)(t-\alpha) + Fu_\alpha(t) = Fu_\alpha(t)\,;$$

thus the function y satisfies the ODE at every continuity of the forcing function Fu_α. Therefore, y given by (6.11) is indeed the solution to (6.9).

• **Summary**: The Laplace transform method may be used to find the solution to second ODE with constant coefficients and discontinuous forcing. In particular, the solution to the IVP

$$y'' + by' + cy = F\mathbf{1}_{(\alpha,\beta)}(t)\,, \qquad y(0) = y_0\,, \quad y'(0) = y_1$$

can be expressed as

$$y(t) = Y(t) + \frac{F}{c}\Big[u_\alpha(t)\big[1 - z(t-\alpha)\big] - u_\beta(t)\big[1 - z(t-\beta)\big]\Big]\,, \tag{6.14}$$

where Y is the solution to (6.9) with $F = 0$ and z is the solution to (6.10).

Example 6.41. Find the solution of the ODE $2y'' + y' + 2y = g(t)$ with initial condition $y(0) = y'(0) = 0$, where

$$g(t) = u_5(t) - u_{20}(t) = \mathbf{1}_{(5,20)}(t)$$

If y is the solution to the ODE, taking the Laplace transform of the ODE we find that

$$2\big[s^2\mathscr{L}(y)(s) - sy(0) - y'(0)\big] + \big[s\mathscr{L}(y)(s) - y(0)\big] + 2\mathscr{L}(y)(s) = \frac{e^{-5t} - e^{-20s}}{s}\,;$$

thus

$$\mathscr{L}(y)(s) = \frac{e^{-5t} - e^{-20t}}{s(2s^2 + s + 2)}\,.$$

Using partial fractions, we obtain that

$$\frac{1}{s(2s^2+s+2)} = \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\Big[\frac{s+\frac{1}{4}}{(s+\frac{1}{4})^2 + \big(\frac{\sqrt{15}}{4}\big)^2} + \frac{1}{\sqrt{15}}\frac{\frac{\sqrt{15}}{4}}{(s+\frac{1}{4})^2 + \big(\frac{\sqrt{15}}{4}\big)^2}\Big]\,.$$

Let $h(t) = \frac{1}{2} - \frac{1}{2}e^{-\frac{1}{4}t}\Big[\cos\big(\frac{\sqrt{15}}{4}t\big) + \frac{1}{\sqrt{15}}\sin\big(\frac{\sqrt{15}}{4}t\big)\Big]$. Then Example 6.6 and Theorem 6.17 show that

$$\mathscr{L}(y)(s) = \big(e^{-5s} - e^{-20s}\big)\mathscr{L}(h)(s)\,;$$

thus Theorem 6.34 further implies that

$$\begin{aligned} y(t) &= u_5(t)h(t-5) - u_{20}(t)h(t-20) \\ &= \frac{1}{2}\big[u_5(t) - u_{20}(t)\big] - \frac{1}{2}\Big[u_5(t)e^{-\frac{1}{4}(t-5)}\cos\big(\frac{\sqrt{15}}{4}(t-5)\big) - u_{20}(t)e^{-\frac{1}{4}(t-20)}\cos\big(\frac{\sqrt{15}}{4}(t-20)\big)\Big] \\ &\quad - \frac{1}{2\sqrt{15}}\Big[u_5(t)e^{-\frac{1}{4}(t-5)}\sin\big(\frac{\sqrt{15}}{4}(t-5)\big) - u_{20}(t)e^{-\frac{1}{4}(t-20)}\sin\big(\frac{\sqrt{15}}{4}(t-20)\big)\Big]\,. \end{aligned}$$

Example 6.42. Find the solution of the ODE $y'' + 4y = g(t)$ with initial data $y(0) = y'(0) = 0$, where the forcing function g is given by

$$g(t) = \begin{cases} 0 & \text{if } 0 < t < 5\,, \\ \dfrac{t-5}{5} & \text{if } 5 < t < 10\,, \\ 1 & \text{if } t > 10\,. \end{cases}$$

We note that $g(t) = \frac{1}{5}\big[u_5(t)(t-5) - u_{10}(t)(t-10)\big]$; thus Example 6.14 and Theorem 6.34 imply that

$$\mathscr{L}(g)(s) = \frac{1}{5}\frac{1}{s^2}(e^{-5s} - e^{-10s}) = \frac{e^{-5s} - e^{-10s}}{5s^2}\,.$$

We also remark that $g'(t) = \frac{1}{5}(u_5(t) - u_{10}(t))$ if $t \neq 5, 10$. Since the value at g' at two points does not affect the Laplace transform, we can use Corollary 6.20 to compute the Laplace transform of g:

$$s\mathscr{L}(g)(s) = s\mathscr{L}(g)(s) - g(0) = \mathscr{L}(g')(s) = \frac{e^{-5s} - e^{-10s}}{5s}\,;$$

thus $\mathscr{L}(g)(s) = \dfrac{e^{-5s} - e^{-10s}}{5s^2}$.

Assume that a solution y to the ODE under consideration exists such that y, y' are continuous and y'' are of exponential order a for some $a \in \mathbb{R}$. Then the Laplace transform implies that

$$s^2\mathscr{L}(y)(s) - sy(0) - y'(0) + 4\mathscr{L}(y)(s) = \frac{e^{-5s} - e^{-10s}}{5s^2}\,.$$

Therefore,

$$\mathscr{L}(y)(s) = \frac{e^{-5s} - e^{-10s}}{5s^2(s^2+4)}\,.$$

Using partial fractions, we assume that $\dfrac{1}{s^2(s^2+4)} = \dfrac{as+b}{s^2} + \dfrac{cs+d}{s^2+4}$, where a, b, c, d satisfy $a + c = 0$, $b + d = 0$, $4a = 0$ and $4b = 1$; thus

$$\mathscr{L}(y)(s) = \frac{e^{-5s} - e^{-10s}}{20}\Big[\frac{1}{s^2} - \frac{1}{2}\frac{2}{s^2+4}\Big]\,.$$

By Theorem 6.17, we find that

$$y(t) = \frac{1}{20}\Big[u_5(t)(t-5) - u_{10}(t)(t-10) - \frac{1}{2}\Big(u_5(t)\sin\big(2(t-5)\big) - u_{10}(t)\sin\big(2(t-10)\big)\Big)\Big]\,.$$

Remark 6.43. The Laplace transform picks up solutions whose derivative of the highest order (which is the same as the order of the ODE under consideration) is of exponential order a for some $a \in \mathbb{R}$.

6.6 Convolution

Definition 6.44. Let f, g be piecewise continuous on $[0, \infty)$. The ***convolution*** of f and g, denoted by $f * g$, is defined by

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau\,. \tag{6.15}$$

Proposition 6.45. *Let f, g, h be piecewise continuous on $[0, \infty)$. Then*

(a) $f * g = g * f$;

(b) $f * (g + h) = (f * g) + (f * h)$;

(c) $(f * g) * h = f * (g * h)$;

(d) $(f * 0) = 0$.

Proof. It is clear that (b) and (d) hold, so we only prove (a) and (c). To see (a), we make a change of variable and find that

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau = \int_t^0 f(u)g(t-u)(-du) = \int_0^t g(t-u)f(u)du = (g * f)(t)\,.$$

To see (c), using (a) and the Fubini theorem,

$$\begin{aligned}\big[(f * g) * h\big](t) &= \int_0^t (g * f)(t-\tau)h(\tau)\,d\tau = \int_0^t \Big(\int_0^{t-\tau} g(t-\tau-u)f(u)\,du\Big)h(\tau)\,d\tau \\ &= \int_0^t \Big(\int_0^{t-\tau} g(t-\tau-u)f(u)h(\tau)\,du\Big)d\tau \\ &= \int_0^t \Big(\int_0^{t-u} g(t-\tau-u)f(u)h(\tau)\,d\tau\Big)du \\ &= \int_0^t f(u)\Big(\int_0^{t-u} g(t-u-\tau)h(\tau)\,d\tau\Big)du \\ &= \int_0^t f(u)(g * h)(t-u)\,du = \big[(g * h) * f\big](t) = \big[f * (g * h)\big](t)\end{aligned}$$

which completes the proof of (d). □

Theorem 6.46. *Let f and g be piecewise continuous on $[0, \infty)$ and are of exponential order α. Then*

$$\mathscr{L}(f * g)(s) = \mathscr{L}(f)(s)\mathscr{L}(g)(s) \qquad \forall\, s > \alpha\,.$$

Proof. Since f is of exponential order α, for some $M_1 > 0$, $|f(t)| \leqslant M_1 e^{\alpha t}$ for all $t > 0$. Therefore, for $s > \alpha$,

$$\Big|\mathscr{L}(f)(s) - \int_0^R e^{-st} f(t)\,dt\Big| \leqslant \int_R^\infty e^{-st}\big|f(t)\big|\,dt \leqslant M_1 \int_R^\infty e^{-(s-\alpha)t}dt \leqslant \frac{M_1}{s-\alpha}e^{(\alpha-s)R}\,.$$

Similarly, for some $M_2 > 0$, $|g(t)| \leqslant M_2 e^{\alpha t}$ for all $t > 0$ and

$$\Big|\mathscr{L}(g)(s) - \int_0^R e^{-st} g(t)\,dt\Big| \leqslant \frac{M_2}{s-\alpha}e^{(\alpha-s)R} \qquad \forall\, s > \alpha\,.$$

By the Fubini theorem,

$$\begin{aligned}&\int_0^R e^{-st}\Big(\int_0^t f(t-\tau)g(\tau)\,d\tau\Big)\,dt = \int_0^R \Big(\int_\tau^R f(t-\tau)g(\tau)e^{-st}\,dt\Big)\,d\tau \\ &\quad= \int_0^R e^{-s\tau}g(\tau)\Big(\int_\tau^R f(t-\tau)e^{-s(t-\tau)}\,dt\Big)\,d\tau = \int_0^R e^{-s\tau}g(\tau)\Big(\int_0^{R-\tau} f(t)e^{-st}\,dt\Big)\,d\tau\,;\end{aligned}$$

thus for $s > \alpha$,

$$\begin{aligned}
&\Big|\int_0^R e^{-st}\Big(\int_0^t f(t-\tau)g(\tau)\,d\tau\Big)\,dt - \mathscr{L}(f)(s)\mathscr{L}(g)(s)\Big| \\
&\quad = \Big|\int_0^R e^{-s\tau}g(\tau)\Big(\int_0^{R-\tau} f(t)e^{-st}\,dt\Big)\,d\tau - \mathscr{L}(f)(s)\mathscr{L}(g)(s)\Big| \\
&\quad = \Big|\int_0^R e^{-s\tau}g(\tau)\Big(\int_0^{R-\tau} f(t)e^{-st}\,dt - \mathscr{L}(f)(s)\Big)\,d\tau + \mathscr{L}(f)(s)\Big(\int_0^R e^{-s\tau}g(\tau)\,d\tau - \mathscr{L}(g)(s)\Big)\Big| \\
&\quad \leqslant \frac{M_1M_2}{s-\alpha}\int_0^R e^{-s\tau}e^{\alpha\tau}e^{(\alpha-s)(R-\tau)}\,d\tau + \frac{M_2}{s-\alpha}\big|\mathscr{L}(f)(s)\big|e^{(\alpha-s)R} \\
&\quad = \frac{M_1M_2}{s-\alpha}Re^{(\alpha-s)R} + \frac{M_2}{s-\alpha}\big|\mathscr{L}(f)(s)\big|e^{(\alpha-s)R}
\end{aligned}$$

which converges to 0 as $R \to \infty$. □

Example 6.47. Find the inverse Laplace transform of $H(s) = \dfrac{a}{s^2(s^2+a^2)}$.

Method 1: Using the partial fractions,

$$\frac{a}{s^2(s^2+a^2)} = \frac{1}{a}\Big[\frac{1}{s^2} - \frac{1}{s^2+a^2}\Big] = \frac{1}{a}\cdot\frac{1}{s^2} - \frac{1}{a^2}\frac{a}{s^2+a^2}\,;$$

thus Example 6.6 and 6.14 imply

$$\mathscr{L}^{-1}(H)(t) = \frac{t}{a} - \frac{1}{a^2}\sin at\,.$$

Method 2: By Theorem 6.46, with F, G denoting the functions $F(s) = \dfrac{1}{s^2}$ and $G(s) = \dfrac{a}{s^2+a^2}$,

$$\begin{aligned}
\mathscr{L}^{-1}(H)(t) &= \big(\mathscr{L}^{-1}(F) * \mathscr{L}^{-1}(G)\big)(t) = \int_0^t (t-\tau)\sin(a\tau)\,d\tau \\
&= t\int_0^t \sin a\tau\,d\tau - \int_0^t \tau\sin a\tau\,d\tau \\
&= -\frac{t}{a}\cos(a\tau)\Big|_{\tau=0}^{\tau=t} - \Big[-\frac{\tau}{a}\cos(a\tau)\Big|_{\tau=0}^{\tau=t} + \frac{1}{a}\int_0^t \cos(a\tau)\,d\tau\Big] \\
&= \frac{t}{a} - \frac{1}{a}\int_0^t \cos(a\tau)\,d\tau = \frac{t}{a} - \frac{\sin a\tau}{a^2}\Big|_{\tau=0}^{\tau=t} = \frac{t}{a} - \frac{\sin at}{a^2}\,.
\end{aligned}$$

Example 6.48. Find the solution of the initial value problem

$$y'' + 4y = g(t)\,, \qquad y(0) = 3\,, \quad y'(0) = -1\,.$$

Taking the Laplace transform of the equation above, we find that

$$\mathscr{L}(y)(s) = \frac{3s-1}{s^2+4} + \frac{\mathscr{L}(g)(s)}{s^2+4} = \frac{3s}{s^2+4} - \frac{1}{2}\frac{2}{s^2+4} + \frac{\mathscr{L}(g)(s)}{2}\frac{2}{s^2+4}\,.$$

Therefore, by Example 6.6 and Theorem 6.46,

$$\begin{aligned}
y(t) &= 3\cos(2t) - \frac{1}{2}\sin(2t) + \frac{1}{2}\int_0^t g(t-\tau)\sin 2\tau\,d\tau \\
&= 3\cos(2t) - \frac{1}{2}\sin(2t) + \frac{1}{2}\int_0^t g(\tau)\sin 2(t-\tau)\,d\tau\,.
\end{aligned}$$

In general, we can consider the second order ODE

$$y'' + by' + cy = g(t)\,, \qquad y(0) = y_0\,, \quad y'(0) = y_1\,.$$

As discussed before, we find that if y is a solution to the ODE above,

$$\mathscr{L}(y)(s) = \frac{(s+b)y_0 + y_1}{s^2+bs+c} + \frac{\mathscr{L}(g)(s)}{s^2+bs+c}\,.$$

Therefore,

1. if $r^2 + br + c = 0$ has two distinct real roots r_1 and r_2, then the solution y is

$$y(t) = \frac{y_1 - r_2 y_0}{r_1 - r_2} e^{r_1 t} + \frac{r_1 y_0 - y_1}{r_1 - r_2} e^{r_2 t} + \int_0^t g(t-\tau)\frac{e^{r_1\tau} - e^{r_2\tau}}{r_1 - r_2}\,d\tau\,.$$

2. if $r^2 + br + c = 0$ has a double root r_1, then the solution y is

$$y(t) = y_0 e^{r_1 t} + (y_1 - r_1 y_0)te^{r_1 t} + \int_0^t g(t-\tau)e^{r_1\tau}\tau\,d\tau\,.$$

3. if $r^2 + br + c = 0$ has two complex roots $\lambda \pm i\mu$, then the solution y is

$$y(t) = y_0 e^{\lambda t}\cos\mu t + \frac{y_1 - \lambda y_0}{\mu} e^{\lambda t}\sin\mu t + \int_0^t g(t-\tau)e^{\lambda\tau}\frac{\sin\mu\tau}{\mu}\,d\tau\,.$$

6.7 Impulse and the Dirac Delta Function

In this section, we are interested in what happens if a moving object in a spring-mass system is hit by an external force which only appears in a very short amount of time period (you can think of hitting an object in a spring-mass system using a hammer in a very short amount of time). In practice, we do not know the exact time period $[\alpha, \beta]$ (with $|\beta - \alpha| \ll 1$) during which the force hits the system, but can assume that the total amount of force which affects the system is known. This kind of phenomena usually can be described by the system

$$y'' + by' + cy = f(t)\,, \qquad y(0) = y_0\,, \quad y'(0) = y_1$$

for some special kind of functions f which has the following properties:

1. f is sign-definite; that is, $f(t) \geqslant 0$ for all $t > 0$ or $f(t) \leqslant 0$ for all $t < 0$;

2. f is and is supported in $[t_0 - \tau, t_0 + \tau]$ for some $t_0 > 0$ and some very small $\tau > 0$;

3. $\displaystyle\int_{t_0-\tau}^{t_0+\tau} f(t)\,dt = F$, where F is a constant independent of τ.

This kind of force is called an ***impulse***.

Example 6.49. Let $d_\tau : \mathbb{R} \to \mathbb{R}$ be a step function defined by

$$d_\tau(t) = \begin{cases} \dfrac{1}{2\tau} & \text{if } t \in (-\tau, \tau)\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{6.16}$$

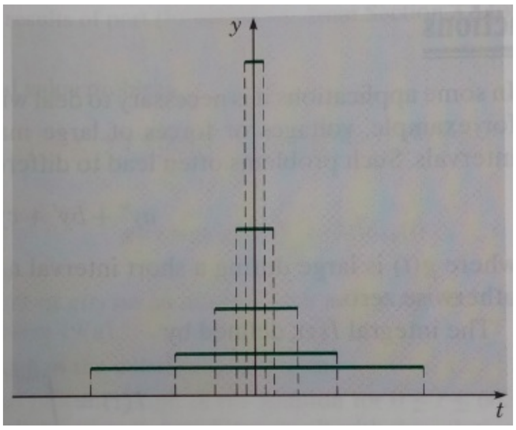

Figure 1: The graph of $y = d_\tau(t)$ as $\tau \to 0^+$.

Then $f(t) = F d_\tau(t)$ is an impulse function. We note that with d denoting the function $\dfrac{1}{2}\mathbf{1}_{(-1,1)}$, then $d_\tau(t) = \dfrac{1}{\tau} d\big(\dfrac{t}{\tau}\big)$. Moreover, if $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous in an open interval containing 0, we must have

$$\lim_{\tau \to 0^+} \int_{-\infty}^{\infty} d_\tau(t)\varphi(t)\,dt = \varphi(0)\,. \tag{6.17}$$

Example 6.50. Let

$$\eta(t) = \begin{cases} C\exp\big(\dfrac{1}{t^2 - 1}\big) & \text{if } |t| < 1\,, \\ 0 & \text{if } |t| \geqslant 1\,, \end{cases}$$

where C is chosen so that the integral of η is 1. Then the sequence $\{\eta_\tau\}_{\tau > 0}$ defined by

$$\eta_\tau(t) = \frac{1}{\tau}\eta\big(\frac{t}{\tau}\big) \tag{6.18}$$

also has the property that

$$\lim_{\tau \to 0} \int_{-\infty}^{\infty} \eta_\tau(t)\varphi(t)\,dt = \varphi(0) \tag{6.19}$$

for all $\varphi : \mathbb{R} \to \mathbb{R}$ which is continuous in an open interval containing 0. To see this, we notice that η_τ is supported in $[-\tau, \tau]$ and the integral of η_τ is still 1. Suppose that $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous on (a, b) for some $a < 0 < b$. Then there exists $0 < \delta < \min\{-a, b\}$ such that

$$\big|\varphi(t) - \varphi(0)\big| < \frac{\varepsilon}{2} \quad \text{whenever} \quad |t| < \delta\,.$$

Therefore, if $0 < \tau < \delta$, by the non-negativity of η_τ we find that

$$\begin{aligned} \Big|\int_{-\infty}^{\infty} \eta_\tau(t)\varphi(t) - \varphi(0)\Big| &= \Big|\int_{-\tau}^{\tau} \eta_\tau(t)\varphi(t)\,dt - \varphi(0)\int_{-\tau}^{\tau} \eta_\tau(t)\,dt\Big| \\ &= \int_{-\tau}^{\tau} \eta_\tau(t)\big[\varphi(t) - \varphi(0)\big]\,dt \\ &\leqslant \int_{-\tau}^{\tau} \eta_\tau(t)\big|\varphi(t) - \varphi(0)\big|\,dt \leqslant \frac{\varepsilon}{2}\int_{-\tau}^{\tau} \eta_\tau(t)\,dt < \varepsilon \end{aligned}$$

which validates (6.19).

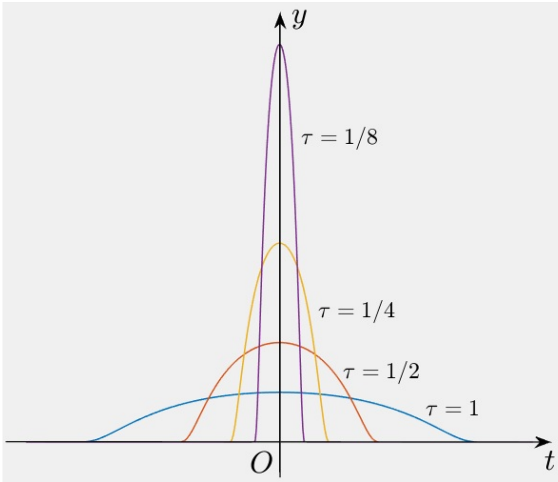

Figure 2: The graph of η_τ for $\tau = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$.

Definition 6.51. A sequence of functions $\{\zeta_\tau\}_{\tau>0}$, where $\zeta_\tau : \mathbb{R} \to \mathbb{R}$ for all $\tau > 0$, is called an ***approximation of the identity*** if $\{\zeta_\tau\}_{\tau>0}$ satisfies

1. $\zeta_\tau(t) \geqslant 0$ for all $t \in \mathbb{R}$.
2. $\displaystyle\lim_{\tau\to 0^+} \int_{-\infty}^{\infty} \zeta_\tau(t)\, dt = 1$.
3. For all $\delta > 0$, $\displaystyle\lim_{\tau\to 0^+} \int_{|t|>\delta} \zeta_\tau(t)\, dt = 0$.

In particular, $\{d_\tau\}_{\tau>0}$ and $\{\eta_\tau\}_{\tau>0}$ are approximations of identity.

Using the same technique of establishing (6.19), one can also prove that if $\{\zeta_\tau\}_{\tau>0}$ is an approximation of the identity, then

$$\lim_{\tau\to 0} \int_{-\infty}^{\infty} \zeta_\tau(t)\varphi(t)\, dt = \varphi(0)\,.$$

Remark 6.52. An approximation of identities does not have to be compactly supported. For example, let $n(t) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$ be the probability density function of the normal distribution $N(0,1)$, then $n_\tau(t) \equiv \dfrac{1}{\sqrt{2\pi\tau}} e^{-\frac{t^2}{2\tau}}$ constitutes an approximation of the identity $\{n_\tau\}_{\tau>0}$.

For $t_0 > 0$ and $0 < \tau < t_0$, let y_τ denote the solution to the IVP

$$y_\tau'' + by_\tau' + cy_\tau = Fd_\tau(t - t_0)\,, \qquad y_\tau(0) = y_0\,, \quad y_\tau'(0) = y_1\,. \tag{6.20}$$

Using (6.14) we find that

$$y_\tau(t) = Y(t) + \frac{F}{2c\tau}\Big[u_{t_0-\tau}(t)\big[1 - z(t - t_0 + \tau)\big] - u_{t_0+\tau}(t)\big[1 - z(t - t_0 - \tau)\big]\Big]\,,$$

where Y is the unique $\mathscr{C}^2$-function solving

$$Y'' + bY' + cY = 0\,, \qquad Y(0) = y_0\,, \quad Y'(0) = y_1\,.$$

and z is the unique $\mathscr{C}^2$-function solving

$$z'' + bz' + cz = 0\,, \qquad z(0) = 1\,, \quad z(0) = 0\,.$$

We remark here that Y, Y', z' and z'' are of exponential order α for some $\alpha > 0$; that is, there exists $M > 0$ such that

$$\big|Y(t)\big| + \big|Y'(t)\big| + |z(t)| + |z'(t)| \leqslant Me^{\alpha t} \qquad \forall\, t > 0\,. \tag{6.21}$$

We also recall that the discussion in Section 6.5 shows that y_τ is continuously differentiable, and y''_τ is piecewise continuous. Our "goal" here is to find a function y which is independent of τ but $|y - y_\tau| \ll 1$ when $\tau \ll 1$. In other words, our goal is to show that $\{y_\tau\}_{\tau>0}$ converges and find the limit of $\{y_\tau\}_{\tau>0}$. We rely on the following theorem:

> Let $a, b \in \mathbb{R}$ and $f_n : [a, b] \to \mathbb{R}$ be a sequence of differentiable functions such that $\{f_n\}_{n=1}^\infty$ and $\{f'_n\}_{n=1}^\infty$ are both ***uniformly bounded***; that is, there exists M such that
>
> $$\big|f_n(x)\big| + \big|f'_n(x)\big| \leqslant M \text{ for all } x \in [a, b] \text{ and } n \in \mathbb{N};$$
>
> Then there is a subsequence $\big\{f_{n_j}\big\}_{j=1}^\infty$ and a continuous function $f : [a, b] \to \mathbb{R}$ such that
>
> $$\lim_{j\to\infty} \sup_{x\in[a,b]} \big|f_{n_j}(x) - f(x)\big| = 0. \tag{$\star$}$$

The convergence in ($\star$) is called the ***uniform convergence***. To be more precise, we say that a sequence of functions $\{g_k\}_{k=1}^\infty$ converges uniformly to g on A if $\lim\limits_{k\to\infty} \sup\limits_{t\in A} |g_k(t) - g(t)| = 0$; thus ($\star$) is the same as saying that $\big\{f_{n_j}\big\}_{j=1}^\infty$ converges uniformly to f on $[a, b]$. The theorem above is a direct consequence of the ***Arzelà-Ascoli theorem***, and the proof of the Arzelà-Ascoli theorem can be found in most of textbooks of Elementary Analysis.

We claim that $\{y_\tau\}_{\tau>0}$ and $\{y'_\tau\}_{\tau>0}$, viewing as functions defined on $[0, T]$ for some $T > 0$, are uniformly bounded (so that we can extract a uniformly convergent subsequence due to the Arzelà-Ascoli theorem). Let $T > 0$ be given such that $t_0 + \tau < T$.

1. If $0 < t < t_0 - \tau$, then $y_\tau(t) = Y(t)$; thus

$$|y_\tau(t)| + |y'_\tau(t)| \leqslant \max_{t\in[0,T]} \big|Y(t)\big| + \max_{t\in[0,T]} \big|Y'(t)\big| \qquad \forall\, t \in (0, t_0 - \tau)\,. \tag{6.22}$$

2. If $t_0 - \tau < t < t_0 + \tau$, then

$$y_\tau(t) = Y(t) + \frac{F}{2c\tau}\big[1 - z(t - t_0 + \tau)\big] \quad \text{and} \quad y'_\tau(t) = Y'(t) - \frac{F}{2c\tau} z'(t - t_0 + \tau)\,.$$

The mean value theorem implies that there exists $\xi_1, \xi_2 \in (t - t_0 + \tau)$ such that

$$\begin{aligned} 1 - z(t - t_0 + \tau) = z(0) - z(t - t_0 + \tau) = -z'(\xi_1)(t - t_0 + \tau)\,, \\ z'(t - t_0 + \tau) = z'(t - t_0 + \tau) - z'(0) = z''(\xi_2)(t - t_0 + \tau)\,. \end{aligned}$$

Since $t_0 - \tau < t < t_0 + \tau$, we must have $|t - t_0 + \tau| < 2\tau$; thus

$$\begin{aligned}
|y_\tau(t)| &\leqslant \big|Y(t)\big| + \frac{|F|}{2|c|\tau}|z'(\xi_1)||t - t_0 + \tau| \\
&\leqslant \max_{t\in[0,T]} \big|Y(t)\big| + \frac{|F|}{|c|} \max_{t\in[0,T]} |z'(t)| \qquad \forall\, t \in (0, t_0 - \tau)\,,
\end{aligned} \tag{6.23a}$$

and similarly,

$$|y'_\tau(t)| \leqslant \max_{t\in[0,T]} \big|Y'(t)\big| + \frac{|F|}{|c|} \max_{t\in[0,T]} \big|z''(t)\big| \qquad \forall\, t \in (0, t_0 - \tau)\,. \tag{6.23b}$$

3. If $t_0 + \tau < t < T$, then

$$\begin{aligned}
y_\tau(t) &= Y(t) - \frac{F}{2c\tau}\big[z(t - t_0 + \tau) - z(t - t_0 - \tau)\big]\,, \\
y'_\tau(t) &= Y'(t) - \frac{F}{2c\tau}\big[z'(t - t_0 + \tau) - z'(t - t_0 - \tau)\big]\,.
\end{aligned}$$

Similar to the argument in the previous case, the mean value theorem provides $\eta_1, \eta_2 \in (t - t_0 - \tau, t - t_0 + \tau)$ such that

$$\begin{aligned}
z(t - t_0 + \tau) - z(t - t_0 - \tau) &= z'(\eta_1) \cdot (2\tau)\,, \\
z'(t - t_0 + \tau) - z'(t - t_0 - \tau) &= z''(\eta_2) \cdot (2\tau)\,;
\end{aligned}$$

thus

$$|y_\tau(t)| \leqslant \max_{t\in[0,T]} \big|Y(t)\big| + \frac{|F|}{|c|} \max_{t\in[0,T]} |z'(t)| \qquad \forall\, t \in (t_0 + \tau, T) \tag{6.24a}$$

$$|y'_\tau(t)| \leqslant \max_{t\in[0,T]} \big|Y'(t)\big| + \frac{|F|}{|c|} \max_{t\in[0,T]} |z''(t)| \qquad \forall\, t \in (t_0 + \tau, T)\,. \tag{6.24b}$$

Noting that there exist $M > 0$ and $\alpha > 0$ such that

$$\big|Y(t)\big| + \big|Y'(t)\big| + |z'(t)| + |z''(t)| \leqslant M e^{\alpha t} \qquad \forall\, t > 0\,,$$

combining (6.22), (6.23) and (6.24) we find that

$$\big|y_\tau(t)\big| + \big|y'_\tau(t)\big| \leqslant 2M\Big(1 + \frac{|F|}{|c|}\Big)e^{\alpha T} \qquad \forall\, t \in (0, T)\backslash\{t_0 - \tau, t_0 + \tau\}\,.$$

Let $\widetilde{M} = 2M\big(1 + \frac{|F|}{|c|}\big)$. By the continuity of y_τ and y'_τ, the inequality above shows that

$$\big|y_\tau(T)\big| + \big|y'_\tau(T)\big| \leqslant \widetilde{M} e^{\alpha T}\,.$$

Since the inequality above holds for all $T > 0$, we conclude that

$$\big|y_\tau(t)\big| + \big|y'_\tau(t)\big| \leqslant \widetilde{M} e^{\alpha t} \qquad \forall\, t > 0\,. \tag{6.25}$$

Therefore, $\{y_\tau\}_{\tau>0}$ and $\{y'_\tau\}_{\tau>0}$ are uniformly bounded on $[0, T]$ for all $T > 0$.

Let $T > 0$ be fixed again. By the Arzelà-Ascoli theorem, there exists a subsequence $\{y_{\tau_j}\}_{j=1}^{\infty}$

which converges to y uniformly on $[0,T]$ as $j\to\infty$. We note that y is a function defined on $[0,T]$. Now, by the uniform boundedness of $\{y_{\tau_j}\}_{j=1}^{\infty}$ and $\{y'_{\tau_j}\}_{j=1}^{\infty}$ on $[0,2T]$, there exists a subsequence $\{y_{\tau_{j_\ell}}\}_{\ell=1}^{\infty}$ which converges to y^* uniformly on $[0,2T]$. Same procedure provides a further subsequence $\{y_{\tau_{j_{\ell_k}}}\}_{k=1}^{\infty}$ which converges to y^{**} uniformly on $[0,3T]$. We note that $y^{**}=y^*$ on $[0,T+1]$ and $y^{**}=y$ on $[0,T]$). We continue this process and obtain a sequence, still denoted by $\{y_{\tau_j}\}_{j=1}^{\infty}$, and a continuous function $y:[0,\infty)\to\mathbb{R}$ such that

$$\lim_{j\to\infty}\sup_{t\in[0,T]}\big|y_{\tau_j}(t)-y(t)\big|=0 \qquad \forall\, T>0\,. \tag{6.26}$$

We note that (6.25) implies that y satisfies

$$|y(t)|\leqslant \widetilde{M}e^{\alpha t} \qquad \forall\, t>0\,;$$

thus the limit function y is of exponential order α for some $\alpha>0$. On the other hand, we also note that it is still possible that there is another convergent subsequence which converges to another limit function, but we will show that there is only one possible limit function.

Note that in Section 6.5 we use the Laplace transform to solve the IVP (6.20) and obtain that

$$(s^2+bs+c)\mathscr{L}(y_\tau)(s)=(s+b)y_0+y_1+F\int_0^\infty d_\tau(t-t_0)e^{-st}\,dt \qquad \forall\, s>\alpha\,,$$

where α is chosen so that y_τ and y are of exponential order α. In particular,

$$(s^2+bs+c)\int_0^\infty y_{\tau_j}(t)e^{-st}\,dt=(s+b)y_0+y_1+F\int_0^\infty d_{\tau_j}(t-t_0)e^{-st}\,dt \qquad \forall\, s>\alpha\,. \tag{6.27}$$

Let $\varepsilon>0$ and $s>\alpha$ be given. Since there exists $M>0$ such that $|y_\tau(t)|+|y(t)|\leqslant Me^{\alpha t}$ for all $t>0$, we can choose $T>0$ such that

$$\int_T^\infty e^{(\alpha-s)t}\,dt=\frac{1}{s-\alpha}e^{(\alpha-s)T}<\frac{\varepsilon}{2M}\,.$$

Then by the convergence (6.26),there is $N>0$ such that if $j\geqslant N$,

$$\sup_{t\in[0,T]}\big|y_{\tau_j}(t)-y(t)\big|<\frac{s\varepsilon}{2(1+e^{-sT})}\,.$$

Then for $j\geqslant N$,

$$\begin{aligned}\Big|\int_0^\infty\big[y_{\tau_j}(t)-y(t)\big]e^{-st}\,dt\Big| &\leqslant \int_0^\infty \big|y_{\tau_j}(t)-y(t)\big|e^{-st}\,dt\\ &=\int_0^T \big|y_{\tau_j}(t)-y(t)\big|e^{-st}\,dt+\int_T^\infty \big|y_{\tau_j}(t)-y(t)\big|e^{-st}\,dt\\ &\leqslant \int_0^T \sup_{t\in[0,T]}\big|y_{\tau_j}(t)-y(t)\big|e^{-st}\,dt+M\int_T^\infty e^{\alpha t}e^{-st}\,dt\\ &\leqslant \frac{s\varepsilon}{2(1+e^{-sT})}\int_0^T e^{-st}\,dt+\frac{\varepsilon}{2}<\varepsilon\end{aligned}$$

which implies that

$$\lim_{j\to\infty}\int_0^\infty y_{\tau_j}(t)e^{-st}\,dt=\int_0^\infty y(t)e^{-st}\,dt=\mathscr{L}(y)(s)\qquad\forall\, s>\alpha\,.$$

On the other hand, the change of variables formula shows that

$$\int_0^\infty d_{\tau_j}(t-t_0)e^{-st}\,dt=\int_{-\infty}^\infty d_{\tau_j}(t-t_0)e^{-st}\,dt=\int_{-\infty}^\infty d_{\tau_j}(t)e^{-s(t+t_0)}\,dt$$

so (6.17) implies that

$$\lim_{j\to\infty}\int_0^\infty d_{\tau_j}(t-t_0)e^{-st}\,dt=e^{-st_0}\,.$$

As a consequence, passing to the limit as $j\to\infty$ in (6.27), we find that

$$(s^2+bs+c)\mathscr{L}(y)(s)=(s+b)y_0+y_1+Fe^{-st_0}\qquad\forall\, s>\alpha\,. \tag{6.28}$$

Since any possible limit of $\{y_\tau\}_{\tau>0}$ has to satisfy the equation above, by Theorem 6.15 we conclude that there is only one uniform limit of $\{y_\tau\}_{\tau>0}$; thus $\{y_\tau\}_{\tau>0}$ converges to y uniformly on $[0,T]$ for every $T>0$; that is,

$$\lim_{\tau\to 0}\sup_{t\in[0,T]}\big|y_\tau(t)-y(t)\big|=0\qquad\forall\, T>0\,. \tag{6.29}$$

The uniform convergence of $\{y_\tau\}_{\tau>0}$ to y implies that if the support of the impulse is really small, even though we might not know the precise value of τ, the solution to (6.20) is very closed to the unique limit function y. We note that the three possible y's given above are continuous but have discontinuous derivatives, and are not differentiable at t_0.

By Theorem 6.34 and 6.17, identity (6.28) implies the following:

1. if $r^2+br+c=0$ has two distinct real roots r_1 and r_2, then the solution y to (6.28) is

$$\begin{aligned}y(t)&=Y(t)+\frac{F}{r_1-r_2}u_{t_0}(t)\big[e^{r_1(t-t_0)}-e^{r_2(t-t_0)}\big]\\&=\frac{y_1-r_2y_0}{r_1-r_2}e^{r_1t}+\frac{r_1y_0-y_1}{r_1-r_2}e^{r_2t}+\frac{F}{r_1-r_2}u_{t_0}(t)\big[e^{r_1(t-t_0)}-e^{r_2(t-t_0)}\big]\,.\end{aligned} \tag{6.30}$$

2. if $r^2+br+c=0$ has a double root r_1, then the solution y to (6.28) is

$$\begin{aligned}y(t)&=Y(t)+Fu_{t_0}(t)(t-t_0)e^{r_1(t-t_0)}\\&=y_0e^{r_1t}+(y_1-r_1y_0)te^{r_1t}+Fu_{t_0}(t)(t-t_0)e^{r_1(t-t_0)}\,.\end{aligned} \tag{6.31}$$

3. if $r^2+br+c=0$ has two complex roots $\lambda\pm i\mu$, then the solution y to (6.28) is

$$\begin{aligned}y(t)&=Y(t)+\frac{F}{\mu}u_{t_0}(t)e^{\lambda(t-t_0)}\sin\mu(t-t_0)\\&=y_0e^{\lambda t}\cos\mu t+\frac{y_1-\lambda y_0}{\mu}e^{\lambda t}\sin\mu t+\frac{F}{\mu}u_{t_0}(t)e^{\lambda(t-t_0)}\sin\mu(t-t_0)\,.\end{aligned} \tag{6.32}$$

6.7.1 The Dirac delta function

Even though we can stop our discussion about second order ODEs with impulse forcing functions here, we would like to go a little bit further by introducing the so-called "Dirac delta function".

Definition 6.53 (Informal definition of the Dirac delta function)**.** For $t_0 > 0$, the ***Dirac delta function*** at t_0, denoted by δ_{t_0}, is the function whose Laplace transform is the function $G(s) = e^{-st_0}$.

Given the definition above, (6.4) and (6.28) imply that y satisfies the ODE

$$y'' + by' + cy = F\delta_{t_0}(t)\,, \qquad y(0) = y_0, \quad y'(0) = y_1\,. \tag{6.33}$$

However, there is **no** such δ_{t_0} for the following reasons:

1. Using (6.30), (6.31) or (6.32), we find that $y'' + by' + cy = 0$ for all $t \neq t_0$. If such δ_{t_0} exists (as a function), then $\delta_{t_0}(t) = 0$ for all $t \neq t_0$ which makes $\mathscr{L}(\delta_{t_0}) = 0$. In other words, if δ_{t_0} is a function of non-negative real numbers, no matter what value is assigned to $\delta_{t_0}(t_0)$, the Laplace transform of δ_{t_0} cannot be e^{-st_0}.

2. Rewriting e^{-st_0} as $s \cdot \dfrac{e^{-st_0}}{s}$, by Theorem 6.19 we find that

$$e^{-st_0} = s\frac{e^{-st_0}}{s} = s\mathscr{L}(u_{t_0}) = \mathscr{L}\big(\frac{d}{dt}u_{t_0}\big)(s) + u_{t_0}(0) = \mathscr{L}\big(\frac{d}{dt}u_{t_0}\big)(s)\,.$$

   Therefore, $\delta_{t_0}(t) = \dfrac{d}{dt}u_{t_0}(t)$ which vanishes as long as $t \neq t_0$.

• **What does $y''+by'+cy = F\delta_{t_0}(t)$ really mean?** Recall that our goal is to find a "representative" of solutions of the sequence of ODEs (6.20). The discussion above shows that such a representative has to satisfies (6.28) which, under the assumption that

$$\mathscr{L}(y'' + by' + cy)(s) = (s^2 + bs + c)\mathscr{L}(y) - sy(0) - y'(0)\,. \tag{6.34}$$

implies the equation $y'' + by' + cy = F\delta_{t_0}(t)$. As we can see from the precise form of the function y in (6.30), (6.31) and (6.32), y' is not even continuous; thus (6.34) is in fact a false assumption.

The way that the ODE $y'' + by' + cy = F\delta_{t_0}(t)$ is understood is through the distribution theory, in which both sides of the ODE are treated as "functions of functions". Let $\varphi : [0,\infty) \to \mathbb{R}$ be a twice continuously differentiable function which vanishes outside $[0, T]$ for some $T > t_0$. Multiplying both sides of (6.20) by φ and then integrating on $[0, T]$, we find that

$$\int_0^T (y''_{\tau_j} + by'_{\tau_j} + cy_{\tau_j})\varphi(t)\,dt = F\int_0^T d_{\tau_j}(t - t_0)\varphi(t)\,dt\,.$$

Integrating by parts (twice if necessary) and making a change of variable on the right-hand side,

$$y_0\varphi'(0) - \big(y_1 + by_0\big)\varphi(0) + \int_0^\infty y_{\tau_j}(t)\big(\varphi''(t) - b\varphi'(t) + c\varphi(t)\big)\,dt = F\int_{-\infty}^\infty d_{\tau_j}(t)\varphi(t + t_0)\,dt \tag{6.35}$$

for all twice continuously differentiable functions φ vanishing outside some interval $[0,T]$. We note that the integral in (6.35) is not an improper integral but indeed an integral on a bounded interval. Passing to the limit as $j \to \infty$ in (6.35), the uniform convergence of $\{y_{\tau_j}\}_{j=1}^{\infty}$ to y on any closed interval $[0,T]$ and (6.17) imply that

$$y_0\varphi'(0) - \big(y_1 + by_0\big)\varphi(0) + \int_0^\infty y(t)\big(\varphi''(t) - b\varphi'(t) + c\varphi(t)\big)\,dt = F\varphi(t_0) \tag{6.36}$$

for all twice continuously differentiable functions φ vanishing outside some interval $[0,T]$.

Definition 6.54. The collection of all k-times continuously differentiable function defined on $[0,\infty$ and vanishing outside some interval $[0,T]$ for some $T > 0$ is denoted by $\mathscr{C}_c^k([0,\infty))$. A function $f : [0,\infty) \to \mathbb{R}$ is said to belong to the space $\mathscr{C}_c^\infty([0,\infty))$ if $f \in \mathscr{C}_c^k([0,\infty))$ for all $k \in \mathbb{N}$. In other words,

$$\mathscr{C}_c^\infty([0,\infty)) \equiv \Big\{ f : [0,\infty) \to \mathbb{R} \,\Big|\, f \in \mathscr{C}_c^k([0,\infty) \;\forall\, k \in \mathbb{N} \Big\}\,.$$

Definition 6.55. Let $f : [0,\infty)$ be a piecewise continuous function. The linear functional induced by f, denoted by $\langle f, \cdot\rangle$, is a function on $\mathscr{C}_c^\infty([0,\infty))$ given by

$$\langle f, \varphi\rangle = \int_0^\infty f(t)\varphi(t)\,dt \qquad \forall\,\varphi \in \mathscr{C}_c^\infty([0,\infty))\,.$$

Consider the following simple ODE

$$y'' + by' + cy = f(t)\,, \qquad y(0) = y_0\,, \quad y'(0) = y_1\,, \tag{6.37}$$

where f is a continuous function of exponential order a for some $a \in \mathbb{R}$. The existence theory implies that there exists a unique twice continuously differentiable solution y to (6.37). Moreover, if $\varphi \in \mathscr{C}_c^2([0,\infty))$,

$$\int_0^\infty \big[y''(t) + by'(t) + cy(t)\big]\varphi(t)\,dt = \int_0^\infty f(t)\varphi(t)\,dt\,, \quad y(0) = y_0\,,\; y'(0) = y_1\,. \tag{6.38}$$

Since y is twice continuously differentiable on $[0,\infty)$, we can integrate by parts and find that the solution y to (6.37) also satisfies

$$y_0\varphi'(0) - \big(y_1 + by_0\big)\varphi(0) + \int_0^\infty y(t)\big(\varphi''(t) - b\varphi'(t) + c\varphi(t)\big)\,dt = \langle f, \varphi\rangle \quad \forall\,\varphi \in \mathscr{C}_c^2([0,\infty))\,. \tag{6.39}$$

On the other hand, if y is a twice continuously differentiable function satisfying (6.39), we can integrate by parts (to put the derivatives on φ back to y) and find that y satisfies

$$\begin{aligned}&\big(y_0 - y(0)\big)\varphi'(0) - \big[y_1 + by_0 - y'(0) - by(0)\big]\varphi(0)\\ &\quad + \int_0^\infty \big[y''(t) + by'(t) + cy(t)\big]\varphi(t)\,dt = \int_0^\infty f(t)\varphi(t)\,dt\end{aligned} \qquad \forall\,\varphi \in \mathscr{C}_c^2([0,\infty))\,.$$

In particular,

$$\int_0^\infty \big[y''(t) + by'(t) + cy(t)\big]\varphi(t)\,dt = \int_0^\infty f(t)\varphi(t)\,dt \quad \forall\,\varphi \in \mathscr{C}_c^2([0,\infty)) \text{ satisfying } \varphi(0) = \varphi'(0) = 0\,.$$

Therefore, $y'' + by' + cy$ must be identical to f since they are both continuous. Having established this, we find that

$$\big(y_0 - y(0)\big)\varphi'(0) - \big[y_1 + by_0 - y'(0) - by(0)\big]\varphi(0) = 0 \qquad \forall\, \varphi \in \mathscr{C}_c^2([0,\infty))\,.$$

Choose $\varphi \in \mathscr{C}_c^2([0,\infty))$ such that $\varphi(0) = 0$ and $\varphi'(0) = 1$, we conclude that $y_0 = y(0)$; thus we arrive at the equality

$$\big[y_1 + by_0 - y'(0) - by(0)\big]\varphi(0) = 0 \qquad \forall\, \varphi \in \mathscr{C}_c^2([0,\infty))\,.$$

The identity above clearly shows that $y_1 = y'(0)$. In other words, if y is twice continuously differentiable and satisfies (6.39), then y satisfies (6.37); thus we establish that given a continuous forcing function f,

y is a solution to (6.37) if and only if y satisfies (6.39).

Thus we change the problem of solving an ODE "in the pointwise sense" to a problem of solving an integral equation which holds "in the sense of distribution" (a distribution means a function of functions). We note that there is one particular advantage of defining solution to (6.37) using (6.39) instead of (6.38): if f is discontinuous somewhere in $[0,\infty)$ (for example, $f = F\mathbf{1}_{(\alpha,\beta)}$ as in the previous section), (6.39) provides a good alternative even if y'' does not always exist.

The discussion above motivates the following

Definition 6.56 (Weak Solutions)**.** Let $f : [0,\infty) \to \mathbb{R}$ be a function of exponential order a for some $a \in \mathbb{R}$. A function $y : [0,\infty) \to \mathbb{R}$ is said to be a ***weak solution*** to (6.37) if y satisfies the integral equation (6.39). The integral equation (6.39) is called the ***weak formulation*** of (6.37).

We remark that the discussion above shows that if $f : [0,\infty) \to \mathbb{R}$ is continuous and of exponential order a for some $a \in \mathbb{R}$, the unique $\mathscr{C}^2$-solution y to (6.37) is also a weak solution.

In view of (6.39), if we define $L : \mathscr{C}_c^2([0,\infty)) \to \mathbb{R}$ by

$$L(\varphi) = y_0\varphi'(0) - \big(y_1 + by_0\big)\varphi(0) + \int_0^\infty y(t)\big(\varphi''(t) - b\varphi'(t) + c\varphi(t)\big)\,dt\,, \tag{6.40}$$

then the integral equation (6.37) is equivalent to that "the two linear functionals L and $\langle f, \cdot\rangle$ are the same on the space $\mathscr{C}_c^2([0,\infty))$". We also note that

$$L(\varphi) = \langle y'' + by' + cy, \varphi\rangle \qquad \text{if } y'' \text{ is piecewise continuous, and } (y(0), y'(0)) = (y_0, y_1)\,;$$

thus if y'' is piecewise continuous, the statement "$L = \langle f, \cdot\rangle$ on $\mathscr{C}_c^2([0,\infty))$" is the same as saying that "the linear functional induced by $y'' + by' + cy$ and the linear functional induced by f are identical". This is what it means by $y'' + by' + cy = f$ in the sense of distribution.

If the right-hand side $\langle f, \cdot\rangle$ is replaced by a general linear functional ℓ, we can still talk about the possibility of finding an integrable function y validating the integral equation (6.39), or more precisely, $L = \ell$ on $\mathscr{C}_c^2([0,\infty))$. In particular, for $F \in \mathbb{R}$ and $t_0 > 0$, it is reasonable to ask whether or not there exists an integrable function y such that

$$y_0\varphi'(0) - \big(y_1 + by_0\big)\varphi(0) + \int_0^\infty y(t)\big(\varphi''(t) - b\varphi'(t) + c\varphi(t)\big)\,dt = F\varphi(t_0) \quad \forall\, \varphi \in \mathscr{C}_c^2([0,\infty))\,, \tag{6.36}$$

where the linear functional $\ell : \mathscr{C}_c^2([0,\infty)) \to \mathbb{R}$ is given by

$$\ell(\varphi) = F\varphi(t_0) \qquad \forall\, \varphi \in \mathscr{C}_c^2([0,\infty))\,. \tag{6.41}$$

This is exactly the integral equation (6.36); thus the ODE $y'' + by' + cy = F\delta_{t_0}(t)$ is understood as $L = \ell$ on $\mathscr{C}_c^2([0,\infty))$, where L and ℓ are defined by (6.40) and (6.41), respectively.

The definition of ℓ motivates the following

Definition 6.57 (Dirac Delta Function)**.** For $t_0 > 0$, the ***Dirac delta function*** at t_0 is a map $\delta_{t_0} : \mathscr{C}_c^2([0,\infty)) \to \mathbb{R}$ defined by

$$\delta_{t_0}(\varphi) = \varphi(t_0)\,.$$

Because of Definition 6.55, one often write $\delta_{t_0}(\varphi) = \int_0^\infty \delta_{t_0}(t)\varphi(t)\,dt$ for $t_0 > 0$.

Under this definition, the ODE $y'' + by' + cy = F\delta_{t_0}$ is understood as "the functional induced by $y'' + by' + cy$ (given by (6.40)) is the same as the functional induced by $F\delta_{t_0}$". The function y given by (6.30), (6.31) or (6.32) is then a ***weak solution*** to (6.33).

• **Summary**:

1. The limit y of the solution y_τ to the IVP (6.20) is the weak solution to the IVP (6.33); that is, y solves (6.33) in the sense of distribution or equivalently, y satisfies (6.36).

2. The limit y can be obtained by solving (6.33) **formally** using the Laplace transform (by treating that $\mathscr{L}(\delta_{t_0})(s) = e^{-st_0}$) and are given by (6.30), (6.31) or (6.32).

Example 6.58. In this example, we would like to find the "anti-derivative" of the Dirac delta function at $t_0 > 0$. In other words, we are looking for a solution to

$$y' = \delta_{t_0}(t)\,, \qquad y(0) = 0\,.$$

Taking the Laplace transform, we find that

$$s\mathscr{L}(y)(s) = e^{-st_0} \quad \text{or equivalently,} \quad \mathscr{L}(y)(s) = \frac{e^{-st_0}}{s}\,. \tag{6.42}$$

As a consequence, by Example 6.5 and Theorem 6.34 we conclude that the (weak) solution to the ODE above is

$$y(t) = u_{t_0}(t)\,.$$

We again emphasize that in principle we are not allowed to use Theorem 6.19 or Corollary 6.20 to compute the Laplace transform of y'; however, the functional induced by y' (by assuming that y is

$$\int_0^\infty y'(t)\varphi(t)\,dt = y(0)\varphi(0) - \int_0^\infty y(t)\varphi'(t)\,dt \qquad \forall\, \varphi \in \mathscr{C}_c^1([0,\infty))$$

so we are in fact solving $y' = \delta_{t_0}(t)$ in the sense of distribution; that is, we look for y satisfying

$$-\int_0^\infty y(t)\varphi'(t)\,dt = \varphi(t_0) \qquad \forall\, \varphi \in \mathscr{C}_c^1([0,\infty))\,.$$

Letting $\varphi(t) = e^{-st}$ leads to (6.42).

7 Series Solutions of Differential Equations

7.1 Properties of Power Series

Definition 7.1. A power series about c is a series of the form $\sum_{k=0}^{\infty} a_k(x-c)^k$ for some sequence $\{a_k\}_{k=0}^{\infty} \subseteq \mathbb{R}$ (or $\mathbb{C}$) and $c \in \mathbb{R}$ (or $\mathbb{C}$).

Proposition 7.2. *If a power series centered at c is convergent at some point $b \neq c$, then the power series converges absolutely for all points in $(c-|b-c|, c+|b-c|)$.*

Proof. Since the series $\sum_{k=0}^{\infty} a_k(b-c)^k$ converges, $|a_k||b-c|^k \to 0$ as $k \to \infty$; thus there exists $M > 0$ such that $|a_k||b-c|^k \leqslant M$ for all k. Then if $x \in (c-|b-c|, c+|b-c|)$, the series $\sum_{k=0}^{\infty} a_k(x-c)^k$ converges absolutely since

$$\sum_{k=0}^{\infty} |a_k(x-c)^k| = \sum_{k=0}^{\infty} |a_k||x-c|^k = \sum_{k=0}^{\infty} |a_k||b-c|^k \frac{|x-c|^k}{|b-c|^k} \leqslant M \sum_{k=0}^{\infty} \Big(\frac{|x-c|}{|b-c|}\Big)^k$$

which converges (because of the geometric series test or ratio test). □

Definition 7.3. A number R is called the ***radius of convergence*** of the power series $\sum_{k=0}^{\infty} a_k(x-c)^k$ if the series converges for all $x \in (c-R, c+R)$ but diverges if $x > c+R$ or $x < c-R$. In other words,

$$R = \sup\Big\{r \geqslant 0 \,\Big|\, \sum_{k=0}^{\infty} a_k(x-c)^k \text{ converges in } [c-r, c+r]\Big\}\,.$$

The ***interval of convergence*** or ***convergence interval*** of a power series is the collection of all x at which the power series converges.

We remark that Proposition 7.2 implies that a power series converges absolutely in the interior of the interval of convergence.

Proposition 7.4. *A power series is continuous in the interior of the convergence interval; that is, if $R > 0$ is the radius of convergence of the power series $\sum_{k=0}^{\infty} a_k(x-c)^k$, then $\sum_{k=0}^{\infty} a_k(x-c)^k$ is continuous in $(c-R, c+R)$.*

Proof. W.L.O.G., we prove that the power series is continuous at $x_0 \in [c, c+R)$. Let $\varepsilon > 0$ be given. Define $r = \dfrac{c+R-x_0}{2}$. Then $x_0 + r \in (c-R, c+R)$; thus there exists $N > 0$ such that

$$\sum_{k=N+1}^{\infty} |a_k||x_0+r-c|^k < \frac{\varepsilon}{3}\,.$$

Moreover, since $\sum_{k=0}^{N} a_k(x-c)^k$ is continuous at x_0, there exists $0 < \delta < r$ such that

$$\Big|\sum_{k=0}^{N} a_k(x-c)^k - \sum_{k=0}^{N} a_k(x_0-c)^k\Big| < \frac{\varepsilon}{3} \qquad \forall\, |x-x_0| < \delta\,.$$

Therefore, if $|x-x_0|<\delta$, we have

$$\begin{aligned}\Big|\sum_{k=0}^{\infty}a_k(x-c)^k-\sum_{k=0}^{\infty}a_k(x_0-c)^k\Big| &\leqslant \Big|\sum_{k=0}^{N}a_k(x-c)^k-\sum_{k=0}^{N}a_k(x_0-c)^k\Big| \\ &\quad+\sum_{k=N+1}^{\infty}|a_k||x_0+r-c|^k\frac{|x-c|^k}{|x_0+r-c|^k}+\sum_{k=N+1}^{\infty}|a_k||x_0+r-c|^k\frac{|x_0-c|^k}{|x_0+r-c|^k} \\ &\leqslant \Big|\sum_{k=0}^{N}a_k(x-c)^k-\sum_{k=0}^{N}a_k(x_0-c)^k\Big|+2\sum_{k=N+1}^{\infty}|a_k|r^k<\varepsilon\end{aligned}$$

which implies that $\sum\limits_{k=0}^{\infty}a_k(x-c)^k$ is continuous at x_0. □

Theorem 7.5. *Let $R>0$ be the radius of convergence of the power series $\sum\limits_{k=0}^{\infty}a_k(x-c)^k$. Then*

$$\int_c^x\sum_{k=0}^{\infty}a_k(t-c)^kdt=\sum_{k=0}^{\infty}\int_c^x a_k(t-c)^kdt=\sum_{k=0}^{\infty}\frac{a_k}{k+1}(x-c)^{k+1}\qquad\forall\,x\in(c-R,c+R)\,.$$

Proof. W.L.O.G., we assume that $x\in(c,c+R)$. Let $\varepsilon>0$ be given. Choose $x_0\in(c-R,c+R)$ such that $|x-c|<|x_0-c|$. Then for $t\in[c,x]$, $\dfrac{|t-c|}{|x_0-c|}\leqslant 1$. Moreover, since $\sum\limits_{k=1}^{\infty}a_k(x_0-c)^k$ converges absolutely, there exists $N\geqslant 0$ such that

$$\sum_{k=N+1}^{\infty}|a_k||x_0-c|^k\leqslant\frac{\varepsilon}{|x_0-c|}\,.$$

Since

$$\begin{aligned}\int_c^x\sum_{k=0}^{\infty}a_k(t-c)^kdt &= \int_c^x\sum_{k=0}^{n}a_k(t-c)^kdt+\int_c^x\sum_{k=N+1}^{\infty}a_k(t-c)^kdt \\ &= \sum_{k=0}^{n}\frac{a_k}{k+1}(x-c)^{k+1}+\int_c^x\sum_{k=n+1}^{\infty}a_k(t-c)^kdt\,,\end{aligned}$$

we have for $n\geqslant N$,

$$\begin{aligned}\Big|\int_c^x\sum_{k=0}^{\infty}a_k(t-c)^kdt-\sum_{k=0}^{n}\frac{a_k}{k+1}(x-c)^{k+1}\Big| &\leqslant \int_c^x\sum_{k=n+1}^{\infty}|a_k||x_0-c|^k\frac{(t-c)^k}{|x_0-c|^k}dt \\ &\leqslant \int_c^x\sum_{k=N+1}^{\infty}|a_k||x_0-c|^kdt\leqslant|x_0-c|\sum_{k=N+1}^{\infty}|a_k||x_0-c|^k<\varepsilon\,.\end{aligned}$$

In other words, $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\dfrac{a_k}{k+1}(x-c)^{k+1}=\displaystyle\int_c^x\sum_{k=0}^{\infty}a_k(t-c)^kdt$ which concludes the corollary. □

Theorem 7.6. *Let $R>0$ be the radius of convergence of the power series $\sum\limits_{k=0}^{\infty}a_k(x-c)^k$. Then*

$$\frac{d}{dx}\sum_{k=0}^{\infty}a_k(x-c)^k=\sum_{k=0}^{\infty}\frac{d}{dx}a_k(x-c)^k=\sum_{k=1}^{\infty}ka_k(x-c)^{k-1}\qquad\forall\,x\in(c-R,c+R)\,.$$

Proof. We first show that the series $\sum\limits_{k=1}^{\infty} ka_k(x-c)^{k-1}$ also converges for all $x \in (c-R, c+R)$. Let $x \in (c-R, c+R)$. Choose $x_0 \in (c-R, c+R)$ such that $|x-c| < |x_0-c|$. Then $\lim\limits_{k\to\infty} k\dfrac{|x-c|^k}{|x_0-c|^k} = 0$. Therefore, there exists $M > 0$ such that

$$k\frac{|x-c|^k}{|x_0-c|^k} \leqslant M \qquad \forall\, k \in \mathbb{N} \cup \{0\}\,;$$

thus

$$\sum_{k=0}^{\infty} k|a_k||x-c|^{k-1} = \sum_{k=0}^{\infty} |a_k||x_0-c|^{k-1}k\frac{|x-c|^{k-1}}{|x_0-c|^{k-1}} \leqslant \frac{M}{|x_0-c|}\sum_{k=0}^{\infty} |a_k||x_0-c|^k < \infty\,.$$

Let $b_k = (k+1)a_{k+1}$. The absolute convergence above implies that the power series $\sum\limits_{k=0}^{\infty} b_k(x-c)^k$ converges absolutely in $(c-R, c+R)$ since

$$\sum_{k=0}^{\infty} b_k(x-c)^k = \sum_{k=1}^{\infty} ka_k(x-c)^{k-1} \qquad \forall\, x \in (c-R, c+R)\,. \tag{7.1}$$

Now, Theorem 7.5 implies that for all $x \in (c-R, c+R)$,

$$\int_c^x \sum_{k=0}^{\infty} b_k(t-c)^k dt = \sum_{k=0}^{\infty} \int_c^x b_k(t-c)^k dt = \sum_{k=0}^{\infty} a_{k+1}(x-c)^{k+1} = \sum_{k=1}^{\infty} a_k(x-c)^k\,;$$

thus by the fact (due to Proposition 7.4) that the power series $\sum\limits_{k=0}^{\infty} b_k|x-c|^k$ is continuous in $(c-R, c+R)$, the fundamental theorem of Calculus implies that

$$\sum_{k=0}^{\infty} b_k(x-c)^k dt = \frac{d}{dx}\sum_{k=0}^{\infty} a_k(x-c)^k \qquad \forall\, x \in (c-R, c+R)\,.$$

The theorem is then concluded because of (7.1). □

Definition 7.7. A function $f : (a, b) \to \mathbb{R}$ is said to be ***analytic*** at $c \in (a, b)$ if f is infinitely many times differentiable at c and there exists $R > 0$ such that

$$f(x) = \sum_{k=0}^{\infty} a_k(x-c)^k \qquad \forall\, x \in (c-R, c+R) \subseteq (a, b)$$

for some sequence $\{a_k\}_{k=0}^{\infty}$.

Remark 7.8. If $f : (a, b) \to \mathbb{R}$ is analytic at $c \in (a, b)$, then Theorem 7.6 implies that

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \qquad \forall\, x \in (c-R, c+R) \subseteq (a, b)$$

for some $R > 0$.

A function which is infinitely many times differentiable at a point c might not be analytic at c. For example, consider the function

$$f(x) = \left\{ \begin{array}{cl} \exp\left(-\dfrac{1}{x^2}\right) & \text{if } x \neq 0\,, \\ 0 & \text{if } x = 0\,. \end{array} \right.$$

Then $f^{(k)}(0) = 0$ for all $k \in \mathbb{N}$ which implies that f cannot be analytic at 0.

7.1.1 Product of Power Series

Definition 7.9. Given two series $\sum\limits_{n=0}^{\infty} a_n$ and $\sum\limits_{n=0}^{\infty} b_n$, the series $\sum\limits_{n=0}^{\infty} c_n$, where $c_n = \sum\limits_{k=0}^{n} a_k b_{n-k}$ for all $n \in \mathbb{N} \cup \{0\}$, is called the ***Cauchy product*** of $\sum\limits_{n=0}^{\infty} a_n$ and $\sum\limits_{n=0}^{\infty} b_n$.

Theorem 7.10. *Suppose that the two series $\sum\limits_{n=0}^{\infty} a_n$ and $\sum\limits_{n=0}^{\infty} b_n$ converge absolutely. Then the Cauchy product of $\sum\limits_{n=0}^{\infty} a_n$ and $\sum\limits_{n=0}^{\infty} b_n$ converges absolutely to $\Big(\sum\limits_{n=0}^{\infty} a_n\Big)\Big(\sum\limits_{n=0}^{\infty} b_n\Big)$; that is,*

$$\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} a_k b_{n-k}\Big) = \Big(\sum_{n=0}^{\infty} a_n\Big)\Big(\sum_{n=0}^{\infty} b_n\Big).$$

Proof. **Claim**: If $\sum\limits_{n=0}^{\infty} a_n$ converges absolutely and $\pi : \mathbb{N} \to \mathbb{N}$ is bijective (that is, one-to-one and onto), then $\sum\limits_{n=0}^{\infty} a_{\pi(n)}$ converges absolutely to $\sum\limits_{n=0}^{\infty} a_n$.

Proof of claim: Let $\sum\limits_{n=0}^{\infty} a_n = a$ and $\varepsilon > 0$ be given. Since $\sum\limits_{n=0}^{\infty} a_n$ converges absolutely, there exists $N > 0$ such that

$$\sum_{n=N+1}^{\infty} |a_n| < \frac{\varepsilon}{2}.$$

Let $K = \max\big\{\pi^{-1}(1), \cdots, \pi^{-1}(N)\big\} + 1$. Then if $k \geqslant K$, $\pi(k) \geqslant N+1$; thus if $k \geqslant K$,

$$\sum_{n=k+1}^{\infty} |a_{\pi(n)}| \leqslant \sum_{n=N+1}^{\infty} |a_n| < \frac{\varepsilon}{2}$$

and

$$\Big|\sum_{n=0}^{k} a_{\pi(n)} - a\Big| \leqslant \Big|\sum_{n=0}^{k} a_{\pi(n)} - \sum_{n=0}^{N} a_n\Big| + \Big|\sum_{n=0}^{N} a_n - a\Big| \leqslant 2\sum_{n=N+1}^{\infty} |a_n| < \varepsilon.$$

Therefore, $\sum\limits_{n=0}^{\infty} a_{\pi(n)}$ converges absolutely to a.

Claim: If $\sum\limits_{n=0}^{\infty} a_n$ and $\sum\limits_{n=0}^{\infty} b_n$ converge absolutely, then $\sum\limits_{n,m=1}^{\infty} a_n b_m$ converges absolutely and

$$\sum_{n,m=1}^{\infty} a_n b_m = \Big(\sum_{n=1}^{\infty} a_n\Big)\Big(\sum_{m=1}^{\infty} b_m\Big),$$

where $\sum\limits_{n,m=1}^{\infty} a_n b_m$ denotes the limit $\lim\limits_{N,M\to\infty} \sum\limits_{n=1}^{N}\sum\limits_{m=1}^{M} a_n b_m$.

Proof of claim: If $N_1 < N_2$ and $M_1 < M_2$,

$$\Big|\Big(\sum_{n=1}^{N_1} |a_n|\Big)\Big(\sum_{m=1}^{M_1} |b_m|\Big) - \Big(\sum_{n=1}^{N_2} |a_n|\Big)\Big(\sum_{m=1}^{M_2} |b_m|\Big)\Big| \leqslant \sum_{n=1}^{N_1} |a_n| \sum_{m=M_1+1}^{M_2} |b_m| + \sum_{n=N_1+1}^{N_2} |a_n| \sum_{m=1}^{M_2} |b_m|$$

which converges as $N_1, M_1 \to \infty$. Therefore, $\sum\limits_{n,m=1}^{\infty} a_n b_m$ converges absolutely, and similar computation shows that

$$
\begin{aligned}
&\Big|\Big(\sum_{n=1}^{N_1} a_n\Big)\Big(\sum_{m=0}^{M_1} b_m\Big) - \sum_{n,m=1}^{\infty} a_n b_m\Big| \\
&\quad = \lim_{N_2, M_2 \to \infty} \Big|\Big(\sum_{n=1}^{N_1} a_n\Big)\Big(\sum_{m=0}^{M_1} b_m\Big) - \Big(\sum_{n=1}^{N_2} a_n\Big)\Big(\sum_{m=0}^{M_2} b_m\Big)\Big| \\
&\quad \leqslant \Big(\sum_{n=1}^{\infty} |a_n| + \sum_{m=1}^{\infty} |b_m|\Big)\Big(\sum_{n=N_1+1}^{\infty} |a_n| + \sum_{m=M_1+1}^{\infty} |b_m|\Big)
\end{aligned}
$$

which converges to zero as $N_1, M_1 \to \infty$. Therefore, we conclude that

$$\sum_{n,m=1}^{\infty} a_n b_m = \Big(\sum_{n=1}^{\infty} a_n\Big)\Big(\sum_{m=1}^{\infty} b_m\Big).$$

The claim is then concluded by passing to the limit as $M_1 \to \infty$ and then $N_1 \to \infty$.

The theorem follows from the fact that the Cauchy product is a special rearrangement of the series $\sum\limits_{n,m=1}^{\infty} a_n b_m$. □

Corollary 7.11. *Let $R_1, R_2 > 0$ be the radius of convergence of the power series $\sum\limits_{k=0}^{\infty} a_k (x-c)^k$ and $\sum\limits_{k=0}^{\infty} b_k (x-c)^k$, respectively. Then with R denoting $\min\{R_1, R_2\}$, we have*

$$\Big(\sum_{k=0}^{\infty} a_k (x-c)^k\Big)\Big(\sum_{k=0}^{\infty} b_k (x-c)^k\Big) = \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} a_k b_{n-k}\Big)(x-c)^n \qquad \forall\, x \in (c-R, c+R).$$

7.2 Power Series Solutions to Linear Differential Equations

The discussion of the power series is for the purpose of solving ODE with analytic coefficients and forcings.

Theorem 7.12 (Cauchy-Kowalevski, Special case)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, and $f : \Omega \times (t_0 - h, t_0 + h) \to \mathbb{R}^n$ be an analytic function in some neighborhood (x_0, t_0) for some $x_0 \in \Omega$; that is, for some $r > 0$,*

$$f(y,t) = f(y_0, t_0) + \sum_{k=1}^{\infty} \sum_{|\alpha|+j=k} c_{\alpha,j} (y - y_0)^{\alpha} (t - t_0)^j \qquad \forall\, (y,t) \in B\big((y_0, t_0), r\big),$$

where $\alpha = (\alpha_1, \cdots, \alpha_{\mathrm{n}})$ is a multi-index satisfying $y^{\alpha} = y_1^{\alpha_1} \cdots y_{\mathrm{n}}^{\alpha_{\mathrm{n}}}$ and $|\alpha| = \alpha_1 + \cdots \alpha_{\mathrm{n}}$. Then there exists $0 < \delta < h$ such that the ODE $y'(t) = f(y,t)$ with initial condition $y(t_0) = y_0$ has a unique analytic solution in the interval $(t_0 - \delta, t_0 + \delta)$.

Remark 7.13. If f is continuous at $(y_0 - k, y_0 + k) \times (t_0 - h, t_0 + h)$, then the general existence and uniqueness theorem guarantees the existence of a unique solution of $y'(t) = f(y,t)$ with initial condition $y(t_0) = y_0$ in some time interval $(t_0 - \delta, t_0 + \delta)$. Theorem 7.12 further implies that the solution is analytic if the “forcing” function f is analytic.

Example 7.14. Find a power series solution to $y' + 2ty = 0$.

Note that the ODE above can be written as $y' = f(y, t)$, where $f(y, t) = -2ty$. Since f is analytic at any point (y_0, t_0), the Cauchy-Kowalevski theorem implies that the solution y is analytic at any t_0. Assume that $y(t) = \sum\limits_{k=0}^{\infty} a_k t^k$ is the power series representation of the solution y at 0 with radius of convergence $R > 0$. Then Theorem 7.6 implies that

$$y'(t) = \sum_{k=1}^{\infty} k a_k t^{k-1} = \sum_{k=0}^{\infty} (k+1) a_{k+1} t^k ;$$

thus the ODE above shows that

$$\begin{aligned} 0 &= \sum_{k=0}^{\infty} (k+1) a_{k+1} t^k + 2t \sum_{k=0}^{\infty} a_k t^k = \sum_{k=0}^{\infty} (k+1) a_{k+1} t^k + 2 \sum_{k=1}^{\infty} a_{k-1} t^k \\ &= a_1 + \sum_{k=1}^{\infty} \big[(k+1) a_{k+1} + 2a_{k-1}\big] t^k . \end{aligned}$$

Therefore, $a_1 = 0$ and $(k+1)a_{k+1} + 2a_{k-1} = 0$ for all $k \in \mathbb{N}$; thus $a_1 = a_3 = \cdots = a_{2k-1} = \cdots = 0$ for all $k \in \mathbb{N}$. Moreover, the fact that $a_{k+1} = -\dfrac{2}{k+1} a_{k-1}$ implies that

$$a_{2k+2} = -\frac{1}{k+1} a_{2k} \qquad \forall\, k \in \mathbb{N} \cup \{0\} ;$$

thus $a_{2k} = \dfrac{(-1)^k}{k!} a_0$ for all $k \in \mathbb{N}$. As a consequence,

$$y(t) = \sum_{k=0}^{\infty} a_{2k} t^{2k} = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^{2k} \Big(= a_0 \sum_{k=0}^{\infty} \frac{(-t^2)^k}{k!} = a_0 e^{-t^2} \Big) .$$

In the remaining chapter we focus on the second order linear homogeneous ODE

$$P(x) \frac{d^2 y}{dx^2} + Q(x) \frac{dy}{dx} + R(x) y = 0 , \tag{7.2}$$

where P, Q, R are assumed to have no common factors. **We note that we change the independent variable from t to x.**

Definition 7.15. A point x_0 is said to be a ***ordinary point*** to ODE (7.2) if $P(x_0) \neq 0$, and the two functions Q/P, R/P are analytic at x_0. It is called a ***singluar point*** if it is not a regular point. It is called a ***regular singular point*** if the two limits

$$\lim_{x \to x_0} (x - x_0) \frac{Q(x)}{P(x)} \quad \text{and} \quad \lim_{x \to x_0} (x - x_0)^2 \frac{R(x)}{P(x)}$$

both exist and are finite. Any singular point that is not a regular singular point is called an ***irregular singular point***.

Example 7.16. 1 is the only singular point for the ODE $xy'' + x(1-x)^{-1} y' + (\sin x) y = 0$.

If x_0 is a regular point to ODE (7.2), then

$$y'' + p(x)y' + q(x)y = 0 \tag{7.3}$$

for some function p and q that are analytic at x_0. Write $y' = z$. Then the vector $\boldsymbol{w} = (y, z)$ satisfies

$$\boldsymbol{w}' = \frac{d}{dx}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} z \\ -p(x)z - q(x)y \end{bmatrix} \equiv f(\boldsymbol{w}, x)\,.$$

It is clear that f is analytic at x_0 if p, q are analytic at x_0; thus the Cauchy-Kowalevski theorem implies that any solutions to (7.3) are analytic. To be more precise, we have the following

Theorem 7.17. *Suppose x_0 is an ordinary point for equation* (7.2). *Then equation* (7.2) *has two linearly independent analytic solutions of the form*

$$y(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k\,.$$

Moreover, the radius of convergence of any power series solution of the form given above is at least as large as the distance from x_0 to the nearest singular point (*possibly a complex number*) *of equation* (7.3).

Example 7.18. The radius of convergence of series solutions about any point $x = x_0$ of the ODE

$$y'' + (\sin x)y' + (1 + x^2)y = 0$$

is infinite; that is, for any $x_0 \in \mathbb{R}$, series solutions about $x = x_0$ of the ODE above converge for all $x \in \mathbb{R}$.

Example 7.19. Find a lower bound for the radius of convergence of series solutions about $x = 0$ of the Legendre equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0\,.$$

Since there are two singular points ± 1, the radius of convergence of the series solution about 0 of the Legendre equation is at least 1. We also note that ± 1 are both regular singular point of the Legendre equation.

Example 7.20. Find a lower bound for the radius of convergence of series solutions about $x = 0$ or about $x = -\frac{1}{2}$ of the ODE

$$(1 + x^2)y'' + 2xy' + 4x^2 y = 0\,.$$

Since there are two singular points, $\pm i$, of the ODE, the radius of convergence of the power series solution about 0 of the ODE is at least 1.

Next, consider the power series solution about $-\frac{1}{2}$. Since the distance between $-\frac{1}{2}$ and $\pm i$ are $\frac{\sqrt{5}}{2}$, the radius of convergence of a power series solution about $-\frac{1}{2}$ is at least $\frac{\sqrt{5}}{2}$.

7.3 Series Solutions Near an Ordinary Point: Part I

In this section, we provide several examples showing how to apply the method of power series to solve ODE (or IVP).

Example 7.21. Find the general solution to the ODE $2y'' + xy' + y = 0$ in the form of a power series about the ordinary point $x = 0$.

Suppose that the solution can be written as $y = \sum\limits_{k=0}^{\infty} a_k x^k$. Then Theorem 7.6 implies that

$$y' = \sum_{k=1}^{\infty} k a_k x^{k-1} \quad \text{and} \quad y'' = \sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} = \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} x^k \,;$$

thus

$$\sum_{k=0}^{\infty} \big[2(k+2)(k+1)a_{k+2} + k a_k + a_k\big] x^k = 0 \,.$$

Therefore, $2(k+2)(k+1)a_{k+2} + (k+1)a_k = 0$ for all $k \in \mathbb{N} \cup \{0\}$ or equivalently,

$$a_{k+2} = -\frac{1}{2(k+2)} a_k \qquad \forall\, k \in \mathbb{N} \cup \{0\} \,.$$

For even $k = 2n$,

$$a_{2n} = -\frac{1}{2n} a_{2n-2} = \frac{1}{2^2 n(2n-2)} a_{2n-4} = \cdots = \frac{(-1)^n}{2^{2n} n!} a_0 \qquad \forall\, n \in \mathbb{N} \,,$$

while for odd $k = 2n+1$,

$$\begin{aligned} a_{2n+1} &= -\frac{1}{2(2n+1)} a_{2n-1} = \frac{1}{2^2(2n+1)(2n-1)} a_{2n-3} = \cdots = \frac{(-1)^n}{2^n (2n+1)(2n-1)\cdots 3} a_1 \\ &= \frac{(-1)^n n!}{(2n+1)!} a_1 \qquad \forall\, n \in \mathbb{N} \,. \end{aligned}$$

Therefore,

$$y = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} n!} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n n!}{(2n+1)!} x^{2n+1} \,.$$

The radius of convergence of the power series given above is also infinite and this coincides with the conclusion in Theorem 7.17.

Note that the function $\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} n!} x^{2n}$ is indeed the function $\exp\big(-\frac{x^2}{4}\big)$; thus using the method of reduction of order, we find that another linearly independent solution is $\exp\big(-\frac{x^2}{4}\big) \int \exp\big(\frac{x^2}{4}\big)\, dx$.

Example 7.22. Find the general solution to Airy's equation $y'' - xy = 0$ in the form of a power series about the ordinary point $x = 0$.

Suppose that the solution can be written as $y = \sum\limits_{k=0}^{\infty} a_k x^k$. Then

$$y'' = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k \,;$$

and

$$xy = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1} x^k\,.$$

Therefore,

$$a_2 + \sum_{k=1}^{\infty} \big[(k+2)(k+1)a_{k+2} - a_{k-1}\big] x^k = 0$$

which implies that $a_2 = 0$ and $a_{k+2} = \dfrac{a_{k-1}}{(k+2)(k+1)}$ for all $k \in \mathbb{N}$. The recurrence relation further implies that $a_5 = a_8 = a_{11} = \cdots = a_{3k-1} = \cdots = 0$ for all $k \in \mathbb{N}$. Furthermore, we have

$$\begin{aligned}
a_{3k} &= \frac{a_{3k-3}}{(3k)(3k-1)} = \frac{a_{3k-6}}{(3k)(3k-1)(3k-3)(3k-4)} = \cdots \\
&= \frac{a_0}{(3k)(3k-1)(3k-3)(3k-4)\cdots 3\cdot 2} = \frac{(3k-2)(3k-5)\cdots 4\cdot 1 a_0}{(3k)!} \\
&= \frac{3^k\big(k-\frac{2}{3}\big)\big(k-\frac{5}{3}\big)\cdots\frac{1}{3}a_0}{(3k)!} = \frac{3^k\Gamma(k+1/3)}{\Gamma(1/3)(3k)!}a_0
\end{aligned}$$

and

$$\begin{aligned}
a_{3k+1} &= \frac{a_{3k-2}}{(3k+1)(3k)} = \frac{a_{3k-5}}{(3k+1)(3k)(3k-2)(3k-3)} = \cdots \\
&= \frac{a_1}{(3k+1)(3k)(3k-2)(3k-3)\cdots 4\cdot 3} = \frac{(3k-1)(3k-4)\cdots 2a_1}{(3k+1)!} \\
&= \frac{3^k\big(k-\frac{1}{3}\big)\big(k-\frac{4}{3}\big)\cdots\frac{2}{3}a_1}{(3k+1)!} = \frac{3^k\Gamma(k+2/3)}{\Gamma(2/3)(3k+1)!}a_1\,.
\end{aligned}$$

Therefore, the solution of Airy's equation is of the form

$$y = a_0 \sum_{k=0}^{\infty} \frac{3^k\Gamma(k+1/3)}{\Gamma(1/3)(3k)!} x^{3k} + a_1 \sum_{k=0}^{\infty} \frac{3^k\Gamma(k+2/3)}{\Gamma(2/3)(3k+1)!} x^{3k+1}\,.$$

Example 7.23. In this example, instead of considering a series solution of Airy's equation $y''-xy=0$ of the form $y = \sum\limits_{k=0}^{\infty} a_k x^k$, we look for a solution of the form $y = \sum\limits_{k=0}^{\infty} a_k (x-1)^k$.

Since

$$y'' = \sum_{k=2}^{\infty} k(k-1)a_k(x-1)^{k-2} = \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}(x-1)^k$$

and

$$xy = (x-1)y + y = \sum_{k=0}^{\infty} a_k(x-1)^{k+1} + \sum_{k=0}^{\infty} a_k(x-1)^k = \sum_{k=1}^{\infty} a_{k-1}(x-1)^k + \sum_{k=0}^{\infty} a_k(x-1)^k\,,$$

we have

$$(2a_2 - a_0) + \big[6a_3 - (a_1+a_0)\big](x-1) + \sum_{k=2}^{\infty}\big[(k+2)(k+1)a_{k+2} - (a_{k-1}+a_k)\big](x-1)^k = 0\,.$$

Therefore, $2a_2 = a_0$, $6a_3 = a_1 + a_0$, $12a_4 = a_2 + a_1$, $20a_5 = a_3 + a_2$, and in general,

$$(k+2)(k+1)a_{k+2} = a_{k+1} + a_k\,.$$

Solving for a few terms, we find that

$$a_2 = \frac{1}{2}a_0\,, \quad a_3 = \frac{1}{6}a_0 + \frac{1}{6}a_1\,, \quad a_4 = \frac{1}{24}a_0 + \frac{1}{12}a_1\,, \quad a_5 = \frac{1}{30}a_0 + \frac{1}{120}a_1\,, \quad \cdots$$

It seems not possible to find a general form the the series solution. Nevertheless, we have

$$\begin{aligned} y = a_0\Big[1 + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{24} + \frac{(x-1)^5}{30} + \cdots\Big] \\ + a_1\Big[(x-1) + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} + \frac{(x-1)^5}{120} + \cdots\Big]\,. \end{aligned}$$

7.4 Series Solution Near an Ordinary Point: Part II

There is another way to computed the coefficients a_k of the series solution to ODE (7.2). The idea is to differentiate the equation (7.2) k-times and then evaluate at an ordinary point x_0 so that $y^{(k+2)}(x_0)$ can be obtained once $y^{(j)}(x_0)$'s are known for $0 \leqslant j \leqslant k+1$. To be more precise, we differentiate (7.2) k-times and use the Leibniz rule to obtain that

$$P(x_0)y^{(k+2)}(x_0) + \sum_{j=0}^{k-1} C^k_j P^{(k-j)}(x_0)y^{(j+2)}(x_0) + \sum_{j=0}^{k} C^k_j \big(Q^{(k-j)}(x_0)y^{(1+j)}(x_0) + R^{(k-j)}(x_0)y^{(j)}(x_0)\big) = 0\,;$$

thus

$$\begin{aligned} P(x_0)y^{(k+2)}(x_0) &= -\sum_{j=2}^{k+1} C^k_{j-2}P^{(k-j+2)}(x_0)y^{(j)}(x_0) - \sum_{j=1}^{k+1} C^k_{j-1}Q^{(k-j+1)}(x_0)y^{(j)}(x_0) \\ &\quad - \sum_{j=0}^{k} C^k_j R^{(k-j)}(x_0)y^{(j)}(x_0) \\ &= -\big[kP'(x_0) + Q(x_0)\big]y^{(k+1)}(x_0) - \big[Q^{(k)}(x_0) + kR^{(k-1)}(x_0)\big]y'(x_0) - R^{(k)}(x_0)y(x_0) \\ &\quad - \sum_{j=2}^{k} \big[C^k_{j-2}P^{(k-j+2)}(x_0) + C^k_{j-1}Q^{(k-j+1)}(x_0) + C^k_j R^{(k-j)}(x_0)\big]y^{(j)}(x_0)\,. \end{aligned}$$

The recurrence relation above can be used to obtain the coefficients $a_{k+2} = \dfrac{y^{(k+2)}(x_0)}{(k+2)!}$ of the series solution $y = \sum\limits_{k=0}^{\infty} a_k(x-x_0)^k$ to (7.2) once $y^{k+1}(x_0), \cdots, f(x_0)$ are known.

Example 7.24. Find the series solution about 1 of Airy's equation $y'' - xy = 0$.

Assume that the series solution is $y = \sum\limits_{k=0}^{\infty} a_k(x-1)^k$. First, we know that $y''(1) - y(1) = 0$. Since $y(1) = a_0$, we know that $a_2 = \dfrac{y''(1)}{2} = \dfrac{a_0}{2}$. Differentiating Airy's equation k-times, we find that

$$y^{(k+2)} - xy^{(k)} - ky^{(k-1)} = 0\,;$$

thus

$$(k+2)!a_{k+2} = y^{(k+2)}(1) = y^{(k)}(1) + ky^{(k-1)} = k!a_k + k(k-1)!a_{k-1} = k!(a_k + a_{k-1})\,.$$

Therefore, $(k+2)(k+1)a_{k+2} = a_k + a_{k-1}$ which is exactly what we use to obtain the series solution about 1 to Airy's equation.

7.5 Cauchy-Euler (Equi-Dimensional) Equations

In this section we consider the Cauchy-Euler (or simply Euler) equation

$$L[y](x) \equiv x^2 y'' + \alpha x y' + \beta y = 0\,. \tag{7.4}$$

Note that $x_0 = 0$ is a regular singular point of (7.4).

Assume that we only consider the solution of the Euler equation in the region $x > 0$. Let $z(t) = y(e^t)$. Then $z'(t) = y'(e^t)e^t$ and $z''(t) = y''(e^t)e^{2t} + y'(e^t)e^t$ which implies that $y''(e^t)e^{2t} = z''(t) - z'(t)$. Therefore,

$$z''(t) + (\alpha - 1)z'(t) + \beta z(t) = 0\,. \tag{7.5}$$

This is a second order ODE with constant coefficients, and can be solved by looking at the multiplicity and complexity of the roots of the characteristic equation

$$r^2 + (\alpha - 1)r + \beta = 0\,. \tag{7.6}$$

We note that (7.6) can also be written as $r(r-1) + \alpha r + \beta = 0$, and is called the ***indicial equation***.

1. Suppose the roots of the characteristic equation are distinct real numbers r_1 and r_2. Then the solution to (7.5) is $z(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$; thus the solution to the Euler equation is
$$y(x) = C_1 e^{r_1 \log x} + C_2 e^{r_2 \log x} = C_1 x^{r_1} + C_2 x^{r_2}\,.$$
2. Suppose the characteristic equation has a real double root r. Then the solution to (7.5) is $z(t) = (C_1 t + C_2)e^{rt}$; thus the solution to the Euler equation is
$$y(x) = (C_1 \log x + C_2)e^{r \log x} = (C_1 \log x + C_2)x^r\,.$$
3. Suppose the roots of the characteristic equation are complex numbers $r_1 = a+bi$ and $r_2 = a-bi$. Then the solution to (7.5) is $z(t) = C_1 e^{at}\cos(bt) + C_2 e^{at}\sin(bt)$; thus the solution to the Euler equation is
$$y(x) = C_1 e^{a \log x}\cos(b \log x) + C_1 e^{a \log x}\sin(b \log x) = C_1 x^a \cos(b \log x) + C_2 x^a \sin(b \log x)\,.$$

Now we consider the solution to (7.4) in the region $x < 0$. We then let $z(x) = y(-x)$ and find that z satisfies also satisfies the same Euler equation; that is,

$$x^2 z'' + \alpha x z' + \beta z = 0\,.$$

We can then solve for z by looking at the multiplicity and complexity of the roots of the characteristic equation, and conclude that

1. Case 1 - Distinct real roots r_1 and r_2:
$$y(x) = C_1 |x|^{r_1} + C_2 |x|^{r_2}\,.$$
2. Case 2 - Double real root r:
$$y(x) = (C_1 |x| + C_2)|x|^r\,.$$
3. Case 3 - Complex roots $a \pm bi$:
$$y(x) = C_1 |x|^{at}\cos(b \log |x|) + C_2 |x|^{at}\sin(b \log |x|)\,.$$

7.5.1 Another way to find solutions to the Cauchy-Euler equations

Assume that the solution is of the form $y(x) = x^r$. Then

$$x^2 r(r-1)x^{r-2} + \alpha x r x^{r-1} + \beta x^r = 0\,.$$

Therefore, r satisfies the indicial equation (7.5).

1. If the indicial equation (7.5) has two distinct real roots r_1 and r_2, then x^{r_1} and x^{r_2} are linearly independent solutions to the Euler equation. Therefore, the solution to the Euler equation (when the indicial equation has two distinct real roots) is given by

$$y(x) = C_1 x^{r_1} + C_2 x^{r_2}\,.$$

2. If the indicial equation (7.5) has two distinct complex roots $a \pm bi$, then x^{a+bi} and x^{a-bi} are linearly independent solutions to the Euler equation. Using the Euler identity,

$$x^{a\pm bi} = e^{(a\pm bi)\log x} = e^{a\log x \pm b\log x i} = x^a\big[\cos(b\log x) \pm i\sin(b\log x)\big]\,;$$

thus the general solution to the Euler equation (when the indicial equation has two comnplex roots) is given by

$$y(x) = C_1 x^a \cos(b\log x) + C_2 x^a \sin(b\log x)\,.$$

3. Suppose that the indicial equation (7.5) has a double root r_0. Then $\beta = \dfrac{(\alpha-1)^2}{4}$, and $x^{\frac{1-\alpha}{2}}$ is a solution to the Euler equation. Suppose that $\beta \neq 0$ (so that the equation does not reduces to a first order one).

   (a) The method of reduction of order: suppose that another linearly independent solution is given by $y(x) = u(x)x^{\frac{1-\alpha}{2}}$ for some function u. Then u satisfies

$$x^2\Big[u'' x^{\frac{1-\alpha}{2}} + (1-\alpha)u' x^{\frac{-1-\alpha}{2}} + \frac{1-\alpha}{2}\frac{-1-\alpha}{2} u x^{\frac{-3-\alpha}{2}}\Big] + \alpha x\Big[u' x^{\frac{1-\alpha}{2}} + \frac{1-\alpha}{2} u x^{\frac{-1-\alpha}{2}}\Big] + \frac{(\alpha-1)^2}{4} u x^{\frac{1-\alpha}{2}} = 0$$

   which can be simplified as

$$xu'' + u' = 0\,.$$

   Therefore, $u'(x) = \dfrac{C}{x}$ which further implies that $u(x) = C\log x + D$. Therefore, another solution is given by $x^{\frac{1-\alpha}{2}}\log x$.

   (b) Let $w(r,x) = x^r$. Then $L[w(r,\cdot)](x) = (r-r_0)^2 x^r$ for all $r \neq 0$. Taking the partial derivative with respect to r variable, we find that

$$\frac{\partial}{\partial r} L[w(r,\cdot)](x) = \frac{\partial}{\partial r}\Big[x^2\frac{\partial^2 w}{\partial x^2} + \alpha x\frac{\partial w}{\partial x} + \frac{(\alpha-1)^2}{4} w\Big] = 2(r-r_0)x^r + (r-r_0)x^r\log x\,.$$

Since the mixed partial derivatives are continuous for all $x, r \neq 0$, we find that $\dfrac{\partial^3 w}{\partial r \partial x^2} = \dfrac{\partial^3 w}{\partial x^2 \partial r}$ and $\dfrac{\partial^2 w}{\partial r \partial x} = \dfrac{\partial^2 w}{\partial x \partial r}$; thus

$$\begin{aligned} 0 &= \frac{\partial}{\partial r}\Big|_{r=r_0} L[w(r,\cdot)](x) = \Big[x^2 \frac{\partial^3 w}{\partial x^2 \partial r} + \alpha x \frac{\partial^2 w}{\partial x \partial r} + \frac{(\alpha-1)^2}{4}\frac{\partial w}{\partial r}\Big](r_0, x) \\ &= L\big[\frac{\partial w}{\partial r}(r_0,\cdot)\big](x)\,. \end{aligned}$$

In other words, $\dfrac{\partial w}{\partial r}(r_0,\cdot)$ is also a solution to the Euler equation. Since $\dfrac{\partial w}{\partial r}(r,x) = x^r \log x$, we find that another linearly independent solution to the Euler equation (when the indicial equation has a double root) is given by $x^{r_0}\log x$.

7.6 Series Solutions Near a Regular Singular Point: Part I

Suppose that x_0 is a regular singular point of the ODE

$$P(x)y'' + Q(x)y' + R(x)y = 0 \qquad x > x_0\,; \tag{7.2}$$

that is, $P(x_0) = 0$, and both limits

$$\lim_{x\to x_0}(x-x_0)\frac{Q(x)}{P(x)} \qquad \text{and} \qquad \lim_{x\to x_0}(x-x_0)^2\frac{R(x)}{P(x)}$$

exist. Suppose that the functions $p(x) \equiv (x-x_0)\dfrac{Q(x)}{P(x)}$ and $q(x) \equiv (x-x_0)^2\dfrac{R(x)}{P(x)}$ are analytic at x_0; that is,

$$(x-x_0)\frac{Q(x)}{P(x)} = \sum_{k=0}^{\infty} p_k (x-x_0)^k \quad \text{and} \quad (x-x_0)^2\frac{R(x)}{P(x)} = \sum_{k=0}^{\infty} q_k (x-x_0)^k$$

in some interval $(x_0 - R, x_0 + R)$. Then by multiplying both side of (7.2) by $\dfrac{(x-x_0)^2}{P(x)}$, we obtain that

$$\begin{aligned} &(x-x_0)^2 y'' + (x-x_0)p(x)y' + q(x)y \\ &\quad = (x-x_0)^2 y'' + (x-x_0)\Big(\sum_{k=0}^{\infty} p_k (x-x_0)^k\Big)y' + \Big(\sum_{k=0}^{\infty} q_k (x-x_0)^k\Big)y = 0\,. \end{aligned} \tag{7.7}$$

We note that if $x_0 = 0$ and $p_k = q_k = 0$ for all $k \in \mathbb{N}$, the equation above is the Euler equation

$$x^2 y'' + p_0 x y' + q_0 y = 0 \tag{7.8}$$

that we discussed in previous section. Therefore, for x near 0, it is "reasonable" to expect that the solution to (7.7) will behave like the solution to the Euler equation

$$x^2 y'' + p_0 x y' + q_0 y = 0\,.$$

The idea due to Frobenius of solving (7.7) is that the solution of (7.7) should be of the form x^r times an analytic function.

To be more precise, the ***method of Frobenius*** provides a way to derive a series solution to (7.2) about the regular singular point x_0:

1. Suppose that a series solution can be written as

$$y(x) = (x - x_0)^r \sum_{k=0}^{\infty} a_k (x - x_0)^k = \sum_{k=0}^{\infty} a_k (x - x_0)^{k+r}$$

for some r and $\{a_k\}_{k=0}^{\infty}$ to be determined. Substitute y into (7.2) to obtain an equation of the form

$$A_0 (x - x_0)^{r+J} + A_1 (x - x_0)^{r+J+1} + A_2 (x - x_0)^{r+J+2} + \cdots = 0 .$$

2. Set $A_0 = A_1 = A_2 = \cdots = 0$. Note that $A_0 = 0$ should correspond to the indicial equation

$$F(r) = r(r-1) + p_0 r + q_0 = 0 ,$$

where $p_0 = \lim_{x \to x_0} (x - x_0) \dfrac{Q(x)}{P(x)}$ and $q_0 = \lim_{x \to x_0} (x - x_0)^2 \dfrac{R(x)}{P(x)}$.

3. Use the system of equations $A_0 = A_1 = A_2 = \cdots = 0$ to find a recurrence relation involving a_k and $a_0, a_1, \cdots, a_{k-1}$.

W.L.O.G., we can assume that $x_0 = 0$ (otherwise make a change of variable $\widetilde{x} = x - x_0$), and only focus the discussion of the solution in the region $x > 0$. Due to the method of Frobenius, we look for solutions of (7.7) of the form

$$y(x) = x^r \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k x^{k+r} , \qquad x > 0 , \tag{7.9}$$

where a_0 is assumed to be non-zero (otherwise we replace r by $1 + r$ if $a_1 \neq 0$). Since

$$y'(x) = r x^{r-1} \sum_{k=0}^{\infty} a_k x^k + x^r \sum_{k=0}^{\infty} k a_k x^{k-1} = \sum_{k=0}^{\infty} (k+r) a_k x^{k+r-1}$$

and accordingly,

$$y''(x) = \sum_{k=0}^{\infty} (k+r)(k+r-1) a_k x^{k+r-2} ,$$

we obtain

$$\sum_{k=0}^{\infty} (k+r)(k+r-1) a_k x^{k+r} + \Big(\sum_{k=0}^{\infty} p_k x^k \Big) \Big(\sum_{k=0}^{\infty} (k+r) a_k x^{k+r} \Big) + \Big(\sum_{k=0}^{\infty} q_k x^k \Big) \Big(\sum_{k=0}^{\infty} a_k x^{k+r} \Big) = 0 ,$$

or cancelling x^r,

$$\sum_{k=0}^{\infty} (k+r)(k+r-1) a_k x^k + \Big(\sum_{k=0}^{\infty} p_k x^k \Big) \Big(\sum_{k=0}^{\infty} (k+r) a_k x^k \Big) + \Big(\sum_{k=0}^{\infty} q_k x^k \Big) \Big(\sum_{k=0}^{\infty} a_k x^k \Big) = 0 .$$

Using the Cauchy product, we further conclude that

$$\sum_{k=0}^{\infty} (k+r)(k+r-1) a_k x^k + \sum_{k=0}^{\infty} \Big(\sum_{j=0}^{k} (j+r) a_j p_{k-j} \Big) x^k + \sum_{k=0}^{\infty} \Big(\sum_{j=0}^{k} q_{k-j} a_j \Big) x^k = 0 .$$

Therefore, we obtain the following recurrence relation:

$$(k+r)(k+r-1)a_k + \sum_{j=0}^{k}(j+r)a_j p_{k-j} + \sum_{j=0}^{k} q_{k-j}a_j = 0 \qquad \forall\, k \in \mathbb{N}\cup\{0\}\,. \tag{7.10}$$

Therefore, with F denoting the function $F(r) = r(r-1) + rp_0 + q_0$, we have

$$F(r+k)a_k + \sum_{j=0}^{k-1}\big[(j+r)p_{k-j} + q_{k-j}\big]a_j = 0 \qquad \forall\, k \in \mathbb{N}\,. \tag{7.11}$$

The case $k = 0$ in (7.10) induces the following

Definition 7.25. If x_0 is a regular singular point of (7.2), then the ***indicial equation*** for the regular singular point x_0 is

$$r(r-1) + p_0 r + q_0 = 0\,, \tag{7.12}$$

where $p_0 = \lim\limits_{x\to x_0}(x-x_0)\dfrac{Q(x)}{P(x)}$ and $q_0 = \lim\limits_{x\to x_0}(x-x_0)^2\dfrac{R(x)}{P(x)}$. The roots of the indicial equation are called the ***exponents*** (***indices***) of the singularity x_0.

Now assume that r_1, r_2 are roots of the indicial equations for a regular singular point x_0.

1. If $r_1, r_2 \in \mathbb{R}$ and $r_1 > r_2$. Since F only has two roots, $F(k + r_1) \neq 0$ for all $k \in \mathbb{N}$. Therefore, for $r = r_1$, (7.11) indeed is a recurrence relation which implies that a_k depends on $a_0, \cdots, a_{k-1}$ and this, in principle, provides a series solution

$$y_1(x) = x^{r_1}\Big[1 + \sum_{k=1}^{\infty}\frac{a_k(r_1)}{a_0}x^k\Big] \tag{7.13}$$

   to (7.7), in which $a_k(r_1)$ denotes the coefficients when $r = r_1$.

   (a) If in addition $r_2 \neq r_1$ and $r_1 - r_2 \notin \mathbb{N}$, the $F(k + r_2) \neq 0$ for all $k \in \mathbb{N}$; thus for $r = r_2$, (7.11) is also a recurrence relation, and this provides another series solution

$$y_2(x) = x^{r_2}\Big[1 + \sum_{k=1}^{\infty}\frac{a_k(r_2)}{a_0}x^k\Big]\,. \tag{7.14}$$

   (b) If $r_1 = r_2$ or $r_1 - r_2 \in \mathbb{N}$, we will discuss later in the next section.

2. If r_1, r_2 are complex roots, then $r_1 - r_2 \notin \mathbb{N}$ and $F(k + r) \neq 0$ for all $k \in \mathbb{N}$ for $r = r_1, r_2$. Since

$$x^{a+bi} = x^a \cdot x^{bi} = x^a e^{ib\log x} = x^a\big[\cos(b\log x) + i\sin(b\log x)\big]\,,$$

   (7.13) and (7.14) provide two solutions of (7.7) or equivalently, the general solution to (7.2) in series form is given by

$$\begin{aligned} y(x) &= C_1(x-x_0)^a\big[\cos(b\log(x-x_0)) + i\sin(b\log(x-x_0))\big]\sum_{k=0}^{\infty}a_k(r_1)(x-x_0)^k \\ &\quad + C_2(x-x_0)^a\big[\cos(b\log(x-x_0)) - i\sin(b\log(x-x_0))\big]\sum_{k=0}^{\infty}a_k(r_2)(x-x_0)^k\,. \end{aligned}$$

In the following discussion, we will only focus on the case that the indicial equation has only real roots.

Example 7.26. Solve the differential equation

$$2x^2y'' - xy' + (1+x)y = 0 \qquad x > 0\,. \tag{7.15}$$

We note that 0 is a regular singular point of the ODE above; thus we look for a series solution to the ODE above of the form

$$y(x) = x^r \sum_{k=0}^{\infty} a_k x^k\,.$$

Then r satisfies the indicial equation for 0

$$2r(r-1) - r + 1 = 0$$

which implies that $r = 1$ or $r = \frac{1}{2}$. Since

$$y'(x) = \sum_{k=0}^{\infty} (k+r)a_k x^{k+r-1} \quad \text{and} \quad y''(x) = \sum_{k=0}^{\infty} (k+r)(k+r-1)a_k x^{k+r-2}\,,$$

we obtain that

$$\sum_{k=0}^{\infty} \big[2(k+r)(k+r-1) - (k+r) + 1\big] a_k x^{k+r} + \sum_{k=0}^{\infty} a_k x^{k+r+1} = 0$$

or cancelling x^r,

$$\sum_{k=0}^{\infty} \big[2(k+r)(k+r-1) - (k+r) + 1\big] a_k x^{k} + \sum_{k=1}^{\infty} a_{k-1} x^{k} = 0\,.$$

Therefore,

$$a_k = -\frac{a_{k-1}}{2(k+r)(k+r-1) - (k+r) + 1} \qquad \forall\, k \in \mathbb{N}\,.$$

1. $r = 1$: $a_k = -\dfrac{a_{k-1}}{k(2k+1)}$ for all $k \in \mathbb{N}$. Therefore,

$$\begin{aligned} a_k &= -\frac{a_{k-1}}{k(2k+1)} = \frac{a_{k-2}}{k(k-1)(2k+1)(2k-1)} = -\frac{a_{k-3}}{k(k-1)(k-2)(2k+1)(2k-1)(2k-3)} \\ &= \frac{(-1)^k}{k!(2k+1)(2k-1)\cdots 1} a_0 = \frac{(2k)(2k-2)(2k-4)\cdots 2(-1)^k}{k!(2k+1)!} a_0 = \frac{(-1)^k 2^k}{(2k+1)!} a_0\,. \end{aligned}$$

   This provides a series solution $y_1(x) = \sum\limits_{k=0}^{\infty} \dfrac{(-1)^k 2^k}{(2k+1)!} x^{k+1}$ whose radius of convergence is ∞.

2. $r = \frac{1}{2}$: $a_k = -\dfrac{a_{k-1}}{k(2k-1)}$ for all $k \in \mathbb{N}$. Therefore,

$$\begin{aligned} a_k &= -\frac{a_{k-1}}{k(2k-1)} = \frac{a_{k-2}}{k(k-1)(2k-1)(2k-3)} = -\frac{a_{k-3}}{k(k-1)(k-2)(2k-1)(2k-3)(2k-5)} \\ &= \frac{(-1)^k}{k!(2k-1)(2k-3)\cdots 1} a_0 = \frac{(-1)^k(2k)(2k-2)\cdots 2}{k!(2k)!} a_0 = \frac{(-1)^k 2^k}{(2k)!} a_0\,. \end{aligned}$$

   This provides a series solution $y_2(x) = \sum\limits_{k=0}^{\infty} \dfrac{(-1)^k 2^k}{(2k)!} x^{k+\frac{1}{2}}$ whose radius of convergence is ∞.

Therefore, the general solution to (7.15) in the series form is $y = C_1 y_1(x) + C_2 y_2(x)$.

Example 7.27. Find a series solution about the regular singular point $x = 0$ of

$$(x+2)x^2 y''(x) - xy'(x) + (1+x)y(x) = 0\,, \qquad x > 0\,.$$

Let $p(x) = -\dfrac{1}{x+2}$ and $q(x) = \dfrac{1+x}{x+2}$. Then

$$p(x) = -\frac{1}{2}\frac{1}{1-\frac{-x}{2}} = -\frac{1}{2}\sum_{k=0}^{\infty}\frac{(-x)^k}{2^k} = \sum_{k=0}^{\infty}\frac{(-1)^{k+1}x^k}{2^{k+1}} = -\frac{1}{2} + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}x^k}{2^{k+1}}\,,$$

$$q(x) = \frac{x+1}{x+2} = 1 - \frac{1}{2}\frac{1}{1-\frac{-x}{2}} = 1 - \sum_{k=0}^{\infty}\frac{(-1)^k x^k}{2^{k+1}} = \frac{1}{2} + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}x^k}{2^{k+1}}\,.$$

Therefore, $(p_0, q_0) = \left(-\frac{1}{2}, \frac{1}{2}\right)$, and $p_k = q_k = \dfrac{(-1)^{k+1}}{2^{k+1}}$ for all $k \in \mathbb{N}$. The indicial equation for 0 is

$$r(r-1) - \frac{1}{2}r + \frac{1}{2} = 0$$

which implies that $r = 1$ or $r = \dfrac{1}{2}$.

1. $r = 1$: Suppose the series solution to the ODE is $y = x\sum\limits_{k=0}^{\infty} a_k x^k = \sum\limits_{k=0}^{\infty} a_k x^{k+1}$. Then

$$(x+2)x^2\sum_{k=0}^{\infty}(k+1)ka_k x^{k-1} - x\sum_{k=0}^{\infty}(k+1)a_k x^k + (1+x)\sum_{k=0}^{\infty}a_k x^{k+1} = 0$$

$$\Rightarrow \sum_{k=0}^{\infty}(k^2+k+1)a_k x^{k+2} + \sum_{k=0}^{\infty}(2k^2+k)a_k x^{k+1} = 0$$

$$\Rightarrow \sum_{k=1}^{\infty}\Big([(k-1)^2 + (k-1) + 1]a_{k-1} + (2k^2+k)a_k\Big)x^{k+1} = 0\,.$$

Therefore, $a_k = -\dfrac{k^2-k+1}{(2k+1)k}\,a_{k-1}$ for all $k \in \mathbb{N}$. Note that

$$\lim_{k\to\infty}\left|\frac{a_k}{a_{k-1}}\right| = \lim_{k\to\infty}\left|\frac{k^2-k+1}{k(2k+1)}\right| = \frac{1}{2}\,;$$

thus the radius of convergence of the series solution $y = \sum\limits_{k=0}^{\infty} a_k x^{k+1}$ is 2.

2. $r = \dfrac{1}{2}$: Suppose the series solution to the ODE is $y = x^{\frac{1}{2}}\sum\limits_{k=0}^{\infty} a_k x^k = \sum\limits_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}$. Then

$$(x+2)\sum_{k=0}^{\infty}\left(k+\frac{1}{2}\right)\left(k-\frac{1}{2}\right)a_k x^{k+\frac{1}{2}} - \sum_{k=0}^{\infty}\left(k+\frac{1}{2}\right)a_k x^{k+\frac{1}{2}} + \sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}} + \sum_{k=0}^{\infty}a_k x^{k+\frac{3}{2}} = 0$$

$$\Rightarrow \sum_{k=0}^{\infty}\left(k^2+\frac{3}{4}\right)a_k x^{k+\frac{3}{2}} + \sum_{k=0}^{\infty}(2k^2-k)a_k x^{k+\frac{1}{2}} = 0$$

$$\Rightarrow \sum_{k=0}^{\infty}\Big(\Big((k-1)^2+\frac{3}{4}\Big)a_{k-1} + (2k^2-k)a_k\Big)x^{k+\frac{1}{2}} = 0\,.$$

Therefore, $a_k = -\dfrac{(k-1)^2+3/4}{k(2k-1)}\,a_{k-1}$ for all $k \in \mathbb{N}$. The radius of convergence of this series solution is also 2.

7.7 Series Solutions Near a Regular Singular Point: Part II

Suppose that r_1 and r_2 are the roots of the indicial equation for a regular singular point. In this section, we discuss how a linearly independent solution y_2 is obtained if $r_1 - r_2 \in \mathbb{N} \cup \{0\}$. In the following, we let $\mathcal{N}(r_1, r_2)$ denote the discrete set

$$\mathcal{N}(r_1, r_2) = \{r - k \,|\, r = r_1 \text{ or } r_2, k \in \mathbb{N}\}\,.$$

Then $F(k+r) \neq 0$ for all $k \in \mathbb{N}$ and $r \notin \mathcal{N}(r_1, r_2)$; thus for some given a_0 the recurrence relation

$$F(k+r)a_k(r) = -\sum_{j=0}^{k-1}\big[(j+r)p_{k-j} + q_{k-j}\big]a_j(r) \qquad \forall\, k \in \mathbb{N} \tag{7.16}$$

can be used to determine a sequence $\{a_k(r)\}_{k=1}^{\infty}$.

7.7.1 The case the indicial equation has a double root

Suppose that $r_1 = r_2$. For $r \notin \mathcal{N}(r_1, r_2)$, we define

$$\varphi(r, x) = x^r \sum_{k=0}^{\infty} a_k(r)x^k\,,$$

where $\{a_k(r)\}_{k=1}^{\infty}$ satisfies the recurrence relation (7.16). Then the computation leading to the recurrence relation (7.11) also yields that

$$\begin{aligned} &x^2\varphi_{xx}(r,x) + xp(x)\varphi_x(r,x) + q(x)\varphi(r,x) \\ &\quad = a_0F(r)x^r + \sum_{k=1}^{\infty}\Big[F(k+r)a_k(r) + \sum_{j=0}^{k-1}\big[(j+r)p_{k-j} + q_{k-j}\big]a_j(r)\Big]x^{k+r} \\ &\quad = a_0(r-r_1)^2x^r\,, \end{aligned}$$

where φ_x and φ_{xx} denote the first and the second partial derivatives of φ w.r.t. x. Differentiating the equation above w.r.t. r variable at $r = r_1$, we find that

$$x^2\varphi_{xxr}(r,x) + xp(x)\varphi_{xr}(r,x) + q(x)\varphi_r(r,x) = \big[2a_0(r-r_1)^2x^r + a_0(r-r_1)^2x^r\log x\big]\Big|_{r=r_1} = 0\,.$$

If $\dfrac{\partial}{\partial r}\varphi_{xx} = \big(\dfrac{\partial\varphi}{\partial r}\big)_{xx}$ and $\dfrac{\partial}{\partial r}\varphi_x = \big(\dfrac{\partial\varphi}{\partial r}\big)_x$ (which in general is not true since it involves exchange of orders of limits), then the equation above implies that

$$x^2\frac{d^2}{dx^2}\Big(\frac{\partial\varphi}{\partial r}(r_1,\cdot)\Big) + xp(x)\frac{d}{dx}\Big(\frac{\partial\varphi}{\partial r}(r_1,x)\Big) + q(x)\frac{\partial\varphi}{\partial r}(r_1,x) = 0\,.$$

In other words, assuming that $\dfrac{\partial}{\partial r}\varphi_{xx} = \big(\dfrac{\partial\varphi}{\partial r}\big)_{xx}$ and $\dfrac{\partial}{\partial r}\varphi_x = \big(\dfrac{\partial\varphi}{\partial r}\big)_x$, $y = \dfrac{\partial\varphi}{\partial r}(r_1,x)$ is also a solution to the ODE (7.7). Formally, we switch the order of the differentiation in r and the infinite sum to obtain that

$$\frac{\partial\varphi}{\partial r}(r_1,x) = x^{r_1}\log x\Big(\sum_{k=0}^{\infty}a_k(r_1)x^k\Big) + x^{r_1}\sum_{k=0}^{\infty}a_k'(r_1)x^k = y_1(x)\log x + \sum_{k=0}^{\infty}a_k'(r_1)x^{k+r_1}\,.$$

In other words, under the assumptions that

$$\frac{\partial}{\partial r}\varphi_{xx} = \Big(\frac{\partial \varphi}{\partial r}\Big)_{xx}, \quad \frac{\partial}{\partial r}\varphi_{x} = \Big(\frac{\partial \varphi}{\partial r}\Big)_{x} \quad \text{and} \quad \frac{\partial}{\partial r}\Big|_{r=r_1} \sum_{k=0}^{\infty} a_k(r) x^k = \sum_{k=0}^{\infty} a_k'(r_1) x^k , \tag{7.17}$$

the function y_2 given by

$$y_2(x) = y_1(x)\log x + \sum_{k=0}^{\infty} a_k'(r_1) x^{k+r_1} \tag{7.18}$$

is indeed a solution to (7.7). In general, it is hard to verify those assumptions in (7.17); however, we can still verify whether (7.18) provides a series solution to (7.7) or not. Let us show that y_2 given by (7.18) is indeed a solution to (7.7) if the radius of convergence of the power series $\sum\limits_{k=0}^{\infty} a_k'(r)x^k$ is not zero. We note that y_2 satisfies

$$xy_2'(x) = xy_1'(x)\log x + y_1(x) + \sum_{k=0}^{\infty}(k+r_1)a_k'(r)x^{k+r_1},$$

$$x^2y_2''(x) = x^2y_1''(x)\log x + 2xy'(x) - y_1(x) + \sum_{k=0}^{\infty}(k+r_1)(k+r_1-1)a_k'(r_1)x^{k+r_1}.$$

Moreover, differentiating (7.16) in r at $r = r_1$, we find that

$$\big[2(k+r_1)-1\big]a_k(r_1) + \sum_{j=0}^{k} p_{k-j}a_j(r_1) + \sum_{j=0}^{k}\big[p_{k-j}(j+r_1)+q_{k-j}\big]a_j'(r_1) = 0 \qquad \forall\, k \in \mathbb{N}\cup\{0\}.$$

Therefore, by the fact that y_1 is a solution to (7.7), we have

$$\begin{aligned}
&x^2y_2'' + xp(x)y_2' + q(x)y_2 \\
&\quad = x^2y_1''(x)\log x + 2xy_1'(x) - y_1(x) + \sum_{k=0}^{\infty}(k+r_1)(k+r-1)a_k'(r_1)x^{k+r_1} \\
&\qquad + xp(x)y_1'(x)\log x + p(x)y_1(x) + \Big(\sum_{k=0}^{\infty}p_kx^k\Big)\Big(\sum_{k=0}^{\infty}(k+r_1)a_k'(r_1)x^{k+r_1}\Big) \\
&\qquad + q(x)y_1(x)\log x + \Big(\sum_{k=0}^{\infty}q_kx^k\Big)\Big(\sum_{k=0}^{\infty}a_k'(r_1)x^{k+r_1}\Big) \\
&\quad = \sum_{k=0}^{\infty}\big[2(k+r_1)-1\big]a_k(r_1)x^{k+r_1} + \sum_{k=0}^{\infty}\Big(\sum_{j=0}^{k}p_{k-j}a_j(r_1)\Big)x^{k+r_1} \\
&\qquad + \sum_{k=0}^{\infty}\Big((k+r_1)(k+r-1)a_k'(r_1) + \sum_{j=0}^{k}\big[p_{k-j}(j+r_1)+q_{k-j}\big]a_j'(r_1)\Big)x^{k+r_1} = 0;
\end{aligned}$$

thus $y_2(x)$ is a solution to (7.7) if the radius of convergence of the power series $\sum\limits_{k=0}^{\infty} a_k'(r)x^k$ is not zero.

Finally, we verify that $\{y_1, y_2\}$ forms a linearly independent set of solutions to (7.7) (or (7.2)). This relies on making sure of that the Wronskian of y_1 and y_2 does not vanish. So we compute the

Wronskian of y_1 and y_2 and obtain that

$$
\begin{aligned}
W[y_1, y_2](x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} &= \begin{vmatrix} y_1(x) & y_1(x)\log x + \sum\limits_{k=0}^{\infty} a_k'(r_1)x^{k+r_1} \\ y_1'(x) & y_1'(x)\log x + \dfrac{y_1(x)}{x} + \sum\limits_{k=0}^{\infty}(k+r_1)a_k'(r_1)x^{k+r-1} \end{vmatrix} \\
&= \frac{y_1^2(x)}{x} + y_1(x)\sum_{k=0}^{\infty}(k+r_1)a_k'(r_1)x^{k+r-1} - y_1'(x)\sum_{k=0}^{\infty} a_k'(r_1)x^{k+r_1} \\
&= \frac{y_1^2(x)}{x} + \sum_{k=0}^{\infty}\sum_{j=0}^{k}\big[a_{k-j}(r_1)(j+r_1)a_j'(r_1) - (k-j+r_1)a_{k-j}(r_1)a_j'(r_1)\big]x^{k+2r-1} \\
&= \sum_{k=0}^{\infty}\Big(\sum_{j=0}^{k}\big[a_j(r_1) + (2j-k)a_j'(r_1)\big]a_{k-j}(r_1)\Big)x^{k+2r-1} \\
&= \Big[a_0^2 + \sum_{k=1}^{\infty}\Big(\sum_{j=0}^{k}\big[a_j(r_1) + (2j-k)a_j'(r_1)\big]a_{k-j}(r_1)\Big)x^k\Big]x^{2r-1}\,.
\end{aligned}
$$

Since $a_0 \neq 0$, $\{y_1, y_2\}$ is a linear independent set of solutions to (7.7) (or (7.2)).

Example 7.28. Find the first few terms in the series expansion about the regular singular point $x_0 = 0$ for two linearly independent solutions to

$$x^2y'' - xy' + (1-x)y = 0 \qquad x > 0\,.$$

The indicial equation to the ODE above is

$$r(r-1) - r + 1 = 0$$

and it has a double root $r_1 = 1$. With $p(x) = -1$ and $q(x) = 1 - x$ in mind, we have $p_0 = -1$ and $p_k = 0$ for all $k \in \mathbb{N}$, and $q_0 = 1$, $q_1 = -1$ and $q_k = 0$ for all $k \geqslant 2$; thus the recurrence relation (7.11) for $\big\{a_k(r)\big\}_{k=1}^{\infty}$ is given by

$$(k+r-1)^2 a_k(r) = -\sum_{j=0}^{k-1}\big[(j+r)p_{k-j} + q_{k-1}\big]a_j(r) = a_{k-1}(r)\,. \tag{7.19}$$

For $r = 1$, with the choice of $a_0 = 1$ we have

$$a_k(1) = \frac{1}{k^2}a_{k-1}(1) = \frac{1}{k^2(k-1)^2}a_{k-2}(1) = \cdots = \frac{1}{(k!)^2}\,;$$

thus a series solution to the ODE is given by

$$y_1(x) = x\Big[1 + \sum_{k=1}^{\infty}\frac{1}{(k!)^2}x^k\Big] = \sum_{k=0}^{\infty}\frac{1}{(k!)^2}x^{k+1}\,.$$

Moreover, (7.19) implies that

$$(k+r-1)^2 a_k'(r) + 2(k+r-1)a_k(r) = a_{k-1}'(r)\,;$$

thus $k^2 a_k'(1) = a_{k-1}'(1) - 2k a_k(1) = a_{k-1}'(1) - \dfrac{2k}{(k!)^2}$ which implies that

$$a_k'(1) = \frac{1}{k^2}\Big[a_{k-1}'(1) - \frac{2k}{(k!)^2}\Big].$$

Take $a_0 = 1$, we find that

$$a_1'(1) = -2\,, \quad a_2'(1) = \frac{1}{2^2}(-2-1) = -\frac{3}{4}\,, \quad a_3'(1) = \frac{1}{3^2}\big(-\frac{3}{4} - \frac{1}{6}\big) = -\frac{11}{108}\,, \quad \cdots.$$

Since $|a_k'(1)| \leqslant 1$ for $k \geqslant 2$ (this can be shown by induction), $\limsup\limits_{k\to\infty} |a_k'(1)|^{\frac{1}{k}} \leqslant 1$; thus the radius of convergence for the series $\sum\limits_{k=1}^{\infty} a_k'(1) x^{k+1}$ is at least 1. Therefore, another linearly independent solution is given by

$$y_2(x) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2} x^{k+1} \log x + \big(-2x^2 - \frac{3}{4}x^3 - \frac{11}{108}x^4 + \cdots\big)\,.$$

7.7.2 The case that the difference between two real roots is an integer

Suppose that $r_1 - r_2 = N \in \mathbb{N}$. Using the recurrence relation (7.16) for $r = r_2$, by the fact that $F(r_2 + N) = F(r_1) = 0$ we cannot find $a_N(r_2)$ so that $a_{N+1}(r_2)$, $a_{N+2}(r_2)$ and so on cannot be determined. In this case, we note that for each $k \in \mathbb{N}$, $a_k(r)$ is a rational function of r. In fact, we can show by induction that

$$a_k(r) = \frac{p_k(r)}{F(k+r)F(k-1+r)\cdots F(1+r)} \qquad r \notin \mathcal{N}(r_1, r_2)$$

for some polynomial $p_k(r)$ (of degree at most k).

1. Suppose that $\sum\limits_{j=0}^{N-1} \big[(j+r)p_{N-j} + q_{N-j}\big]a_j(r)$ is divisible by $r - r_2 = r + N - r_1$. Since (7.11) implies that

$$(r - r_2)(r + N - r_2)a_N(r) = -\sum_{j=0}^{N-1} \big[(j+r)p_{N-j} + q_{N-j}\big]a_j(r)\,.$$

we can compute $a_N(r_2)$ by

$$a_N(r_2) = \lim_{r\to r_2} a_N(r) = -\frac{1}{N}\lim_{r\to r_2} \frac{\sum\limits_{j=0}^{N-1} \big[(j+r)p_{N-j} + q_{N-j}\big]a_j(r)}{r - r_2}\,;$$

thus the recurrence relation can be used to determine $a_{N+1}(r_2)$, $a_{N+2}(r_2)$ and so on. In such a case, another solution can be written by (7.14) as well.

2. What if the rational function $\sum\limits_{j=0}^{N-1} \big[(j+r)p_{N-j} + q_{N-j}\big]a_j(r)$ is not divisible by $r - r_2$? Note that then (7.16) implies that $a_N(r)$ is unbounded as r approaches r_2, and $(r - r_2)a_N(r)$ is bounded in a neighborhood of r_2. In this case, we let $\widetilde{a}_k(r) = (r - r_2)a_k(r)$ and

$$\psi(r, x) = (r - r_2)x^r \sum_{k=0}^{\infty} a_k(r)x^k = x^r \sum_{k=0}^{\infty} \widetilde{a}_k(r)x^k\,,$$

where $a_k(r)$ satisfies the recurrence relation (7.11). Then

$$x^2\psi_{xx}(r,x) + xp(x)\psi_x(r,x) + q(x)\psi(r,x) = a_0(r-r_1)(r-r_2)^2x^r .$$

Differentiating in r variable at $r = r_2$, we find that

$$x^2\psi_{xxr}(r_2,x) + xp(x)\psi_{xr}(r_2,x) + q(x)\psi_r(r_2,x) = 0 ,$$

which, as discussed before, we expect that

$$\psi_r(r_2,x) = \frac{\partial}{\partial r}\sum_{k=0}^{\infty} \widetilde{a}_k(r)x^{k+r}$$

or to be more precise,

$$y_2(x) = \sum_{k=0}^{\infty} \widetilde{a}_k(r_2)x^{k+r_2}\log x + \sum_{k=0}^{\infty} \widetilde{a}_k'(r_2)x^{k+r_2} \tag{7.20}$$

is also a solution to (7.7), where $\widetilde{a}_k(r_2) \equiv \lim\limits_{r\to r_2} \widetilde{a}_k(r)$. Note that $\widetilde{a}_0(r_2) = 0$, and if $N \neq 1$, the recurrence relation (7.11) implies that

$$\lim_{r\to r_2}(r-r_2)a_1(r) = -\lim_{r\to r_2}\frac{\big(rp_1+q_1\big)(r-r_2)a_0}{F(r+1)} = 0 .$$

Similarly, for $k < N$,

$$\lim_{r\to r_2}(r-r_2)a_k(r) = -\lim_{r\to r_2}\frac{\sum\limits_{j=0}^{k-1}\big[(j+r)p_{k-j}+q_{k-j}\big]a_j(r)(r-r_2)}{F(k+r)} = 0 .$$

In other words, $\widetilde{a}_k(r_2) = 0$ for $0 \leqslant k \leqslant N-1$. Now we consider $\lim\limits_{r\to r_2}(r-r_2)a_N(r)$. Since $F(r+N) = (r-r_2)(r+N-r_2)$, we have

$$(r-r_2)a_N(r) = -\frac{\sum\limits_{j=0}^{N-1}\big[(j+r)p_{k-j}+q_{k-j}\big]a_j(r)}{(r+N-r_2)} ;$$

thus

$$\widetilde{a}_N(r_2) \equiv \lim_{r\to r_2}(r-r_2)a_N(r) = -\frac{1}{N}\sum_{j=0}^{N-1}\big[(j+r_2)p_{k-j}+q_{k-j}\big]a_j(r_2)$$

which exists and does not vanish $\Big($since $\sum\limits_{j=0}^{N-1}\big[(j+r)p_{N-j}+q_{N-j}\big]a_j(r)$ is not divisible by $r-r_2\Big)$. Then for $k > N$, we have

$$\begin{aligned}
\widetilde{a}_k(r_2) &= \lim_{r\to r_2}(r-r_2)a_k(r) = -\lim_{r\to r_2}\frac{\sum\limits_{j=0}^{k-1}\big[(j+r)p_{k-j}+q_{k-j}\big]a_j(r)(r-r_2)}{F(k+r)} \\
&= -\lim_{r\to r_2}\frac{\sum\limits_{j=N}^{k-1}\big[(j+r)p_{k-j}+q_{k-j}\big]a_j(r)(r-r_2)}{F(k+r)} = -\frac{\sum\limits_{j=N}^{k-1}\big[(j+r_2)p_{k-j}+q_{k-j}\big]\widetilde{a}_j(r_2)}{F(k+r_2)} \\
&= -\frac{\sum\limits_{j=0}^{k-N-1}\big[(j+r_1)p_{k-j-N}+q_{k-j-N}\big]\widetilde{a}_{j+N}(r_2)}{F(k-N+r_1)} .
\end{aligned}$$

Let $b_j = \widetilde{a}_{j+N}(r_2)$. Then the identity above implies that the sequence $\{b_j\}_{j=0}^{\infty}$ satisfies

$$F(k+r_1)b_k + \sum_{j=0}^{k-1}\big[(j+r_1)p_{k-j} + q_{k-j}\big]b_j = 0 \qquad \forall\, k \in \mathbb{N}\,.$$

In other words, $\{b_k\}_{k=0}^{\infty}$ satisfies the same recurrence relation as $\big\{a_k(r_1)\big\}_{k=0}^{\infty}$. By the fact that $\dfrac{a_k(r)}{a_0}$ is independent of a_0, we must have $\dfrac{b_k}{b_0} = \dfrac{a_k(r_1)}{a_0}$. As a consequence, (7.20) implies that

$$\begin{aligned} y_2(x) &= \sum_{k=N}^{\infty} \widetilde{a}_k(r_2)x^{k+r_2}\log x + \sum_{k=0}^{\infty}\widetilde{a}_k'(r_2)x^{k+r_2} = \sum_{k=0}^{\infty} b_k x^{k+r_1}\log x + \sum_{k=0}^{\infty}\widetilde{a}_k'(r_2)x^{k+r_2} \\ &= \frac{b_0}{a_0}\sum_{k=0}^{\infty} a_k(r_1)x^{k+r_1}\log x + \sum_{k=0}^{\infty}\widetilde{a}_k'(r_2)x^{k+r_2} \\ &= \frac{b_0}{a_0} y_1(x)\log x + \sum_{k=0}^{\infty} c_k x^{k+r_2}\,, \end{aligned} \tag{7.21}$$

where $b_0 = \lim\limits_{r\to r_2}(r-r_2)a_N(r)$ and $c_k = \dfrac{\partial}{\partial r}\Big|_{r=r_2}(r-r_2)a_k(r)$.

Remark 7.29. We note that (7.21) is also a solution even if $\sum\limits_{j=0}^{N-1}\big[(j+r)p_{N-j}+q_{N-j}\big]a_j(r)$ is divisible by $r-r_2$. In fact, if $\sum\limits_{j=0}^{N-1}\big[(j+r)p_{N-j}+q_{N-j}\big]a_j(r)$ is divisible by $r-r_2$, then $b_0 = 0$ which implies that all b_k's are zeros for all k. Moreover, in this case $a_k(r_2)$ exists; thus

$$c_k = \frac{\partial}{\partial r}\Big|_{r=r_2}(r-r_2)a_k(r) = a_k(r_2)$$

which implies that (7.21) agrees with (7.14).

Example 7.30. Find the general series solution about the regular singular point $x_0 = 0$ of

$$xy'' + 3y' - xy = 0 \qquad x > 0\,.$$

Rewrite the equation above as

$$x^2y'' + 3xy' - x^2y = 0\,,$$

and let $p(x) = 3$ and $q(x) = -x^2$. Then $x^2y'' + xp(x)y' + q(x)y = 0$, where we note that

1. $p_0 = 3$ and $p_j = 0$ for all $j \in \mathbb{N}$.
2. $q_2 = -1$, q_j for all $j \in \mathbb{N}\cup\{0\}$ and $j \neq 2$.

The indicial equation of the ODE above is $F(r) = r(r-1)+3r = 0$ which has two distinct roots $r_1 = 0$ and $r_2 = -2$.

Using (7.11), we find that $\{a_k(r_1)\}_{k=1}^{\infty}$ satisfies the recurrence relation

$$2a_1(r_1) = 0 \qquad \text{and} \qquad k(k+2)a_k(r_1) - a_{k-2}(r_1) = 0 \qquad \forall\, k \geqslant 2\,.$$

Therefore, $a_1(r_1) = a_3(r_1) = \cdots = a_{2n+1}(r_1) = 0$ and

$$a_{2n} = \frac{1}{2n(2n+2)} a_{2n-2} = \frac{1}{2n(2n+2)(2n-2)2n} a_{2n-4} = \cdots = \frac{1}{2^{2n} n!(n+1)!} a_0 \qquad \forall\, n \in \mathbb{N};$$

thus a series solution (with $a_0 = 1$) is given by

$$y_1(x) = 1 + \sum_{k=1}^{\infty} \frac{1}{2^{2k} k!(k+1)!} x^{2k} = \sum_{k=0}^{\infty} \frac{1}{2^{2k} k!(k+1)!} x^{2k}.$$

Now we look for a second series solution as discussed above. Note that $N = r_1 - r_2 = 2$. Since $a_0 = 1$, using (7.11) we obtain that

$$(r+3)(r+1)a_1(r) = 0$$
$$(k+r+2)(k+r)a_k(r) - a_{k-2}(r) = 0 \qquad \forall\, k \geqslant 2.$$

Therefore,

$$a_1(r) = a_3(r) = a_5(r) = \cdots = a_{2k-1}(r) = 0 \qquad \text{for } r \approx -2 \text{ and } k \in \mathbb{N} \tag{7.22}$$

and

$$(r+2)a_k(r) = \frac{1}{(k+r+2)(k+r)} (r+2)a_{k-2}(r) \qquad \forall\, k \geqslant 2. \tag{7.23}$$

We first compute b_0. By definition,

$$b_0 = \lim_{r \to r_2} (r - r_2) a_N(r) = \lim_{r \to -2} (r+2) a_2(r) = \lim_{r \to -2} \frac{1}{r+4} = \frac{1}{2}.$$

Now we compute $c_k(r_2)$. With $\widetilde{a}_k(r) \equiv (r+2)a_k(r)$, the recurrence relation (7.23) implies that $\widetilde{a}_{2k-1}(r) = 0$ for all $k \in \mathbb{N}$. Moreover,

$$\widetilde{a}_0(r) = (r+2)a_0 = r+2$$

and for $k \in \mathbb{N}$,

$$\begin{aligned}
\widetilde{a}_{2k}(r) &= \frac{1}{(2k+r+2)(2k+r)} \widetilde{a}_{2k-2}(r) = \frac{1}{(2k+r+2)(2k+r)^2(2k+r-2)} \widetilde{a}_{2k-4}(r) = \cdots \\
&= \frac{1}{(2k+r+2)(2k+r)^2(2k+r-2)^2 \cdots (4+r)^2(2+r)} \widetilde{a}_0 \\
&= \frac{1}{(2k+r+2)(2k+r)^2(2k+r-2)^2 \cdots (4+r)^2};
\end{aligned}$$

thus

$$\log \widetilde{a}_{2k}(r) = -\log(2k+r+2) - 2\big[\log(2k+r) + \log(2k+r-2) + \cdots + \log(4+r)\big] \qquad k \in \mathbb{N}.$$

Differentiating $\widetilde{a}_{2k}$ at $r = -2$, we obtain that $c_0(-2) = \widetilde{a}_0(-2) = 1$ and for all $k \in \mathbb{N}$,

$$\begin{aligned}
c_{2k}(-2) = \widetilde{a}_{2k}'(-2) &= -\Big[\frac{1}{2k} + 2\Big(\frac{1}{2k-2} + \frac{1}{2k-4} + \cdots + \frac{1}{2}\Big)\Big] \widetilde{a}_{2k}(-2) \\
&= -\Big[\frac{1}{2k} + \Big(\frac{1}{k-1} + \frac{1}{k-2} + \cdots + \frac{1}{1}\Big)\Big] \frac{1}{(2k)(2k-2)^2(2k-4)^2 \cdots 2^2} \\
&= -\frac{1}{2^{2k-1} k!(k-1)!} \Big(H_k - \frac{1}{2k}\Big),
\end{aligned}$$

where $H_k = \sum\limits_{\ell=1}^{k} \frac{1}{\ell}$ is the k-th partial sum of the harmonic series. Therefore,

$$\begin{aligned} y_2(x) &= \frac{1}{2} y_1(x) \log x + x^{-2} - \sum_{k=1}^{\infty} \frac{1}{2^{2k-1} k!(k-1)!} \big(H_k - \frac{1}{2k}\big) x^{2k-2} \\ &= \frac{1}{2} y_1(x) \log x + x^{-2} - \frac{1}{4} - \frac{5}{64} x^2 \cdots - \frac{5}{1152} x^4 + \cdots , \end{aligned}$$

and the general series solution is given by

$$y(x) = C_1 y_1(x) + C_2 y_2(x) = C_1 y_1(x) + C_2 \Big[\frac{1}{2} y_1(x) \log x + x^{-2} - \frac{1}{4} - \frac{5}{64} x^2 \cdots - \frac{5}{1152} x^4 + \cdots \Big] .$$

We summarize the discussions above into the following

Theorem 7.31. *Let x_0 be a regular singular point for*

$$(x - x_0)^2 y'' + (x - x_0) p(x) y' + q(x) y = 0 \qquad x > x_0 ,$$

and let r_1, r_2 be the roots of the associated indicial equation $r(r-1) + p(x_0) r + q(x_0) = 0$, where $Re(r_1) \geqslant Re(r_2)$.

1. *If $r_1 - r_2$ is not an integer, then there exist two linearly independent solutions of the form*

$$\begin{aligned} y_1(x) &= \sum_{k=0}^{\infty} a_k (x - x_0)^{k + r_1} , \qquad a_0 \neq 0 , \\ y_2(x) &= \sum_{k=0}^{\infty} b_k (x - x_0)^{k + r_2} , \qquad b_0 \neq 0 . \end{aligned}$$

2. *If $r_1 = r_2$, then there exists two linearly independent solutions of the form*

$$\begin{aligned} y_1(x) &= \sum_{k=0}^{\infty} a_k (x - x_0)^{k + r_1} , \qquad a_0 \neq 0 , \\ y_2(x) &= y_1(x) \log(x - x_0) + \sum_{k=0}^{\infty} b_k (x - x_0)^{k + r_2} . \end{aligned}$$

3. *If $r_1 - r_2 \in \mathbb{N}$, then there exists two linearly independent solutions of the form*

$$\begin{aligned} y_1(x) &= \sum_{k=0}^{\infty} a_k (x - x_0)^{k + r_1} , \qquad a_0 \neq 0 , \\ y_2(x) &= C y_1(x) \log(x - x_0) + \sum_{k=0}^{\infty} b_k (x - x_0)^{k + r_2} , \qquad b_0 \neq 0 , \end{aligned}$$

   where C is a constant that could be zero.

Example 7.32. Let us use Theorem 7.31 to find solutions

$$xy'' + 3y' - xy = 0 \qquad x > 0$$

in the form of series about the regular singular point $x_0 = 0$.

Following Example 7.30, the indicial equation of the ODE above has two distinct roots $r_1 = 0$ and $r_2 = -2$. Therefore, by Theorem 7.31 there exists two linearly independent solutions:

$$y_1(x) = \sum_{k=0}^{\infty} a_k x^k \qquad \text{and} \qquad y_2(x) = C y_1(x) \log x + \sum_{k=0}^{\infty} b_k x^{k-2},$$

where $\{a_k\}_{k=1}^{\infty}$ satisfies the recurrence relation

$$2a_1 = 0 \qquad \text{and} \qquad k(k+2)a_k - a_{k-2} = 0 \qquad \forall\, k \geqslant 2$$

and, by taking $a_0 = 1$, is given by $a_{2n} = \dfrac{1}{2^{2n} n!(n+1)!}$ and $a_{2n-1} = 0$ for all $n \in \mathbb{N}$.

Now we determine the constant C and the sequence $\{b_k\}_{k=0}^{\infty}$. Since y_2 is also a solution and

$$\begin{aligned}
y_2'(x) &= C y_1'(x) \log x + C\frac{y_1(x)}{x} + \sum_{k=0}^{\infty}(k-2)b_k x^{k-3} \\
&= C y_1'(x) \log x + C \sum_{k=0}^{\infty} a_k x^{k-1} + \sum_{k=0}^{\infty}(k-2)b_k x^{k-3}, \\
y_2''(x) &= C y_1''(x) \log x + C\frac{y_1'(x)}{x} + C\sum_{k=0}^{\infty}(k-1)a_k x^{k-2} + \sum_{k=0}^{\infty}(k-2)(k-3)b_k x^{k-4} \\
&= C y_1''(x) \log x + C\sum_{k=0}^{\infty}(2k-1)a_k x^{k-2} + \sum_{k=0}^{\infty}(k-2)(k-3)b_k x^{k-4},
\end{aligned}$$

we have (with $a_0 = 1$ in mind)

$$\begin{aligned}
0 &= x y_2'' + 3y_2' - x y_2 \\
&= C\sum_{k=0}^{\infty}(2k+2)a_k x^{k-1} + \sum_{k=0}^{\infty} k(k-2)b_k x^{k-3} - \sum_{k=0}^{\infty} b_k x^{k-1} \\
&= 2Cx^{-1} - b_1 x^{-2} - b_0 x^{-1} + C\sum_{k=0}^{\infty}(2k+4)a_{k+1}x^k + \sum_{k=0}^{\infty}\big[(k+3)(k+1)b_{k+3} - b_{k+1}\big]x^k \\
&= -b_1 x^{-2}(2C - b_0)x^{-1} + \sum_{k=0}^{\infty}\Big[(k+3)(k+1)b_{k+3} - b_{k+1} + C(2k+4)a_{k+1}\Big]x^k.
\end{aligned}$$

Therefore, $b_0 = 2C$, $b_1 = 0$ and

$$(k+3)(k+1)b_{k+3} - b_{k+1} + C(2k+4)a_{k+1} = 0.$$

If $C = 0$, then $b_k = 0$ for all $k \in \mathbb{N} \cup \{0\}$; thus for y_2 being non-trivial $C \neq 0$. W.L.O.G. we can assume that $C = \dfrac{1}{2}$. Since $a_{2k-1} = 0$ for all $k \in \mathbb{N}$ and $b_1 = 0$, we find that $b_{2k-1} = 0$ for all $k \in \mathbb{N}$; thus

$$2n(2n+2)b_{2n+2} - b_{2n} + C(4n+2)a_{2n} = 2n(2n+2)b_{2n+2} - b_{2n} + \frac{(2n+1)}{2^{2n}n!(n+1)!} = 0 \qquad \forall\, n \in \mathbb{N}.$$

This implies that

$$b_4 = \frac{b_2}{8} - \frac{3}{64}, \quad b_6 = \frac{b_4}{24} - \frac{5}{24 \cdot 192} = \frac{b_2}{192} - \frac{7}{2304},$$

and this further implies that

$$\begin{aligned}y_2(x) &= \frac{1}{2}y_1(x)\log x + x^{-2} + b_2 + b_4x^2 + b_6x^4 + \cdots \\ &= \Big(\frac{1}{2}y_1(x)\log x + x^{-2} - \frac{3}{64}x^2 - \frac{7}{2304}x^4 + \cdots\Big) + b_2\Big(1 + \frac{1}{8}x^2 + \frac{1}{192}x^4 + \cdots\Big).\end{aligned}$$

We note that y_2 in Example 7.30 is given by $b_2 = -\frac{1}{4}$ in the expression above.

7.7.3 The radius of convergence of series solutions

The radius of convergence of the series solution (7.9) cannot be guaranteed by Theorem 7.17; however, we have the following

Theorem 7.33 (Frobenius)**.** *If x_0 is a regular singular point of ODE* (7.2)*, then there exists at least one series solution of the form*

$$y(x) = (x - x_0)^r \sum_{k=0}^{\infty} a_k(x - x_0)^k\,,$$

where r is the largest root or any complex root of the associated indicial equation. Moreover, the series solution converges for all $x \in 0 < x - x_0 < R$, where R is the distance from x_0 to the nearest other singular point (*real or complex*) *of* (7.2)*.*

7.8 Special Functions

7.8.1 Bessel's Equation

We consider three special cases of Bessel's equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0\,, \tag{7.24}$$

where ν is a constant. It is easy to see that $x = 0$ is a regular singular point of (7.24) since

$$\lim_{x\to 0} x \cdot \frac{x}{x^2} = 1 = p_0 \qquad \text{and} \qquad \lim_{x\to 0} x^2 \cdot \frac{x^2 - \nu^2}{x^2} = -\nu^2 = q_0\,.$$

Therefore, the indicial equation for the regular singular point $x = 0$ is

$$r(r-1) + r - \nu^2 = 0$$

which implies that $r = \pm\nu$. The ODE (7.24) is called ***Bessel's equation of order ν***.

To find series solution to (7.24), we first note that in the case of Bessel's equation of order ν, $F(r) = r^2 - \nu^2$, $p(x) = 1$ (which implies that $p_0 = 1$ while $p_k = 0$ for all $k \in \mathbb{N}$) and $q(x) = x^2 - \nu^2$ (which implies that $q_0 = -\nu^2$ and $q_2 = 1$ and $q_k = 0$ otherwise). Therefore, the recurrence relation (7.11) implies that

$$\big[(k+r)^2 - \nu^2\big]a_k(r) + \sum_{j=0}^{k-1} q_{k-j}a_j(r) = 0 \qquad \forall\, k \in \mathbb{N}\,.$$

This implies that

$$\big[(1+r)^2-\nu^2\big]a_1(r)=0 \tag{7.25a}$$

$$\big[(k+r)^2-\nu^2\big]a_k(r)+a_{k-2}(r)=0 \qquad \forall\, k \geqslant 2 \tag{7.25b}$$

• **Bessel's Equation of Order Zero:** Consider the case $\nu=0$. Then the roots of the indicial equation are identical: $r_1=r_2=0$. Using (7.25a), $a_1(r)\equiv 0$ (in a small neighborhood of 0) and (7.25b) implies that

$$a_k(r)=-\frac{1}{(k+r)^2}a_{k-2}(r) \qquad \forall\, k\geqslant 2\,; \tag{7.26}$$

thus $a_3(r)=a_5(r)=\cdots=a_{2m+1}(r)=\cdots=0$ for all $m\in\mathbb{N}$. Note that $a_{2m-1}(r)=0$ for all $m\in\mathbb{N}$ also implies that $a'_{2m-1}(r)=0$ for all $m\in\mathbb{N}$.

On the other hand, recurrence relation (7.26) also implies that

$$\begin{aligned} a_{2m}(r) &= -\frac{1}{(2m+r)^2}a_{2m-2}(r)=\frac{1}{(2m+r)^2(2m+r-2)^2}a_{2m-4}(r)\\ &=\cdots=\frac{(-1)^{m-1}}{(2m+r)^2(2m+r-2)^2\cdots(4+r)^2}a_2(r)\\ &=\frac{(-1)^m}{(2m+r)^2(2m+r-2)^2\cdots(4+r)^2(2+r)^2}a_0\,; \end{aligned}$$

thus $a_{2m}(0)=\dfrac{(-1)^m}{2^{2m}(m!)^2}a_0$ and rearranging terms, we obtain that

$$\log\frac{(-1)^m a_{2m}(r)}{a_0}=-2\big[\log(2m+r)+\log(2m+r-2)+\cdots+\log(4+r)+\log(2+r)\big]\,.$$

Differentiating both sides above in r,

$$\frac{a'_{2m}(r)}{a_{2m}(r)}=-2\Big[\frac{1}{2m+r}+\frac{1}{2m+r-2}+\cdots+\frac{1}{4+r}+\frac{1}{2+r}\Big]\,,$$

and evaluating the equation above at $r=0$ we conclude that

$$a'_{2m}(0)=-H_m a_{2m}(0)=\frac{(-1)^{m+1}H_m}{2^{2m}(m!)^2}a_0\,,$$

where $H_m=\sum\limits_{k=1}^{m}\dfrac{1}{k}$. As a consequence, the first series solution is given by

$$y_1(x)=\sum_{k=0}^{\infty}a_{2k}(0)x^{2k}=a_0\Big[1+\sum_{k=1}^{\infty}\frac{(-1)^k x^{2k}}{2^{2k}(k!)^2}\Big]\,,$$

and the second solution is given by

$$y_2(x)=a_0\Big[J_0(x)\log x+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}H_k x^{2k}}{2^{2k}(k!)^2}\Big]\,,$$

where $J_0=a_0^{-1}y_1$ is called the ***Bessel function of the first kind of order zero***. We note that y_1 and y_2 can be defined for all $x>0$ since the radius of convergence of the series involved in y_1 and y_2 are infinite.

Any linear combinations of y_1 and y_2 is also a solution to Bessel's equation (7.24) of order zero. Consider the ***Bessel function of the second kind of order zero***

$$Y_0(x) = \frac{2}{\pi}\Big[\frac{1}{a_0}y_2(x) + (\gamma - \log 2)J_0(x)\Big], \tag{7.27}$$

where $\gamma = \lim\limits_{k\to\infty}(H_k - \log k) \approx 0.5772$ is called the ***Euler-Máscheroni constant***. Substituting for y_2 in (7.27), we obtain

$$Y_0(x) = \frac{2}{\pi}\Big[\Big(\gamma + \log\frac{x}{2}\Big)J_0(x) + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}H_k}{2^{2k}(k!)^2}x^{2k}\Big], \qquad x > 0. \tag{7.28}$$

A general solution to Bessel's equation (7.24) of order zero then can be written as

$$y(x) = C_1 J_0(x) + C_2 Y_0(x).$$

• **Bessel's Equation of Order One-Half:** Now suppose that $\nu = \frac{1}{2}$ (thus $r_1 = \frac{1}{2}$ and $r_2 = -\frac{1}{2}$). To obtain solutions to Bessel's equation (7.24) of order one-half, we need to compute the coefficients $a_k(r)$ for all $k \in \mathbb{N}$ (given a_0), and $b_0 = \lim\limits_{r\to-\frac{1}{2}}(r-r_2)a_1(r)$ as well as $c_k = \frac{\partial}{\partial r}\Big|_{r=r_2}(r-r_2)a_k(r)$.

Using (7.25b), we find that

$$a_k(r) = \frac{-1}{(k+r)^2 - \frac{1}{4}}a_{k-2}(r) = \frac{-1}{(k+r+\frac{1}{2})(k+r-\frac{1}{2})}a_{k-2}(r) \qquad \forall\, k \geqslant 2,$$

while if $r \approx r_1 = \frac{1}{2}$, (7.25a) implies that $a_1(r) = 0$ which further implies that $a_3(r) = a_5(r) = \cdots = a_{2m-1}(r) = \cdots = 0$ for all $m \in \mathbb{N}$ if $r \approx \frac{1}{2}$. In particular, we have

$$a_{2m}(\tfrac{1}{2}) = \frac{(-1)^m a_0}{(2m+1)!} \quad \text{and} \quad a_{2m-1}(\tfrac{1}{2}) = 0 \qquad \forall\, m \in \mathbb{N};$$

thus a series solution of (7.24) is

$$y_1(x) = a_0 x^{\frac{1}{2}}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k+1)!} = a_0 x^{-\frac{1}{2}}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!} = a_0\frac{\sin x}{\sqrt{x}}.$$

The ***Bessel function of the first kind of order one-half*** is defined by (letting $a_0 = \sqrt{\frac{2}{\pi}}$ in the expression of y_1 above)

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}}\frac{\sin x}{\sqrt{x}} = \sqrt{\frac{2}{\pi x}}\sin x.$$

Now we compute the limit of $(r-r_2)a_1(r)$ as r approaches r_2. Since (7.25a) implies that $(r+\frac{3}{2})(r+\frac{1}{2})a_1(r) = 0$, we have $(r-r_2)a_1(r) = 0$ for all $r \approx r_2 = -\frac{1}{2}$. Therefore,

$$b_0 = \lim_{r\to r_2}(r-r_2)a_1(r) = 0$$

which implies that there will be no logarithmic term in the second solution y_2 given by (7.14).

Now we compute $\frac{\partial}{\partial r}\Big|_{r=r_2}(r-r_2)a_k(r)$. Since

$$\begin{aligned} a_{2m}(r) &= \frac{-1}{(2m+r+\frac{1}{2})(2m+r-\frac{1}{2})}a_{2m-2}(r) = \cdots \\ &= \frac{(-1)^m}{(2m+r+\frac{1}{2})(2m+r-\frac{1}{2})\cdots(2+r+\frac{1}{2})(2+r-\frac{1}{2})}a_0 \\ &= \frac{(-1)^m}{(2m+r+\frac{1}{2})(2m+r-\frac{1}{2})\cdots(r+\frac{5}{2})(r+\frac{3}{2})}a_0 \end{aligned}$$

which implies that $|a'_{2m}(r_2)| < \infty$. Therefore,

$$c_{2m}(r_2) = \frac{\partial}{\partial r}\Big|_{r=r_2}(r-r_2)a_{2m}(r) = a_{2m}(r_2) = \frac{(-1)^m}{(2m)!}a_0 \,.$$

On the other hand, using (7.25a) again, we find that $a_1(r_2)$ is not necessary zero; thus we let a_1 be a free constant and use (7.25b) to obtain that

$$a_{2m+1}(r) = \frac{(-1)^m}{(2m+1+r+\frac{1}{2})(2m+1+r-\frac{1}{2})\cdots(3+r+\frac{1}{2})(3+r-\frac{1}{2})}a_1 \,.$$

Since $|a'_{2m+1}(r_2)| < \infty$, we find that

$$c_{2m+1}(r_2) = \frac{\partial}{\partial r}\Big|_{r=r_2}(r-r_2)a_{2m+1}(r) = a_{2m}(r_2) = \frac{(-1)^m}{(2m+1)!}a_1 \,.$$

Therefore,

$$y_2(x) = \sum_{k=0}^{\infty} c_k(r_2)x^{k+r_2} = x^{-\frac{1}{2}}\Big[a_0\sum_{k=1}^{\infty}\frac{(-1)^m}{(2k)!}x^{2k} + a_1\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k-1)!}x^{2k-1}\Big] = a_0\frac{\cos x}{\sqrt{x}} + a_1\frac{\sin x}{\sqrt{x}} \,.$$

This produces the Bessel function of the second kind of order one-half

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x \,,$$

and the general solution of Bessel's equation of order one-half can be written as $y = C_1 J_{\frac{1}{2}}(x) + C_2 J_{-\frac{1}{2}}(x)$.

• **Bessel's Equation of Order One:** Now we consider the case that $\nu = 1$ (thus $r_1 = 1$ and $r_2 = -1$). Again, we need to compute $\{a_k(r_1)\}_{k=1}^{\infty}$, $\lim\limits_{r\to r_2}(r-r_2)a_2(r)$ and $c_k(r_2) = \frac{\partial}{\partial r}\Big|_{r=r_2}(r-r_2)a_k(r)$.

Note that (7.25a) implies that $a_1(r_1) = 0$ (which implies that $a_{2m-1}(r_1) = 0$ for all $m \in \mathbb{N}$). Moreover,

$$\begin{aligned} a_{2m}(r_1) &= \frac{-1}{(2m+2)2m}a_{2m-2}(r) = \frac{1}{(2m+2)(2m)^2(2m-2)}a_{2m-4}(r) \\ &= \cdots = \frac{(-1)^m}{(2m+2)(2m)^2(2m-4)^2\cdots 4^2\cdot 2}a_0 = \frac{(-1)^m}{2^{2m}(m+1)!m!}a_0 \,; \end{aligned}$$

thus

$$y_1(x) = a_0 x\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k}(k+1)!k!}x^{2k} \,.$$

Now we focus on finding b_0 and $\{c_k(r_2)\}_{k=0}^{\infty}$. Note that by (7.25a),

$$F(2+r)a_2(r) = -a_0\,;$$

thus $(r+1)a_2(r) = -\dfrac{1}{2(r+3)}$ which implies that $b_0 = \lim\limits_{r\to r_2}(r-r_2)a_2(r) = -\dfrac{a_0}{2}$.

To compute $\{c_k(r_2)\}_{k=0}^{\infty}$, we first note that (7.25a) implies that $a_1(r) \equiv 0$; thus we use (7.25b) to conclude that $a_{2m-1}(r) = 0$ for all $m \in \mathbb{N}$ and $r \approx r_2$. This implies that $c_{2m-1}(r_2) = 0$ for all $m \in \mathbb{N}$. On the other hand, for $m \in \mathbb{N}$ and $r \approx r_2$,

$$a_{2m}(r) = \frac{(-1)^m}{(2m+r+1)(2m+r-1)^2\cdots(r+3)^2(r+1)}a_0\,;$$

thus

$$(r-r_2)a_{2m}(r) = \frac{(-1)^m}{(2m+r+1)(2m+r-1)^2\cdots(r+3)^2}a_0\,.$$

Therefore, using the formula $\dfrac{d}{dr}f(r) = f(r)\dfrac{d}{dr}\log f(r)$ if $f(r) > 0$, we find that

$$\begin{aligned}
c_{2m}(r_2) &= \frac{(-1)^{m+1}a_0}{(2m)(2m-2)^2\cdots 2^2}\Big[\frac{1}{2m+r+1} + \frac{2}{2m+r-1} + \cdots + \frac{2}{r+3}\Big]\Big|_{r=r_2} \\
&= \frac{(-1)^{m+1}a_0}{2^{2m-1}m!(m-1)!}\Big[\frac{1}{2m} + \frac{2}{2m-2} + \cdots + \frac{2}{2}\Big] \\
&= \frac{(-1)^{m+1}a_0}{2^{2m}m!(m-1)!}\Big[\frac{1}{m} + \frac{2}{m-1} + \cdots + \frac{2}{1}\Big] = \frac{(-1)^{m+1}\big(H_m + H_{m-1}\big)}{2^{2m}m!(m-1)!}a_0\,.
\end{aligned}$$

Moreover, $c_0(r_2) = \dfrac{\partial}{\partial r}\Big|_{r=r_2}(r-r_2)a_0 = a_0$. Then the second solution to Bessel's equation of order one is

$$\begin{aligned}
y_2(x) &= \frac{b_0}{a_0}y_1(x)\log x + \sum_{k=0}^{\infty} c_k(r_2)x^{k+r_2} = -J_1(x)\log x + x^{-1}\Big[a_0 + \sum_{k=1}^{\infty} c_{2k}(r_2)x^{2k}\Big] \\
&= -\frac{1}{2}y_1(x)\log x + \frac{a_0}{x}\Big[1 - \sum_{k=1}^{\infty}\frac{(-1)^k(H_k + H_{k-1})}{2^{2k}k!(k-1)!}x^{2k}\Big]\,.
\end{aligned}$$

This produces the ***Bessel function of the first kind of order one***:

$$J_1(x) \equiv \frac{1}{2}y_1(x) = \frac{x}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k}(k+1)!k!}x^{2k}$$

and the ***Bessel function of the second kind of order one***:

$$Y_1(x) \equiv \frac{2}{\pi}\big[-y_2(x) + (\gamma - \log 2)J_1(x)\big]\,,$$

where γ is again the Euler-Máscheroni constant. The general solution to Bessel's equation of order one then can be written as

$$y = C_1J_1(x) + C_2Y_1(x)\,.$$

- **The General Case:** In general, we have the following

Definition 7.34 (Bessel's function of the first kind)**.** For $\nu \geqslant 0$, the ***Bessel function of the first kind of order*** $\boldsymbol{\nu}$, denoted by J_ν, is defined as the series solution $\sum\limits_{k=0}^{\infty} a_k(\nu)x^{k+\nu}$ to the Bessel equation of order ν

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \tag{7.24}$$

with a specific $a_0(\nu) = \dfrac{1}{\Gamma(\nu+1)2^\nu}$, where $\Gamma : (0,\infty) \to \mathbb{R}$ is the Gamma-function. In other words, the Bessel function of the first kind of order ν is the series solution to (7.24) of the form $J_\nu(x) = x^\nu\Big[\dfrac{1}{\Gamma(\nu+1)2^\nu} + \sum\limits_{k=1}^{\infty} a_k(\nu)x^k\Big]$.

For $J_\nu(x)$ to be a solution to (7.24), the coefficients $\{a_k(\nu)\}_{k=1}^{\infty}$ must satisfy (7.25) (with $r = \nu$) and this implies that

$$\begin{aligned} F(1+\nu)a_1(\nu) &= 0\,, \\ F(k+\nu)a_k(\nu) + a_{k-2}(\nu) &= 0 \qquad \forall\, k \geqslant 2\,. \end{aligned}$$

Therefore, we conclude that $a_1(\nu) = 0$ and

$$a_k(\nu) = \frac{-1}{(k+\nu-\nu)(k+\nu+\nu)}a_{k-2}(\nu) = \frac{-1}{k(k+2\nu)}a_{k-2}(\nu) \qquad \forall\, k \geqslant 2\,;$$

thus $a_{2m+1}(\nu) = 0$ for all $m \in \mathbb{N} \cup \{0\}$ and

$$\begin{aligned} a_{2k}(\nu) &= \frac{1}{2k(2k+2\nu)(2k-2)(2k-2+2\nu)}a_{2k-4}(\nu) = \cdots \\ &= \frac{(-1)^k}{2k(2k-2)(2k-4)\cdots 2(2k+2\nu)(2k+2\nu-2)\cdots(2+2\nu)}a_0 \\ &= \frac{(-1)^k}{2^{2k}k!(k+\nu)(k+\nu-1)\cdots(\nu+1)} \cdot \frac{1}{\Gamma(\nu+1)2^\nu}\,. \end{aligned} \tag{7.29}$$

Using the property that $\Gamma(x+1) = x\Gamma(x)$ for all $x > 0$, we find that

$$(k+\nu)(k+\nu-1)\cdots(\nu+1)\Gamma(\nu+1) = \Gamma(k+\nu+1)\,;$$

thus

$$a_{2k}(\nu) = \frac{(-1)^k\Gamma(\nu+1)}{2^{2k}k!\Gamma(k+\nu+1)} \cdot \frac{1}{\Gamma(\nu+1)2^\nu} = \frac{(-1)^k}{2^{2k+\nu}k!\Gamma(k+\nu+1)}\,.$$

Therefore,

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+\nu}k!\,\Gamma(k+\nu+1)}x^{2k+\nu} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)}\Big(\frac{x}{2}\Big)^{2k+\nu}.$$

A second solution may be found using reduction of order, but it is not of the same form as a Bessel function of the first kind. Therefore, we refer to it as a ***Bessel function of the second kind***, which is also known as a ***Neumann function*** or ***Weber function***.

When $2\nu \notin \mathbb{N}$, discussion in Section 7.7 shows that

$$y_2(x) = \sum_{k=0}^{\infty} a_{2k}(-\nu)x^{2k-\nu}$$

is a linearly independent (w.r.t. J_ν) solution to Bessel's equation of order ν, where $a_{2k}(-\nu)$ is given by (7.29). Let $\Gamma : \mathbb{C}\backslash\{0,-1,-2,\cdots\} \to \mathbb{C}$ be the analytic continuation of $\Gamma : \mathbb{R}^+ \to \mathbb{R}$ satisfying $\Gamma(z+1) = z\Gamma(z)$ for $-z \notin \mathbb{N} \cup \{0\}$ (and $1/\Gamma(z) = 0$ for all $-z \in \mathbb{N} \cup \{0\}$). The function $J_{-\nu}$ is the function y_2 with the choice of $a_0 = \dfrac{1}{2^{-\nu}\Gamma(1-\nu)}$; that is,

$$J_{-\nu}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k-\nu+1)} x^{2k-\nu} .$$

We note that when $\nu \in \mathbb{N} \cup \{0\}$, using the property that $1/\Gamma(z) = 0$ for all $-z \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} J_{-\nu}(x) &= \sum_{k=\nu}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k-\nu+1)} \Big(\frac{x}{2}\Big)^{2k-\nu} = \sum_{k=0}^{\infty} \frac{(-1)^{k+\nu}}{(k+\nu)!\,\Gamma(k+1)} \Big(\frac{x}{2}\Big)^{2k+\nu} \\ &= (-1)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)} \Big(\frac{x}{2}\Big)^{2k+\nu} = (-1)^\nu J_\nu(x) . \end{aligned}$$

Definition 7.35 (Bessel's function of the second kind)**.**

1. For $\nu \notin \mathbb{N} \cup \{0\}$, the ***Bessel function of the second kind of order ν*** is the function Y_ν defined as the following linear combination of J_ν and $J_{-\nu}$:

$$Y_\nu(x) = \frac{\cos(\nu\pi)J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)} .$$

2. For $m \in \mathbb{N} \cup \{0\}$, the ***Bessel function of the second kind of order*** m is the function Y_m defined by

$$Y_m(x) = \lim_{\nu\to m} \frac{\cos(\nu\pi)J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)} .$$

• **Properties of Bessel's functions:** Here we lists some properties of Bessel's functions.

1. **Some recurrence relation**: Using the series representation of Bessel's function J_ν, it is not difficult to show that J_ν satisfies

$$\frac{d}{dx}[x^\nu J_\nu(x)] = x^\nu J_{\nu-1}(x) , \qquad \frac{d}{dx}[x^{-\nu} J_\nu(x)] = -x^{-\nu} J_{\nu+1}(x) ,$$
$$J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x) - J_{\nu-1}(x) , \qquad J_{\nu+1}(x) = J_{\nu-1}(x) - 2J_\nu'(x) .$$

2. **Some asymptotic behaviors**:

$$\begin{aligned} J_\nu(x) &\approx \sqrt{\frac{2}{\pi x}} \cos\Big(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\Big) \quad x \gg 1 , \\ Y_\nu(x) &\approx \sqrt{\frac{2}{\pi x}} \sin\Big(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\Big) \quad x \gg 1 . \end{aligned}$$

8 Matrix Methods for Linear Systems

8.1 Introduction

There are several reasons that we should consider system of first order ODEs, and here we provide two of them.

1. In real life, a lot of phenomena can be modelled by system of first order ODE. For example, the Lotka–Volterra equation or the predator-prey equation:

$$\begin{aligned} p' &= \gamma p - \alpha pq\,, \\ q' &= \beta q + \delta pq\,. \end{aligned}$$

   in Example 1.9 can be used to described a predator-prey system. Let $\boldsymbol{x} \equiv (x_1, x_2) = (p, q)^{\mathrm{T}}$ and $\boldsymbol{F}(t, \boldsymbol{x}) = (\gamma x_1 - \alpha x_1 x_2, \beta x_2 + \delta x_1 x_2)^{\mathrm{T}}$. Then the Lotka-Volterra equation can also be written as

$$\boldsymbol{x}'(t) = \boldsymbol{F}\big(t, \boldsymbol{x}(t)\big)\,. \tag{8.1}$$

2. Suppose that we are considering a scalar n-th order ODE

$$y^{(n)}(t) = f\big(t, y(t), y'(t), \cdots, y^{(n-1)}(t)\big)\,.$$

   Let $x_1(t) = y(t)$, $x_2(t) = y'(t)$, $\cdots$, $x_n(t) = y^{(n-1)}(t)$. Then $(x_1, \cdots, x_n)$ satisfies

$$x_1'(t) = x_2(t)\,, \tag{8.2a}$$

$$x_2'(t) = x_3(t)\,, \tag{8.2b}$$

$$\vdots \;=\; \vdots \tag{8.2c}$$

$$x_n'(t) = f\big(t, x_1(t), x_2(t), \cdots, x_n(t)\big)\,. \tag{8.2d}$$

   Let $\boldsymbol{x} = (x_1, \cdots, x_n)^{\mathrm{T}}$ be an n-vector, and $\boldsymbol{F}(t, \boldsymbol{x}) = \big(x_2, x_3, \cdots, x_n, f(t, x_1, x_2, \cdots, x_n)\big)^{\mathrm{T}}$ be a vector-valued function. Then (8.2) can also be written as (8.1).

Definition 8.1. The system of ODE (8.1) is said to be ***linear*** if $\boldsymbol{F}$ is of the form

$$\boldsymbol{F}(t, \boldsymbol{x}) = \boldsymbol{P}(t)\boldsymbol{x} + \boldsymbol{g}(t)$$

for some matrix-valued function $\boldsymbol{P} = \big[p_{ij}(t)\big]_{n\times n}$. (8.1) is said to be homogeneous if $\boldsymbol{g}(t) = 0$.

Example 8.2. Consider the second order ODE

$$y'' - y' - 2y = \sin t\,. \tag{8.3}$$

Let $x_1(t) = y(t)$ and $x_2(t) = y'(t)$. Then $\boldsymbol{x} = (x_1, x_2)^{\mathrm{T}}$ satisfies

$$\boldsymbol{x}'(t) = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ \sin t \end{bmatrix}\,. \tag{8.4}$$

Therefore, the second order linear ODE (8.3) corresponds to a system of first order linear ODE (8.4).
Review: to solve (8.3), we use the method of variation of parameters and assume that the solution to (8.3) can be written as

$$y(t) = u_1(t)e^{2t} + u_2(t)e^{-t},$$

where $\{e^{2t}, e^{-t}\}$ is a fundamental set of (8.3). By the additional assumption $u_1'(t)e^{2t} + u_2'(t)e^{-t} = 0$, we find that

$$\begin{bmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \sin t \end{bmatrix}.$$

Therefore, with $W(t)$ denoting the Wronskian of $\{e^{2t}, e^{-t}\}$, we have

$$u_1'(t) = \frac{1}{W(t)} \det\left(\begin{bmatrix} 0 & e^{-t} \\ \sin t & -e^{-t} \end{bmatrix} \right) = \frac{-e^{-t}\sin t}{-3e^t} = \frac{1}{3}e^{-2t}\sin t$$

and

$$u_2'(t) = \frac{1}{W(t)} \det\left(\begin{bmatrix} e^{2t} & 0 \\ 2e^{2t} & \sin t \end{bmatrix} \right) = \frac{e^{2t}\sin t}{-3e^t} = -\frac{1}{3}e^{t}\sin t$$

which further implies that a particular solution is

$$\begin{aligned} y(t) &= -\frac{2e^{-2t}\sin t + e^{-2t}\cos t}{15}e^{2t} + \frac{e^t\cos t - e^t \sin t}{6}e^{-t} \\ &= -\frac{2\sin t + \cos t}{15} + \frac{\cos t - \sin t}{6} = \frac{\cos t - 3\sin t}{10}. \end{aligned}$$

This particular solution provides a particular solution to (8.4):

$$\boldsymbol{x}(t) = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} \dfrac{\cos t - 3\sin t}{10} \\ -\dfrac{\sin t + 3\cos t}{10} \end{bmatrix}.$$

Example 8.3. The ODE

$$\boldsymbol{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \boldsymbol{x} \tag{8.5}$$

is a system of first order linear homogeneous ODE. Suppose the initial condition is given by $\boldsymbol{x}(0) = (x_{10}, x_{20})^{\mathrm{T}}$.

1. Let $\boldsymbol{x} = (x_1, x_2)^{\mathrm{T}}$. Then

$$x_1'(t) = x_1(t) + x_2(t), \tag{8.6a}$$

$$x_2'(t) = 4x_1(t) + x_2(t). \tag{8.6b}$$

Note that (8.6a) implies $x_2 = x_1' - x_1$; thus replacing x_2 in (8.6) by $x_2 = x_1' - x_1$ we find that

$$x_1'' - x_1' = 4x_1 + x_1' - x_1 \qquad \text{or} \qquad x_1'' - 2x_1' - 3x_1 = 0.$$

Therefore, $x_1(t) = C_1e^{3t} + C_2e^{-t}$ and this further implies that $x_2(t) = 2C_1e^{3t} - 2C_2e^{-t}$; thus the solution to (8.5) can be expressed as

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + C_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}.$$

2. Let $\boldsymbol{x}_h(k) \approx \boldsymbol{x}(kh) = (x_1(kh), x_2(kh))^{\mathrm{T}}$ be the approximated value of $\boldsymbol{x}$ at the k-th step. Since

$$\boldsymbol{x}((k+1)h) \approx \boldsymbol{x}(kh) + h \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \boldsymbol{x}_h(k) \,,$$

we consider the (explicit) Euler scheme

$$\boldsymbol{x}_h(k+1) = \boldsymbol{x}_h(k) + h \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \boldsymbol{x}_h(k) = \Big(\mathrm{Id} + h \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\Big)^k \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \,,$$

and we expect that for $t > 0$ and $k = t/h$, then $\boldsymbol{x}_h(k) \to \boldsymbol{x}(t)$ as $h \to 0$.

To compute the k-th power of the matrix $\mathrm{Id} + h \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$, we diagonize the matrix and obtain that

$$\mathrm{Id} + h \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1+h & h \\ 4h & 1+h \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1-h & 0 \\ 0 & 1+3h \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}^{-1} ;$$

thus

$$\Big(\mathrm{Id} + h \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\Big)^k = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} (1-h)^k & 0 \\ 0 & (1+3h)^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}^{-1} .$$

As a consequence, using the limit $(1-h)^{\frac{t}{h}} \to e^{-t}$ and $(1+3h)^{\frac{t}{h}} \to e^{3t}$ as $t \to 0$, we find that

$$\begin{aligned}
\boldsymbol{x}(t) = \lim_{h \to 0} \boldsymbol{x}_h\big(\tfrac{t}{h}\big) &= \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \\
&= \frac{1}{4} \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \\
&= \frac{1}{4} \begin{bmatrix} 2e^{-t} + 2e^{3t} & -e^{-t} + e^{3t} \\ -4e^{-t} + 4e^{3t} & 2e^{-t} + 2e^{3t} \end{bmatrix} \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \\
&= \frac{1}{4} \begin{bmatrix} 2x_{10} + x_{20} \\ 4x_{10} + 2x_{20} \end{bmatrix} e^{3t} + \frac{1}{4} \begin{bmatrix} 2x_{10} - x_{20} \\ -4x_{10} + 2x_{20} \end{bmatrix} e^{-t} .
\end{aligned}$$

Choose $\boldsymbol{x}_0 = (1,2)^{\mathrm{T}}$ and $\boldsymbol{x}_0 = (1,-2)^{\mathrm{T}}$, we find that

$$\boldsymbol{x}_1(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} \qquad \text{and} \qquad \boldsymbol{x}_2(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$$

are both solution to (8.5).

Remark 8.4. For $a, b, c, d \in \mathbb{R}$ being given constants, suppose that x_1 and x_2 satisfy the system of first order linear ODE

$$x_1' = ax_1 + bx_2 \,, \tag{8.7a}$$

$$x_2' = cx_1 + dx_2 \,. \tag{8.7b}$$

Using (8.7a), we have $bx_2 = x_1' - ax_2$; thus (8.7b) implies that x_1 satisfies

$$x_1'' - (a+d)x_1' + (ad - bc)x_1 = 0 \,.$$

We note that the characteristic equation for the ODE above is exactly the characteristic equation of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Moreover, suppose that $\lambda_1 \neq \lambda_2$ are distinct zeros of the characteristic equation, then

$$x_1(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}\,.$$

Similarly, $x_2(t) = C_3 e^{\lambda_1 t} + C_4 e^{\lambda_2 t}$ for some C_3, C_4 satisfying

$$\begin{aligned} \lambda_1 C_1 e^{\lambda_1 t} + \lambda_2 C_2 e^{\lambda_2 t} &= (aC_1 + bC_3) e^{\lambda_1 t} + (aC_2 + bC_4) e^{\lambda_2 t}\,, \\ \lambda_1 C_3 e^{\lambda_1 t} + \lambda_2 C_2 e^{\lambda_2 t} &= (cC_1 + dC_3) e^{\lambda_1 t} + (cC_2 + dC_4) e^{\lambda_2 t}\,. \end{aligned}$$

Since $\{e^{\lambda_1 t}, e^{\lambda_2 t}\}$ are linearly independent, we must have that C_1, C_2, C_3, C_4 satisfy

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} C_1 \\ C_3 \end{bmatrix} = \lambda_1 \begin{bmatrix} C_1 \\ C_3 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} C_2 \\ C_4 \end{bmatrix} = \lambda_2 \begin{bmatrix} C_2 \\ C_4 \end{bmatrix}.$$

In other words, $(C_1, C_3)^{\mathrm{T}}$ and $(C_2, C_4)^{\mathrm{T}}$ are the eigenvectors of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ associated with eigenvalues λ_1 and λ_2, respectively. Therefore,

$$\boldsymbol{x}(t) = \begin{bmatrix} C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \\ C_3 e^{\lambda_1 t} + C_4 e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} C_1 \\ C_3 \end{bmatrix} e^{\lambda_1 t} + \begin{bmatrix} C_2 \\ C_4 \end{bmatrix} e^{\lambda_2 t} = \boldsymbol{u}_1 e^{\lambda_1 t} + \boldsymbol{u}_2 e^{\lambda_2 t}\,,$$

where $\boldsymbol{u}_1 = (C_1, C_3)^{\mathrm{T}}$ and $\boldsymbol{u}_2 = (C_2, C_4)^{\mathrm{T}}$.

8.2 Basic Theory of Systems of First Order Equations

Similar to Theorem 1.24, we have the following

Theorem 8.5. *Let $\boldsymbol{x}_0 = (x_{10}, x_{20}, \cdots, x_{n0})$ be a point in $\mathbb{R}^n$, $\mathcal{V} \subseteq \mathbb{R}^n$ be an open set containing $\boldsymbol{x}_0$, and $\boldsymbol{F} : (\alpha, \beta) \times \mathcal{V} \to \mathbb{R}^n$ be a vector-valued function of t and $\boldsymbol{x}$ such that $\boldsymbol{F} = (F_1, \cdots, F_n)$ and the partial derivative $\dfrac{\partial F_i}{\partial x_j}$ is continuous in $(\alpha, \beta) \times \mathcal{V}$ for all $i, j \in \{1, 2, \cdots, n\}$. Then in some interval $t \in (t_0 - h, t_0 + h) \subseteq (\alpha, \beta)$, there exists a unique solution $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ to the initial value problem*

$$\boldsymbol{x}' = \boldsymbol{F}(t, \boldsymbol{x}) \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0\,. \tag{8.8}$$

Moreover, if (8.8) is linear and $\mathcal{V} = \mathbb{R}^n$, then the solution exists throughout the interval (α, β).

The proof of this theorem is almost the same as the proof of Theorem 1.24 (by simply replacing $|\cdot|$ with $\|\cdot\|_{\mathbb{R}^n}$), and is omitted.

Theorem 8.6 (Principle of Superposition)**.** *If the vector $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are solutions of the linear system $\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x}$, then the linear combination $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2$ is also a solution for any constants c_1 and c_2.*

Example 8.7. Consider the system of ODE

$$\boldsymbol{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \boldsymbol{x} \tag{8.5}$$

and note that $\boldsymbol{x}_1(t) = \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$ and $\boldsymbol{x}_2(t) = \begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$ are solutions to this ODE; that is,

$$\boldsymbol{x}_1'(t) = \begin{bmatrix} 3 \\ 6 \end{bmatrix} e^{3t} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \boldsymbol{x}_1(t)$$

and

$$\boldsymbol{x}_2'(t) = \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-t} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \boldsymbol{x}_2(t)\,.$$

Therefore, $y = c_1\boldsymbol{x}_1(t) + c_2\boldsymbol{x}_2(t)$ is also a solution to (8.5).

Theorem 8.8. *Let $\mathcal{M}_{n\times n}$ denote the space of $n\times n$ real matrices, and $\boldsymbol{P} : (\alpha,\beta) \to \mathcal{M}_{n\times n}$ be a matrix-valued function. If the vector-valued functions $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n$ are linearly independent solutions to*

$$\boldsymbol{x}'(t) = \boldsymbol{P}(t)\boldsymbol{x}(t) \tag{8.9}$$

then each solution $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ to (8.9) can be expressed as a linear combination of $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n$ in exact one way; that is, there exists a unique vector $(c_1, \cdots, c_n)$ such that

$$\boldsymbol{\varphi}(t) = c_1\boldsymbol{x}_1(t) + \cdots + c_n\boldsymbol{x}_n(t)\,. \tag{8.10}$$

Proof. By Theorem 8.5, for each $\boldsymbol{e}_i = (\underbrace{0, \cdots, 0}_{(i-1)\text{ slots}}, 1, 0, \cdots, 0)$, there exists a unique solution $\boldsymbol{x} = \boldsymbol{\varphi}_i(t)$ to (8.9) satisfying the initial data $\boldsymbol{x}(0) = \boldsymbol{e}_i$. The set $\{\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \cdots, \boldsymbol{\varphi}_n\}$ are linearly independent for otherwise there exists a non-zero vector $(c_1, \cdots, c_n)$ such that

$$c_1\boldsymbol{\varphi}_1(t) + c_2\boldsymbol{\varphi}_2(t) + \cdots + c_n\boldsymbol{\varphi}_n(t) = \mathbf{0}$$

which, by setting $t = 0$, would imply that $(c_1, c_2, \cdots, c_n) = \mathbf{0}$, a contradiction.

We note that every solution $\boldsymbol{x}(t)$ to (8.9) can be uniquely expressed by

$$\boldsymbol{x}(t) = x_1(0)\boldsymbol{\varphi}_1(t) + x_2(0)\boldsymbol{\varphi}_2(t) + \cdots + x_n(0)\boldsymbol{\varphi}_n(t)\,. \tag{8.11}$$

In fact, $\boldsymbol{x}(t)$ and $x_1(0)\boldsymbol{\varphi}_1(t) + \cdots + x_n(0)\boldsymbol{\varphi}_n(t)$ are both solutions to (8.9) satisfying the initial data

$$\boldsymbol{x}(0) = \big(x_1(0), \cdots, x_n(0)\big)^{\mathrm{T}};$$

thus by uniqueness of the solution, (8.11) holds.

Now, since $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n$ are solution to (8.9), we find that

$$\operatorname{span}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n) \subseteq \operatorname{span}(\boldsymbol{\varphi}_1, \cdots, \boldsymbol{\varphi}_n)\,.$$

Since $\{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n\}$ are linearly independent, $\dim\big(\operatorname{span}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n)\big) = n$; thus by the fact that $\dim\big(\operatorname{span}(\boldsymbol{\varphi}_1, \cdots, \boldsymbol{\varphi}_n)\big) = n$, we must have

$$\operatorname{span}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n) = \operatorname{span}(\boldsymbol{\varphi}_1, \cdots, \boldsymbol{\varphi}_n)\,.$$

Therefore, every solution $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ of (8.9) can be (uniquely) expressed by (8.10). □

Definition 8.9. Let $\mathcal{P}(t) \in \mathcal{M}_{n\times n}$, and $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n$ be linearly independent solutions to (8.9). Then $\{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n\}$ is called a ***fundamental set*** of (8.9), the matrix $\boldsymbol{\Psi}(t) = \Big[[\boldsymbol{x}_1(t)] \vdots [\boldsymbol{x}_2(t)] \vdots \cdots \vdots [\boldsymbol{x}_n(t)]\Big]$ is called the ***fundamental matrix*** of (8.9), and $\boldsymbol{\varphi}(t) = c_1\boldsymbol{x}_1(t) + \cdots + c_n\boldsymbol{x}_n(t)$ is called the ***general solution*** of (8.9).

Theorem 8.10. *Let $\boldsymbol{P} : (\alpha, \beta) \to \mathcal{M}_{n\times n}$ be continuous matrix-valued function, $\boldsymbol{x}_p$ be a particular solution to the non-homogeneous system*

$$\boldsymbol{x}'(t) = \boldsymbol{P}(t)\boldsymbol{x}(t) + \boldsymbol{g}(t) \tag{8.12}$$

on (α, β), and $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n\}$ be a fundamental set of the ODE $\boldsymbol{x}'(t) = \boldsymbol{P}(t)\boldsymbol{x}(t)$. Then every solution to (8.12) can be expressed in the form

$$\boldsymbol{x}(t) = C_1\boldsymbol{x}_1(t) + C_2\boldsymbol{x}_2(t) + \cdots + C_n\boldsymbol{x}_n(t) + \boldsymbol{x}_p(t).$$

Theorem 8.11. *If $\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \cdots, \boldsymbol{\varphi}_n$ are solutions to (8.9), then*

$$\det\Big(\Big[[\boldsymbol{\varphi}_1] \vdots [\boldsymbol{\varphi}_2] \vdots \cdots \vdots [\boldsymbol{\varphi}_n]\Big]\Big)$$

is either identically zero or else never vanishes.

Recall Theorem 5.4 that for a collection of solutions $\{\varphi_1, \cdots, \varphi_n\}$ to a n-th order ODE

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1y' + p_0y = 0,$$

the derivative of Wronskian $W(t) = \begin{vmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1' & \varphi_2' & \cdots & \varphi_n' \\ \vdots & & \ddots & \vdots \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} & \cdots & \varphi_n^{(n-1)} \end{vmatrix}$ satisfies $\dfrac{d}{dt}W(t) = -p_{n-1}(t)W(t)$ which can be used to show that $W(t)$ is identically zero or else never vanishes. We use the same idea and try to find the derivative of the determinant $\mathrm{W}(t) \equiv \det\Big(\Big[[\boldsymbol{\varphi}_1] \vdots [\boldsymbol{\varphi}_2] \vdots \cdots \vdots [\boldsymbol{\varphi}_n]\Big]\Big)$.

Proof. Let $\mathrm{W}(t) \equiv \det\Big(\Big[[\boldsymbol{\varphi}_1] \vdots [\boldsymbol{\varphi}_2] \vdots \cdots \vdots [\boldsymbol{\varphi}_n]\Big]\Big)$, $\boldsymbol{P} = [p_{ij}]_{n\times n}$, and the i-th component of $\boldsymbol{\varphi}_j$ be $\varphi_j^{(i)}$; that is,

$$[\boldsymbol{\varphi}_j] = \Big[\varphi_j^{(1)}, \cdots, \varphi_j^{(n)}\Big]^{\mathrm{T}}.$$

Since $\varphi_j^{(i)\prime} = \sum\limits_{k=1}^{n} p_{ik}\varphi_j^{(k)}$, using the properties of the determinants we find that

$$\begin{vmatrix} \varphi_1^{(1)} & \varphi_2^{(1)} & \cdots & \cdots & \varphi_n^{(1)} \\ \vdots & \vdots & & & \vdots \\ \varphi_1^{(j-1)} & \varphi_2^{(j-1)} & \cdots & \cdots & \varphi_n^{(j-1)} \\ \varphi_1^{(j)\prime} & \varphi_2^{(j)\prime} & \cdots & \cdots & \varphi_n^{(j)\prime} \\ \varphi_1^{(j+1)} & \varphi_2^{(j+1)} & \cdots & \cdots & \varphi_n^{(j+1)} \\ \vdots & \vdots & & & \vdots \\ \varphi_1^{(n)} & \varphi_2^{(n)} & \cdots & \cdots & \varphi_n^{(n)} \end{vmatrix} = \begin{vmatrix} \varphi_1^{(1)} & \varphi_2^{(1)} & \cdots & \cdots & \varphi_n^{(1)} \\ \vdots & \vdots & & & \vdots \\ \varphi_1^{(j-1)} & \varphi_2^{(j-1)} & \cdots & \cdots & \varphi_n^{(j-1)} \\ \sum\limits_{k=1}^{n} p_{jk}\varphi_1^{(k)} & \sum\limits_{k=1}^{n} p_{jk}\varphi_2^{(k)} & \cdots & \cdots & \sum\limits_{k=1}^{n} p_{jk}\varphi_n^{(k)} \\ \varphi_1^{(j+1)} & \varphi_2^{(j+1)} & \cdots & \cdots & \varphi_n^{(j+1)} \\ \vdots & \vdots & & & \vdots \\ \varphi_1^{(n)} & \varphi_2^{(n)} & \cdots & \cdots & \varphi_n^{(n)} \end{vmatrix}$$

$$\overset{\text{“row operations”}}{=} \begin{vmatrix} \varphi_1^{(1)} & \varphi_2^{(1)} & \cdots & \cdots & \varphi_n^{(1)} \\ \vdots & \vdots & & & \vdots \\ \varphi_1^{(j-1)} & \varphi_2^{(j-1)} & \cdots & \cdots & \varphi_n^{(j-1)} \\ p_{jj}\varphi_1^{(j)} & p_{jj}\varphi_2^{(j)} & \cdots & \cdots & p_{jj}\varphi_n^{(j)} \\ \varphi_1^{(j+1)} & \varphi_2^{(j+1)} & \cdots & \cdots & \varphi_n^{(j+1)} \\ \vdots & \vdots & & & \vdots \\ \varphi_1^{(n)} & \varphi_2^{(n)} & \cdots & \cdots & \varphi_n^{(n)} \end{vmatrix} = p_{jj}\mathrm{W}.$$

Therefore,

$$\frac{d}{dt}\mathrm{W} = \begin{vmatrix} \varphi_1^{(1)\prime} & \varphi_2^{(1)\prime} & \cdots & \varphi_n^{(1)\prime} \\ \varphi_1^{(2)} & \varphi_2^{(2)} & \cdots & \varphi_n^{(2)} \\ \vdots & & \ddots & \vdots \\ \varphi_1^{(n)} & \varphi_2^{(n)} & \cdots & \varphi_n^{(n)} \end{vmatrix} + \begin{vmatrix} \varphi_1^{(1)} & \varphi_2^{(1)} & \cdots & \varphi_n^{(1)} \\ \varphi_1^{(2)\prime} & \varphi_2^{(2)\prime} & \cdots & \varphi_n^{(2)\prime} \\ \varphi_1^{(3)} & \varphi_2^{(3)} & \cdots & \varphi_n^{(3)} \\ \vdots & & \ddots & \vdots \\ \varphi_1^{(n)} & \varphi_2^{(n)} & \cdots & \varphi_n^{(n)} \end{vmatrix} + \cdots + \begin{vmatrix} \varphi_1^{(1)} & \varphi_2^{(1)} & \cdots & \varphi_n^{(1)} \\ \varphi_1^{(2)} & \varphi_2^{(2)} & \cdots & \varphi_n^{(2)} \\ \vdots & & \ddots & \vdots \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} & \cdots & \varphi_n^{(n-1)} \\ \varphi_1^{(n)\prime} & \varphi_2^{(n)\prime} & \cdots & \varphi_n^{(n)\prime} \end{vmatrix}$$
$$= (p_{11} + \cdots + p_{nn})\mathrm{W} = \mathrm{tr}(\boldsymbol{P})\mathrm{W}\,;$$

thus

$$\mathrm{W}(t) = \exp\Big(\int_{t_0}^t \mathrm{tr}(\boldsymbol{P})(s)\,ds\Big)\mathrm{W}(t_0)$$

which implies that W is identically zero (if $\mathrm{W}(t_0)$ is zero) or else never vanishes (if $\mathrm{W}(t_0) \neq 0$). □

Definition 8.12. If $\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \cdots, \boldsymbol{\varphi}_n$ are n solutions to (8.9), the determinant

$$\mathrm{W}(\boldsymbol{\varphi}_1, \cdots, \boldsymbol{\varphi}_n)(t) \equiv \det\Big(\Big[[\boldsymbol{\varphi}_1] \vdots [\boldsymbol{\varphi}_2] \vdots \cdots \vdots [\boldsymbol{\varphi}_n]\Big]\Big)$$

is called the ***Wronskian*** of $\{\boldsymbol{\varphi}_1, \cdots, \boldsymbol{\varphi}_n\}$.

Theorem 8.13. *Let $\boldsymbol{u}, \boldsymbol{v} : (\alpha, \beta) \to \mathbb{R}^n$ be real vector-valued functions. If $\boldsymbol{x}(t) = \boldsymbol{u}(t) + i\boldsymbol{v}(t)$ is a solution to* (8.9)*, so are $\boldsymbol{u}$ and $\boldsymbol{v}$.*

Proof. Since $\boldsymbol{x}(t) = \boldsymbol{u}(t) + i\boldsymbol{v}(t)$ is a solution to (8.9), $\boldsymbol{x}'(t) - \boldsymbol{P}(t)\boldsymbol{x}(t) = \mathbf{0}$; thus

$$\begin{aligned} \mathbf{0} &= \boldsymbol{u}'(t) + i\boldsymbol{v}'(t) - \boldsymbol{P}(t)\big(\boldsymbol{u}(t) + i\boldsymbol{v}(t)\big) = \boldsymbol{u}'(t) + i\boldsymbol{v}'(t) - \boldsymbol{P}(t)\boldsymbol{u}(t) - i\boldsymbol{P}(t)\boldsymbol{v}(t) \\ &= \boldsymbol{u}'(t) - \boldsymbol{P}(t)\boldsymbol{u}(t) + i\big(\boldsymbol{v}'(t) - \boldsymbol{P}(t)\boldsymbol{v}(t)\big)\,. \end{aligned}$$

Since $\boldsymbol{u}'(t) - \boldsymbol{P}(t)\boldsymbol{u}(t)$ and $\boldsymbol{v}'(t) - \boldsymbol{P}(t)\boldsymbol{v}(t)$ are both real vectors, we must have

$$\boldsymbol{u}'(t) - \boldsymbol{P}(t)\boldsymbol{u}(t) = \boldsymbol{v}'(t) - \boldsymbol{P}(t)\boldsymbol{v}(t) = \mathbf{0}\,.$$

Therefore, $\boldsymbol{u}$ and $\boldsymbol{v}$ are both solutions to (8.9). □

8.3 Homogeneous Linear Systems with Constant Coefficients

In this section, we consider the equation

$$\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t)\,, \tag{8.13}$$

where $\boldsymbol{A}$ is a constant $n \times n$ matrix.

8.3.1 The case that $\boldsymbol{A}$ has n linearly independent eigenvectors

By Remark 8.4, it is natural to first look at the eigenvalues and eigenvectors of $\boldsymbol{A}$. Suppose that $\boldsymbol{A}$ has real eigenvalues $\lambda_1, \cdots, \lambda_n$ with corresponding eigenvectors $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n$ such that $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n$ are linearly independent. Let $\mathbf{\Lambda} = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n) = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$ and $\boldsymbol{P} = \Big[[\boldsymbol{v}_1] \vdots [\boldsymbol{v}_2] \vdots \cdots \vdots [\boldsymbol{v}_n]\Big]$.

Then $\boldsymbol{A} = \boldsymbol{P}\mathbf{\Lambda}\boldsymbol{P}^{-1}$ which implies that

$$\boldsymbol{x}'(t) = \boldsymbol{P}\mathbf{\Lambda}\boldsymbol{P}^{-1}\boldsymbol{x}(t)\,.$$

Therefore, with $\boldsymbol{y}(t)$ denoting the vector $\boldsymbol{P}^{-1}\boldsymbol{x}(t)$, by the fact that $\boldsymbol{y}'(t) = \boldsymbol{P}\boldsymbol{x}'(t)$ (since $\boldsymbol{P}$ is a constant matrix), we have

$$\boldsymbol{y}'(t) = \mathbf{\Lambda}\boldsymbol{y}(t)\,. \tag{8.14}$$

In components, we obtain that for $1 \leqslant j \leqslant n$,

$$y_j'(t) = \lambda_j y_j(t)$$

if $\boldsymbol{y}(t) = \big(y_1(t), \cdots, y_n(t)\big)^{\mathrm{T}}$. As a consequence, if $\boldsymbol{y}(t_0) = \boldsymbol{y}_0 = (y_{01}, \cdots, y_{0n})^{\mathrm{T}}$ is given, we obtain that the solution to (8.14) (with initial data $\boldsymbol{y}(t_0) = \boldsymbol{y}_0$) can be written as

$$\boldsymbol{y}(t) = \begin{bmatrix} e^{\lambda_1(t-t_0)}y_{01} \\ e^{\lambda_2(t-t_0)}y_{02} \\ \vdots \\ e^{\lambda_n(t-t_0)}y_{0n} \end{bmatrix} = \begin{bmatrix} e^{\lambda_1(t-t_0)} & & & \\ & e^{\lambda_2(t-t_0)} & & \\ & & \ddots & \\ & & & e^{\lambda_n(t-t_0)} \end{bmatrix} \boldsymbol{y}_0\,;$$

thus the solution of (8.13) with initial data $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ (which implies that $\boldsymbol{y}_0 = \boldsymbol{P}^{-1}\boldsymbol{x}_0$) can be written as

$$\boldsymbol{x}(t) = \boldsymbol{P}\boldsymbol{y}(t) = \boldsymbol{P} \begin{bmatrix} e^{\lambda_1(t-t_0)} & & & \\ & e^{\lambda_2(t-t_0)} & & \\ & & \ddots & \\ & & & e^{\lambda_n(t-t_0)} \end{bmatrix} \boldsymbol{P}^{-1}\boldsymbol{x}_0\,. \tag{8.15}$$

Defining the exponential of an $n \times n$ matrix $\boldsymbol{M}$ by

$$e^{\boldsymbol{M}} = \mathbf{I}_{n\times n} + \boldsymbol{M} + \frac{1}{2!}\boldsymbol{M}^2 + \frac{1}{3!}\boldsymbol{M}^3 + \cdots + \frac{1}{k!}\boldsymbol{M}^k + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}\boldsymbol{M}^k\,,$$

by the fact that $(t\mathbf{\Lambda})^k = \begin{bmatrix} (\lambda_1 t)^k & & \\ & \ddots & \\ & & (\lambda_n t)^k \end{bmatrix}$, we find that

$$e^{t\mathbf{\Lambda}} = \begin{bmatrix} \sum\limits_{k=0}^{\infty} \frac{1}{k!}(\lambda_1 t)^k & & \\ & \ddots & \\ & & \sum\limits_{k=0}^{\infty} \frac{1}{k!}(\lambda_n t)^k \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}.$$

Therefore, (8.15) implies that the solution to (8.13) with initial data $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ can be expressed as

$$\boldsymbol{x}(t) = \boldsymbol{P}e^{(t-t_0)\mathbf{\Lambda}}\boldsymbol{P}^{-1}\boldsymbol{x}_0\,.$$

Moreover, (8.15) also implies that the solution to (8.13) with initial data $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ can be written as

$$\begin{aligned}
\boldsymbol{x}(t) &= \Big[[\boldsymbol{v}_1] \vdots \cdots \vdots [\boldsymbol{v}_n]\Big] \begin{bmatrix} e^{\lambda_1(t-t_0)} & & & \\ & e^{\lambda_2(t-t_0)} & & \\ & & \ddots & \\ & & & e^{\lambda_n(t-t_0)} \end{bmatrix} \begin{bmatrix} y_{01} \\ \vdots \\ y_{0n} \end{bmatrix} \\
&= \Big[e^{\lambda_1(t-t_0)}[\boldsymbol{v}_1] \vdots \cdots \vdots e^{\lambda_n(t-t_0)}[\boldsymbol{v}_n]\Big] \begin{bmatrix} y_{01} \\ \vdots \\ y_{0n} \end{bmatrix} \\
&= y_{01} e^{\lambda_1(t-t_0)} \boldsymbol{v}_1 + y_{02} e^{\lambda_2(t-t_0)} \boldsymbol{v}_2 + \cdots + y_{0n} e^{\lambda_n(t-t_0)} \boldsymbol{v}_n \,. \qquad (8.16)
\end{aligned}$$

In other words, solutions to (8.13) are linear combination of vectors $\big\{e^{\lambda_1(t-t_0)}\boldsymbol{v}_1, \cdots, e^{\lambda_n(t-t_0)}\boldsymbol{v}_n\big\}$.

On the other hand, using that $t\boldsymbol{A} = \boldsymbol{P}(t\boldsymbol{\Lambda})\boldsymbol{P}^{-1}$, we have $(t\boldsymbol{A})^k = \boldsymbol{P}(t\boldsymbol{\Lambda})^k\boldsymbol{P}^{-1}$; thus the definition of exponential of matrices provides that

$$\begin{aligned}
e^{(t-t_0)\boldsymbol{A}} &= \sum_{k=0}^{\infty} \frac{1}{k!}\big((t-t_0)\boldsymbol{A}\big)^k = \sum_{k=0}^{\infty} \frac{1}{k!}\big(\boldsymbol{P}((t-t_0)\boldsymbol{\Lambda})^k\boldsymbol{P}^{-1}\big) = \boldsymbol{P}\Big[\sum_{k=0}^{\infty} \frac{1}{k!}\big((t-t_0)\boldsymbol{\Lambda}\big)^k\Big]\boldsymbol{P}^{-1} \\
&= \boldsymbol{P}e^{(t-t_0)\boldsymbol{\Lambda}}\boldsymbol{P}^{-1} \,.
\end{aligned}$$

Therefore, the solution to (8.13) with initial data $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ can also be expressed as

$$\boldsymbol{x}(t) = e^{(t-t_0)\boldsymbol{A}}\boldsymbol{x}_0 \,. \qquad (8.17)$$

We remark that in contrast the solution to $x'(t) = ax(t)$, where a is a constant, can be written as

$$x(t) = e^{a(t-t_0)}x_0 \,,$$

where $x_0 = x(t_0)$ is the initial condition.

8.3.2 The case that $\boldsymbol{A}$ has complex eigenvalues

Now we consider the system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ when $\boldsymbol{A}$ has complex eigenvalues.

Example 8.14. Find a fundamental set of real-valued solution of the system

$$\boldsymbol{x}' = \begin{bmatrix} -1/2 & 1 \\ -1 & -1/2 \end{bmatrix} \boldsymbol{x} \,. \qquad (8.18)$$

We first diagonalize the matrix $\boldsymbol{A} \equiv \begin{bmatrix} -1/2 & 1 \\ -1 & -1/2 \end{bmatrix}$ and find that

$$\begin{bmatrix} -1/2 & 1 \\ -1 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} -1/2+i & 0 \\ 0 & -1/2-i \end{bmatrix} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}^{-1} .$$

Therefore, Remark 8.4 implies that

$$\boldsymbol{x}_1(t) = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(-1/2+i)t} = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{-\frac{t}{2}}(\cos t + i \sin t) = \begin{bmatrix} e^{-\frac{t}{2}}\cos t \\ -e^{-\frac{t}{2}}\sin t \end{bmatrix} + i \begin{bmatrix} e^{-\frac{t}{2}}\sin t \\ e^{-\frac{t}{2}}\cos t \end{bmatrix}$$

and

$$\boldsymbol{x}_2(t) = \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(-1/2-i)t} = \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{-\frac{t}{2}}(\cos t - i \sin t) = \begin{bmatrix} e^{-\frac{t}{2}} \cos t \\ -e^{-\frac{t}{2}} \sin t \end{bmatrix} - i \begin{bmatrix} e^{-\frac{t}{2}} \sin t \\ e^{-\frac{t}{2}} \cos t \end{bmatrix}$$

are both solutions to the ODE. By Theorem 8.13, $\boldsymbol{\varphi}_1(t) = \begin{bmatrix} e^{-\frac{t}{2}} \cos t \\ -e^{-\frac{t}{2}} \sin t \end{bmatrix}$ and $\boldsymbol{\varphi}_2(t) = \begin{bmatrix} e^{-\frac{t}{2}} \sin t \\ e^{-\frac{t}{2}} \cos t \end{bmatrix}$ are also solutions to (8.18).

To see the linear independence of $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$, we note that the Wronskian of $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ is

$$\mathrm{W}(t) = \begin{vmatrix} e^{-\frac{t}{2}} \cos t & e^{-\frac{t}{2}} \sin t \\ -e^{-\frac{t}{2}} \sin t & e^{-\frac{t}{2}} \cos t \end{vmatrix} = e^{-t}$$

which never vanishes. Therefore, $\{\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2\}$ is a fundamental set of (8.18).

In general, if the constant matrix $\boldsymbol{A}$ has complex eigenvalues $r_\pm = \lambda \pm i\mu$ with corresponding eigenvectors $\boldsymbol{u}_\pm$. Then

$$(\boldsymbol{A} - r_\pm \mathbf{I})\boldsymbol{u}_\pm = \mathbf{0} \Leftrightarrow (\boldsymbol{A} - \overline{r_\pm} \mathbf{I})\overline{\boldsymbol{u}_\pm} = \mathbf{0} \Leftrightarrow (\boldsymbol{A} - r_\mp \mathbf{I})\overline{\boldsymbol{u}_\pm} = \mathbf{0}\,.$$

Therefore, $\boldsymbol{u}_-$ could be chosen as the complex conjugate of $\boldsymbol{u}_+$. Let $\boldsymbol{u}_+ = \boldsymbol{a} + i\boldsymbol{b}$ and $\boldsymbol{u}_- = \boldsymbol{a} - i\boldsymbol{b}$ be eigenvectors associated with r_+ and r_-, respective, where $\boldsymbol{a}$, $\boldsymbol{b}$ are real vectors. Let $\boldsymbol{x}_1(t) = \boldsymbol{u}_+ e^{r_+ t}$ and $\boldsymbol{x}_2(t) = \boldsymbol{u}_- e^{r_- t}$. Then $\boldsymbol{x}_1, \boldsymbol{x}_2$ are both solutions to $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ since

$$\begin{aligned} \boldsymbol{x}_1'(t) &= r_+ \boldsymbol{u}_+ e^{r_+ t} = e^{r_+ t}(\boldsymbol{A}\boldsymbol{u}_+) = \boldsymbol{A}\boldsymbol{x}_1(t)\,, \\ \boldsymbol{x}_2'(t) &= r_- \boldsymbol{u}_- e^{r_- t} = e^{r_- t}(\boldsymbol{A}\boldsymbol{u}_-) = \boldsymbol{A}\boldsymbol{x}_2(t)\,. \end{aligned}$$

On the other hand, using the Euler identity we have

$$\begin{aligned} \boldsymbol{x}_1(t) &= (\boldsymbol{a} + i\boldsymbol{b})e^{(\lambda+i\mu)t} = (\boldsymbol{a} + i\boldsymbol{b})e^{\lambda t}(\cos \mu t + i \sin \mu t) \\ &= (\boldsymbol{a} \cos \mu t - \boldsymbol{b} \sin \mu t)e^{\lambda t} + i(\boldsymbol{a} \sin \mu t + \boldsymbol{b} \cos \mu t)e^{\lambda t}\,, \\ \boldsymbol{x}_2(t) &= (\boldsymbol{a} - i\boldsymbol{b})e^{(\lambda-i\mu)t} = (\boldsymbol{a} - i\boldsymbol{b})e^{\lambda t}(\cos \mu t - i \sin \mu t) \\ &= (\boldsymbol{a} \cos \mu t - \boldsymbol{b} \sin \mu t)e^{\lambda t} - i(\boldsymbol{a} \sin \mu t + \boldsymbol{b} \cos \mu t)e^{\lambda t}\,. \end{aligned}$$

Therefore, Theorem 8.13 implies that $\boldsymbol{\varphi}_1(t) \equiv (\boldsymbol{a} \cos \mu t - \boldsymbol{b} \sin \mu t)e^{\lambda t}$ and $\boldsymbol{\varphi}_2(t) \equiv (\boldsymbol{a} \sin \mu t + \boldsymbol{b} \cos \mu t)e^{\lambda t}$ are also solutions to $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$.

Now suppose that $\boldsymbol{A}$ is an $n \times n$ matrix which has k distinct complex eigenvalues denoted by $r_\pm^{(1)}, r_\pm^{(2)}, \cdots, r_\pm^{(k)}$ and $n - 2k$ distinct real eigenvalues $r_{2k+1}, \cdots, r_n$ with corresponding eigenvectors $\boldsymbol{u}_\pm^{(1)}$, $\boldsymbol{u}_\pm^{(2)}$, $\cdots$, $\boldsymbol{u}_\pm^{(k)}$, $\boldsymbol{u}_{2k+1}, \cdots, \boldsymbol{u}_k$, where

$$r_\pm^{(j)} = \lambda_j \pm i\mu_j \text{ for some } \lambda_j, \mu_j \in \mathbb{R}, \text{ and } \boldsymbol{u}_+^{(j)} = \overline{\boldsymbol{u}_-^{(j)}} = \boldsymbol{a}^{(j)} + i\boldsymbol{b}^{(j)}\,.$$

Then the general solutions of $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ is of the form

$$\boldsymbol{x}(t) = \sum_{j=1}^{k} \Big[C_1^{(j)} \big(\boldsymbol{a}^{(j)} \cos \mu_j t - \boldsymbol{b}^{(j)} \sin \mu_j t \big) + C_2^{(j)} \big(\boldsymbol{a}^{(j)} \sin \mu_j t + \boldsymbol{b}^{(j)} \cos \mu_j t \big) \Big] e^{\lambda_j t} + \sum_{j=2k+1}^{n} C_j \boldsymbol{u}_j e^{\lambda_j t}\,.$$

If $\boldsymbol{A}$ is a 2×2 matrix which has complex eigenvalues, then $\det(\boldsymbol{A}) \neq 0$; thus $\mathbf{0}$ is the only equilibrium of the system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$. Now we check the stability of this equilibrium. Let $\boldsymbol{u}, \boldsymbol{v}$ be given as above. Then the Wronskian of $\boldsymbol{u}, \boldsymbol{v}$ never vanishes. In fact,

$$
\begin{aligned}
& W[\boldsymbol{u},\boldsymbol{v}](t) \\
&= \begin{vmatrix} (a_1\cos\mu t - b_1\sin\mu t)e^{\lambda t} & (a_1\sin\mu t + b_1\cos\mu t)e^{\lambda t} \\ (a_2\cos\mu t - b_2\sin\mu t)e^{\lambda t} & (a_2\sin\mu t + b_2\cos\mu t)e^{\lambda t} \end{vmatrix} \\
&= e^{2\lambda t}\big[(a_1\cos\mu t - b_1\sin\mu t)(a_2\sin\mu t + b_2\cos\mu t) - (a_2\cos\mu t - b_2\sin\mu t)(a_1\sin\mu t + b_1\cos\mu t)\big] \\
&= e^{2\lambda t}(a_1b_2 - a_2b_1) \neq 0\,;
\end{aligned}
$$

thus $\{\boldsymbol{u},\boldsymbol{v}\}$ is a linearly independent set. Moreover, Theorem 8.8 implies that every solution to $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ can be expressed as a unique linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$ (thus every solution to $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ can be expressed as a unique linear combination of $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$). Therefore, we immediately find that $\mathbf{0}$ is an asymptotically stable equilibrium if and only if $\lambda < 0$.

Example 8.15. Consider the two-mass three-spring system

$$
\begin{aligned}
m_1\frac{d^2x_1}{dt^2} &= -(k_1+k_2)x_1 + k_2x_2 + F_1(t)\,, \\
m_2\frac{d^2x_2}{dt^2} &= k_2x_1 - (k_2+k_3)x_2 + F_2(t)
\end{aligned}
$$

which is used to model the motion of two objects shown in the figure below.

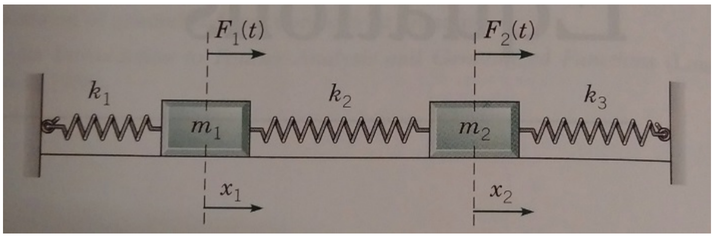

Figure 3: A two-mass three-spring system

Letting $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_1'$ and $y_4 = x_2'$, we find that $\boldsymbol{y} = (y_1, y_2, y_3, y_4)^{\mathrm{T}}$ satisfies

$$
\boldsymbol{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{k_1+k_2}{m_1} & \dfrac{k_2}{m_1} & 0 & 0 \\ \dfrac{k_2}{m_2} & -\dfrac{k_2+k_3}{m_2} & 0 & 0 \end{bmatrix}\boldsymbol{y} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{F_1(t)}{m_1} \\ \dfrac{F_2(t)}{m_2} \end{bmatrix}.
$$

Now suppose that $F_1(t) = F_2(t) = 0$, and $m_1 = 2$, $m_2 = \frac{9}{4}$, $k_1 = 1$, $k_2 = 3$, $k_3 = \frac{15}{4}$. Letting $\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & \frac{3}{2} & 0 & 0 \\ \frac{4}{3} & -3 & 0 & 0 \end{bmatrix}$, then $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$. The eigenvalue r of $\boldsymbol{A}$ satisfies

$$\det(\boldsymbol{A}-r\mathbf{I}) = \begin{vmatrix} -r & 0 & 1 & 0 \\ 0 & -r & 0 & 1 \\ -2 & \frac{3}{2} & -r & 0 \\ \frac{4}{3} & -3 & 0 & -r \end{vmatrix} = -r \begin{vmatrix} -r & 0 & 1 \\ \frac{3}{2} & -r & 0 \\ -3 & 0 & -r \end{vmatrix} + \begin{vmatrix} 0 & -r & 1 \\ -2 & \frac{3}{2} & 0 \\ \frac{4}{3} & -3 & -r \end{vmatrix}$$
$$= -r(-r^3 - 3r) + (6 - 2 + 2r^2) = r^4 + 5r^2 + 4 = 0\,.$$

Therefore, $\pm i, \pm 2i$ are eigenvalues of $\boldsymbol{A}$. Let $r_1 = i$, $r_2 = -i$, $r_3 = 2i$ and $r_4 = -2i$. Corresponding eigenvectors can be chosen as

$$\boldsymbol{u}_1 = \begin{bmatrix} 3 \\ 2 \\ 3i \\ 2i \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 0 \\ 3 \\ 2 \end{bmatrix}, \ \boldsymbol{u}_2 = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix} - i \begin{bmatrix} 0 \\ 0 \\ 3 \\ 2 \end{bmatrix}, \ \boldsymbol{u}_3 = \begin{bmatrix} 3 \\ -4 \\ 6i \\ -8i \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 0 \\ 6 \\ -8 \end{bmatrix}, \text{ and } \boldsymbol{u}_4 = \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \end{bmatrix} - i \begin{bmatrix} 0 \\ 0 \\ 6 \\ -8 \end{bmatrix}.$$

Therefore, with $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ denoting the vectors $(3, 2, 0, 0)^{\mathrm{T}}$, $(0, 0, 3, 2)^{\mathrm{T}}$, $(3, -4, 0, 0)^{\mathrm{T}}$ and $(0, 0, 6, -8)^{\mathrm{T}}$, respectively, the general solution to $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$ is

$$\boldsymbol{y}(t) = C_1(\boldsymbol{a}\cos t - \boldsymbol{b}\sin t) + C_2(\boldsymbol{a}\sin t + \boldsymbol{b}\cos t) + C_3(\boldsymbol{c}\cos 2t - \boldsymbol{d}\sin 2t) + C_4(\boldsymbol{c}\sin 2t + \boldsymbol{d}\cos 2t)\,.$$

In particular,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = C_1 \begin{bmatrix} 3\cos t \\ 2\cos t \end{bmatrix} + C_2 \begin{bmatrix} 3\sin t \\ 2\sin t \end{bmatrix} + C_3 \begin{bmatrix} 3\cos 2t \\ -4\cos 2t \end{bmatrix} + C_4 \begin{bmatrix} 3\sin 2t \\ -4\sin 2t \end{bmatrix}.$$

8.3.3 The case that $\boldsymbol{A}$ is not diagonalizable

In this case, there must be at least one eigenvalue λ of $\boldsymbol{A}$ such that the dimension of the eigenspace $\{\boldsymbol{v} \in \mathbb{C}^n \,|\, (\boldsymbol{A} - \lambda\mathbf{I})\boldsymbol{v} = \boldsymbol{0}\}$ is smaller than the algebraic multiplicity of λ.

Example 8.16. Let $\boldsymbol{A} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$ and consider the system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$. We first compute the eigenvalues (and the corresponding eigenvectors) and find that 2 is the only eigenvalue (with algebraic multiplicity 2), while $\boldsymbol{u} = [1, -1]^{\mathrm{T}}$ is the only eigenvector associated with this eigenvalue. Therefore, $\boldsymbol{A}$ is not diagonalizable.

Let $\boldsymbol{x} = [x, y]^{\mathrm{T}}$. Then x, y satisfy

$$x' = x - y\,, \tag{8.19a}$$
$$y' = x + 3y\,. \tag{8.19b}$$

Using (8.19a) we obtain $y = x - x'$; thus applying this identity to (8.19b) we find that x satisfies

$$x' - x'' = x + 3(x - x') \qquad \text{or equivalently,} \qquad x'' - 4x' + 4x = 0\,.$$

The characteristic equation to the ODE above is $r^2 - 4r + 4 = 0$ (which should be the same as the characteristic equation for the matrix $\boldsymbol{A}$); thus 2 is the only zero. From the discussion in Section 4.6, we find that the solution to ODE (that x satisfies) is

$$x(t) = C_1 e^{2t} + C_2 t e^{2t}\,.$$

Using $y = x - x'$, we find that the general solution to $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ is

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} C_1 e^{2t} + C_2 t e^{2t} \\ -(C_1 + C_2) e^{2t} - C_2 t e^{2t} \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} t e^{2t}.$$

Letting $\boldsymbol{v} = [0, 1]^{\mathrm{T}}$, we have $\boldsymbol{x} = (C_1 + C_2 t) e^{2t} \boldsymbol{u} + C_2 e^{2t} \boldsymbol{v}$.

Given an large non-diagonalizable square matrix $\boldsymbol{A}$, it is almost impossible to carry out the same computation as in Example 8.16, so we need to find another systematic way to find the solution to $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$. The following theorem states that $\boldsymbol{x}(t)$ given by (8.17) is always the solution to $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ with initial data $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$, even if $\boldsymbol{A}$ is not diagonalizable.

Theorem 8.17. *Let $\boldsymbol{A}$ be a square real constant matrix. Then the solution to $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ with initial data $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ is given by*

$$\boldsymbol{x}(t) = e^{(t-t_0)\boldsymbol{A}} \boldsymbol{x}_0 . \tag{8.17}$$

Proof. Let $\boldsymbol{y}(t) = e^{(t-t_0)\boldsymbol{A}} \boldsymbol{x}_0$. Then

$$\begin{aligned} \boldsymbol{y}(t) &= \Big(\mathbf{I} + (t-t_0)\boldsymbol{A} + \frac{(t-t_0)^2}{2!}\boldsymbol{A}^2 + \cdots\Big)\boldsymbol{y}_0 \\ &= \boldsymbol{y}_0 + (t-t_0)\boldsymbol{A}\boldsymbol{y}_0 + \frac{(t-t_0)^2}{2!}\boldsymbol{A}^2\boldsymbol{y}_0 + \cdots + \frac{(t-t_0)^k}{k!}\boldsymbol{A}^k\boldsymbol{y}_0 + \cdots . \end{aligned}$$

Therefore,

$$\begin{aligned} \boldsymbol{y}'(t) &= \boldsymbol{A}\boldsymbol{y}_0 + (t-t_0)\boldsymbol{A}\boldsymbol{y}_0 + \cdots + \frac{(t-t_0)^{k-1}}{k!}\boldsymbol{A}^k\boldsymbol{y}_0 + \cdots \\ &= \boldsymbol{A}\Big(\mathbf{I} + (t-t_0)\boldsymbol{A} + \frac{(t-t_0)^2}{2!}\boldsymbol{A}^2 + \cdots\Big)\boldsymbol{y}_0 = \boldsymbol{A}\boldsymbol{y} \end{aligned}$$

which implies that $\boldsymbol{y}$ is a solution to $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ with initial data $\boldsymbol{y}(t_0) = e^{0\cdot\boldsymbol{A}}\boldsymbol{x}_0 = \boldsymbol{x}_0$. By the uniqueness of the solution, we know that the solution to (8.13) with initial data $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ is given by (8.17). □

Having established Theorem 8.17, we now focus on how to compute the exponential of a square matrix if it is not diagonizable.

For a 2×2 matrix $\boldsymbol{A}$ with repeated eigenvalue λ whose corresponding eigenvector is $\boldsymbol{u}$ (but not more linearly independent eigenvector), by Example 8.16 we can conjecture that the general solution to $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ is

$$\boldsymbol{x}(t) = (C_1 + C_2 t) e^{\lambda t} \boldsymbol{u} + C_2 e^{\lambda t} \boldsymbol{v}$$

for some unknown vector $\boldsymbol{v}$. Now let us see what role $\boldsymbol{v}$ plays.

Since $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$, we must have

$$\lambda(C_1 + C_2 t) e^{\lambda t} \boldsymbol{u} + C_2 e^{\lambda t} \boldsymbol{u} + C_2 \lambda e^{\lambda t} \boldsymbol{v} = (C_1 + C_2 t) e^{\lambda t} \boldsymbol{A}\boldsymbol{u} + C_2 e^{\lambda t} \boldsymbol{A}\boldsymbol{v} .$$

By the fact that $\boldsymbol{A}\boldsymbol{u} = \lambda\boldsymbol{u}$ and C_2 is a general constant, the identity above implies that

$$\boldsymbol{u} = (\boldsymbol{A} - \lambda\mathbf{I})\boldsymbol{v} .$$

As a consequence, $\boldsymbol{v}$ satisfies $(\boldsymbol{A}-\lambda\mathbf{I})^2\boldsymbol{v}=0$. Moreover, we must have $\boldsymbol{v} \nparallel \boldsymbol{u}$ (for otherwise $\boldsymbol{u}=\mathbf{0}$) which implies that $\boldsymbol{u}, \boldsymbol{v}$ are linearly independent.

Let $\boldsymbol{P}=\begin{bmatrix}\boldsymbol{u} \vdots \boldsymbol{v}\end{bmatrix}$, and $\boldsymbol{\Lambda}=\begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix}$. Then $\boldsymbol{AP}=\boldsymbol{P\Lambda}$. Since $\boldsymbol{u}, \boldsymbol{v}$ are linearly independent, $\boldsymbol{P}$ is invertible; thus

$$\boldsymbol{A}=\boldsymbol{P\Lambda P}^{-1}\,.$$

Therefore, the same computations used in Section 8.3.1 shows that

$$e^{(t-t_0)\boldsymbol{A}}=\boldsymbol{P}e^{(t-t_0)\boldsymbol{\Lambda}}\boldsymbol{P}^{-1}\,.$$

Finally, taking $t_0=0$ (since the initial time could be translated to 0), then observing that

$$\Lambda^k=\begin{bmatrix}\lambda^k & k\lambda^{k-1}\\ 0 & \lambda^k\end{bmatrix}, \tag{8.20}$$

we conclude that

$$e^{t\boldsymbol{\Lambda}}=\sum_{k=0}^{\infty}\frac{t^k}{k!}\boldsymbol{\Lambda}^k=\begin{bmatrix}\sum\limits_{k=0}^{\infty}\dfrac{t^k}{k!}\lambda^k & \sum\limits_{k=1}^{\infty}\dfrac{t^k}{(k-1)!}\lambda^{k-1}\\ 0 & \sum\limits_{k=0}^{\infty}\dfrac{t^k}{k!}\lambda^k\end{bmatrix}=\begin{bmatrix}e^{\lambda t} & te^{\lambda t}\\ 0 & e^{\lambda t}\end{bmatrix}. \tag{8.21}$$

Having obtained the identity above, using (8.17) one immediately see that the general solution to $\boldsymbol{x}'=\boldsymbol{Ax}$ is given by

$$\boldsymbol{x}(t)=\begin{bmatrix}\boldsymbol{u} \vdots \boldsymbol{v}\end{bmatrix}\begin{bmatrix}e^{\lambda t} & te^{\lambda t}\\ 0 & e^{\lambda t}\end{bmatrix}\begin{bmatrix}C_1\\ C_2\end{bmatrix}.$$

In the following, we develop a general theory to compute $e^{(t-t_0)\boldsymbol{A}}$ for a square matrix $\boldsymbol{A}$.

Definition 8.18. A square matrix $\boldsymbol{A}$ is said to be of ***Jordan canonical form*** if

$$\boldsymbol{A}=\begin{bmatrix}\boldsymbol{A}_1 & \boldsymbol{O} & \cdots & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{A}_2 & \ddots & \boldsymbol{O}\\ \vdots & \ddots & \ddots & \vdots\\ \boldsymbol{O} & \cdots & \boldsymbol{O} & \boldsymbol{A}_\ell\end{bmatrix}, \tag{8.22}$$

where each $\boldsymbol{O}$ is zero matrix, and each $\boldsymbol{A}_i$ is a square matrix of the form $[\lambda]$ or

$$\begin{bmatrix}\lambda & 1 & 0 & \cdots & \cdots & \cdots & 0\\ 0 & \lambda & 1 & 0 & \cdots & \cdots & 0\\ \vdots & 0 & \ddots & \ddots & 0 & \cdots & \vdots\\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 & \vdots\\ \vdots & \vdots & \ddots & \ddots & \ddots & 1 & 0\\ \vdots & \vdots & \ddots & \ddots & 0 & \lambda & 1\\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & \lambda\end{bmatrix}$$

for some eigenvalue λ of $\boldsymbol{A}$.

We note that the diagonal elements of different $\boldsymbol{A}_i$ might be the same, and a diagonal matrix is of Jordan canonical form. Moreover, if $\boldsymbol{A}$ is of Jordan canonical form given by (8.22), then

$$\boldsymbol{A}^k = \begin{bmatrix} \boldsymbol{A}_1^k & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{A}_2^k & \ddots & \boldsymbol{O} \\ \vdots & \ddots & \ddots & \vdots \\ \boldsymbol{O} & \cdots & \boldsymbol{O} & \boldsymbol{A}_\ell^k \end{bmatrix} \qquad \text{and} \qquad e^{t\boldsymbol{A}} = \begin{bmatrix} e^{\boldsymbol{A}_1} & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{O} & e^{\boldsymbol{A}_2} & \ddots & \boldsymbol{O} \\ \vdots & \ddots & \ddots & \vdots \\ \boldsymbol{O} & \cdots & \boldsymbol{O} & e^{\boldsymbol{A}_\ell} \end{bmatrix}. \tag{8.23}$$

Example 8.19. Let $\boldsymbol{\Lambda} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$. Then $\boldsymbol{\Lambda}$ is of Jordan canonical form, and using (8.20) and (8.21) we conclude that

$$e^{t\boldsymbol{\Lambda}} = \begin{bmatrix} e^{\lambda t} & 0 & 0 \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}.$$

Example 8.20. Let $\boldsymbol{\Lambda} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$. Then $\boldsymbol{\Lambda}$ is of Jordan canonical form, and

$$\boldsymbol{\Lambda}^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} & \dfrac{k(k-1)}{2}\lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{bmatrix}.$$

Therefore,

$$e^{t\boldsymbol{\Lambda}} = \begin{bmatrix} \sum\limits_{k=0}^{\infty} \dfrac{1}{k!}t^k\lambda^k & \sum\limits_{k=1}^{\infty} \dfrac{1}{(k-1)!}t^k\lambda^{k-1} & \sum\limits_{k=2}^{\infty} \dfrac{1}{2(k-2)!}t^k\lambda^{k-1} \\ 0 & \sum\limits_{k=0}^{\infty} \dfrac{1}{k!}t^k\lambda^k & \sum\limits_{k=1}^{\infty} \dfrac{1}{(k-1)!}t^k\lambda^{k-1} \\ 0 & 0 & \sum\limits_{k=0}^{\infty} \dfrac{1}{k!}t^k\lambda^k \end{bmatrix} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{1}{2}t^2e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}.$$

In general, if $\boldsymbol{\Lambda} = \begin{bmatrix} \lambda & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & 0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 & \lambda & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & \lambda \end{bmatrix}$ is an $m \times m$ matrix, then with C_m^k denoting the number $\dfrac{k!}{m!(k-m)!}$ (if $k \geqslant m$, and 0 if $k < m$), we have

$$\boldsymbol{\Lambda}^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} & C_2^k\lambda^{k-2} & \cdots & \cdots & C_{m-1}^k\lambda^{k-m+1} \\ 0 & \lambda^k & k\lambda^{k-1} & \ddots & \ddots & C_{m-2}^k\lambda^{k-m+2} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & 0 & \lambda^k & k\lambda^{k-1} \\ 0 & \cdots & \cdots & \cdots & 0 & \lambda^k \end{bmatrix}$$

(which can be shown by induction using Pascal's formula). As a consequence,

$$e^{t\boldsymbol{\Lambda}} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{1}{2}t^2 e^{\lambda t} & \cdots & \cdots & \frac{t^{m-1}}{(m-1)!}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} & \ddots & \ddots & \frac{t^{m-2}}{(m-2)!}e^{\lambda t} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & \cdots & \cdots & \cdots & 0 & e^{\lambda t} \end{bmatrix}. \tag{8.24}$$

The reason for introducing the Jordan canonical form and computing the exponential of matrices of Jordan canonical form is because of the following

Theorem 8.21. *Every square matrix is similar to a matrix of Jordan canonical form. In other words, if $\boldsymbol{A} \in \mathcal{M}_{n\times n}$, then there exists an invertible $n \times n$ matrix $\boldsymbol{P}$ and a matrix $\boldsymbol{\Lambda}$ of Jordan canonical form such that*

$$\boldsymbol{A} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1}.$$

Given a Jordan decomposition $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1}$, we have $e^{t\boldsymbol{A}} = \boldsymbol{P}e^{t\boldsymbol{\Lambda}}\boldsymbol{P}^{-1}$ in which the exponential of $e^{t\boldsymbol{\Lambda}}$ can be obtained using (8.23) and (8.24); thus the computation of the exponential of a general square matrix $\boldsymbol{A}$ becomes easier as long as we know how to find the decomposition $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1}$.

- **How to obtain a Jordan decomposition of a square matrix $\boldsymbol{A}$?**

Definition 8.22 (Generalized Eigenvectors)**.** Let $\boldsymbol{A} \in \mathcal{M}_{n\times n}$. A vector $\boldsymbol{v} \in \mathbb{C}^n$ is called a generalized eigenvector of $\boldsymbol{A}$ associated with λ if $(\boldsymbol{A} - \lambda\mathbf{I})^p\boldsymbol{v} = \mathbf{0}$ for some positive integer p.

If $\boldsymbol{v}$ is a generalized eigenvector of $\boldsymbol{A}$ associated with λ, and p is the smallest positive integer for which $(\boldsymbol{A} - \lambda\mathbf{I})^p\boldsymbol{v} = \mathbf{0}$, then $(\boldsymbol{A} - \lambda\mathbf{I})^{p-1}\boldsymbol{v}$ is an eigenvector of $\boldsymbol{A}$ associated with λ. Therefore, λ is an eigenvalue of $\boldsymbol{A}$.

Definition 8.23 (Generalized Eigenspaces)**.** Let $\boldsymbol{A} \in \mathcal{M}_{n\times n}$ and λ be an eigenvalue of $\boldsymbol{A}$. The generalized eigenspace of $\boldsymbol{A}$ associated with λ, denoted by $\mathbf{K}_\lambda$, is the subset of $\mathbb{C}^n$ given by

$$\mathbf{K}_\lambda \equiv \left\{\boldsymbol{v} \in \mathbb{C}^n \,\middle|\, (\boldsymbol{A} - \lambda\mathbf{I})^p\boldsymbol{v} = \mathbf{0} \text{ for some positive integer } p\right\}.$$

- **The construction of Jordan decompositions**: Let $\boldsymbol{A} \in \mathcal{M}_{n\times n}$ be given.

Step 1: Let $\lambda_1, \lambda_2, \cdots, \lambda_k$ be all the eigenvalues of $\boldsymbol{A}$ with multiplicity $m_1, m_2, \cdots, m_k$. We first focus on how to determine the block

$$\boldsymbol{\Lambda}_j = \begin{bmatrix} \boldsymbol{\Lambda}_j^{(1)} & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{\Lambda}_j^{(2)} & \ddots & \boldsymbol{O} \\ \vdots & \ddots & \ddots & \vdots \\ \boldsymbol{O} & \cdots & \boldsymbol{O} & \boldsymbol{\Lambda}_j^{(r_j)} \end{bmatrix},$$

whose diagonal is a fixed eigenvalue λ_j with multiplicity m_j for some $j \in \{1, 2, \cdots, k\}$, and the size of $\mathbf{\Lambda}_j^{(i)}$ is not smaller than the size of $\mathbf{\Lambda}_j^{(i+1)}$ for $i = 1, \cdots, r_j - 1$. Once all $\mathbf{\Lambda}'_j s$ are obtained, then

$$\mathbf{\Lambda} = \begin{bmatrix} \mathbf{\Lambda}_1 & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{O} & \mathbf{\Lambda}_2 & \ddots & \boldsymbol{O} \\ \vdots & \ddots & \ddots & \vdots \\ \boldsymbol{O} & \cdots & \boldsymbol{O} & \mathbf{\Lambda}_k \end{bmatrix}.$$

Step 2: Let $\mathbf{E}_j$ and $\mathbf{K}_j$ denote the eigenspace and the generalized eigenspace associated with λ_j, respectively. Then $r_j = \dim(\mathbf{E}_j)$ and $m_j = \dim(\mathbf{K}_j)$. Determine the smallest integer n_j such that

$$m_j = \dim\left(\mathrm{Ker}(\boldsymbol{A} - \lambda_j \mathbf{I})^{n_j}\right).$$

Find the value

$$p_j^{(\ell)} = \dim(\mathrm{Ker}(\boldsymbol{A} - \lambda_j \mathbf{I})^{\ell}) \quad \text{for} \quad \ell \in \{1, 2, \cdots, n_j\}$$

and set $p_j^{(0)} = 0$. Construct an $r_j \times n_j$ matrix whose entries only takes the value 0 or 1 and for each $\ell \in \{1, \cdots, n_j\}$ only the first $p_j^{(\ell)} - p_j^{(\ell-1)}$ components takes value 1 in the ℓ-th column of this matrix. Let $s_j^{(i)}$ be the sum of the i-th row of the matrix just obtained. Then $\mathbf{\Lambda}_j^{(i)}$ is a $s_j^{(i)} \times s_j^{(i)}$ matrix.

Step 3: Next, let us determine matrix $\boldsymbol{P}$. Suppose that

$$\boldsymbol{P} = \left[\boldsymbol{u}_1^{(1)} \vdots \cdots \vdots \boldsymbol{u}_1^{(m_1)} \vdots \boldsymbol{u}_2^{(1)} \vdots \cdots \vdots \boldsymbol{u}_2^{(m_2)} \vdots \boldsymbol{u}_3^{(1)} \vdots \cdots \vdots \boldsymbol{u}_k^{(n)}\right].$$

Then $\boldsymbol{A}\left[\boldsymbol{u}_j^{(1)} \vdots \cdots \vdots \boldsymbol{u}_j^{(m_j)}\right] = \left[\boldsymbol{u}_j^{(1)} \vdots \cdots \vdots \boldsymbol{u}_j^{(m_j)}\right]\mathbf{\Lambda}_j$. Divide $\left\{\boldsymbol{u}_j^{(1)}, \cdots, \boldsymbol{u}_j^{(m_j)}\right\}$ into r_j groups:

$$\left\{\boldsymbol{u}_j^{(1)}, \cdots, \boldsymbol{u}_j^{(s_j^{(1)})}\right\}, \left\{\boldsymbol{u}_j^{(s_j^{(1)}+1)}, \cdots, \boldsymbol{u}_j^{(s_j^{(1)}+s_j^{(2)})}\right\}, \cdots, \text{ and } \left\{\boldsymbol{u}_j^{(s_j^{(1)}+\cdots+s_j^{(r_j-1)}+1)}, \cdots, \boldsymbol{u}_j^{(m_j)}\right\}.$$

For each $\ell \in \{1, \cdots, r_j\}$, we let the ℓ-th group refer to the group of vectors

$$\left\{\boldsymbol{u}_j^{(s_j^{(1)}+\cdots+s_j^{(\ell-1)}+1)}, \cdots, \boldsymbol{u}_j^{(s_j^{(1)}+\cdots+s_j^{(\ell)})}\right\}.$$

We then set up the first group by picking up an arbitrary non-zero vectors $\boldsymbol{v}_1 \in \mathrm{Ker}\big((\boldsymbol{A} - \lambda_j \mathbf{I})^{s_j^{(1)}} \backslash \mathrm{Ker}\big((\boldsymbol{A} - \lambda_j \mathbf{I})^{s_j^{(1)}-1}\big)$ and let

$$\boldsymbol{u}_j^{(i)} = (\boldsymbol{A} - \lambda_j \mathbf{I})^{s_j^{(1)}-i}\boldsymbol{v}_1 \quad \text{for } i \in \{1, \cdots, s_j^{(1)} - 1\}.$$

Inductively, once the first ℓ groups of vectors are set up, pick up an arbitrary non-zero vectors $\boldsymbol{v}_{\ell+1} \in \mathrm{Ker}\big((\boldsymbol{A} - \lambda_j \mathbf{I})^{s_j^{(\ell+1)}} \backslash \mathrm{Ker}\big((\boldsymbol{A} - \lambda_j \mathbf{I})^{s_j^{(\ell+1)}-1}\big)$ such that $\boldsymbol{v}_{\ell+1}$ is not in the span of the vectors from the first ℓ groups, and define

$$\boldsymbol{u}_j^{(s_j^{(1)}+\cdots+s_j^{(\ell)}+i)} = (\boldsymbol{A} - \lambda_j \mathbf{I})^{s_j^{(\ell+1)}-i}\boldsymbol{v}_{\ell+1} \quad \text{for } i \in \{1, \cdots, s_j^{(\ell+1)} - 1\}.$$

This defines the $(\ell + 1)$-th group. Keep on doing so for all $\ell \leqslant r_j$ and for $j \in \{1, \cdots, k\}$, we complete the construction of $\boldsymbol{P}$.

Example 8.24. Find the Jordan decomposition of the matrix $\boldsymbol{A} = \begin{bmatrix} 4 & -2 & 0 & 2 \\ 0 & 6 & -2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -2 & 0 & 6 \end{bmatrix}$.

If λ is an eigenvalue of $\boldsymbol{A}$, then λ satisfies

$$0 = \det(\boldsymbol{A} - \lambda \mathbf{I}) = \begin{vmatrix} 4-\lambda & -2 & 0 & 2 \\ 0 & 6-\lambda & -2 & 0 \\ 0 & 2 & 2-\lambda & 0 \\ 0 & -2 & 0 & 6-\lambda \end{vmatrix} = (4-\lambda) \begin{vmatrix} 6-\lambda & -2 & 0 \\ 2 & 2-\lambda & 0 \\ -2 & 0 & 6-\lambda \end{vmatrix}$$
$$= (4-\lambda)\big[(6-\lambda)^2(2-\lambda) + 4(6-\lambda)\big] = (6-\lambda)(4-\lambda)\big[(6-\lambda)(2-\lambda)+4\big]$$
$$= (\lambda-4)^3(\lambda-6)\,.$$

Let $\lambda_1 = 4$, $\lambda_2 = 6$, $m_1 = 3$ and $m_2 = 1$. Note that

$$\dim\big(\mathrm{Ker}(\boldsymbol{A} - 4\mathbf{I})\big) = 2 \quad \text{and} \quad \dim\big(\mathrm{Ker}(\boldsymbol{A} - 4\mathbf{I})^2\big) = 3\,.$$

Therefore, $n_1 = 2$ and $p_1^{(1)} = 2$, $p_1^{(2)} = 4$. We then construct the matrix according to Step 2 above, and the matrix is a 2×2 matrix given by $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. This matrix provides that $s_1 = 2$ and $s_2 = 1$; thus the block associated with the eigenvalue $\lambda = 4$, is $\begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Therefore, $\boldsymbol{\Lambda} = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$.

First, we note that the eigenvector associated with $\lambda = 6$ can be chosen as $(1,0,0,1)^{\mathrm{T}}$. Computing $\mathrm{Ker}\big((\boldsymbol{A} - 4\mathbf{I})\big)$ and $\mathrm{Ker}\big((\boldsymbol{A} - 4\mathbf{I})^2\big)$, we find that

$$\mathrm{Ker}\big((\boldsymbol{A} - 4\mathbf{I})\big) = \mathrm{span}\big((1,0,0,0)^{\mathrm{T}}, (0,1,1,1)^{\mathrm{T}}\big)\,,$$
$$\mathrm{Ker}\big((\boldsymbol{A} - 4\mathbf{I})^2\big) = \mathrm{span}\big((1,0,0,0)^{\mathrm{T}}, (0,1,0,2)^{\mathrm{T}}, (0,1,2,0)^{\mathrm{T}}\big)\,.$$

We note that either $(0,1,0,2)^{\mathrm{T}}$ or $(0,1,2,0)^{\mathrm{T}}$ is in $\mathrm{Ker}\big((\boldsymbol{A} - 4\mathbf{I})\big)$, we can choose $\boldsymbol{v} = (0,1,0,2)^{\mathrm{T}}$. Then $(\boldsymbol{A} - 4\mathbf{I})\boldsymbol{v} = (2,2,2,2)^{\mathrm{T}}$. Finally, for the third column of $\boldsymbol{P}$ we can choose either $(1,0,0,0)^{\mathrm{T}}$ or $(0,1,1,1)^{\mathrm{T}}$ (or even their linear combination) since these vectors are not in the span of $(2,2,2,2)^{\mathrm{T}}$ and $(0,1,0,2)$. Therefore,

$$\boldsymbol{P} = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \boldsymbol{P} = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 1 \end{bmatrix}$$

satisfies $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1}$.

Example 8.25. Let $\boldsymbol{A}$ be given in Example 8.24, and consider the system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$. Let $\boldsymbol{u}_1 = (2,2,2,2)^{\mathrm{T}}$, $\boldsymbol{u}_2 = (0,1,0,2)^{\mathrm{T}}$, $\boldsymbol{u}_3 = (1,0,0,0)^{\mathrm{T}}$ and $\boldsymbol{u}_4 = (1,0,0,1)^{\mathrm{T}}$. Then the general solution to

$\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ is given by

$$
\begin{aligned}
\boldsymbol{x}(t) &= \big[\boldsymbol{u}_1 \,\vdots\, \boldsymbol{u}_2 \,\vdots\, \boldsymbol{u}_3 \,\vdots\, \boldsymbol{u}_4\big] e^{t\boldsymbol{\Lambda}}(\boldsymbol{P}^{-1}\boldsymbol{x}_0) \\
&= \big[\boldsymbol{u}_1 \,\vdots\, \boldsymbol{u}_2 \,\vdots\, \boldsymbol{u}_3 \,\vdots\, \boldsymbol{u}_4\big] \begin{bmatrix} e^{4t} & te^{4t} & 0 & 0 \\ 0 & e^{4t} & 0 & 0 \\ 0 & 0 & e^{4t} & 0 \\ 0 & 0 & 0 & e^{6t} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} \\
&= \big[\boldsymbol{u}_1 \,\vdots\, \boldsymbol{u}_2 \,\vdots\, \boldsymbol{u}_3 \,\vdots\, \boldsymbol{u}_4\big] \begin{bmatrix} C_1e^{4t} + C_2te^{4t} \\ C_2e^{4t} \\ C_3e^{4t} \\ C_4e^{6t} \end{bmatrix} \\
&= (C_1e^{4t} + C_2te^{4t})\boldsymbol{u}_1 + C_2e^{4t}\boldsymbol{u}_2 + C_3e^{4t}\boldsymbol{u}_3 + C_4e^{6t}\boldsymbol{u}_4 \,,
\end{aligned}
$$

where $\boldsymbol{\Lambda}$ is given in Example 8.24, $\boldsymbol{x}_0$ is the value of $\boldsymbol{x}$ at $t = 0$ (which can be arbitrarily given), and $(C_1, C_2, C_3, C_4)^{\mathrm{T}} = \boldsymbol{P}^{-1}\boldsymbol{x}_0$.

Example 8.26. Let $\boldsymbol{A} = \begin{bmatrix} a & 0 & 1 & 0 & 0 \\ 0 & a & 0 & 1 & 0 \\ 0 & 0 & a & 0 & 1 \\ 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & a \end{bmatrix}$. Then the characteristic equation of $\boldsymbol{A}$ is $(a - \lambda)^5$; thus $\lambda = a$ is the only eigenvalue of $\boldsymbol{A}$. First we compute the kernel of $(\boldsymbol{A} - a\mathbf{I})^p$ for various p. With $\boldsymbol{e}_i = (\underbrace{0, \cdots, 0}_{(i-1)\text{-slots}}, 1, 0, \cdots, 0)^{\mathrm{T}}$ denoting the i-th vector in the standard basis of $\mathbb{R}^5$, we find that

$$
\begin{aligned}
\operatorname{Ker}((\boldsymbol{A} - a\mathbf{I})) &= \big\{\boldsymbol{e}_1 \,\big|\, x_1, x_2 \in \mathbb{R}\big\} = \operatorname{span}(\boldsymbol{e}_1, \boldsymbol{e}_2) \,, \\
\operatorname{Ker}((\boldsymbol{A} - a\mathbf{I})^2) &= \big\{(x_1, x_2, x_3, x_4, 0)^{\mathrm{T}} \,\big|\, x_1, x_2, x_3, x_4 \in \mathbb{R}\big\} = \operatorname{span}(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4) \,, \\
\operatorname{Ker}((\boldsymbol{A} - a\mathbf{I})^3) &= \mathbb{R}^5 = \operatorname{span}(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4, \boldsymbol{e}_5) \,.
\end{aligned}
$$

The matrix obtained by Step 2 is $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ which implies that the two Jordan blocks is of size 3×3 and 2×2. Therefore,

$$
\Lambda = \begin{bmatrix} a & 1 & 0 & 0 & 0 \\ 0 & a & 1 & 0 & 0 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & a & 1 \\ 0 & 0 & 0 & 0 & a \end{bmatrix}.
$$

We note that $\boldsymbol{e}_5 \in \operatorname{Ker}\big((\boldsymbol{A} - a\mathbf{I})^3\big) \backslash \operatorname{Ker}\big((\boldsymbol{A} - a\mathbf{I})^2\big)$; thus the first three column of $\boldsymbol{P}$ can be chosen as

$$
\boldsymbol{P}(1:3) = \big[(\boldsymbol{A} - a\mathbf{I})^2\boldsymbol{e}_5 \,\vdots\, (\boldsymbol{A} - a\mathbf{I})\boldsymbol{e}_5 \,\vdots\, \boldsymbol{e}_5\big] = \big[\boldsymbol{e}_1 \,\vdots\, \boldsymbol{e}_3 \,\vdots\, \boldsymbol{e}_5\big] \,.
$$

To find the last two columns, we try to find a vector $\boldsymbol{w} \in \operatorname{Ker}\big((\boldsymbol{A} - a\mathbf{I})^2\big) \backslash \operatorname{Ker}\big((\boldsymbol{A} - a\mathbf{I})\big)$ so that $\boldsymbol{w}$ is not in the span of $\{\boldsymbol{e}_1, \boldsymbol{e}_3, \boldsymbol{e}_5\}$. Therefore, we may choose $\boldsymbol{w} = \boldsymbol{e}_4$; thus the last two columns of $\boldsymbol{P}$ is

$$
\boldsymbol{P}(4:5) = \big[(\boldsymbol{A} - a\mathbf{I}\boldsymbol{e}_4 \,\vdots\, \boldsymbol{e}_4\big] = \big[\boldsymbol{e}_2 \,\vdots\, \boldsymbol{e}_4\big]
$$

which implies that

$$\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Example 8.27. Let $\boldsymbol{A}$ be given in Example 8.24, and consider the system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$. Following the procedure in Example 8.25, we find that the general solution to $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ is given by

$$\begin{aligned} \boldsymbol{x}(t) &= \big[\boldsymbol{e}_1 \vdots \boldsymbol{e}_3 \vdots \boldsymbol{e}_5 \vdots \boldsymbol{e}_2 \vdots \boldsymbol{e}_4\big] \begin{bmatrix} e^{at} & te^{at} & \frac{t^2}{2}e^{at} & 0 & 0 \\ 0 & e^{at} & te^{at} & 0 & 0 \\ 0 & 0 & e^{at} & 0 & 0 \\ 0 & 0 & 0 & e^{at} & te^{at} \\ 0 & 0 & 0 & 0 & e^{at} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{bmatrix} \\ &= \Big(C_1 e^{at} + C_2 te^{at} + \frac{C_3}{2} t^2 e^{at}\Big)\boldsymbol{e}_1 + (C_2 e^{at} + C_3 te^{at})\boldsymbol{e}_3 + C_3 e^{at}\boldsymbol{e}_5 + (C_4 e^{at} + C_5 te^{at})\boldsymbol{e}_2 + C_5 e^{at}\boldsymbol{e}_4 . \end{aligned}$$

8.4 Fundamental Matrices

In Definition 8.9 we have talked about the fundamental matrix of system $\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x}$. It is defined as a square matrix whose columns form an linearly independent set of solutions to the ODE $\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x}$. Let $\boldsymbol{\Psi}$ be a fundamental matrix of $\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x}$. Since each column of $\boldsymbol{\Psi}$ is a solution to the ODE, we must have

$$\boldsymbol{\Psi}'(t) = \boldsymbol{P}(t)\boldsymbol{\Psi}(t) .$$

By the linearly independence of columns of $\boldsymbol{\Psi}$, we must have

$$\boldsymbol{\Psi}'(t)\boldsymbol{\Psi}(t)^{-1} = \boldsymbol{P}(t) \qquad \text{for all } t \text{ in the interval of interest.} \tag{8.25}$$

A special kind of fundamental matrix $\boldsymbol{\Phi}$, whose initial value $\boldsymbol{\Phi}(t_0)$ is the identity matrix, is in particular helpful for constructing solutions to

$$\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x} , \tag{8.26a}$$

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0 . \tag{8.26b}$$

In fact, if $\boldsymbol{\Phi}$ is a fundamental matrix of system $\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x}$ satisfying $\boldsymbol{\Phi}(t_0) = \mathbf{I}$, then the solution to (8.26) is given by

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t)\boldsymbol{x}_0 .$$

It should be clear to the readers that the i-th column of $\boldsymbol{\Phi}$ is the solution to

$$\begin{aligned} \boldsymbol{x}' &= \boldsymbol{P}(t)\boldsymbol{x} , \\ \boldsymbol{x}(t_0) &= \boldsymbol{e}_i , \end{aligned}$$

where $\boldsymbol{e}_i = (\underbrace{0, \cdots, 0}_{(i-1)\text{-slots}}, 1, 0, \cdots, 0)^{\mathrm{T}}$ is the i-th vector in the standard basis of $\mathbb{R}^n$ (here we assume that the size of $\boldsymbol{P}$ is $n \times n$). Moreover, for each fundamental matrix $\boldsymbol{\Psi}$ of (8.26a), we have the relation

$$\boldsymbol{\Psi}(t) = \boldsymbol{\Phi}(t)\boldsymbol{\Psi}(t_0) .$$

Therefore, given a fundamental matrix $\mathbf{\Psi}$, we can easily construct the fundamental matrix $\mathbf{\Phi}(t)$ by

$$\mathbf{\Phi}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}(t_0)^{-1}.$$

Caution: Based on the discussions above and the information that the solution to the scalar equation $x' = p(t)x$ with initial data $x(t_0) = x_0$ is $x(t) = \exp\big(\int_{t_0}^t p(s)\,ds\big)x_0$, one might start guessing that the solution to (8.26) is

$$\boldsymbol{x}(t) = \exp\Big(\int_{t_0}^t \boldsymbol{P}(s)\,ds\Big)\boldsymbol{x}_0. \tag{8.27}$$

This is in fact **NOT TRUE** because in general $\boldsymbol{P}(s)\boldsymbol{P}(t) \neq \boldsymbol{P}(t)\boldsymbol{P}(s)$. Nevertheless, if $\boldsymbol{P}(s)\boldsymbol{P}(t) = \boldsymbol{P}(t)\boldsymbol{P}(s)$ for all s and t, then the solution to (8.26) is indeed given by (8.27). To see this, we first notice that

$$\boldsymbol{P}(t)\Big(\int_{t_0}^t \boldsymbol{P}(s)\,ds\Big) = \int_{t_0}^t \boldsymbol{P}(t)\boldsymbol{P}(s)\,ds = \int_{t_0}^t \boldsymbol{P}(s)\boldsymbol{P}(t)\,ds = \Big(\int_{t_0}^t \boldsymbol{P}(s)\,ds\Big)\boldsymbol{P}(t);$$

thus

$$\begin{aligned}\frac{d}{dt}\Big(\int_{t_0}^t \boldsymbol{P}(s)\,ds\Big)^k &= \boldsymbol{P}(t)\Big(\int_{t_0}^t \boldsymbol{P}(s)\,ds\Big)^{k-1} + \Big(\int_{t_0}^t \boldsymbol{P}(s)\,ds\Big)\boldsymbol{P}(t)\Big(\int_{t_0}^t \boldsymbol{P}(s)\,ds\Big)^{k-2} + \cdots \\ &\quad + \Big(\int_{t_0}^t \boldsymbol{P}(s)\,ds\Big)^{k-2}\boldsymbol{P}(t)\Big(\int_{t_0}^t \boldsymbol{P}(s)\,ds\Big) + \Big(\int_{t_0}^t \boldsymbol{P}(s)\,ds\Big)^{k-1}\boldsymbol{P}(t) \\ &= k\boldsymbol{P}(t)\Big(\int_{t_0}^t \boldsymbol{P}(s)\,ds\Big)^{k-1}.\end{aligned}$$

Therefore, the function given by (8.27) satisfies that

$$\begin{aligned}\frac{d}{dt}\exp\Big(\int_{t_0}^t \boldsymbol{P}(s)\,ds\Big)\boldsymbol{x}_0 &= \frac{d}{dt}\Big[\sum_{k=0}^\infty \frac{1}{k!}\Big(\int_{t_0}^t \boldsymbol{P}(s)\,ds\Big)^k\Big]\boldsymbol{x}_0 = \sum_{k=1}^\infty \frac{1}{(k-1)!}\boldsymbol{P}(t)\Big(\int_{t_0}^t \boldsymbol{P}(s)\,ds\Big)^{k-1}\boldsymbol{x}_0 \\ &= \boldsymbol{P}(t)\Big(\sum_{k=0}^\infty \frac{1}{k!}\Big(\int_{t_0}^t \boldsymbol{P}(s)\,ds\Big)^k\Big)\boldsymbol{x}_0 = \boldsymbol{P}(t)\exp\Big(\int_{t_0}^t \boldsymbol{P}(s)\,ds\Big).\end{aligned}$$

On the other hand, $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$. As a consequence, $\boldsymbol{x}(t)$ given by (8.27) is the solution to (8.26).

Now suppose that $\boldsymbol{P}(t) = \boldsymbol{A}$ is time-independent. Then by Theorem 8.17 we find that the fundamental matrix $\mathbf{\Phi}(t)$ is given by

$$\mathbf{\Phi}(t) = \boldsymbol{P}e^{(t-t_0)\mathbf{\Lambda}}\boldsymbol{P}^{-1},$$

where $\boldsymbol{P}\mathbf{\Lambda}\boldsymbol{P}^{-1}$ is a Jordan decomposition of $\boldsymbol{A}$. Moreover,

$$\mathbf{\Phi}(t)\mathbf{\Phi}(s) = \mathbf{\Phi}(s)\mathbf{\Phi}(t) \qquad \forall\, t, s \in \mathbb{R}. \tag{8.28}$$

To see this, let t_1, t_2 be given real number, and $\boldsymbol{x}_0 \in \mathbb{R}^n$ be a vector. By the existence and uniqueness theorem (Theorem 8.5), the solution to system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ with initial data $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ is given by $\boldsymbol{x}(t) = \mathbf{\Phi}(t)\boldsymbol{x}_0$ for all $t \in \mathbb{R}$.

On the other hand, again by the uniqueness of the solution, the solution $\boldsymbol{\varphi}_1$ to

$$\begin{aligned}\boldsymbol{\varphi}' &= \boldsymbol{A}\boldsymbol{\varphi}\,,\\ \boldsymbol{\varphi}(t_0) &= \boldsymbol{x}(t_1)\,,\end{aligned}$$

and the solution $\boldsymbol{\varphi}_2$ to

$$\begin{aligned}\boldsymbol{\varphi}' &= \boldsymbol{A}\boldsymbol{\varphi}\,,\\ \boldsymbol{\varphi}(t_0) &= \boldsymbol{x}(t_2)\,,\end{aligned}$$

satisfy that $\boldsymbol{\varphi}_1(t) = \boldsymbol{x}(t-t_0+t_1)$ and $\boldsymbol{\varphi}_2(t) = \boldsymbol{x}(t-t_0+t_2)$. Moreover, using the fundamental matrix $\boldsymbol{\Phi}$ we also have $\boldsymbol{\varphi}_1(t) = \boldsymbol{\Phi}(t)\boldsymbol{x}(t_1)$ and $\boldsymbol{\varphi}_2(t) = \boldsymbol{\Phi}(t)\boldsymbol{x}(t_2)$. Therefore,

$$\boldsymbol{\Phi}(t_2)\boldsymbol{\Phi}(t_1)\boldsymbol{x}_0 = \boldsymbol{\Phi}(t_2)\boldsymbol{x}(t_1) = \boldsymbol{\varphi}_1(t_2) = \boldsymbol{x}(t_1+t_2-t_0) = \boldsymbol{\varphi}_2(t_1) = \boldsymbol{\Phi}(t_1)\boldsymbol{\Phi}(t_2)\boldsymbol{x}_0\,.$$

Since $\boldsymbol{x}_0$ is arbitrary, we must have $\boldsymbol{\Phi}(t_2)\boldsymbol{\Phi}(t_1) = \boldsymbol{\Phi}(t_1)\boldsymbol{\Phi}(t_2)$; thus (8.28) is concluded.

8.5 Non-homogeneous Linear Systems

Now we consider the non-homogeneous linear system

$$\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x} + \boldsymbol{g}(t)\,, \tag{8.29a}$$

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0\,, \tag{8.29b}$$

for some non-zero vector-valued forcing $\boldsymbol{g}$. As in Definition 4.14 we said that a vector-valued function $\boldsymbol{x}_p(t)$ is called a ***particular solution*** to (8.29a) if $\boldsymbol{x}_p$ satisfies (8.29a). As long as a particular solution to (8.29a) is obtained, then the general solution to (8.29a) is given by

$$\boldsymbol{x}(t) = \boldsymbol{\Psi}(t)\boldsymbol{C} + \boldsymbol{x}_p(t)\,,$$

where $\boldsymbol{\Psi}$ is a fundamental matrix of $\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x}$, and $\boldsymbol{C}$ is an arbitrary constant vector. to satisfy the initial data (8.29b), we let $\boldsymbol{C} = \boldsymbol{\Psi}(t_0)^{-1}\big(\boldsymbol{x}_0 - \boldsymbol{x}_p(t_0)\big)$ and the solution to (8.29) is

$$\boldsymbol{x}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}(t_0)^{-1}\big(\boldsymbol{x}_0 - \boldsymbol{x}_p(t_0)\big) + \boldsymbol{x}_p(t)\,.$$

To get some insight of solving (8.29), let us first assume that $\boldsymbol{P}(t) = \boldsymbol{A}$ is a time-independent matrix. In such a case,

$$e^{-t\boldsymbol{A}}\boldsymbol{x}' = e^{-t\boldsymbol{A}}\big(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{g}(t)\big) \quad \text{or} \quad e^{-t\boldsymbol{A}}(\boldsymbol{x}' - \boldsymbol{A}\boldsymbol{x}) = e^{-t\boldsymbol{A}}\boldsymbol{g}(t)\,.$$

Since $\dfrac{d}{dt}e^{-t\boldsymbol{A}} = -\boldsymbol{A}e^{-t\boldsymbol{A}} = -e^{-t\boldsymbol{A}}\boldsymbol{A}$, the equality above implies that

$$\big(e^{-t\boldsymbol{A}}\boldsymbol{x}\big)' = e^{-t\boldsymbol{A}}\boldsymbol{g}(t) \quad \Rightarrow \quad e^{-t\boldsymbol{A}}\boldsymbol{x}(t) - e^{-t_0\boldsymbol{A}}\boldsymbol{x}(t_0) = \int_{t_0}^{t} \boldsymbol{e}^{-s\boldsymbol{A}}\boldsymbol{g}(s)\,ds\,.$$

Therefore, the solution to (8.29) is

$$\boldsymbol{x}(t) = e^{t\boldsymbol{A}}e^{-t_0\boldsymbol{A}}\boldsymbol{x}_0 + \int_{t_0}^{t} e^{t\boldsymbol{A}}e^{-s\boldsymbol{A}}\boldsymbol{g}(s)\,ds\,.$$

Using fundamental matrices $\boldsymbol{\Psi}$ of system $\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x}$, we have the following similar result.

Theorem 8.28. *Let $\boldsymbol{\Psi}(t)$ be a fundamental matrix of system $\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x}$, and $\boldsymbol{\varphi}(t)$ be the solution to the non-homogeneous linear system*

$$\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x} + \boldsymbol{g}(t)\,, \tag{8.30a}$$

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0\,. \tag{8.30b}$$

Then $\boldsymbol{\varphi}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}(t_0)^{-1}\boldsymbol{x}_0 + \int_{t_0}^{t} \boldsymbol{\Psi}(t)\boldsymbol{\Psi}(s)^{-1}\boldsymbol{g}(s)\,ds$*.*

Proof. We directly check that the solution $\boldsymbol{\varphi}$ given above satisfies (8.30). It holds trivially that $\boldsymbol{\varphi}(t_0) = \boldsymbol{x}_0$, so it suffices to show the validity of (8.30a) with $\boldsymbol{\varphi}$ replacing $\boldsymbol{x}$.

Differentiating $\boldsymbol{\varphi}$ and using (8.25), we find that

$$\begin{aligned}
\boldsymbol{\varphi}'(t) &= \boldsymbol{\Psi}'(t)\boldsymbol{\Psi}(t_0)^{-1}\boldsymbol{x}_0 + \boldsymbol{\Psi}(t)\boldsymbol{\Psi}(t)^{-1}\boldsymbol{g}(t) + \int_{t_0}^{t} \boldsymbol{\Psi}'(t)\boldsymbol{\Psi}(s)^{-1}\boldsymbol{g}(s)\,ds \\
&= \boldsymbol{\Psi}'(t)\boldsymbol{\Psi}(t)^{-1}\Big(\boldsymbol{\Psi}(t)\boldsymbol{\Psi}(t_0)^{-1}\boldsymbol{x}_0 + \int_{t_0}^{t} \boldsymbol{\Psi}(t)\boldsymbol{\Psi}(s)^{-1}\boldsymbol{g}(s)\,ds\Big) + \boldsymbol{g}(t) \\
&= \boldsymbol{P}(t)\boldsymbol{\varphi}(t) + \boldsymbol{g}(t)
\end{aligned}$$

which shows that $\boldsymbol{\varphi}$ satisfies (8.30a). □

• Another point of view - variation of parameters: Let $\boldsymbol{\Psi}$ be a fundamental matrix of $\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x}$. We look for a particular solution to $\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x} + \boldsymbol{g}(t)$. By the method of variation of parameters we can assume that a particular solution can be expressed as

$$\boldsymbol{x}(t) = \boldsymbol{\Psi}(t)\boldsymbol{u}(t)$$

for some vector-valued function $\boldsymbol{u}$. Since $\boldsymbol{x}$ is a solution, we must have

$$\boldsymbol{\Psi}'(t)\boldsymbol{u}(t) + \boldsymbol{\Psi}(t)\boldsymbol{u}'(t) = \boldsymbol{P}(t)\boldsymbol{\Psi}(t)\boldsymbol{u}(t) + \boldsymbol{g}(t)\,.$$

Since $\boldsymbol{\Psi}' = \boldsymbol{P}(t)\boldsymbol{\Psi}$, we obtain that $\boldsymbol{u}$ satisfies

$$\boldsymbol{u}'(t) = \boldsymbol{\Psi}(t)^{-1}\boldsymbol{g}(t)\,. \tag{8.31}$$

Therefore, we can choose $\boldsymbol{u}(t) = \int \boldsymbol{\Psi}(t)^{-1}\boldsymbol{g}(t)\,dt$ and a particular solution to $\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x} + \boldsymbol{g}(t)$ is given by

$$\boldsymbol{x}_p(t) = \boldsymbol{\Psi}(t)\Big(\int \boldsymbol{\Psi}(t)^{-1}\boldsymbol{g}(t)\,dt\Big)\,. \tag{8.32}$$

On the other hand, (8.31) implies that $\boldsymbol{u}(t) = \int_{t_0}^{t} \boldsymbol{\Psi}(s)^{-1}\boldsymbol{g}(s)\,ds + \boldsymbol{u}(t_0)$, where $\boldsymbol{u}(t_0)$ is the value of $\boldsymbol{u}$ at the initial time given by $\boldsymbol{u}(t_0) = \boldsymbol{\Psi}(t_0)^{-1}\boldsymbol{x}(t_0)$; thus the solution to $x' = \boldsymbol{P}(t)\boldsymbol{x} + \boldsymbol{g}(t)$ with initial data $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ is

$$\begin{aligned}
\boldsymbol{x}(t) &= \boldsymbol{\Psi}(t)\Big(\int_{t_0}^{t} \boldsymbol{\Psi}(s)^{-1}\boldsymbol{g}(s)\,ds + \boldsymbol{u}(t_0)\Big) \\
&= \boldsymbol{\Psi}(t)\boldsymbol{\Psi}(t_0)^{-1}\boldsymbol{x}_0 + \int_{t_0}^{t} \boldsymbol{\Psi}(t)\boldsymbol{\Psi}(s)^{-1}\boldsymbol{g}(s)\,ds\,.
\end{aligned}$$

Example 8.29. Let $\boldsymbol{A} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ and $\boldsymbol{g}(t) = \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}$. Find a particular solution of $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{g}(t)$.

We first find the Jordan decomposition of $\boldsymbol{A}$. The characteristic equation of $\boldsymbol{A}$ is $(-2-r)^2 - 1 = 0$ which implies that $\lambda = -1$ and $\lambda = -3$ are eigenvalues of $\boldsymbol{A}$. The corresponding eigenvectors are $(1,1)^{\mathrm{T}}$ and $(1,-1)^{\mathrm{T}}$; thus

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{\mathrm{T}};$$

thus

$$e^{t\boldsymbol{A}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{\mathrm{T}}.$$

The general solution to $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ is

$$\boldsymbol{x}(t) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = C_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

1. To obtain a particular solution, we can use (8.32) and find that

$$\begin{aligned} \boldsymbol{x}_p(t) &= \begin{bmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{bmatrix} \int \begin{bmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{bmatrix}^{-1} \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} dt \\ &= \frac{1}{2} \begin{bmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{bmatrix} \int \begin{bmatrix} e^{t} & e^{t} \\ e^{3t} & -e^{3t} \end{bmatrix} \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} dt \\ &= \frac{1}{2} \begin{bmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{bmatrix} \int \begin{bmatrix} 2 + 3te^{t} \\ 2e^{2t} - 3te^{3t} \end{bmatrix} dt\,. \end{aligned}$$

   Since $\displaystyle\int te^{\lambda t}\,dt = \frac{t}{\lambda}e^{\lambda t} - \frac{1}{\lambda^2}e^{\lambda t}$, we obtain that

$$\boldsymbol{x}_p(t) = \frac{1}{2} \begin{bmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{bmatrix} \begin{bmatrix} 2t + 3(te^{t} - e^{t}) \\ e^{2t} - (te^{3t} - \frac{1}{3}e^{3t}) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2te^{-t} + 3(t-1) + e^{-t} - (t - \frac{1}{3}) \\ 2te^{-t} + 3(t-1) - e^{-t} + (t - \frac{1}{3}) \end{bmatrix}$$

2. Without memorizing the formula (8.32) for a particular solution, we can use the method of variation of parameters by assuming that

$$\boldsymbol{x}_p(t) = C_1(t)e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2(t)e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

   for some scalar functions C_1, C_2. Then the equation $\boldsymbol{x}_p' = \boldsymbol{A}\boldsymbol{x}_p + \boldsymbol{g}(t)$ implies that

$$\begin{aligned} &C_1'(t)e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - C_1(t)e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2'(t)e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} - 3C_2(t)e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= -C_1(t)e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 3C_2(t)e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}. \end{aligned}$$

   As a consequence

$$C_1'(t)e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2'(t)e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}$$

which implies that

$$\begin{bmatrix} C_1'(t) \\ C_2'(t) \end{bmatrix} = \begin{bmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{bmatrix}^{-1} \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}.$$

The computation above (in 1) can be used to conclude that

$$C_1(t) = 2t + 3(te^t - e^t) \qquad \text{and} \qquad C_2(t) = e^{2t} - \left(te^{3t} - \frac{1}{3}e^{3t}\right);$$

thus a particular solution is given by

$$\boldsymbol{x}_p(t) = \left[2t + 3(te^t - e^t)\right]e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left[e^{2t} - \left(te^{3t} - \frac{1}{3}e^{3t}\right)\right]e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$